Smooth Ambiguity Preferences and the Continuous-Time Limit

Ronald Gindrat*†and Jean Lefoll‡

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Abstract

Klibanoff, Marinacci and Mukerji (2005) propose a quite general and elegant model of smooth ambiguity preferences that embeds SEU and maxmin preferences as special cases. Unfortunately, Skiadas (2008) shows that KMM preferences cannot be distinguished from SEU preferences when considering Brownian or Poissonian uncertainty. This drawback suggests that KMM preferences cannot be used to model the ambiguity attitude in a continuous-time setting. This paper proposes an alternative representation of preferences under ambiguity that overcomes the problem highlighted by Skiadas (2008) while preserving the advantages of KMM preferences. The proposed preferences are shown to also have SEU and maxmin preferences as special cases.

1 Introduction

The fundamental assumption of subjective expected utility (SEU henceforth) is that agents with an expected utility representation act as if they can attach probabilities to all relevant states with absolute confidence (Savage, 1954; Anscombe and Aumann, 1963). Therefore, ambiguity about the evaluation of a prospect is indistinguishable from the risk inherent in that prospect[1]. In other words, the SEU axioms imply that the probability

*University of Geneva and Swiss Finance Institute. E-mail: ronald.gindrat@unige.ch. Address: Ronald Gindrat, HEC Geneva, University of Geneva, 40 Bd. du Pont d'Arve, CH-1211 Geneva, Switzerland. Phone: +4122/3798804. Fax: +4122/3798104.

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‡University of Geneva and Swiss Finance Institute. E-mail: jean.lefoll@unige.ch. Address: Jean Lefoll, HEC Geneva, University of Geneva, 40 Bd. du Pont d'Arve, CH-1211 Geneva, Switzerland. Phone: +4122/3798133. Fax: +4122/3798104.

[1]Ambiguity and risk together characterize uncertainty.

weighting across possible evaluations (ambiguity) and the probabilities for payoffs (risk) can be collapsed to represent the decision maker's preferences with a single probability measure for the relevant states.

Although SEU allows to set problems in a simple and tractable way, this comes at a cost: The confusion of ambiguity with risk sources leads to discrepancies between predictions implied by SEU and actual behavior. Emblematic are the results in Ellsberg (1961) who shows that the amount and quality of the information a decision maker has about the relevant events affects his preferences. In particular, he observes that decision makers display a dislike for ambiguous situations, they prefer to choose when probabilities are known and are willing to pay in order to avoid choosing in an ambiguous context. As a consequence, individuals can be seen as displaying ambiguity aversion, just as they display risk aversion.

In the spirit of Ellsberg's results, many decision theory models of ambiguity have been put forward. The most prominent are the multiple priors maxmin expected utility (MEU) proposed by Gilboa and Schmeidler (1989) and the Choquet expected utility (CEU) model of Schmeidler (1989). The former allows the decision maker's beliefs to be represented by multiple probabilities, and represents his preferences by the maxmin on the set of the expected utilities. The non-singleton nature of the priors set reflects the limited information available (ambiguity) to the decision maker, which may not be enough detailed to summarize his beliefs in a single probability measure. The "min" represents the decision maker's (extreme) aversion to ambiguity. The inter-temporal versions of MEU preferences are proposed and analyzed in Epstein and Wang (1994) for the discrete-time case and in Chen and Epstein (2002) for the continuous-time case. An extension of MEU to weighted MEU (α-MEU) is provided in Ghirardato, Maccheroni and Marinacci (2004). On the other hand, CEU models the decision maker's beliefs on the state space as non-additive probabilities (i.e. capacities) and his preferences by Choquet integrals. Reference is made for instance to Ghirardato and LeBreton (2000) or Zhang (2002) for investigations on Choquet rationality and Bassett, Koenker and Kordas (2004) for application of Choquet rationality to portfolio selection.

More recently, to account for ambiguity and aversion to ambiguity, researchers in economics and finance have used robust control theory whose application to economics was pioneered by Hansen and Sargent (and coauthors). In the robust control approach, the decision maker problem is seen as the interplay of two agents with opposed interests. On one hand, using a reference model, a benevolent agent tries to maximize the satisfaction of the decision maker by selecting the best act among a set of accessible acts. On the other hand, a malevolent nature chooses the probability measure that, associated with the act chosen by the benevolent agent, minimizes the satisfaction of the decision maker. The malevolent nature cannot, however, choose any possible probability measure. Her choice is restricted by a convex cost function that makes deviations from the reference model of the decision maker less attractive for her, the more confident about the correctness of the reference model the decision maker is. Entropic preferences have been recently axiomatized by Maccheroni, Marinacci and Rustichini (2006).

Also MEU was successfully used to explain financial market phenomena in disagreement with the predictions of models based on (S)EU preferences. For instance, Epstein and Miao (2003) use MEU preferences to explain the home bias regarding investment decisions, Routledge and Zin (2004) show that ambiguity aversion can lead to situations with lack of liquidity and trade on a market, Cao et al. (2005) show that limited participation (under-diversification) may arise in an equilibrium with heterogeneous ambiguity averse agents in the presence of ambiguity, Epstein and Schneider (2008) and Illeditsch (2008) show that MEU preferences can generate excess volatility, negative skewness and excess kurtosis in returns moments. In order to obtain the desired results, these contributions exploit the presence of a kink in the MEU preferences as documented in Dow and Werlang (1992). This non-differentiable point generates a so called first-order risk aversion.

However, whereas the kink in MEU preferences can lead to interesting results, it should not be a necessary feature of ambiguity averse preferences. To this end, Klibanoff, Marinacci and Mukerji (2005) (KMM henceforth) propose a smooth representation of preferences under ambiguity. Whereas ambiguity is still modeled by a set of priors, called Δ, their basic idea is to model the attitude toward ambiguity with a function $\phi = v \circ u^{-1}$ operating on the set of expected utilities of an *ambiguous* act generated by all probability measures in Δ, with v being a *second-order* utility function and u the classic VNM utility function characterizing the risk attitude. Beside smoothness, their representation offers other advantages. First, in contrast to MEU and CEU, there is a full separation between ambiguity and ambiguity aversion. Second, their representation allows to use the same well-developed tools that are applied to the analysis of decision makers' risk attitude to quantify ambiguity aversion. Third, risk and ambiguity premia are distinctly determined. Unfortunately, as pointed out by Skiadas (2008), KMM preferences have the drawback that, if Brownian or Poissonian uncertainty is considered, they generate an ambiguity premium which, compared to the risk premium, is of second-order importance only, so that KMM preferences cannot be distinguished from SEU preferences. This also suggests that KMM preferences cannot be used in a continuous-time setting to gain new insights. The problem highlighted by Skiadas (2008) is given by the fact that KMM preferences model the attitude toward ambiguity on the dispersion of expected utilities. As a consequence, the implied ambiguity premium will depend on the variance of these expected utilities: A negligible term in a continuous-time setting.

In our opinion this drawback stems from the fact that KMM (2005) treat simultaneously the phase of assessment of the distribution (over outcomes) of the ambiguous act and the prospect evaluation phase[2]. In order to overcome the problem, we suggest that the assessment of the distribution (over outcomes) of the ambiguous act should be treaded before the evaluation phase. We assume that the decision maker is able to extrapolate the source of ambiguity from the context and then, once he has resolved the ambiguity problem by selecting a single distribution over outcomes that suits his ambiguity preferences, he evaluates the prospect taking into account the context (e.g. his current wealth, the time elapsing to payoff realization) and his risk attitude; this even in continuous-time if considered as the limit of the discrete-time case.

This way of reasoning seems to us more natural and there is also neuroscientifical evidence in this sense. Hsu, Bhatt, Adolphs, Tranel and Camerer (2005) show with functional magnetic resonance imaging (fMRI) that regions of the brain more sensitive to ambiguity (amygdala and orbitofrontal cortex, OFC) react rapidly and activity in regions related to reward anticipation builds up slowly and peaks significantly later than that of the amygdala and the OFC.

The purpose of this paper is therefore to propose an alternative representation of preferences under ambiguity that overcomes the problem raised by Skiadas (2008) while trying to preserve the most of the advantages and the most of the intuition of KMM preferences. To this end, whereas KMM model ambiguity aversion on the set of expected utilities associated with an ambiguous act, called f, we propose to model the decision maker's attitude toward ambiguity directly on the ambiguous act f. For a single period choice, the proposed preferences representation takes the following form

$$V(f) = \mathbb{E}\left[u\left(\ell_{re}^{f}\right)\right], \tag{1}$$

where f is an ambiguous act, u is a von Neumann-Morgenstern (VNM henceforth) utility index and ℓ_{re}^{f} is what we call a *risk-equivalent* lottery belonging to the set of merely risky acts. The attitude toward ambiguity is taken into account in the distribution ℓ_{re}^{f} under which the expected utility of the act f is calculated. Because our main interest is the Brownian setting, we present the case where there is only ambiguity about the first

[2]In their model, the ambiguity about the distribution over outcomes of an act is taken into account through the possible certainty equivalents (evaluation phase) of the ambiguous act (refer to Definition 1 and 2, pp. 1854, 1857, regarding second-order acts and Assumption 2, p. 1855, regarding the evaluation of second-order acts).

moment of the distributions over outcomes of the ambiguous acts since, in such an environment, ambiguity can only concern the first moment of the prospected payoffs. It is also shown that the proposed preferences representation embeds SEU and MEU preferences as special cases.

Finally, we use the expression *single period choice* to denote a decision taking into account *today*, denoted by time t, and *tomorrow*, denoted by $t + dt$ in a continuous-time setting or $t + 1$ is a discrete-time setting. Single period choices may be analyzed also in a dynamic environment by focusing on a single time interval and the information set *today* and neglecting what might happen after *tomorrow*. In contrast, *inter-temporal* choices take into account a sequence of time intervals. The length of the time interval is dt in the continuous-time case.

The remainder of this paper is as follows. The next section reviews the problem highlighted by Skiadas (2008) both in the single period and with recursive KMM preferences. Section 3 introduces the proposed preferences representation in a single period choice setting. An inter-temporal version, and its continuous-time limit, is outlined in section 4. Section 5 relates the inter-temporal preferences representation to the generalized stochastic differential utility of Lazrak and Quenez (2003). Section 6 concludes.

2 KMM preferences and the continuous-time limit

This section provides a brief review of the problem highlighted by Skiadas (2008). Namely, that the ambiguity premium implied in KMM preferences vanishes when Brownian or Poissonian uncertainty are considered. Only Brownian uncertainty is considered here. It has to emphasized here that the problem highlighted by Skiadas (2008) does not extend to all small risks. For small risks à la Pratt (discrete-time), KMM preferences generate a *non*-negligible ambiguity premium that is mainly driven by the dispersion of the means of the plausible payoff distributions. Chen, Ju and Miao (2008) and Ju and Miao (2008), for instance, apply KMM preferences in such a context. In a continuous-time setting, however, the notion of "local" is confounded with the notion of "instantaneous". Consequently, *all* (instantaneous) moments are constrained to be "small" and the dispersions thereof are constrained to be negligibly "small".

2.1 Single period choice

The KMM approach is presented in a setting called single period choice, where the decision maker evaluates at time t a payoff prospect X at time $t + dt$. The payoff prospect is modeled as a diffusion process, providing the decision maker with an information set $\mathcal{F}_t$ at time t which restricts the anticipations of the decision maker as the possible payoff realizations.

The smooth ambiguity preferences proposed by KMM (2005) have the following form

$$V(f) = \int_{\Delta} \phi \left(\int_{S} u(f) d\pi \right) d\psi = \mathbb{E}_{\psi} \left[\phi \left(\mathbb{E}_{\pi} \left[u \left(f \right) \right] \right) \right], \tag{2}$$

where Δ represents a set of countably additive probability measures π, ψ is a *second-order* countably additive probability measure that assigns weights (beliefs) to the different probability measures $\pi \in \Delta$, f is an ambiguous act, u is a VNM utility function, and $\phi = v \circ u^{-1}$ is a function operating on the set of expected utilities and characterizing ambiguity aversion, with v being a *second-order* utility function. Note that SEU preferences are embedded in KMM preferences: If ϕ is linear ψ and π's can be combined in a single probability measure. In addition, KMM (2005) show that MEU preferences are a special case of KMM preferences when ambiguity aversion (the concavity of ϕ) tends to infinity[3].

Consider now a probability space $(\Omega, \mathcal{F}, P)$ and a standard, one-dimensional Brownian motion $B = (B_t)$ defined on $(\Omega, \mathcal{F}, P)$. The Brownian filtration $(\mathcal{F}_t)$ is generated by the realizations of $\{B_s\}_{0 \leq s \leq t}$ and the P-null sets of $\mathcal{F}$. Note that P is neither the true, objective measure, nor the subjective measure used by the decision maker. It has just the function of determining what is possible and what is not, i.e. defining the null sets.

Consider also a diffusion process $X = (X_t)$ that is adapted to $(\mathcal{F}_t)$, with $dX_t = \mu dt + \sigma dBt$ under P, where μ and σ are finite constants. Given the "history" $\mathcal{F}_t$, consider a decision maker facing, at t, a prospect that pays off a monetary amount $X_{t+dt} = x_t + dX_t = x_t + \mu dt + \sigma dB_t$, at $t + dt$, where $x_t = X(\mathcal{F}_t)$[4].

Following Chen and Epstein (2002), ambiguity in such an environment is about whether B is a standard Brownian motion. The decision maker considers the prospect under several probability measures, denoted by Q^{θ_i} with $i = 1, ..., k$, that are plausibly describing the history represented by $\mathcal{F}_t$ and are equivalent to P in the sense that they agree with P on what is possible and impossible almost surely. The set comprising these measures is denoted by Δ and P belongs to it. Assuming that all measures in Δ are mutually absolutely continuous with respect to P, they can be defined using their densities which, in turn, are defined by density generators $\theta_i = (\theta_{i,t})$ such that $Z_t^{\theta_i}$ is a P-martingale $dZ_t^{\theta_i} = -Z_t^{\theta_i} \theta_{i,t} dB_t$ with $\left. \frac{dQ^{\theta_i}}{dP} \right|_{\mathcal{F}_t} \equiv Z_t^{\theta_i}$. For the sake of simplicity $\theta_{i,t}$ is considered to be constant in time for $i = 1, ..., k$.

[3] Refer to KMM(2005), Proposition 3 and Appendix A.3.

[4] x_t can be interpreted as the current wealth of the decision maker at t and X_{t+dt} as the future wealth at $t + dt$.

Of course, P makes the process B a standard Brownian motion. Therefore, there is ambiguity whether B is a Q^{θ_i}-Brownian motion, $i = 1, ..., k$. Specifically Girsanov's theorem implies that for Q^{θ_i}, $dB_t^i = dB_t + \theta_i dt$ is a Q^{θ_i}-Brownian motion. Consequently, ambiguity concerns exclusively the drift of dX_t. This is due to the Brownian environment and the assumption of absolute continuity.

Considering the measures in Δ and Girsanov's theorem, the decision maker considers k potential distributions for dX_t. Specifically, $dX_t^i \sim N(\mu_i dt, \sigma^2 dt)$ where $\mu_i = \mu - \sigma\theta_i$ for $i = 1, ..., k$. The decision maker assigns weights $\psi = (\psi_1, .., \psi_k)$ to the elements in Δ or, equivalently, to the k plausible distributions of dX_t.

Having defined the setting in which the decision maker takes his decisions, we can verify the point made by Skiadas (2008) that, when the anticipation interval is converging to zero, the KMM certainty equivalent reduces to an expected utility certainty equivalent implying that ambiguity can be disregarded.

The certainty equivalent of KMM preferences is defined as follows

$$CE_{KMM} = u^{-1}\left(\phi^{-1}\left(\mathbb{E}_\psi\left[\phi\left(\mathbb{E}_i\left[u\left(X_{t+dt}\right)\right]\right)\right]\right)\right),$$

or, since $\phi = v \circ u^{-1}$,

$$CE_{KMM} = v^{-1}\left(\mathbb{E}_\psi\left[v\left(CE_X\left(i\right)\right)\right]\right),$$

where $CE_X(i)$ is the certainty equivalent of X_{t+dt} computed with respect to the distribution of dX_t induced by $Q^{\theta_i} \in \Delta$

$$CE_X(i) \approx X_t + \mu_i dt - \frac{1}{2}A^u\left(X_t + \mu_i dt\right)\sigma^2 dt, \quad A^u(\cdot) \equiv -\frac{u''(\cdot)}{u(\cdot)}.$$

We can therefore compute the KMM certainty equivalent

$$CE_{KMM} \approx \mathbb{E}_\psi\left[CE_X(i)\right] - \frac{1}{2}A^v\left(\mathbb{E}_\psi\left[CE_X(i)\right]\right)\mathbb{E}_\psi\left[\left(CE_X(i) - \mathbb{E}_\psi\left[CE_X(i)\right]\right)^2\right], \quad A^v(\cdot) \equiv -\frac{v''(\cdot)}{v'(\cdot)}.$$

Proposition 1 [KMM certainty equivalent]. By letting $\mathbb{E}_\psi\left[CE_X(i)\right] = \overline{CE_X}$ and $\mathbb{E}_\psi[\mu_i] = \overline{\mu}$, we have that

$$\begin{aligned} CE_{KMM} \approx\ & X_t + \overline{\mu}dt - \frac{1}{2}\mathbb{E}_\psi\left[A^u\left(X_t + \mu_i dt\right)\right]\sigma^2 dt \\ & - \frac{1}{2}A^v\left(\overline{CE_X}\right)\mathbb{E}_\psi\left[\left(\mu_i - \overline{\mu} - \frac{1}{2}\left(A^u\left(X_t + \mu_i dt\right) - \mathbb{E}_\psi\left[A^u\left(X_t + \mu_i dt\right)\right]\right)\sigma^2\right)^2\right] dt^2. \end{aligned}$$

Proof. Omitted.

As it can be noted, the ambiguity premium is a dt^2 term and therefore it is a negligible term as compared to the risk premium which is a dt term.

The intuition is as follows. KMM assume that for every ambiguous act f there exists a corresponding second-order act f^2 mapping the set of priors, Δ, into the set of consequences. According to Definition 2, p. 1857, KMM (2005), the consequences associated to the second-order act, f^2, correspond to the certainty equivalents of the objective lotteries that are associated to the ambiguous act f. In a Brownian setting, the certainty equivalents ($CE_X(i)$) are functions not only of the moments but also of dt. In particular, the ambiguous moment of dX_t, μ, is multiplied by dt. Consequently, since dt is infinitesimal, the certainty equivalents, $CE_X(i)$'s, are so close to each other that the second-order act associated to X_{t+dt} is an *almost* constant act and the decision maker considers the degree of ambiguity in the lottery X_{t+dt}, i.e. the dispersion of the $CE_X(i)$'s, as negligible.

2.2 Inter-temporal choice

To further emphasize the isomorphism between KMM preferences and SEU in a continuous-time setting, it is shown that the stochastic differential utility (SDU henceforth) implied by KMM preferences is equivalent to that of SEU preferences. According to KMM (2009), the recursive form of their smooth ambiguity preferences is given by

$$V_t(f) = u(f_t) + \beta\phi^{-1}\left(\mathbb{E}_{\psi,t}\left[\phi\left(\mathbb{E}_{\pi,t}\left[V_{t+1}\right]\right)\right]\right), \tag{3}$$

where β is a discount factor, f is now an ambiguous *plan* with payoff f_t at time t, and the rest has the same interpretation as for equation (2).

As primitives consider the Brownian environment and the decision maker with multiple priors described in the preceding subsection[5]. Additionally, let the time set be $\mathcal{T} = [0, T]$, with $T < \infty$. Consider now an ambiguous payoff plan $C = (C_t)$ taking values in $\mathbb{R}$. The payoff process C is progressively measurable with respect to $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and squared integrable. The SDU for KMM preferences considering the process C is provided in the following Proposition.

Proposition 2 [KMM SDU]. Assume that the SDU follows the generic law of motion $dV_t^C = \mu_t^{V^C}dt + \sigma_t^{V^C}dB_t$ under the probability measure P where $\mu_t^{V^C}$ and $\sigma_t^{V^C}$ are suitable, adapted processes. Then, for the ambiguous payoff plan C, the KMM SDU is given by

$$dV_t^C \approx \left(-u\left(C_t\right) + \delta V_t^C\right)dt + \overline{\theta}\sigma_t^{V^C}dt + \sigma_t^{V^C}dB_t, \quad \mathbb{E}_\psi[\theta_i] = \overline{\theta}, \quad V_T^C = 0. \tag{4}$$

[5]We abstract from restrictions on the set of priors others than those described in the preceding section. We are aware that, in an inter-temporal setting, learning may impose additional restrictions on Δ.

PROOF. Refer to Appendix A.1.

In other words, assuming that the ambiguous plan follows the dynamics $dC_t = \mu_t^C dt + \sigma_t^C dB_t$ under the probability measure P, the result in Proposition 2 suggests that a KMM decision maker cannot be distinguished from a SEU decision maker that considers the process $dC_t = \left(\mu_t^C - \overline{\theta}\sigma_t^C\right) dt + \sigma_t^C dB_t^{\overline{\theta}}$ with $\mathbb{E}_\psi[\theta_i] = \overline{\theta}$, as describing the likely dynamics of C. This is because the function $\phi(\cdot)$, characterizing the ambiguity attitude of the KMM decision maker, does not play any role in the KMM SDU. Consequently, in a continuous-time environment, inter-temporal KMM preferences cannot provide any additional insights compared to SEU preferences. Given the result in Section 2.1, this conclusion is not surprising since KMM model their inter-temporal preferences as a sequence of single period choices (quasi-myopic preferences).

3 Risk-equivalent preferences: Single period choice

An alternative representation of preferences under ambiguity is introduced in this section. This preferences representation is mainly based on three assumptions that will be elicited in subsection 3.2. The main idea is to separate in two stages the evaluation of ambiguous acts. In the first stage, the decision maker deducts the possible distributions over consequences that the ambiguous act may be associated to and, by focusing on the source of ambiguity, he selects the *risk-equivalent* distribution over consequences that is compatible with his ambiguity attitude. In the second stage, using the determined *risk-equivalent* distribution and VNM's expected utility, the decision maker evaluates the ambiguous act according to his risk attitude. Of course, for merely risky acts, only the second stage applies.

In other words, the decision maker expresses two kind of preferences. On one hand, as usual, he states his preferences over acts through a VNM index u. On the other hand, he also states preferences over the set of expected values of ambiguous acts through a VNM index ν. While the former has the usual interpretation in terms of risk attitude, the latter has the function of describing the decision maker's ambiguity attitude. Specifically, using standard tools, it is possible to determine the amount, in terms of expected value, that the decision maker is ready to give up to exchange the ambiguous situation with a *just* risky situation. Consequently, it is possible to derive the associated *risk-equivalent* distribution over consequences. The role of this distribution is to aggregate probabilistic information about the ambiguous act with the tastes of the decision maker over ambiguity, so that, in a second step, the utility associated with the ambiguous act can be computed with this

single, risk-equivalent distribution in mind.
Note that the proposed preferences focus on ambiguity about the first moment. Ambiguity on the second moment is not considered here since in a diffusion setting with ambiguity, ambiguity is exclusively associated to the drift of any prospect.
While the formulation of preferences directly on probabilistic characteristics of the distribution of the ambiguous act is unusual, it has an intuitive appeal. By doing so, the decision maker displays a general preference for distributions yielding high expected values compared to others associated with low expected values. However, the decision maker may display decreasing marginal satisfaction in the expected value, so that he would feel great if the expectation value were large, but he would be glad to exchange some of that large expected value to avoid facing ambiguity and, therefore, potentially facing a low expected value which he deeply dislikes.

3.1 Preliminaries

We consider the same setting as KMM (2005). States of nature and events are represented by the pair $(\mathcal{S}, \Sigma_{\mathcal{S}})$, where $\mathcal{S} = \Omega \times (0,1]$ and $\Sigma_{\mathcal{S}} = \mathcal{A} \otimes \mathcal{B}_1$. $\mathcal{A}$ and $\mathcal{B}_1$ are the σ-algebras on Ω and $(0,1]$, respectively.
An act is a mapping from $\mathcal{S}$ to the set of consequences $\mathcal{C} \subset \mathbb{R}$ and it is generically denoted by $f : \mathcal{S} \to \mathcal{C}$. The set of all acts is denoted by $\mathcal{F}$.
In this setting a merely risky lottery is defined as an act $\ell \in \mathcal{F}$ that is invariant to events in Ω: $\ell(A_1, B) = \ell(A_2, B)$, for any $A_1, A_2 \in \mathcal{A}$ and $B \subseteq (0,1]$. The set of these particular acts is denoted by $\mathcal{L}^r \subset \mathcal{F}$.
Let Δ be the set of countably additive product probability measures on $\mathcal{S}$. An element in Δ is denoted by $\pi_i : \Sigma_{\mathcal{S}} \to \mathcal{C}$, $i = 1, ..., k$. Elements in Δ are such that, given a Lebesgue measure $\lambda : \mathcal{B}_1 \to [0,1]$, $\pi(A \times B) = \pi(A \times (0,1])\lambda(B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}_1$. Consequently, for lotteries in $\mathcal{L}^r$, $\pi(A \times B) = \lambda(B) = q$, for any $A \in \mathcal{A}$, $B \subseteq (0,1]$ and $q \in [0,1]$. In other words, for any probability $q \in [0,1]$, the KMM (2005) setting allows to construct an act (risky lottery) that yield a consequence with probability q. The set $\mathcal{F}$ is, therefore, populated with both ambiguous acts, generically denoted by $f, g \in \mathcal{F}$ and merely risky lotteries, generically denoted by $\ell \in \mathcal{L}^r \subset \mathcal{F}$.
An act f combined with a probability measure π_i induces a probability distribution π_i^f over the consequences set $\mathcal{C}$. Formally, $\pi_i^f : \Sigma_{\mathcal{C}} \to [0,1]$, where $\Sigma_{\mathcal{C}}$ is the σ-algebra on $\mathcal{C}$ with $\pi_i^f(c) = \pi_i(f^{-1}(c))$, $c \in \Sigma_{\mathcal{C}}$. KMM (2005) show that, given any $f \in \mathcal{F}$ and any $\pi_i \in \Delta$, there exists a lottery $\ell \in \mathcal{L}^r$ that has the same distribution of π_i^f[6], this

[6]Refer to KMM (2005), Lemma 1.

risky lottery is denoted by ℓ_i^f. Accordingly, an act f can be also considered as a set $L^f = \{\ell_1^f, ..., \ell_k^f\} \subset \mathcal{L}^r$ of merely risky lotteries where ℓ_i^f is generated by combining π_i with f, for $i = 1, ..., k$. This makes ambiguous acts similar to Anscombe and Aumann (1963) horse-roulette lotteries.

As in KMM (2005), the decision maker has beliefs as the likelihood of the different probability measures in the set Δ to be the *right* probability measure. These beliefs are represented by a *second-order* countably additive probability measure which is denoted by $\psi : \Sigma_\Delta \to [0, 1]$, where Σ_Δ represents the partition on the set Δ.

Since the focus of this analysis is on the ambiguity about the first moment as implied by the diffusion setting, and in particular by the Girsanov's theorem, we make the following assumption.

Assumption 0. For an ambiguous act $f \in \mathcal{F}$, the (objective) lotteries $\ell_i^f \in L^f$, $i = 1, ..., k$, are all equal in distribution up to a right or left translation such that the distributions differ only in their first moments[7].

For each ambiguous act f, $\widetilde{\mathcal{M}}^f = \left\{\overline{\mu}_1^f, ..., \overline{\mu}_k^f\right\}$ denotes the set comprising the diverging parameter between $\ell_1^f, ..., \ell_k^f$[8]. In other words the set $\widetilde{\mathcal{M}}^f$ represents the source of ambiguity faced by the decision maker when considering the act f. The set $\widetilde{\mathcal{M}}$ collects the sets $\widetilde{\mathcal{M}}^f$ for all $f \in \mathcal{F}$. To better understand the meaning of the set $\widetilde{\mathcal{M}}^f$, let us consider an example.

Example 1. Consider again the ambiguous prospect X_{t+dt} of section 2.1. Under any probability measure $\pi_i \in \Delta$ the prospect X_{t+dt} satisfies

$$X_{t+dt} = x_t + (\mu - \theta_i \sigma)dt + \sigma dB_t^i = x_t + \overline{\mu}_i^{X_{t+dt}} dt + \sigma dB_t^i.$$

In this case $\overline{\mu}_i^{X_{t+dt}}$ represents the ambiguous drift of the monetary payoff dX_t. The set $\widetilde{\mathcal{M}}^{X_{t+dt}}$ is the collection of all $\overline{\mu}_i^{X_{t+dt}}$, $i = 1, ..., k$.

3.2 Behavioral assumptions

This subsection presents the behavioral underpinnings of the proposed representation. The essence of the proposed preferences can be summarized by three assumptions.

Assumption 1 [Expected Utility on lotteries]. There exists a continuous, strictly in-

[7]This is not a particularly restricting assumption since most of the studies in finance and economics using preferences with multiple priors make this assumption.

[8]$\overline{\mu}_i^f$ has to be interpreted as the distribution drift parameter which can be (but not necessarily) the expected gain in the discrete case and the usual process drift in a Brownian setting.

creasing index $u : \mathcal{C} \rightarrow \mathbb{R}$ such that for all lotteries ℓ_i, $\ell_j \in \mathcal{L}^r$

$$\ell_i \succeq \ell_j \Leftrightarrow \mathbb{E}\left[u\left(\ell_i\right)\right] \geq \mathbb{E}\left[u\left(\ell_j\right)\right].$$

In Assumption 1, the set of risky lotteries $\mathcal{L}^r$ is exploited to determine the risk attitude of the decision maker so that he expresses his risk preferences in the standard way. This same assumption is made in KMM (2005).

Assumption 2 [Subjective Expected Utility on $\widetilde{\mathcal{M}}$]. There exists a unique, countably additive, *second-order* probability measure ψ and a continuous, strictly increasing index $\nu : \widetilde{\mathcal{M}} \rightarrow \mathbb{R}$ such that for $f, g \in \mathcal{F}$

$$f \succeq^{\mathcal{M}} g \Leftrightarrow \int_{\Delta} \nu\left(\widetilde{\mu}^f\right) d\psi(\pi) \geq \int_{\Delta} \nu\left(\widetilde{\mu}^g\right) d\psi(\pi).$$

Note that the preference order $\succeq^{\mathcal{M}}$ on the set of ambiguous acts $\mathcal{F} \backslash \mathcal{L}^r$ is just a preliminary preference order. If we asked the decision maker about his preferences at this stage, he would say: "Look, for the moment I have just assessed the ambiguity associated to the ambiguous acts f and g. Given the fact that ambiguity makes me uncomfortable, I would rather prefer f to g. However, this is not my final word. Before stating my ultimate preference order, I have to further consider the riskiness of the prospects. It may be also possible that I will change my mind."
This assumption allows to determine a unique "certainty equivalent" for all sets $\widetilde{\mathcal{M}}^f \in \widetilde{\mathcal{M}}$ and, therefore, the risk-equivalent distribution ℓ^f_{re} for all $f \in \mathcal{F}$.

Definition 1 [Risk-equivalent distribution]. Consider an ambiguous act $f \in \mathcal{F}$. Given the assumed index ν and beliefs ψ on Δ, the risk-equivalent distribution ℓ^f_{re} is the one for which the following condition holds

$$\nu\left(\overline{\mu}^f_{re}\right) = \mathbb{E}_{\psi}\left[\nu\left(\widetilde{\mu}^f\right)\right].$$

The risk-equivalent distribution allows the decision maker to aggregate all his information, beliefs and attitude toward ambiguity in a *single* distribution over outcomes so that the decision maker can now formulate a preference ordering over acts in $\mathcal{F}$ using the risk-equivalent distribution as the *single* probabilistic reference for the ambiguous prospects. With this in mind, we can state our last behavioral assumption.

Assumption 3 [Expected Utility on equivalent lotteries]. The decision maker ranks ambiguous acts according to the expected utility criterion applied to risk-equivalent lotteries

ℓ_{re}. Therefore, for $f, g \in \mathcal{F}$

$$f \succeq g \Leftrightarrow \mathbb{E}\left[u\left(\ell_{re}^{f}\right)\right] \geq \mathbb{E}\left[u\left(\ell_{re}^{g}\right)\right].$$

Note that the preference $\succeq$ represents the ultimate decision of the decision maker, the one, that we can ascertain by observing the actions of the decision maker.

3.3 Characterization of ambiguity aversion

Following the definition of *absolute* ambiguity aversion proposed in Ghirardato and Marinacci (2002)[9], and applied by KMM (2005)[10], the ambiguity aversion is characterized by the properties of the index ν, the function representing preferences over the set $\widetilde{\mathcal{M}}$.
An ambiguity averse decision maker will (weakly) prefer the certainty-equivalent (CE henceforth) of an ambiguous act f calculated under SEU assumptions to the act f itself. In other words, the SEU decision maker is taken as the *ambiguity neutrality* reference.

Definition 2 [Absolute ambiguity aversion]. A decision maker displays (*absolute*) *ambiguity aversion* if, for all $f \in \mathcal{F}$,

$$CE\left(f\right)_{SEU} \succeq f.$$

For a SEU decision maker $CE\left(f\right)_{SEU} \sim f$. Furthermore, he aggregates all the information at his disposal and his beliefs in a single probability measure. Consequently, the SEU decision maker evaluates the utility associated to an act f with the unique probability measure $\overline{\pi}$ such that $\mathbb{E}_{\overline{\pi}}[f] = \mathbb{E}_{\psi}\left[\mathbb{E}_{\pi}\left[f\right]\right]$ for all $f \in \mathcal{F}$. In the proposed preferences representation, this is equivalent to saying that, for an SEU decision maker, the index ν is linear. For strict preference $\succ$, as usual, the index ν must be strictly concave. This is stated in the following Proposition.

Proposition 3 [Absolute ambiguity aversion]. A decision maker with a preference index ν and beliefs ψ on $\pi \in \Delta$ is said to be *averse* to ambiguity, if for all $f \in \mathcal{F}$

$$\nu\left(\mathbb{E}_{\psi}\left[\widetilde{\mu}^{f}\right]\right) > \mathbb{E}_{\psi}\left[\nu\left(\widetilde{\mu}^{f}\right)\right]. \tag{5}$$

Proof. Omitted.

Proposition 3 suggests that in order to display ambiguity aversion, the decision maker

[9]Refer to Ghirardato and Marinacci (2002), Definition 9.
[10]Refer to KMM (2005), Definition 4.

must have a strictly increasing, concave index ν: $\nu'(\cdot)$ captures the decision maker's preferences for rosy scenarios (distributions giving a large expected payoff) and $\nu''(\cdot)$ his discomfort with the possibility of having to face a bad scenario (characterized by a low expected payoff). Using conventional tools, it is possible to quantify this trade-off in terms of expected payoff that the decision maker is willing to give up in order to face *just* risk.

3.4 Uncertainty premium

We have now all the ingredients to analyze the decision maker's reward for bearing uncertainty. The uncertainty premium can be derived by a two-steps procedure. The first aims at determining the risk-equivalent distribution ℓ_{re}. The second derives the uncertainty premium implied by the decision maker's ordering over acts.

PROPOSITION 4 [Risk equivalent distribution]. Assume a decision maker facing an ambiguous act f. Assume further that the decision maker has beliefs ψ over $\pi \in \Delta$. Let the source of ambiguity be represented by the set $\widetilde{\mathcal{M}}^f = \left\{\overline{\mu}_1^f, ..., \overline{\mu}_k^f\right\}$. Furthermore, let $\overline{\mu}^f$ and $\sigma^2_{\widetilde{\mu}^f}$ be the mean and the variance of the ambiguous component measured with respect to the decision maker's beliefs ψ. Then, assuming that $\sigma^2_{\widetilde{\mu}}$ is local, the risk-equivalent distribution ℓ^f_{re} is the one satisfying the condition

$$\overline{\mu}^f_{re} \approx \overline{\mu}^f - \frac{1}{2} A^\nu\left(\overline{\mu}^f\right)\sigma^2_{\widetilde{\mu}^f}, \quad A^\nu\left(\overline{\mu}^f\right) \equiv -\frac{\nu''\left(\overline{\mu}^f\right)}{\nu'\left(\overline{\mu}^f\right)}. \tag{6}$$

PROOF. Omitted.

By analogy to risk theory, A^ν will be called throughout the paper the *coefficient of* (*absolute*) *ambiguity aversion*. Note, furthermore, that the uniqueness of ℓ^f_{re} is ensured by Assumption 0 constraining ambiguity to be only about the first moment. Since we have now a distribution that summarizes all the available information and the decision maker's attitude toward ambiguity, it is possible to determine the uncertainty premium.
Let us first consider a discrete case where $\overline{\mu}_1^f, ..., \overline{\mu}_k^f$ are interpreted as expected gains.

PROPOSITION 5a [Uncertainty premium]. Assume a decision maker facing an ambiguous act f. Then, for a local (small) risk in f, σ^2, the uncertainty premium is given by

$$\widehat{\kappa} \approx \frac{1}{2} A^\nu\left(\overline{\mu}^f\right)\sigma^2_{\widetilde{\mu}^f} + \frac{1}{2} A^u\left(\overline{\mu}^f_{re}\right)\sigma^2, \quad A^u(\overline{\mu}^f_{re}) \equiv -\frac{u''\left(\overline{\mu}^f_{re}\right)}{u'\left(\overline{\mu}^f_{re}\right)}. \tag{7}$$

PROOF. Refer to Appendix A.2.

Equation (7) implies that the decision maker not only requires a reward for bearing risk, σ^2, according to his risk attitude, $A^u(\cdot)$, but he requires also to be rewarded for the poor information quality (or ambiguous information), $\sigma^2_{\widetilde{\mu}}$, related to act f according to his attitude toward ambiguity (or information quality), $A^\nu(\cdot)$.
Note that if the available information on the occurrence of the underlying states is very accurate, $\sigma^2_{\widetilde{\mu}^f}$ tends to zero for all $f \in \mathcal{F}$ and the first term in equation (7), i.e. the ambiguity premium, disappears. This is not because the decision maker is not averse to ambiguity, but rather because there is no ambiguity about the occurrence of the states. In other words, in an environment without ambiguity the actions undertaken by a risk-equivalent decision maker are not distinguishably different from those of a VNM expected utility decision maker.
Let us now turn to the continuous case where $\overline{\mu}^f_1, ..., \overline{\mu}^f_k$ are interpreted as drifts.

Proposition 5b [Uncertainty premium]. Assume a decision maker facing an ambiguous prospect X_{t+dt} as described in Section 2.1. Then the uncertainty premium is given by

$$\widehat{\kappa} \approx \frac{1}{2}A^\nu(\mu - \overline{\theta}\sigma)\sigma^2_\theta\sigma^2 dt + \frac{1}{2}A^u(\mathbb{E}_{re}[X_{t+dt}])\sigma^2 dt, \quad \sigma^2_\theta = \mathbb{E}_\psi[(\theta_i - \overline{\theta})^2]. \tag{8}$$

Proof. Refer to Appendix A.3.

Note that the product $\sigma^2_\theta\sigma^2$ is not necessarily converging to zero as fast as dt and, therefore, the ambiguity premium and the risk premium can be of the same order dt. In our framework, there is no a priori reason that, locally, the risk-equivalent distribution overlaps the true distribution which is unknown. Equation (8) implies that when there is no risk, $\sigma^2 = 0$, both premia disappear because the prospect is deterministic, that is, riskless and unambiguous.

3.5 Comparison of ambiguity attitudes

The following definition specifies the comparative notion of ambiguity aversion in the present context.

Definition 3 [Comparative ambiguity aversion]. Assume two decision makers A and B characterized by $\succeq_A$ and $\succeq_B$. Assume further that the two decision makers share the same set of priors Δ and they additionally share the same beliefs ψ on probability measures in Δ. Then, A is said to be *more* ambiguity averse than B if, for all $f \in \mathcal{F}$,

$$\ell^f_{re,B} \succeq_A \ell^f_{re,A} \quad \text{and} \quad \ell^f_{re,B} \succeq_B \ell^f_{re,A}, \quad \ell^f_{re,A}, \ell^f_{re,B} \in \mathcal{L}^r.$$

Given this definition, the following Proposition determines the notion of comparative ambiguity aversion in cardinal terms.

Proposition 6 [Comparative ambiguity aversion]. Assume two decision makers A and B facing the same ambiguous act f. Then, given Definition 3, saying that A is *more* ambiguity averse than B implies that

$$A_A^{\nu}\left(\overline{\mu}^f\right) \geq A_B^{\nu}\left(\overline{\mu}^f\right) \Leftrightarrow \overline{\mu}_{re,A}^f \leq \overline{\mu}_{re,B}^f.$$

Proof. Omitted.

Given the monotonicity of the utility indexes u_A and u_B the result is straightforward. In a nutshell, Proposition 6 states that the *more* ambiguity averse decision maker will be the one displaying *more* cautiousness when evaluating the utility associated to the ambiguous prospect f.

Note that the comparison of ambiguity across decision makers can be made *independently* of their risk attitudes. This is in contrast to KMM preferences since, in their setting, the function ϕ, characterizing ambiguity aversion, is related to the VNM utility function u, characterizing risk aversion, through the second-order utility function v.

3.6 Extreme attitudes toward ambiguity

This section studies the implications of extreme ambiguity attitudes. Namely, we analyze the situations of neutrality and extreme aversion toward ambiguity. As it will be shown, neutrality corresponds to the classic case of SEU preferences, while extreme ambiguity aversion corresponds to Gilboa and Schmeidler's MEU.

Neutrality toward ambiguity implies a linear index ν. As a consequence, the risk-equivalent probability measure corresponds to the simple aggregation of beliefs ψ with the probability measures in the set Δ and, therefore, his preferences cannot be distinguished from SEU preferences. The following Proposition formalizes the implications of ambiguity neutrality.

Proposition 7 [Neutrality toward ambiguity]. Combining a linear preference index ν on probability measures with Propositions 4 and 5(a,b), the following statements are equivalent:

1. the risk-equivalent distribution ℓ_{re}^f implies that $\overline{\mu}_{re}^f = \overline{\mu}^f$;

2. the required uncertainty premium $\widehat{\kappa}$ corresponds to the risk premium;

3. an ambiguity neutral decision maker cannot be distinguished from a SEU decision maker.

PROOF. Omitted.

On the other hand, in the case of an extreme ambiguity aversion, the decision maker's attitude toward ambiguity is described by an extremely concave index ν. The next proposition shows the isomorphism between MEU preferences and extreme ambiguity averse risk-equivalent preferences.

PROPOSITION 8. [Extreme ambiguity aversion]. Assume that for a decision maker n $A_n^\nu \to \infty$. Then the following statements are equivalent:

1. the risk-equivalent distribution ℓ_{re}^f implies that $\overline{\mu}_{re}^f = \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \widetilde{\mu}_i^f$;
2. the required uncertainty premium $\widehat{\kappa}$ corresponds to $\widehat{\kappa} \approx \overline{\mu}^f - \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f + \frac{1}{2} A^u \left(\overline{\mu}_{re}^f\right) \sigma^2$ (in the discrete case, Proposition 5a) and $\widehat{\kappa} \approx (\overline{\mu}^f - \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f) dt + \frac{1}{2} A^u (\mathbb{E}_{re}[X_{t+dt}]) \sigma^2 dt$ (in the continuous case, Proposition 5b);
3. an extremely ambiguity averse decision maker cannot be distinguished from a MEU decision maker.

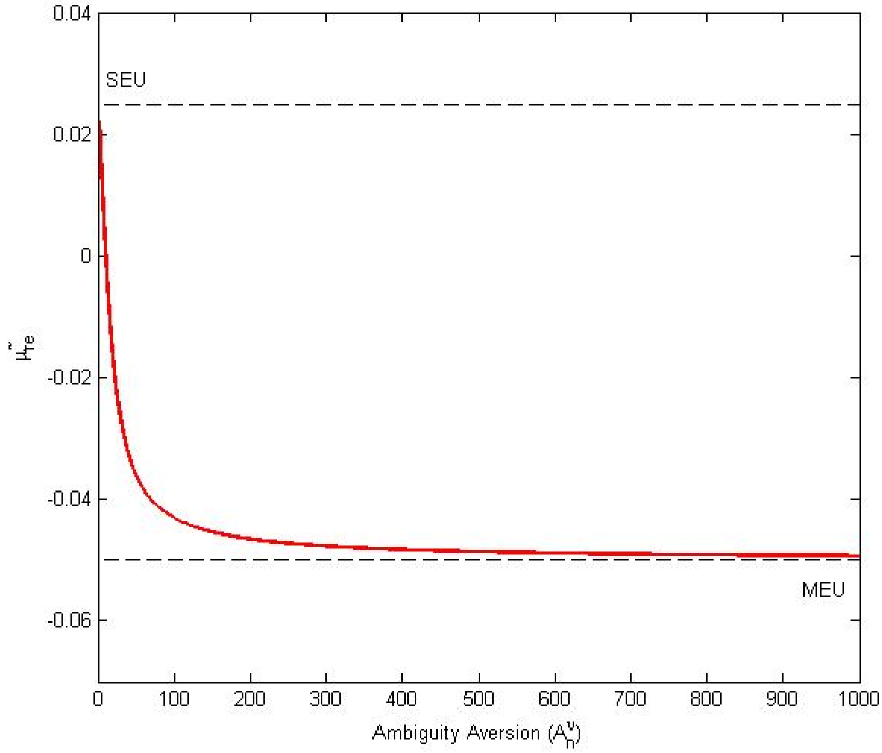
PROOF. Refer to Appendix A.4.

In order to better understand the convergence mechanisms toward SEU and MEU implied by ambiguity neutrality and extreme ambiguity aversion respectively, let us consider the following example.

EXAMPLE 2. Assume a decision maker facing an ambiguous act f. The decision maker ambiguity aversion is denoted by A^ν as defined by equation (5). The decision maker's preferences over probability measures are represented by a negative exponential $\nu(\widetilde{\mu}) = -\exp(-A\widetilde{\mu})$ so that $A^\nu = A$. Assume further that the decision maker's set of prior probability measures contains only two elements $\Delta = [\pi_1, \pi_2]$ such that $\widetilde{\mathcal{M}}^f = \left[\overline{\mu}_1^f, \overline{\mu}_2^f\right]$ with $\overline{\mu}_1^f < \overline{\mu}_2^f$. Finally, the decision maker has beliefs $\psi = [0.5, 0.5]$ on the probability measures π_1 and π_2, that is, he thinks that the right probability measure may be π_1 with probability 0.5 and π_2 with probability 0.5.
Applying Definition 1, we have that the decision maker will consider as risk-equivalent the lottery ℓ_{re}^f that satisfies

$$\overline{\mu}_{re}^f = -\frac{1}{A} \ln \left(0.5 \exp\left(-A\overline{\mu}_1^f\right) + 0.5 \exp\left(-A\overline{\mu}_2^f\right)\right).$$

The following figure plots $\overline{\mu}_{re}^f$ for $\overline{\mu}_1^f = -0.05$ and $\overline{\mu}_2^f = 0.10$ as a function of A.



As it can be noted, for $A \to 0$, the risk-equivalent lottery ℓ_{re}^f tends to the one implied by ambiguity neutrality, that is,

$$\lim_{A \to 0} \overline{\mu}_{re}^f = \overline{\mu}^f = 0.5\overline{\mu}_1^f + 0.5\overline{\mu}_2^f = 0.025.$$

Consequently, according to Proposition 7, the decision maker can be hardly distinguished from a SEU decision maker.
On the other hand, as $A \to \infty$, the decision maker will be almost indistinguishable from a MEU decision maker, that is,

$$\lim_{A \to \infty} \overline{\mu}_{re}^f = \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f = \overline{\mu}_1^f = -0.05.$$

In other words, a decision maker with an extremely concave index ν considers exclusively the most disadvantageous distribution as risk-equivalent. This is in accordance with Proposition 8.

4 Risk-equivalent preferences: Inter-temporal choice

The *recursive* representation of risk-equivalent preferences is now introduced. In order to have a simple, recursive preferences formulation, additional assumptions are needed.

Namely, these assumptions impose the *temporal consistency* of choices, the *irrelevance of the past* and the *invariance* of the shape of u and ν and of the discount factor β across states and trough time. Under these assumptions the proposed preferences have the form

$$V_{s^t}(f) = u(f_{s^t}(s^t)) + \beta \mathbb{E}_{p^{f_{s^t}}_{re(s^t)}}[V_{s^t,x_{t+1}}(f)].$$

4.1 Preliminaries

In an inter-temporal setting, the decision maker's objects of choice are no more acts but rather *plans*. A plan is still a map from the states space to the consequences space. However, the states space describes now all possible paths on an event tree. The states space is modeled as in KMM (2009). Consider an infinite time horizon $\mathcal{T} = \{1, ..., t, ...\}$. Let $\{\mathcal{X}_t\}_{t\in\mathcal{T}}$ be finite observations spaces on which the sequence of random variables $\{X_t\}_{t\in\mathcal{T}}$ is defined. A realization of the random variable X_t is denoted by x_t. The random variables $\{X_t\}_{t\in\mathcal{T}}$ are endowed with their power sets $\mathcal{A}_t = 2^{\mathcal{X}_t}$, $t \in \mathcal{T}$.
A path up to time t is denoted by $s^t = (x_1, ..., x_t)$ and the collection of all finite paths up to time t is given by $S^t = \prod_{\tau=1}^{t} \mathcal{X}_\tau$ and, in general, the set of all possible paths is $S = \prod_{t\in\mathcal{T}} \mathcal{X}_t$. An observation path s^t identifies a node, that is, the history of observations up to time t. The set of all nodes is $\mathcal{S} = \bigcup_{t\in\mathcal{T}} \mathcal{X}_t$.
The product σ-algebra on $\mathcal{S}$ is defined by $\Sigma = \bigotimes_{t\in\mathcal{T}} \mathcal{A}_t$ and Δ represents the set of probability measures defined on Σ. As in the single period choice setting, $\pi_i : \Sigma \to [0, 1]$ denotes a generic element of Δ. Given $B \in \Sigma$ and an observation path s^t, $\pi_i(B \mid s^t)$ represents the probability, under i, that the observation path will belong to B given that node s^t was reached (provided that s^t is a possible node, i.e. $\pi_i(s^t) > 0$). Moreover, for all $t \in \mathcal{T}$, the one-step-ahead probability $\pi_i(\cdot; s^t) : \mathcal{A}_{t+1} \to [0, 1]$ given $s^t \in S^t$ is

$$\pi_i(x_{t+1}; s^t) = \frac{\pi_i(x_1, .., x_t, x_{t+1})}{\pi_i(x_1, .., x_t)}, \text{ for any } x_{t+1} \in \mathcal{X}_{t+1}, \quad \pi_i \in \Delta.$$

$\pi_i(x_{t+1}; s^t)$ is the probability that is assigned, under i, to the observation x_{t+1} given that node s^t was attained.
The decision maker assigns a degree of belief to all measures in Δ. The beliefs $\psi : \Sigma_\Delta \to [0, 1]$ are indexed by the reached observation node s^t, $t \in \mathcal{T}$, such that

$$\psi_{s^t}(\pi_i) = \frac{\pi_i(s^t)\psi(\pi_i)}{\int_\Delta \pi(s^t) d\psi(\pi)}{}^{11}, i = 1, ..., k.$$

[11] We recognize that the equation contains an inconsistency. Given the uncountable nature of the objects in Δ, writing $\psi(\pi_i)$ and $\psi_{s^t}(\pi_i)$ is imprecise. This is done, however, to maintain some unity of notation with Section 3.1.

At time 0 and $t \in \mathcal{T}$, the decision maker chooses a payoff plan. At time t the available information is given by the history of observations s^t. Denoting by $\mathcal{C} \subset \mathbb{R}$ the set of consequences, a plan f is a mapping $f : \{0\} \times \mathcal{T} \times S \to \mathcal{C}$ associating a payoff stream to all possible observation paths. The payoff of plan f at time t is denoted by $f(s^t) \in \mathcal{C}$ and it represents the payoff the decision maker receives if he chooses plan f and the node s^t is reached. f can, therefore, be regarded also as a function $f : \mathcal{S} \to \mathcal{C}$. Payoff plan f is an adapted payoff process meaning that the payoff $f(s^t)$ is measurable with respect to the filtration $\Sigma_t = \sigma(x_1, ..., x_t)$ and $\Sigma_0 = \{\mathcal{S}, \emptyset\}$, for $t \in \{0\} \cup \mathcal{T}$. The set of such plans is denoted by $\mathcal{F}$.

A special class of payoff plans are the deterministic plans. A deterministic plan is generically denoted by d and the set of all deterministic plans by $\mathcal{D}$. Deterministic plans are characterized by the fact that their payoff streams do not depend on the reached node, that is, for each $t \in \mathcal{T}$ the payoff of plan d at t is $d(s^t) = c(t) \in \mathcal{C}$ for any $s^t \in S^t$. Considered are also (objective) randomizations of deterministic plans, called mixed deterministic plans, paying off at time $t \in \mathcal{T}$ $d_i(s^t) = c_i(t)$ with probability p_i where $d_i \in \mathcal{D}$, $i = 1, ..., n$ and $0 \geq p_i \geq 1$, $\sum_{i=1}^n p_i = 1$. The set of mixed deterministic plans is denoted by $\mathcal{P}$. Mixed deterministic plans are the counterpart of the merely risky lotteries in $\mathcal{L}^r$ in the above single period choice setting.

Fixing the node s^t, the set of continuation plans at s^t, denoted by $\mathcal{F}_{s^t}$ comprises plans in $\mathcal{F}$ only at the possible succeeding nodes $\bigcup_{\tau \geq t} S^\tau(s^t)$. A continuation plan is generically denoted by $f_{s^t} : \bigcup_{\tau \geq t} S^\tau(s^t) \to \mathcal{C}$. An important subset of continuation plans are one-step-ahead continuation plans characterized by a constant payoff stream from time $t+1$ onward depending on the realization x_{t+1}, that is, on the node reached at time $t+1$. In other words, for these special plans all the uncertainty is resolved between time t and $t+1$ with the realization of the random variable X_{t+1}. A one-step-ahead continuation plan is generically denoted by $f^*_{s^t}$ and is characterized by the following general payoff structure

$$f^*_{s^t}(s^\tau) = \begin{cases} f_{s^t}(s^t), & \tau = t, \\ f_{s^t}(s^{t+1}), & \tau \geq t+1. \end{cases} \tag{9}$$

The set of such plans is denoted by $\mathcal{F}^*_{s^t} \subset \mathcal{F}_{s^t}$.

For deterministic plans, the set $\mathcal{D}_{s^t}$ represents the set of deterministic continuation plans and a generic deterministic continuation plan is denoted by d_{s^t}. We consider also continuation plans for mixed deterministic payoff plans. A mixed deterministic continuation plan is denoted by $p_{s^t} \in \mathcal{P}_{s^t}$. Note that deterministic and mixed deterministic continuation plans do not depend on the reached node s^t, such continuation plans are indexed with s^t

just for uniformity of notation.

We consider one-step-ahead mixed deterministic continuation plans denoted generically by $p^*_{s^t} \in \mathcal{P}^*_{s^t} \subset \mathcal{P}_{s^t}$. Such plans are randomizations of deterministic plans that agree on the payoff at time t and are characterized by constant payoff streams from $t+1$ onward. The peculiarity of these mixed plans is that their implied one-step-ahead distributions over consequences are the same at every point in time $\tau \geq t$. A one-step-ahead mixed deterministic continuation plan is characterized by the following general payoff structure

$$p^*_{s^t}(s^\tau) = \begin{cases} c(t) = d_{j,s^t}(s^t), \quad d_{j,s^t}(s^t) = d_{k,s^t}(s^t), \quad j = 1, ..., n, \quad k = 1, ..., n, \quad \tau = t, \\ c(\tau) = d_{j,s^t}(s^\tau) \quad \text{with probability } p_j, \quad j = 1, ..., n, \quad \tau \geq t+1. \end{cases}$$

In general, (one-step-ahead) mixed deterministic continuation plans may represent randomizations over uncountably many plans in $\mathcal{D}_{s^t} = \prod_{\tau \geq t} \mathcal{C}$. Consequently, a randomization over deterministic continuation plans $p_{s^t}(i) \in \mathcal{P}_{s^t}$ is characterized by a function $p_i : \Sigma_{\mathcal{D}_{s^t}} \to [0,1]$ where $\Sigma_{\mathcal{D}_{s^t}}$ is the σ-algebra induced by the product topology on $\mathcal{D}_{s^t}$.

Combining $f^*_{s^t}$ with the one-step-ahead probability $\pi_i(x_{t+1}; s^t)$, $\pi_i \in \Delta$, as above defined, determines a one-step-ahead distribution over $\mathcal{C}$ which can be replicated by some one-step-ahead mixed deterministic plan denoted by $p^*_{s^t}(f^*_{s^t}, \pi_i(x_{t+1}; s^t))$ with $c(t) = f_{s^t}(s^t)$. Therefore, $f^*_{s^t}$ can be considered as a set $P^{f^*_{s^t}} = \{p^*_{s^t}(f^*_{s^t}, \pi_1), ..., p^*_{s^t}(f^*_{s^t}, \pi_k)\} \subset \mathcal{P}^*_{s^t}$.

Recalling that we are focusing on ambiguity on the first moment, Assumption 0 is rephrased for the present context.

Assumption 0*. For an ambiguous one-step-ahead continuation plan $f^*_{s^t} \in \mathcal{F}^*_{s^t}$, the associated one-step-ahead distributions implied by the one-step-ahead mixed deterministic continuation plans in $P^{f^*_{s^t}}$ differ only in their drifts.

Example 3. Consider a decision maker facing the ambiguous plan $f^*_{s^t}$. Suppose that the one-step-ahead mixed deterministic continuation plans in $P^{f^*_{s^t}}$ imply the following one-step-ahead distribution over consequences

$$c(t + \Delta t) = c(t) + \overline{\mu}_i^{f^*_{s^t}}(s^t)\Delta t + \sigma(s^t)\widetilde{\epsilon}\Delta t, \quad i = 1, ..., k, \quad c(t) = f_{s^t}(s^t),$$

where $\tilde{\epsilon}$ is a compact white noise.

The set $\widetilde{\mathcal{M}}^{f^*_{s^t}} = \{\overline{\mu}_1^{f^*_{s^t}}(s^t), ..., \overline{\mu}_k^{f^*_{s^t}}(s^t)\}$ collects the $\overline{\mu}_i^{f^*_{s^t}}(s^t)$ implied by all $\pi_i(x_{t+1}; s^t) \in \Delta$. The mean over all $\overline{\mu}_i^{f^*_{s^t}}(s^t) \in \widetilde{\mathcal{M}}^{f^*_{s^t}}$ measured with respect to the decision makers beliefs at node s^t is denoted by $\mathbb{E}_{\psi(s^t)}[\widetilde{\mu}^{f^*_{s^t}}(s^t)] = \overline{\mu}^{f^*_{s^t}}(s^t)$.

4.2 Assumptions

In order to obtain a simple, recursive preferences representation, additional assumptions on the inter-temporal process of decision making are needed. To this end, it seems reasonable to consider the same assumptions made by KMM (2009). While Assumptions 4 and 5 are quite standard for an inter-temporal setting and are not specific to the model of preferences under ambiguity presented above, Assumptions 6 and 7 allow to embed the single period choice preferences representation in the inter-temporal setting. While some of these assumptions may be not necessary in general, they are considered to make the recursive representation as simple as possible and to be able to apply standard tools in dynamic programming.

Assumption 4 [Consequentialism]. In evaluating plans at a node s^t, the decision maker considers only payoffs from that point onward.

This assumption ensures that the decision maker's problem can be approached at the time at which it occurs without considering what happened or did not happen at all precedent nodes $(S^1, ..S^{t-1})$.

Assumption 5 [Dynamic consistency]. Given two ambiguous plans f and g that yield the same payoff today, and, no matter what happens between t and $t+1$, f is always preferred to g at $t+1$, then f is preferred to g also at time t.

Assumption 5 essentially imposes that, if we consider two plans that differ (in terms of payoffs) only from tomorrow onward, no matter what might occur from today to tomorrow, then the plan associated with the highest payoff tomorrow, will not only be preferred tomorrow but already today.

Assumption 6 [Discounting]. Consider the decision maker facing a deterministic plan $d \in \mathcal{D}$ at s^t. Then the utility associated with this plan is represented by the utility index $U_{(s^t)} : \mathcal{C} \to \mathbb{R}$ which has the form

$$U_{s^t}\left(d_{s^t}\right) = \sum_{\tau \geq t} \beta_{s^t}^{\tau - t} u_{s^t}\left(d_{s^t}\left(s^\tau\right)\right).$$

Assumption 7 [Invariance]. There is $\beta \in (0,1)$ and $u : \mathcal{C} \to \mathbb{R}$ continuous and strictly increasing such that, in Assumption 6, $\beta_{s^t} = \beta$ and $u_{s^t} = u$ for all s^t. Additionally, there is $\nu : \widetilde{\mathcal{M}} \to \mathbb{R}$ continuous and strictly increasing such that $\nu_{s^t} = \nu$ for all s^t.

4.3 Representation

Given Assumptions 2, 7 and the above notion of one-step-ahead mixed deterministic plan, we are able to determine the s^t-conditional one-step-ahead risk-equivalent distribution for all $f^*_{s^t} \in \mathcal{F}^*_{s^t}$. By analogy to Definition 1 and adapting Proposition 4, the s^t-conditional one-step-ahead risk-equivalent distribution is defined as follows.

Definition 4. [s^t-conditional risk-equivalent distribution]. Consider an ambiguous act $f^*_{s^t} \in \mathcal{F}^*_{s^t}$. the s^t-conditional risk-equivalent distribution is the one for which the following condition holds

$$\nu\left(\overline{\mu}^{f^*_{s^t}}_{re}(s^t)\right) = \mathbb{E}_{\psi(s^t)}\left[\nu\left(\widetilde{\mu}^{f^*_{s^t}}(s^t)\right)\right].$$

The one-step-ahead mixed deterministic plan implying the s^t-conditional one-step-ahead risk-equivalent distribution for $f^*_{s^t}$ is denoted by $p^*_{s^t}(f^*_{s^t}, re(s^t))$.

Given Assumptions 3, 4, 6 and 7, the continuation value $V_{s^t}(\cdot)$ representing $\succeq_{s^t}$ on $\mathcal{F}_{s^t} \cup \mathcal{P}_{s^t} \cup \mathcal{D}_{s^t}$ associated to $f^*_{s^t}$ is given by

$$V_{s^t}(f^*_{s^t}) = \int_{\mathcal{D}_{s^t}} \left(\sum_{\tau \geq t} \beta^{\tau - t} u(d_{s^t}(s^\tau))\right) dp^{f^*_{s^t}}_{re(s^t)}(d_{s^t}), \tag{10}$$

$$= u(f^*_{s^t}(s^t)) + \beta \frac{\mathbb{E}_{p^{f^*_{s^t}}_{re(s^t)}}\left[u(d_{s^t}(s^{t+1}))\right]}{1-\beta}, \tag{11}$$

where the second equation uses the payoff structure of $f^*_{s^t}$ detailed in equation (9). Given the generality of the payoff structure in equation (9), the result in equation (11) holds for all plans in $\mathcal{F}^*_{s^t}$.

To find the recursive preference representation, we need finally the notion of continuation certainty equivalent.

Definition 5. Given $f \in \mathcal{F}$ and $s^t \in S^t$, the continuation certainty equivalent $\vec{c}_{f,s^t} \in \mathcal{D}_{s^t}$ of f at s^t is a constant payoff stream with $c_{f,s^t} \in \mathcal{C}$ such that $\vec{c}_{f,s^t} \sim_{s^t} f_{s^t}$.

Combining Assumptions 6 and 7 with Definition 5, the continuation value of f at s^t satisfies

$$V_{s^t}(f) = U_{s^t}(\vec{c}_{f,s^t}) = \frac{u(c_{f,s^t})}{1-\beta}.$$

We can now state recursive preference representation.

Proposition 9. Given Assumptions 0*-7, Definitions 4 and 5, the inter-temporal risk-equivalent preferences representation has the following form

$$V_{s^t}(f) = u(f_{s^t}(s^t)) + \beta \mathbb{E}_{p^{f_{s^t}}_{re(s^t)}} [V_{s^{t+1}}(f)]. \tag{12}$$

Proof. Refer to Appendix A.5.

4.4 Continuous-time limit

In this section, the continuous-time version of the recursive utility representation in equation (12) is presented. To this end the setting in Section 4.1 has to be specialized. The finite observation space is now assumed to be the same over time $\{\mathcal{X}_t\}_{t\in\mathcal{T}} = \mathcal{X}$ and it comprises only two elements, $\mathcal{X} = \{H, T\}$. Moreover, let $\pi_i \in \Delta$ be such that the random variables $\{X_t\}_{t\in\mathcal{T}}$ are i.i.d., that is,

$$\pi_i(x_{t+1}; s^t) = \frac{\pi_i(x_1, ..., x_{t+1})}{\pi_i(x_1, ..., x_t)} = \frac{\prod_{\tau=1}^{t+1} q_i(x_\tau)}{\prod_{\tau=1}^{t} q_i(x_\tau)} = q_i(x_{t+1}), \quad x_{t+1} = H, T,$$

where $q_i(\cdot) = \mathcal{A}_t \to [0, 1]$, $t \in \{0\} \cup \mathcal{T}$, is the marginal distribution associated to π_i.
Let us define the random variable $Y_t : \mathcal{X} \to \{-\sqrt{\Delta t}, \sqrt{\Delta t}\}$, $t \in \mathcal{T}$, where Δt represents the (discrete) time increment, such that

$$Y_t = \begin{cases} \sqrt{\Delta t} \text{ if } x_t = H, \\ -\sqrt{\Delta t} \text{ if } x_t = T. \end{cases}$$

Consider also $M_t = \sum_{\tau=1}^{t} Y_\tau$ with $M_0 = 0$. We assume that there exists a probability measure in Δ, denoted by $\hat{\pi}$ such that stochastic process M_t is a symmetric random walk. In other words, we assume that there exists $\hat{\pi} \in \Delta$ such that $q_{\hat{\pi}}(H) = q_{\hat{\pi}}(T) = \frac{1}{2}$.
By letting Δt tend to the infinitesimal time increment dt, we can approach the continuous-time limit and by the Central Limit Theorem the $\hat{\pi}$-random walk becomes a $\hat{\pi}$-standard Brownian motion $B = (B_t)$ defined on $(\mathcal{S}, \Sigma, \hat{\pi})$. Note that $\hat{\pi}$ is neither the objective measure nor the subjective measure used by the decision maker, it has the function, beside that of making B a standard Brownian motion, of determining null sets.
As in Section 2.1, we assume that measures in Δ are absolutely continuous with respect to $\hat{\pi}$ and, therefore, they can be defined by density generators $\theta_i = (\theta_{i,t})$ such that $Z_t^{\theta_i} = \frac{d\pi_i}{d\hat{\pi}}\big|_{\Sigma_t}$ is a $\hat{\pi}$-martingale. For simplicity, we assume that θ_i is constant for all $\pi_i \in \Delta$.
Consider now a square-integrable payoff process (plan) $C = (C_t) \in \mathcal{F}$ taking values in $\mathcal{C}$. Assuming that C is a Σ_t-adapted process that evolves according to the SDE $dC_t =$

$\mu_t^C dt + \sigma_t^C dB_t$ under $\hat{\pi}$ and $dC_t = (\mu_t^C - \theta_i \sigma_t^C)dt + \sigma_t^C dB_t$ under π_i, with μ_t^C and σ_t^C being finite parameters, the following Proposition delivers the risk-equivalent preferences SDU associated to plan C.

Proposition 10 [Risk-equivalent preferences SDU]. Consider a decision maker endowed with risk-equivalent preferences facing the ambiguous plan C. Assume that the SDU follows the generic law of motion $dV_t^C = \mu_t^{V^C} dt + \sigma_t^{V^C} dB_t$ under the probability measure $\hat{\pi}$. Then, the risk-equivalent preferences SDU associated to C is given by

$$dV_t^C \approx \left(-u\left(C_t\right) + \delta V_t^C\right) dt + \overline{\theta}\left(s^t\right) \sigma_t^{V^C} dt + \frac{1}{2} A^\nu \left(\overline{\mu}_t^C\left(s^t\right)\right) \sigma_\theta^2(s^t) \sigma_t^C \sigma_t^{V^C} dt + \sigma_t^{V^C} dB_t, \quad V_T^C = 0. \tag{13}$$

Proof. Refer to Appendix A.6.

By looking at equation (13), it is possible to emphasize the difference between decision makers with SEU and risk-equivalent preferences. The difference has a twofold interpretation.

On one hand, both the SEU and the risk-equivalent decision makers can be seen as considering the process $dC_t = \left(\mu_t^C - \overline{\theta}\left(s^t\right)\sigma_t^C\right) dt + \sigma_t^C dB_t^{\overline{\theta}\left(s^t\right)}$ to be the likely payoff process of plan C. However, while the SEU decision maker is ambiguity neutral[12] and, therefore, he does not require any reward for bearing ambiguity, the risk-equivalent decision maker demands an ambiguity premium approximatively equal to $\frac{1}{2} A^\nu \left(\overline{\mu}_t^C\left(s^t\right)\right) \sigma_\theta^2(s^t)\sigma_t^C \sigma_t^{V^C} dt > 0$.

On the other hand, the risk-equivalent decision maker can be considered as more cautious than the SEU decision maker. That is, while the SEU decision maker deems the above process dC_t under $\pi_{\overline{\theta}(s^t)}$ as the likely one, the risk-equivalent decision makers considers the process C as evolving according to $dC_t \approx \left(\mu_t^C - \overline{\theta}\left(s^t\right)\sigma_t^C - \frac{1}{2} A^\nu \left(\overline{\mu}_t^C\left(s^t\right)\right) \sigma_\theta^2(s^t)\sigma_t^{C2}\right) dt + \sigma_t^C dB_t^{re(s^t)}$. Since $\frac{1}{2} A^\nu \left(\overline{\mu}_t^C\left(s^t\right)\right) \sigma_\theta^2(s^t)\sigma_t^{C2} dt > 0$, the drift of dC_t is lower for the risk-equivalent decision maker and, therefore, he displays cautiousness.

Note that, for $A^\nu \to 0$, that is, for a linear index ν, we find again the same isomorphism between SEU and risk-equivalent preferences stated in Proposition 7. Similarly, for $A^\nu \to \infty$, that is, for an extremely concave index ν, the same isomorphism of Proposition 8 between MEU and risk-equivalent preferences is obtained. These claims are stated formally in the following Corollary to Proposition 10.

Corollary 10 [SDU isomorphisms]. *(a)* For $A^\nu \to \infty$, the risk-equivalent SDU con-

[12]Refer to Proposition 7.

verges to the MEU SDU

$$dV_t^C = \left(-u\left(C_t\right) + \delta V_t^C\right) dt + \sup_{\pi_i \in \Delta} \theta_i\left(s^t\right) \sigma_t^{V^C} dt + \sigma_t^{V^C} dB_t, \quad V_T^C = 0. \tag{14}$$

(b) For $A^\nu \to 0$, the risk-equivalent SDU converges to the SEU SDU

$$dV_t^C = \left(-u(C_t) + \delta V_t^C\right) dt + \overline{\theta}\left(s^t\right) \sigma_t^{V^C} dt + \sigma_t^{V^C} dB_t, \quad V_T^C = 0. \tag{15}$$

PROOF. For part *(a)* refer to Appendix A.7, for part *(b)* the proof is omitted.

Equation (14) corresponds to the backward stochastic differential equation (BSDE henceforth) of Theorem 2.2(a) in Chen and Epstein (2002), where the inter-temporal aggregator has the form $f\left(C_t, V_t^C\right) = u\left(C_t\right) - \delta V_t^C$. Note that $\sup_{\pi_i \in \Delta} \theta_i\left(s^t\right)$ is the largest θ associated to a $\pi \in \Delta$ such that $\psi\left(s^t(\pi)\right) > 0$.

5 Generalized SDU and risk-equivalent SDU

Lazrak and Quenez (2003) present a general form for the SDU (GSDU henceforth) that accounts for a dependency of the inter-temporal aggregator $f(\cdot)$ with respect to the intensity process ($\sigma_t^{V^C}$ in the above notation). Introducing such a dependency implies a form of local non-affinity with the utility process and therefore increases risk aversion. The GSDU is defined as the solution of the BSDE

$$-dV_t^C = f(C_t, V_t^C, Z_t)dt - Z_t dB_t, \quad V_T^C = 0,$$

where Z_t is the intensity process in Lazrak and Quenez (2003) notation. To gain further insights, their inter-temporal aggregator can be decomposed into two component

$$f(C_t, V_t^C, Z_t) = g(C_t, V_t^C) - h(C_t, V_t^C, Z_t),$$

where

$$g(C_t, V_t^C) = f(C_t, V_t^C, 0) \quad \text{and} \quad h(C_t, V_t^C, Z_t) = f(C_t, V_t^C, 0) - f(C_t, V_t^C, Z_t).$$

The decomposition tells us that each GSDU can be associated with a function g that is interpreted as the inter-temporal aggregator of the usual SDU and a function $h(\cdot)$ that represents a risk penalization (if $h(\cdot) > 0$) of the GSDU with respect to its associated SDU. This penalization comes from the aversion of the decision maker to the "intensity" of the variability in the utility process and non-affinity with surprises in the utility process. The authors show that the GSDU can accommodate the SEU SDU and the MEU SDU as specific cases[13]. As it can be readily seen from equation (13) and since SEU and MEU are special cases of risk-equivalent preferences[14], risk-equivalent preferences can be also embedded in the GSDU of Lazrak and Quenez (2003). Namely, for risk-equivalent preferences we have that

$$f(C_t, V_t^C, Z_t) \approx u\left(C_t\right) - \delta V_t^C - \left(\overline{\theta} + \frac{1}{2} A^{\nu} \left(\overline{\mu}_t^C\right) \sigma_\theta^2 \sigma_t^C\right) Z_t.^{15} \tag{16}$$

This is important since it allows to use the properties of GSDU developed in Lazrak and Quenez (2003) to characterize the *generalized* risk attitude of the decision maker implied by risk-equivalent preferences. In other words, this allows us to confirm the statements in Propositions 3 and 6.

5.1 Absolute generalized risk aversion

In line with the definition of risk aversion in Duffie and Epstein (1992), Lazrak and Quenez (2003) give the following definition of a risk averse GSDU.

Definition 6. A GSDU is called risk averse iff for any feasible payoff process C, the deterministic process $\overline{C}$ given for all t by $\overline{C}_t = \mathbb{E}\left[C_t\right]$ is initially preferred to C

$$V_0^{\overline{C}} \geq V_0^C.$$

Applying this definition, Lazrak and Quenez (2003) show that if $f(\cdot)$, i.e. the inter-temporal aggregator of the GSDU, is (1) concave with respect to C and V and (2)

$$f\left(C_t, V_t^C, Z_t\right) \leq f\left(C_t, V_t^C, 0\right), \tag{17}$$

then the associated GSDU is risk averse in the *generalized* sense. While the concavity of $f(\cdot)$ in C and V captures the *classical* risk aversion, the second condition captures the non-affinity with surprises in the utility function, which, in the present context, is named ambiguity aversion.

Equation (17) implies that $\frac{\partial f}{\partial Z} < 0$. Combining equation (16) with Proposition 3, it can be readily checked that, in the present context, this is indeed the case when $A^{\nu} > 0$, that

[13]Refer to Lazrak and Quenez (2003), Section 3, p. 158-159.

[14]Refer to Corollary 10.

[15]To make notation less heavy, the node s^t is omitted.

is, when the decision maker is ambiguity averse.

5.2 Comparative generalized risk aversion

According to Proposition 4 in Lazrak and Quenez (2003), a decision maker A is more risk averse (in the general sense) than another decision maker B if

1. $f^A\left(C_t, V_t^{C,A}, 0\right) = f^B\left(C_t, V_t^{C,B}, 0\right)$, and
2. $f^A\left(C_t, V_t^{C,A}, Z_t\right) \leq f^B\left(C_t, V_t^{C,B}, Z_t\right)$.

Applying these conditions to the specification of $f(\cdot)$ in equation (16), and assuming that A and B have the same beliefs and share the same set of priors we have that

$$f^A\left(C_t, V_t^{C,A}, Z_t\right) \leq f^B\left(C_t, V_t^{C,B}, Z_t\right) \Rightarrow A_A^{\nu}(\cdot) \geq A_B^{\nu}(\cdot). \tag{18}$$

This confirms the statement in Proposition 6. Note that assumptions made on beliefs and set of priors were also present in Proposition 6, here they ensure that the term $\overline{\theta}Z_t$ in equation (16) is equal for both decision makers.

6 Conclusion

Focusing on the Brownian setting, the paper proposes a representation of preferences under ambiguity that tries to preserve the most of the intuition of KMM preferences while overcoming their drawback as shown by Skiadas (2008). Specifically, for our representation of preferences, the implied ambiguity premium does not necessarily evaporate with the time increment $dt \to 0$. We also show that risk-equivalent preferences may be indistinguishable from SEU and MEU preferences when the index ν is linear and extremely concave, respectively. Moreover, risk-equivalent preferences have the advantage that they allow to compare ambiguity attitudes across decision makers without imposing any restriction on their risk attitudes. Finally, the paper advances an inter-temporal version of the proposed preferences representation. Exploiting the properties of the Lazrak and Quenez (2003) GSDU, the analysis of the inter-temporal version and of its continuous-limit confirms the insights gained in the single period choice setting.

A Proofs

A.1 Proposition 2

Considering that the discount factor in continuous-time is $\beta = \exp(-\delta dt)$, the recursive continuous-time version of equation (3), given the payoff process C, is

$$V_t^C = \exp(-\delta dt) u\left(C_t\right) dt + \exp(-\delta dt)\phi^{-1}\left(\mathbb{E}_{\psi,t}\left[\phi\left(\mathbb{E}_{\pi,t}\left[V_t^C + dV_t^C\right]\right)\right]\right), \qquad (19)$$

where $dV_t^C = \mu_t^{V^C} dt + \sigma_t^{V^C} dBt$ under probability measure P. Let us now focus on the second term on the right-hand side. Approximating by Taylor expansion $\phi^{-1}\left(\mathbb{E}_{\psi,t}\left[\phi\left(\mathbb{E}_{\pi,t}\left[V_t^C + dV_t^C\right]\right)\right]\right)$ around V_t^C we have that

$$\phi^{-1}\left(\mathbb{E}_{\psi,t}\left[\phi\left(\mathbb{E}_{\pi,t}\left[V_t^C + dV_t^C\right]\right)\right]\right) \approx V_t^C + \mathbb{E}_{\psi,t}\left[\mathbb{E}_{\pi,t}\left[dV_t^C\right]\right] - \frac{1}{2}A^\phi\left(V_t^C\right)\mathbb{E}_{\psi,t}\left[\left(\mathbb{E}_{\pi,t}\left[dV_t^C\right]\right)^2\right].$$

Injecting this approximation in equation (19), regrouping V_t^C terms, and multiplying both sides by $\exp(\delta dt)$

$$\left(\exp(\delta dt) - 1\right) V_t^C \approx u\left(C_t\right) dt + \mathbb{E}_{\psi,t}\left[\mathbb{E}_{\pi,t}\left[dV_t^C\right]\right] - \frac{1}{2}A^\phi\left(V_t^C\right)\mathbb{E}_{\psi,t}\left[\left(\mathbb{E}_{\pi,t}\left[dV_t^C\right]\right)^2\right].$$

Considering that $dV_t^C = \mu_t^{V^C} dt + \sigma_t^{V^C} dB_t$ under P and that the third term on the right-hand-side is negligible (dt^2 term)

$$\delta V_t^C dt \approx u\left(C_t\right) dt + \mu_t^{V^C} dt - \overline{\theta}\sigma_t^{V^C} dt, \quad \mathbb{E}_{\psi,t}[\mathbb{E}_{\pi,t}[\theta_i]] = \overline{\theta},$$

and, therefore,

$$dV_t^C \approx (-u(C_t) + \delta V_t^C)dt + \overline{\theta}\sigma_t^{V^C} dt + \sigma_t^{V^C} dB_t.$$

Q.E.D.

A.2 Proposition 5a

Considering that $\overline{\mu}_{re}^f$ in equation (6) is the expected gain of the risk-equivalent distribution and that by Assumption 3 the utility of act f is given by

$$V(f) = \mathbb{E}\left[u\left(\ell_{re}^f\right)\right]. \qquad (20)$$

We just need to apply conventional tools to determine the certainty equivalent of $V(f)$ and replace the expected gain $\overline{\mu}_{re}^f$ by the right-hand side of equation (6) to find equation (7).

Q.E.D.

A.3 Proposition 5b

The ambiguous prospect is given by

$$X_{t+dt} = x_t + dX_t = x_t + (\mu - \theta_i \sigma)dt + \sigma dB_t^i = x_t + \overline{\mu}_i^{X_{t+dt}} dt + \sigma dB_t^i.$$

As already mentioned in Example 2, the source of ambiguity for this prospect is represented by $\widetilde{\mathcal{M}}^{X_{t+dt}} = \{\overline{\mu}_1^{X_{t+dt}}, ..., \overline{\mu}_k^{X_{t+dt}}\}$.
Using Proposition 4, the risk-equivalent distribution of X_{t+dt} satisfies the condition

$$\overline{\mu}_{re}^{X_{t+dt}} \approx \overline{\mu}^{X_{t+dt}} - \frac{1}{2} A^\nu (\overline{\mu}^{X_{t+dt}}) \sigma^2_{\widetilde{\mu}^{X_{t+dt}}},$$

and the risk-equivalent lottery is therefore

$$\begin{aligned} X_{t+dt} &\approx x_t + (\overline{\mu}^{X_{t+dt}} - \frac{1}{2} A^\nu (\overline{\mu}^{X_{t+dt}}) \sigma^2_{\widetilde{\mu}^{X_{t+dt}}}) dt + \sigma dB_t^{re}, \\ &\approx x_t + (\mu - \overline{\theta}\sigma - \frac{1}{2} A^\nu (\mu - \overline{\theta}\sigma) \sigma_\theta^2 \sigma^2) dt + \sigma dB_t^{re}, \quad \sigma_\theta^2 = \mathbb{E}_\psi[(\theta_i - \overline{\theta})^2]. \end{aligned}$$

Knowing the risk-equivalent distribution we can finally compute the certainty equivalent associated to X_{t+dt}

$$\begin{aligned} CE(X_{t+dt}) &= u^{-1}(\mathbb{E}_{re}[u(X_{t+dt})]) \approx \mathbb{E}_{re}[X_{t+dt}] - \frac{1}{2} A^u (\mathbb{E}_{re}[X_{t+dt}]) \mathbb{E}_{re}[(X_{t+dt} - \mathbb{E}_{re}[X_{t+dt}])^2], \\ &\approx x_t + (\mu - \overline{\theta}\sigma) dt - \frac{1}{2} A^\nu (\mu - \overline{\theta}\sigma) \sigma_\theta^2 \sigma^2 dt - \frac{1}{2} A^u (\mathbb{E}_{re}[X_{t+dt}]) \sigma^2 dt. \end{aligned}$$

Therefore, the uncertainty premium is

$$\widehat{\kappa} \approx \frac{1}{2} A^\nu (\mu - \overline{\theta}\sigma) \sigma_\theta^2 \sigma^2 dt + \frac{1}{2} A^u (\mathbb{E}_{re}[X_{t+dt}]) \sigma^2 dt, \sigma_\theta^2 = \mathbb{E}_\psi[(\theta_i - \overline{\theta})^2].$$

Q.E.D.

A.4 Proposition 8

The proof of Proposition 8 is mainly based on Lemma 8 in KMM (2005), p. 1886.

Lemma. Let ξ be a mapping $\xi : \Delta \to \mathbb{R}$ and let η be a countably additive probability measure on Δ. Suppose $\{\rho_n\}_n$ is a sequence of real-valued functions $\rho_n : I \to \mathbb{R}$ defined on an interval I of $\mathbb{R}$ with Arrow-Pratt coefficients $A_n^\rho : I \to \mathbb{R}$ such that

$\lim_{n\to\infty}\left(\inf_{x\in I} A_n^{\rho}(x)\right) = +\infty$ and $A_n^{\rho} \leq A_{n+1}^{\rho}$ for each $x \in I$ and each n. Then

$$\lim_{n\to\infty} \rho_n^{-1}\left(\int_{\Delta} \rho_n(\xi)\, d\eta\right) = \inf \xi. \tag{21}$$

Proof. Refer to KMM (2005), pp. 1886-1887.

Translating the notation of this Lemma in the present context, for an ambiguous act $f \in \mathcal{F}$, the set $I \subset \mathbb{R}$ corresponds to the set $\widetilde{\mathcal{M}}^f$, the real-valued ρ is $\nu : \widetilde{\mathcal{M}}^f \to \mathbb{R}$, η corresponds to the decision maker's beliefs ψ on $\pi \in \Delta$, and, finally, ξ is $\overline{\mu}^f : \Delta \to \widetilde{\mathcal{M}}^f$. The Lemma implies that if we consider a set of decision makers that are sorted according to their ambiguity aversion $A_0^{\nu} \geq A_1^{\nu} \geq .. \geq .. \geq A_n^{\nu} \geq ..$ and we have that $\lim_{n\to\infty}\left(\inf_{x\in I} A_n^{\nu}(x)\right) = +\infty$, that is, as n grows very large we have extremely ambiguity averse decision makers, then

$$\lim_{n\to\infty} \nu_n^{-1}\left(\int_{\Delta} \nu_n\left(\widetilde{\mu}^f\right) d\psi\right) = \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f. \tag{22}$$

It follows that for an extremely ambiguity averse decision maker n facing act f, the risk-equivalent distribution satisfies

$$\overline{\mu}_{re}^f = \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f, \tag{23}$$

and, by following the steps of the proofs of Propositions 5a and 5b, the required uncertainty premia are

$$\begin{aligned}
\widehat{\kappa} &\approx \overline{\mu}^f - \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f + \frac{1}{2} A^u\left(\overline{\mu}_{re}^f\right)\sigma^2, \\
\widehat{\kappa} &\approx (\overline{\mu}^f - \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f)dt + \frac{1}{2} A^u(\mathbb{E}_{re}[X_{t+dt}])\sigma^2 dt,
\end{aligned}$$

respectively.

Q.E.D.

A.5 Proposition 9

By Definition 5, $f_{s^t} \sim_{s^t} \vec{c}_{f,s^t}$ and, therefore, $f_{s^t}, x_{t+1} \sim_{s^t,x_{t+1}} \vec{c}_{f,s^t,x_{t+1}}$ for all $x_{t+1} \in \mathcal{X}_{t+1}$. Consider a continuation plan $\tilde{f}_{s^t} \in \mathcal{F}_{s^t}^*$ characterized by the following payoff structure

$$\tilde{f}_{s^t}(s^{\tau}) = \begin{cases} f_{s^t}(s^t), & \tau = t, \\ c_{f,s^t,x_{t+1}}(s^{\tau}), & \tau \geq t+1. \end{cases}$$

By Definition 5, $f_{s^t} \sim_{s^t} \tilde{f}_{s^t}$. Given that $V_{s^t}(f) = U_{s^t}(\vec{c}_{f,s^t}) = \frac{u(c_{f,s^t})}{1-\beta}$,

$$V_{s^t,x_{t+1}}(f)(1-\beta) = u(c_{f,s^t,x_{t+1}}).$$

Since $\tilde{f}_{s^t} \in \mathcal{F}^*_{s^t}$, we can use the result in equation (11) to write

$$\begin{aligned}
V_{s^t}(\tilde{f}_{s^t}) &= u(f_{s^t}(s^t)) + \beta \frac{\mathbb{E}_{p^{\tilde{f}_{s^t}}_{re(s^t)}}[u(d_{s^t}(s^{t+1}))]}{1-\beta}, \\
&= u(f_{s^t}(s^t)) + \beta \frac{\mathbb{E}_{p^{\tilde{f}_{s^t}}_{re(s^t)}}[u(u^{-1}(\mathbb{E}_{p^{\tilde{f}_{s^t}}_{re(s^t)}}[u(d_{s^t}(s^{t+1}))]))]}{1-\beta}, \\
&= u(f_{s^t}(s^t)) + \beta \frac{\mathbb{E}_{p^{\tilde{f}_{s^t}}_{re(s^t)}}[u(c_{f,s^t,x_{t+1}})]}{1-\beta}, \\
&= u(f_{s^t}(s^t)) + \beta \mathbb{E}_{p^{\tilde{f}_{s^t}}_{re(s^t)}}[V_{s^t,x_{t+1}}(\tilde{f}_{s^t})].
\end{aligned}$$

Since $f_{s^t} \sim_{s^t} \tilde{f}_{s^t}$ and by Assumption 4, we find that

$$V_{s^t}(f) = u(f_{s^t}(s^t)) + \beta \mathbb{E}_{p^{f_{s^t}}_{re(s^t)}}[V_{s^t,x_{t+1}}(f)].$$

Q.E.D.

A.6 Proposition 10

Considering that the discount factor in continuous-time is $\beta = \exp(-\delta dt)$, the recursive continuous-time version of equation (12), given the payoff process C, is

$$V_t^C = \exp(-\delta dt) u(C_t) dt + \exp(-\delta dt) \mathbb{E}_{re^C(s^t)}[V_t^C + dV_t^C]. \tag{24}$$

Regrouping V_t terms, the above equation may be rewritten as

$$\left(\exp(\delta dt) - 1\right) V_t^C = u\left(C_t\right) dt + \mathbb{E}_{re^C(s^t)}\left[dV_t\right]. \tag{25}$$

Consider now the ambiguous process C whose payoffs evolves according to $dC_t = \left(\mu_t^C - \theta_i \sigma_t^C\right) dt + \sigma_t^C dB_t^i$ for $\pi_i \in \Delta$. The source of ambiguity is represented by the set $\widetilde{\mathcal{M}}_t^C = \{\overline{\mu}_{t,1}^C, ..., \overline{\mu}_{t,k}^C\}$ with $\overline{\mu}_{t,i}^C = \mu_t^C - \theta_i \sigma_t^C$. Following Proposition 4, it can be shown that, upon having reached node s^t, the risk-equivalent distribution is characterized by

$$\overline{\mu}_{t,re}^C(s^t) \approx \underbrace{\mu_t^C - \overline{\theta}\left(s^t\right) \sigma_t^C}_{\overline{\mu}_t^C(s^t)} - \frac{1}{2} A^\nu \left(\overline{\mu}_t^C\left(s^t\right)\right) \sigma_\theta^2(s^t) \sigma_t^{C2}, \tag{26}$$

where $\overline{\theta}\left(s^t\right) = \mathbb{E}_{\psi(s^t)}\left[\theta_i\right]$and $\sigma_\theta^2(s^t) = \mathbb{E}_{\psi(s^t)}[(\theta_i - \overline{\theta}(s^t))^2]$.
By Girsanov's theorem, this implies that $dB_t^{re^C\left(s^t\right)} \approx dB_t + \left(\overline{\theta}\left(s^t\right) + \frac{1}{2}A^\nu\left(\overline{\mu}_t^C\left(s^t\right)\right)\sigma_\theta^2(s^t)\sigma_t^C\right)dt$.
With the conjecture that the SDU evolves according to $dV_t = \mu_t^{V^C}dt + \sigma_t^{V^C}dB_t$ under $\hat{\pi}$, we have that

$$\mathbb{E}_{re^C(s^t)}\left[dV_t^C\right] \approx \left(\mu^{V^C}{}_t - \overline{\theta}\left(s^t\right)\sigma_t^V - \frac{1}{2}A^\nu\left(\overline{\mu}_t^C\left(s^t\right)\right)\sigma_\theta^2(s^t)\sigma_t^C\sigma_t^{V^C}\right)dt. \tag{27}$$

Combining equation (25) with equation (27), we finally find the SDU for recursive risk-equivalent preferences stated in Proposition 10

$$dV_t^C \approx \left(-u\left(C_t\right) + \delta V_t^C\right)dt + \overline{\theta}\left(s^t\right)\sigma_t^{V^C}dt + \frac{1}{2}A^\nu\left(\overline{\mu}_t^C\left(s^t\right)\right)\sigma_\theta^2(s^t)\sigma_t^C\sigma_t^{V^C}dt + \sigma_t^{V^C}dB_t.$$

Q.E.D.

A.7 Corollary 10(a)

From Proposition 8(1) we know that an extremely ambiguity averse decision maker deems as risk-equivalent the distribution yielding the expectation

$$\overline{\mu}_{re}^f = \inf_{\overline{\mu}_i^f \in \widetilde{\mathcal{M}}^f} \overline{\mu}_i^f. \tag{28}$$

If now this decision makers considers an ambiguous plan C whose payoff under some probability measure $\pi_i \in \Delta$ evolves according to

$$dC_t = \underbrace{\left(\mu_t^C - \theta_i\sigma_t^C\right)}_{\overline{\mu}_t^C}dt + \sigma_t^C dB_t^i, \tag{29}$$

then he considers as risk-equivalent the distribution that minimizes $\overline{\mu}_t^C$, and he perceives the payoff of C as evolving according to

$$dC_t = \underbrace{\left(\mu_t^C - \sup_{\pi_i\in\Delta}\theta_i\left(s^t\right)\sigma_t^C\right)}_{\inf_{\overline{\mu}_{i,t}^C \in \widetilde{\mathcal{M}}_t^C}\overline{\mu}_t^C}dt + \sigma_t^C dB_t^{\arg\sup_{\pi_i\in\Delta}\theta_i}. \tag{30}$$

Note that $\sup_{\pi_i\in\Delta}\theta_i\left(s^t\right)$ is the largest θ associated to a $\pi \in \Delta$ such that $\psi\left(s^t(\pi)\right) > 0$. By Girsanov's theorem we have, therefore, that $dB_t^{\arg\sup_{\pi_i\in\Delta}\theta_i} = dB_t + \sup_{\pi_i\in\Delta}\theta_i\left(s^t\right)dt$. Conjecturing that the SDU follows the differential equation $dV_t^C = \mu_t^{V^C}dt + \sigma_t^{V^C}dB_t$ under

$\hat{\pi}$, we find that

$$\mathbb{E}_{re^{C}(s^t)}\left[dV_t^C\right] = \left(\mu_t^{V^C} - \sup_{\pi_i \in \Delta} \theta_i \left(s^t\right) \sigma^{V^C}\right) dt. \tag{31}$$

Combining equation (25) with equation (31), we have that the SDU for an extremely ambiguity averse decision maker follows the process

$$dV_t^C = \left(-u\left(C_t\right) + \delta V_t^C\right) dt + \sup_{\pi_i \in \Delta} \theta_i \left(s^t\right) \sigma_t^{V^C} dt + \sigma_t^{V^C} dB_t, \tag{32}$$

which corresponds to the SDU in Theorem 2.2(a) in Chen and Epstein (2002), with the inter-temporal aggregator $f\left(C_t, V_t^C\right) = u\left(C_t\right) - \delta V_t^C$.

Q.E.D.

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