Managing Inventories of Perishable Goods: The Effect of Substitution

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We consider a discrete-time supply chain for perishable goods where there are separate demand streams for items of different ages. We propose two practical replenishment policies: replenishing inventory according to order-up-to level policies based on either (i) total inventory in system or (ii) new items in stock. Given these policies, we concentrate on four different ways of fulfilling demand: (1) demand for an item can only be satisfied by an item of that age (No-Substitution); (2) demand for new items can only be satisfied by new ones, but excess demand for old items can be satisfied by new (Downward-Substitution); (3) demand for old items can only be satisfied by old, but excess demand for new items can be satisfied by old (Upward-Substitution); (4) both downward and upward substitution are employed (Full-Substitution). We compare these substitution options analytically in terms of the infinite horizon expected costs, providing conditions on cost parameters that determine when (if at all) one substitution option is more profitable than the others, for an item with a two-period lifetime. We also prove that inventory is "fresher" whenever downward substitution is employed. Our results are based on sample-path analysis, and as such make *no* assumptions on demand. We complement our results with numerical experiments exploring the effect of problem parameters on performance.

Keywords: Perishable goods, Inventory management, Substitution, Stochastic models, Infinite horizon, Sample path analysis.

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1 Introduction

Effective management of a supply chain calls for getting the right product at the right time to the right place and in the right condition. This has never been a simple task, and in many sectors recent developments have made it more complicated still: Increasing numbers of products are becoming subject to obsolescence, losing value over time as new items are introduced while older versions are phased out. Even products previous thought of as "durable", such as furniture, high-technology goods and biotechnology products (drugs, vitamins, cosmetics) are falling prey to perishability. Managing such "perishable" supply chains is not for the faint of heart - matching supply with demand can be very challenging, and their mismatch may be extremely costly: In a 2003 survey of distributors to supermarkets and drug stores, overall unsalable costs within the consumer packaged goods industry were estimated at $2.57 billion, for the branded segment 22% of these costs were attributed to expiration (Grocery Manufacturers of America, 2004).

Further complicating management of these chains is the fact that often products of different "ages" – at different phases in their life cycles – co-exist in the market at the same time. Differences in consumer preferences, price points, functionality of the products and/or rate of adoption (e.g. Norton and Bass, 1987, Mahajanan and Muller, 1996), have made such overlapping of product life cycles not only viable, but often advisable. For example, in the early 1990's 31% to 37% of the electronics products in a *single* product line were at the earlier stages of their life-cycles (Mendelson and Pillai, 1999). In such a case, demand for both new and old goods may be significant; a seller must manage multiple demand streams for different-aged, often *substitutable* products.

A similar situation exists in the $70 billion fresh produce market in the United States. In this market large retail stores and food service establishments share 48.4% and 50% of total sales (Kaufman et al. 2000). While large retail stores can hold stocks of goods in their supply chains (Pressler, 2004), food service establishments such as restaurants, fast food stores and institutional services buy goods in smaller quantities for immediate use. Wholesalers are the major supplier for both segments; from a wholesaler's perspective, these two segments have different preferences when it comes to purchasing fresh produce. Retailers prefer longer shelf-life items (i.e. not fully ripe), while food-service establishments prefer items ready-to-use, creating demand streams differentiated by the age of the produce. Another important example occurs in blood bank management: Certain critical surgeries and treatments (trauma surgery and platelet transfusions for chemotherapy patients) specifically ask for product with maximal freshness, while others do not, creating demand streams for different ages of blood products (Angle, 2003).

The produce and blood-product industries tend to substitute freely between products of different ages – sending product that is a little older or younger than requested is better than sending nothing at all. This is far from the only strategy practiced though. In the computer industry, where faster computer chips drive slower chips toward obsolescence, manufacturers may fuse fast chips and sell them to satisfy slow chip demand, using a new product as a substitute for an old, but not vice-versa. Such new-to-old substitution is also inherently practiced when there is a single demand stream and items are supplied according to age (oldest first). Exclusive old-to-new substitution is likewise practiced: your local bagel shop may pour bagels fresh from the oven on top of bagels already in the bins, serving customers the freshest bagels first. (They serve older product only when the fresh bagels are gone.) Finally, companies may also utilize strategies to minimize or prevent substitution: In 2002 previous season's lines comprised 50% of Bloomingdale's total annual sales (about $400M) with 9% going to salvage retailers, in order to avoid excessive substitution between new and old goods (Berman, 2003).

We investigate the general question of substitution; specifically what costs and demand characteristics may drive certain industries to make very different substitution choices from among their (possibly practically limited) substitution options. And, since substitution decisions cannot be made without considering the policy used to replenish inventory, we simultaneously examine two simple replenishment policies for the supplier: Base-stock policies based on either the total amount of inventory of all ages, or the amount of new items. Thus the supplier must choose an inventory policy *and* a substitution policy. The former policy governs the supplier's reordering, and the latter policy specifies whether in each period he will use excess stock of new (old) items to fulfill the excess demand of old (new). We address these questions in a periodic, infinite horizon setting, considering a single product with two periods of lifetime. In each period the supplier can replenish stock of "new" products, and after one period in inventory unsold items become "old". Any product that is unsold in two periods perishes. In each period, there may be random demand for both new and old items.

Our contributions include the following: (*i*) We study two practical replenishment policies for perishable goods, featured in literature and practice; (*ii*) Under these policies, we show that substitution may *not* always be economically advisable for the supplier; (*iii*) We determine the conditions under which a supplier will indeed benefit from (which forms of) substitution; (*iv*) We provide a comparative analysis of substitution options with respect to the average age of goods in inventory; (*v*) We perform computational work quantifying the sensitivity of our results to problem parameters; and (*vi*) All our analytical results are based on sample-path analysis, hence no assumptions, save for ergodicity, are made regarding demand: We allow demand to be correlated over time, across products, or both, as is common in practical settings involving substitutable products (if not the literature).

We first provide a review of the related literature in Section 2. In Section 3, we formally define the problem, introducing the notation and the formulation for different replenishment and substitution policies. Cost comparisons of substitution rules for our two different inventory policies are provided in Section 4, while analysis on freshness of goods in inventory is given in Section 5. We provide several numerical examples in Section 6 complementing our analytical results. Conclusions and directions for future research are discussed in Section 7.

2 Literature Review

Most of the work on inventory problems for perishable goods focuses on optimal or near-optimal ordering policies to minimize operating costs, under a single demand stream (thus rendering substitution decisions moot). Even in this simplified setting the problem is very difficult: Unlike standard inventory control theory, where generally the only information needed is the inventory position, the optimal ordering policy for perishables requires information about the amount of inventory of every age. Therefore the state space (and problem complexity) grows with the lifetime of the product. For two-period lifetime problem, Nahmias and Pierskalla (1973) determined some of the properties of the stationary, state-dependent optimal policy; this was extended by Fries (1975) and Nahmias (1975), independently, to the general m-period lifetime problem. They characterized some properties of the optimal ordering policy, but its exact structure was not found. This difficulty motivated exploration of heuristic methods. Cohen (1976), Nahmias (1976) and Chazan and Gal (1977) proposed the *fixed-critical number* (order-up-to) policy, in which orders are placed at the end of each period to bring the total inventory summed across all ages to a specific level, S. (This is one of the inventory policies we use in our model; we call it the Total-Inventory-to S policy, TIS.) For the two-period lifetime problem, Cohen (1976) found a closed-form method for computing the optimal order-up-to level.

Still for a single demand stream, Nahmias (1976, 1977) and Nandakumar and Morton (1993) show that order-up-to policies perform very well compared to other methods, including optimal policies; they develop and analyze heuristics to choose the best order-up-to level. Cooper (2001) provides further analysis of the properties of the TIS policy, while Nahmias (1978) shows that when the ordering cost is high, an (s, S) type heuristic is better than order-up-to policies. Liu and Lian (1999) analyze such an (s, S) continuous review inventory system. More recent papers on inventory management of perishables are by Ketzenberg and Ferguson (2003), and Ferguson and Ketzenberg (2004) - they focus on information sharing in a supply chain. A complete review of the research on perishable goods is available Goyal and Giri (2001), and a summary focused on blood bank supply chains is provided by Pierskella (2004). As mentioned previously, in all works cited above there is a single demand stream fulfilled according to FIFO (oldest items issued first); FIFO is known to be optimal for many fixed-life inventory problems with a single demand stream (Pierskalla and Roach, 1972).

There are a few papers that model multiple types of customers or demand streams for perishables. Ishii et al. (1996) focus on two types of customers (high and low priority), items of m different ages with different prices, and only a *single-period* decision horizon. High priority customers only buy the freshest products, so the freshest products are first sold to the high priority customers and the remaining items are issued according to a FIFO policy. They provide the optimal ordering policy under a warehouse capacity constraint. Parlar (1985) considers a perishable product that has two-periods of lifetime, where a fixed proportion of unmet demand for new items is fulfilled by unsold old items (and vice-versa). Goh et al. (1993) consider a two-stage perishable inventory problem. Their model has random supply and separate demand modeled as a Poisson process. They *computationally* compare a restricted policy (equivalent to our No-Substitution option) and an unrestricted policy (equivalent to our Downward-Substitution option). Considering only shortage and outdating costs they conclude that the unrestricted policy is less costly, unless the shortage cost for fresh units is very high. Ferguson and Koenigsberg (2004) study a problem which is similar to ours in a two-period setting with pricing and internal competition between new and old items. In our model, we have a more general cost structure (crucially, we include *substitution costs*), no assumptions on the demand distribution, and we *analytically* provide conditions guaranteeing dominance between substitution options over infinite horizon.

There is a second stream of research related to our work: Inventory management of substitutable products, where perishability or age is irrelevant. McGillivray and Silver (1978) provide analytical results for two products with similar unit and penalty costs and unmet demand can be fulfilled from the other product's inventory (if there is any inventory left). Parlar and Goyal (1984) study two differentiated products with a fixed substitution fraction and analyze structural properties of the expected profit function. Their work is extended by Pasternack and Drezner (1991) by adding revenue earned from a substitution (different from the original selling price), shortage cost, and salvage value. Rajaram and Tang (2001) subsumes the existing models for the single-period two-product problem. They show that demand substitution between products always leads to higher expected profits than the no-substitution case, and provide a heuristic to determine the order quantities. We show such a dominance relation need *not* hold in an infinite-horizon, perishable setting. There are several other papers looking at single-period inventory problems with multiple products that are substitutable: Parlar (1988), Wang and Parlar (1994), Bassok et al. (1999), Ernst and Kouvelis (1999) Smith and Agrawal (2000), Mahajan and van Ryzin (2001), Avsar et al. (2002), Eynan and Fouque (2003); the reader can refer to Netessine and Rudi (2003) and the references therein.

Our work is unique with respect to these two streams of research in that (*i*) we consider substitution of *perishable* products, formally defining four substitution options; (*ii*) we compare *different* forms of substitution based on two measures (total cost and average age of inventory in the system), providing analytical proofs or counter-examples for dominance relations among the substitution options; (*iii*) our results are *free* of distributional assumptions on demand, save for ergodicity; demand can be auto-correlated or correlated with the product of different age; (*iv*) we consider an *infinite horizon problem* as opposed to single period models commonly used; and (*v*) we analyze *two* distinct, practical replenishment policies. The model with a single stream of demand and FIFO service – standard in the literature on perishable goods – remains a special case of ours.

3 Problem Formulation

3.1 Problem Definitions

We have a single product with a lifetime of two periods; the value of the product decreases as it ages, and there may be random demand for both new and old items. A single supplier may replenish new items periodically, with zero lead time. At the end of each period, any remaining old items are outdated, while any unused new items become old. The supplier has to decide how much to order in each period, and choose a policy to fulfill demand. Depending on the product and/or preferences of the supplier, he can employ one of four substitution options: The No-Substitution policy (denoted $\mathcal{N}$) restricts the use of items of a certain age to only fulfill the demand for products of that age. Using the Upward-Substitution policy (denoted $\mathcal{U}$), excess demand of new items is fulfilled by excess stock of old items, but not vice-versa. Downward-Substitution (denoted $\mathcal{D}$) does the opposite; excess demand of old items is satisfied by excess stock of new items.[1] Finally, in the Full-Substitution policy (denoted $\mathcal{F}$), the supplier uses excess stock of new or old items to satisfy the excess demand of the other-aged item (i.e under policy $\mathcal{F}$ the supplier utilizes both

[1] Throughout the paper "Upward-Substitution" (or $\mathcal{U}$) and "Downward-Substitution" (or $\mathcal{D}$) denote substitution policies. By "downward substitution" or "upward substitution" we mean the substitution event itself.

downward and upward substitution). Any substitution occurs only as a "recourse, " i.e. demand for a new (old) item is satisfied from the stock of new (old) unless there is a stock-out. We model the substitution decision as being made by the supplier with the implicit consent of the customer; examples of this situation include blood, technological products and wholesale produce. In many practical cases as well as in the literature, a supplier commits to using one form of substitution in all periods. We assume this as well.

We denote by X_i^n $(i = 1, 2)$ the amount of product with i periods of lifetime remaining at the beginning of period n. There are different demand streams for the items of different ages, denoted D_i^n $(i = 1, 2)$. Demand is stochastic and may have arbitrary joint distribution; dependence between demand for different ages and over time is allowed, but demand is assumed ergodic and independent of supplier decisions. Unsatisfied demand for both new and old items are lost; backordering new item demand does not change our results. The order of events is as follows: First inventory is replenished, then demand occurs and is fulfilled or lost. Stocks are aged; new goods become old and old goods perish. Finally, costs are assessed and the order arriving in the next period is placed.

Our analysis is based on the supplier's cost function, which includes explicit costs associated with upward or downward substitution: The supplier incurs a downward substitution cost of γ per unit and an upward substitution cost of α per unit. These may represent additional production or shipping costs for the supplier (e.g. fusing fast chips, shipping goods from different locations), potential loss of customer goodwill and/or loss in revenue (if price discounts are given for substitution). In addition, there is a unit lost sale cost of p_i for age i product; we assume $p_2 > p_1$, new item demand is more profitable than old item demand. There is also an inventory carrying cost of h per unit, and an outdating cost of m per unit. All cost parameters are assumed to be non-negative. The cost function defined using these parameters allows us to represent the problem in a very general form: Our analysis does not change, for example, when there are unit ordering costs, or sales prices are incorporated as in a profit-based model (a discussion on the equivalence of these models is available in Deniz, 2004).

3.2 Replenishment Policies

We examine two simple, practical replenishment policies for the supplier:

TIS (Total-Inventory-to-S): At the beginning of every period the total inventory level (old plus new items) is brought up to S. That is, for all n,

$$X_1^n + X_2^n = S. \tag{1}$$

NIS (New-Inventory-to-S): At the beginning of every period the inventory level of new items is brought up to S. Hence, for all n,

$$X_2^n = S. \tag{2}$$

The TIS policy is simple, common and has a long tradition in the literature. To our best knowledge NIS, used in practice (Angle 2003), has not been investigated in the literature. In either policy, the order-up-to level S can be determined by optimization or by practical rules of thumb. TIS has been shown to have good performance compared to optimal (e.g. Nahmias, 1976, Nandakumar and Morton, 1993) when a single demand stream is fulfilled according to FIFO. (In our model, this would represent the case with no demand for new items.) However, when demand for new items is present, no such results are known.

The expected total cost as a function of the order-up-to level S can be written in a compact and unified way for all the policies. To accomplish this, we define parameters π_D and π_U which denote the fraction of customers offered downward or upward substitution, respectively. Then, by defining $(\pi_D, \pi_U) = (0, 0)$, we represent policy $\mathcal{N}$. Similarly, $(\pi_D, \pi_U) = (1, 1)$ represents $\mathcal{F}$, $(\pi_D, \pi_U) = (1, 0)$ represents $\mathcal{D}$, and $(\pi_D, \pi_U) = (0, 1)$ represents $\mathcal{U}$. Given these definitions, the expected cost function[2] of the supplier is :

$$E[C(S)] = \lim_{N\to\infty} \frac{1}{N}\, E[\ \sum_{n=0}^{N} hX_1^{n+1} + p_1 L_1^n + p_2 L_2^n + mO^n + \alpha\ us^n + \gamma\ ds^n\ ], \tag{3}$$

where

$L_1^n = [(D_1^n - X_1^n) - ds^n]^+$ is the amount of lost sales for old items;

$L_2^n = [(D_2^n - X_2^n) - us^n]^+$ is the amount of lost sales for new items;

$O^n = [(X_1^n - D_1^n) - us^n]^+$ is the amount that outdates at the end of the period;

$X_1^{n+1} = [(X_2^n - D_2^n) - ds^n]^+$ is the amount of inventory carried to the next period;

$ds^n = \min\{\pi_D(D_1^n - S + X_2^n)^+,\ (X_2^n - D_2^n)^+\}$ is the downward substitution amount; and

$us^n = \min\{\pi_U(D_2^n - X_2^n)^+,\ (S - X_2^n - D_1^n)^+\}$ is the upward substitution amount.

In the above formulation, the only difference between TIS and NIS is in the inventory recursions regarding the inventory levels of new items (see equations (1) and (2)). Note that, along any sample path within either TIS or NIS, $\mathcal{F}$ and $\mathcal{D}$ have equal amounts of stock in any period for a given S, and so do $\mathcal{U}$ and $\mathcal{N}$. Also, for all the substitution policies, (X_2^n, X_1^n) forms an ergodic process; thus all costs can be expressed as functions of the time-average distribution of the inventory, which converges.

[2]All "expectations" refer to the time-averages, eliminating concerns about the existence of stationary distributions.

Even though we assume that the customers accept substitution whenever offered, the above formulation allows for modeling more general cases, by allowing $0 \leq \pi_D, \pi_U \leq 1$. Some of our analytical results hold for this general case; these will be noted in the text. We investigate the broad effects of more general customer behavior on expected policy costs in Section 6.

4 Cost Comparison of Substitution Policies

In this section, we identify conditions on cost parameters that guarantee that a specific substitution policy is less costly compared to other policies. We show that reasonable parameter settings exist that lead to any one of our four policies being superior, in an almost sure sense. We also identify parameter regions where no such dominance conditions can exist.

We provide cost comparisons under TIS in Section 4.1, and under NIS in Section 4.2, summarizing these in Section 4.3. All of our comparisons are based on key results regarding inventory levels of different substitution policies, when a common order-up-to level S is used. Since our pairwise cost-dominance results do not change if optimal order-up-to levels are used, the choice of S is not a concern here. Throughout, we use the notation $\succ$ to denote dominance between two substitution options (e.g. $\mathcal{F} \succ \mathcal{D}$ means the expected cost of $\mathcal{F}$ is lower than that of $\mathcal{D}$). All proofs are in the Appendix, unless noted otherwise.

4.1 Cost Comparisons under TIS

In this section we provide pairwise comparisons of $\mathcal{N}$, $\mathcal{D}$, $\mathcal{U}$ and $\mathcal{F}$ when the supplier replenishes inventory using TIS. We begin with preliminary results, in Section 4.1.1, before moving to pairwise comparison of policies in Sections 4.1.2-4.1.5.

4.1.1 Preliminary Results under TIS

As we are using TIS, we have $X_1^n = S - X_2^n$ in each period n; thus it suffices to track only X_2^n. We use the following notation for ease of exposition: X_I^n denotes the stock level of new items at the beginning of period n, i.e. X_2^n under policy I for $I = \mathcal{F}, \mathcal{N}, \mathcal{D}, \mathcal{U}$.

Proposition 1 *Along any sample path of demand and for the same S, if $X_I^n < X_J^n$, then $X_I^{n-1} > X_J^{n-1}$ for $I = \mathcal{F}, \mathcal{D}$, $J = \mathcal{U}, \mathcal{N}$.*

Proof : We only provide the proof for $I = \mathcal{F}$ and $J = \mathcal{N}$; the others follow from the equivalence of the inventory recursions. X_F^n and X_N^n are both non-negative by definition.

$$X_N^n = S - (X_N^{n-1} - D_2^{n-1})^+ \quad > \quad S - [(X_F^{n-1} - D_2^{n-1})^+ - (D_1^{n-1} - S + X_F^{n-1})^+]^+ = X_F^n$$

$$
\begin{aligned}
(X_N^{n-1} - D_2^{n-1})^+ &< [(X_F^{n-1} - D_2^{n-1})^+ - (D_1^{n-1} - S + X_F^{n-1})^+]^+ \\
&< (X_F^{n-1} - D_2^{n-1})^+.
\end{aligned}
$$

$(X_F^{n-1} - D_2^{n-1})^+$ must be greater than zero in order to satisfy the last inequality above, implying $X_F^{n-1} > X_N^{n-1}$. ■

Proposition 1 states that along any sample path, if the inventory of new items under policy $\mathcal{F}$ (or $\mathcal{D}$) is less than the inventory of new items under policy $\mathcal{N}$ (or $\mathcal{U}$), then in the previous period the new items in $\mathcal{F}$ (or $\mathcal{D}$) must have been more than the new items in $\mathcal{N}$ (or $\mathcal{U}$). Using this fact we can divide the horizon into two classes of periods: Pairs of periods $n-1$ and n for the case when $X_F^n < X_N^n$; and periods not in these pairs (i.e. runs of consecutive periods in which $X_F^k \geq X_N^k$, for $k, k+1, ...$) . In the remainder of the paper, we use the term *pair* to denote two consecutive periods with the above property. If we refer to a period with more new items under full-substitution as F and otherwise as N, a sequence of periods could look like this, with the "pairs" bracketed:

$$
...FFF\,\overbrace{FN}\,\overbrace{FN}\,FF\,\overbrace{FN}\,FFF...
$$

Note that due to Proposition 1 there are never two consecutive Ns.

We next provide two more results which are used later in our comparative analysis.

Proposition 2 *If an F period follows another F period, then in the first one there must be downward substitution.*

Proposition 3 *Suppose an F is followed by another F in periods n and $n+1$. Let $\tilde{\Delta}_n$ be the amount of downward substitution in period n. Then, $X_F^{n+1} - X_N^{n+1} = \tilde{\Delta}_n - (X_F^n - X_N^n)$.*

4.1.2 Value of upward substitution under TIS: $\mathcal{F}$ vs. $\mathcal{D}$ and $\mathcal{U}$ vs. $\mathcal{N}$

When an upward substitution takes place, an unsold old item is used as a substitute to satisfy the excess demand of a new item. Options $\mathcal{F}$ and $\mathcal{U}$ employ this rule. For every item substituted, the supplier saves outdating (m) and penalty (p_2) costs, but incurs the substitution cost (α). Therefore, in order for upward substitution to provide additional value, the following must hold:

$$
\alpha \quad < \quad m + p_2. \tag{4}
$$

Lemma 1 *On every sample path, under TIS, $\mathcal{F} \succ \mathcal{D}$ and $\mathcal{U} \succ \mathcal{N}$, if and only if condition (4) holds. The same dominance relations hold when not all the customers accept upward substitution (i.e. $0 \leq \pi_U \leq 1$).*

Proof : Inventory levels of $\mathcal{D}$ and $\mathcal{F}$ are defined by the same recursions. Thus their costs that are independent of upward substitution (i.e. those related to h, p_1 and γ) are equal. Therefore $\mathcal{F}$ has a lower cost than $\mathcal{D}$ if and only if upward substitution is profitable, i.e. when $\alpha < m + p_2$. This remains true for all $\pi_U \in [0, 1]$. The same argument holds for $\mathcal{U}$ vs. $\mathcal{N}$. ■

This result is simple and intuitive: The cost α can be taken as a proxy for the amount of discount the supplier offers to the customer to accept a substitute. The supplier benefits from offering an old item for a new one when the discount offered does not exceed a critical value - in this case the lost sale cost of new items and the outdating cost of old items.

4.1.3 Value of upward and downward substitution under TIS: $\mathcal{F}$ vs. $\mathcal{N}$

The next question is whether the supplier benefits from offering both upward and downward substitution. Condition (4) is necessary and sufficient for $\mathcal{F}$ to be less costly compared to $\mathcal{D}$, however, the total benefit of $\mathcal{F}$ over $\mathcal{N}$, depends on both γ and α. Together with condition (4), the following inequalities ensure that $\mathcal{F}$ is more profitable than $\mathcal{N}$:

$$\gamma < m + 2p_1 - p_2, \tag{5}$$

$$\gamma < m + p_1 - \alpha, \tag{6}$$

$$\gamma < \frac{h + p_1}{2}. \tag{7}$$

We formally prove these conditions below, but provide some intuitive explanation first. Using downward substitution, a vendor saves $h + p_1$ and incurs γ for every item substituted in the period preceding a *pair*. (Such a substitution is necessary to form a *pair*.) Conditions (5) and (6) arise as follows: If demand is low in the first period of the *pair*, $\mathcal{N}$ incurs $m - h$ more cost than $\mathcal{F}$ per item substituted upward, because $\mathcal{N}$ has fewer new items in the first period of the *pair*. In the second period of the *pair*, $\mathcal{N}$ has more new items, which may give $\mathcal{N}$ a benefit of $p_2 - p_1$ per item (if $\mathcal{F}$ cannot substitute upward), or α per item (if $\mathcal{F}$ can substitute upward). Therefore if $h + p_1 - \gamma + m - h - (p_2 - p_1)$ is positive (corresponding to the first case) and $h + p_1 - \gamma + m - h - \alpha$ is positive (corresponding to the second case), then $\mathcal{N}$ is more costly than $\mathcal{F}$ under either situation. Note that (4) is redundant when (6) holds, as we assume $p_1 < p_2$.

Given the immediate cost difference due to downward substitution, if $\gamma < h + p_1$ holds then - intuitively - $\mathcal{F}$ should be less costly. However, $\gamma < (h + p_1)/2$ is in fact required. The idea behind (7) is as follows: Downward substitution causes $\mathcal{F}$ to have fewer old items (and more new items) than $\mathcal{N}$ in the subsequent period, which may lead $\mathcal{F}$ to substitute downward again in this next period, incurring upward substitution cost γ twice in the same *pair*.

Combining (5) - (7) we have the following bound on the downward substitution cost:

$$\gamma \quad < \quad \min\{m + 2p_1 - p_2, \;\; m + p_1 - \alpha, \;\; \frac{h + p_1}{2}\}. \tag{8}$$

Below is the formal statement of these properties, proof is in the appendix.

Lemma 2 *If (4) and (8) hold, then $\mathcal{F} \succ \mathcal{N}$.*

The above result is intuitive: Full substitution is beneficial when substitution costs are low. The condition is sufficient, but not necessary, for $\mathcal{F}$ to dominate $\mathcal{N}$. We now explore necessity: What happens when condition (8) does not hold; is there a critical value of γ, beyond which $\mathcal{N}$ dominates $\mathcal{F}$? Our next result provides the answer.

Lemma 3 *When $m + p_1 > 0$, there is <u>no</u> value of γ that guarantees $\mathcal{N} \succ \mathcal{F}$.*

Proof : We provide a counter-example. There are no old items in stock, and the inventory state (X_2^n, X_1^n) for both $\mathcal{F}$ and $\mathcal{N}$ is initially $(S, 0)$. Consider a sample path of demand for new and old items $\{(D_2^n, D_1^n), n = 1, 2, ...\}$, where the sequence $\{(S - \epsilon, S), (0, 0), (0, S), (0, 0), (0, S), (S, 0)\}$ repeats every six periods for any $0 \le \epsilon \le S$. The resulting inventory levels at the beginning of each of the next six periods are $\{(S, 0), (0, S), (S, 0), (0, S), (S, 0), (S, 0)\}$ for $\mathcal{F}$ and $\{(S - \epsilon, \epsilon), (\epsilon, S - \epsilon), (S - \epsilon, \epsilon), (\epsilon, S - \epsilon), (S - \epsilon, \epsilon), (S, 0)\}$ for $\mathcal{N}$. Note that the six periods starting with the initial period, when both policies are at $(S, 0)$, form an $FFNFNF$ sequence, where there are two *pairs*, and the inventory in the seventh period is the same as in the initial period. Let $C_i, \; i = \mathcal{F}, \mathcal{N}$ denote the total cost of policy i in this cycle. The cost difference is $C_{\mathcal{F}} - C_{\mathcal{N}} = \epsilon[\gamma - (m + p_1)(J + 1) - h - p_2]$, where J is the number of pairs in the cycle ($J = 2$ in this example). By appropriately selecting a demand stream, one can construct a sample path where J is arbitrarily large. Thus $C_{\mathcal{F}} - C_{\mathcal{N}}$ can always be made non-positive, independent of the magnitude of γ, so long as $m + p_1 > 0$. ■

Note that the above result requires the outdating or old item lost sales cost to be positive. If this is not true, i.e. when old item demand is irrelevant and there is no outdating cost, the nature of the problem may be fundamentally different.

Lemma 3 is counter-intuitive, it says that even when $\gamma \to \infty$ there can be a benefit to using substitution (or more precisely substitution cannot be ruled out almost surely). While extreme, this example illustrates real potential dangers of using the strict replenishment rule TIS with policy $\mathcal{N}$ when demand is intermittent or periodic: these factors can result in a chain reaction of inventory carrying, outdating and lost sales. This suggests that using TIS, as advocated in the literature, may be inadvisable without the additional flexibility offered by substitution.

4.1.4 Value of downward substitution under TIS: $\mathcal{F}$ vs. $\mathcal{U}$ and $\mathcal{D}$ vs. $\mathcal{N}$

Downward substitution – providing a new item as a substitute for excess demand of an old item – is one of the most common substitution practices. When substitution costs are below critical values, downward substitution provides additional benefit when combined with upward substitution (Lemma 4), but unlike upward substitution, no such guarantee can be made with respect to practicing downward substitution alone (Lemma 6).

Lemma 4 *If conditions (4) and (8) hold, then* $\mathcal{F} \succ \mathcal{U}$.

Similar to our analysis of $\mathcal{F}$ vs. $\mathcal{N}$, the above condition is only sufficient; there is no guarantee that $\mathcal{U}$ will dominate $\mathcal{F}$ even when the condition fails to hold.

Lemma 5 *When* $m + p_1 > 0$*, there is* <u>*no*</u> *condition on* γ *that guarantees* $\mathcal{U} \succ \mathcal{F}$.

Proof : The proof is identical to the proof of Lemma 3 because $\mathcal{U}$ and $\mathcal{N}$ have the same cost (as the recursions that define their inventory levels are the same). ■

Lemma 6 *When* $m + p_1 > 0$*, there is* <u>*no*</u> *condition on* γ *that guarantees* $\mathcal{N} \succ \mathcal{D}$. *There is* <u>*no*</u> *condition on* γ *that guarantees* $\mathcal{D} \succ \mathcal{N}$.

Proof : This is proved using two examples. The first statement is true due to the example in the proof of Lemma 3 (because the costs of $\mathcal{D}$ and $\mathcal{F}$ are the same). For the second statement, we give an example where $\mathcal{N}$ is less costly than $\mathcal{D}$ for any $\gamma \geq 0$: Suppose there are no old items in stock and the inventory state of both $\mathcal{D}$ and $\mathcal{N}$ is $(S, 0)$ initially. Consider a sample path of demand where the sequence $\{(0, S), (0, 0), (S, 0)\}$ repeats every three periods. Corresponding inventory levels are $\{(S, 0), (0, S), (S, 0)\}$ for $\mathcal{N}$ and $\{(S, 0), (S, 0), (0, S)\}$ for $\mathcal{D}$. The total costs of $\mathcal{N}$ and $\mathcal{D}$ in these cycles are $S(h + m + p_1)$ and $S(\gamma + h + m + p_2)$, respectively. ■

The result is again counter-intuitive: The form of substitution that is commonly used in practice may not be beneficial, even if the substitution cost is zero. On the other hand, no matter how high the substitution cost is, downward substitution may be better than not substituting. Again, this counter intuitive behavior is partly due to the use of TIS, which constrains reordering behavior.

4.1.5 Value of upward vs. downward substitution under TIS: $\mathcal{U}$ vs. $\mathcal{D}$

Given our results so far, it is not surprising that there is no dominance relation between $\mathcal{D}$ and $\mathcal{U}$.

Lemma 7 *There is* <u>*no*</u> *condition on* γ *that guarantees* $\mathcal{D} \succ \mathcal{U}$. *When* $m + p_1 > 0$*, there is* <u>*no*</u> *condition on* γ *that guarantees* $\mathcal{U} \succ \mathcal{D}$.

Proof : The examples proving Lemma 6 are sufficient, as $\mathcal{N}$ and $\mathcal{U}$ have identical inventory recursions. ■

Note, surprisingly, that the above result is independent of the upward substitution cost α.

4.2 Cost Comparison of Substitution Policies under NIS

In this section we continue our analysis, comparing substitution policies under the NIS policy. We begin again with a preliminary result showing that there are fewer old items in stock if downward substitution is practiced (Proposition 4), before moving to pairwise comparison of the policies (Lemmas 8 - 12). Since $X_2^n = S$ for all n under NIS, we track Y_I^n, denoting the stock level of old items at the beginning of period n (i.e. X_1^n) under policy I, for $I = \mathcal{F}, \mathcal{N}, \mathcal{D}, \mathcal{U}$.

Proposition 4 *For a given S, $Y_{\mathcal{F}}^n = Y_{\mathcal{D}}^n \leq Y_{\mathcal{U}}^n = Y_{\mathcal{N}}^n$ for all n along any sample path.*

Proof : Due to the recursions that define inventory levels, we know $Y_{\mathcal{F}}^n = Y_{\mathcal{D}}^n$ and $Y_{\mathcal{U}}^n = Y_{\mathcal{N}}^n$. If there is no downward substitution in period n, then $Y_{\mathcal{F}}^n$ will be equal to $Y_{\mathcal{N}}^n$. In case of any downward substitution, there will be fewer old items carried to the next period under $\mathcal{F}$ (or $\mathcal{D}$), i.e. $Y_{\mathcal{F}}^n < Y_{\mathcal{N}}^n$. Therefore $Y_{\mathcal{F}}^n = Y_{\mathcal{D}}^n \leq Y_{\mathcal{U}}^n = Y_{\mathcal{N}}^n$. ■

The result is intuitive: under NIS, no substitution or only upward substitution result in higher levels of old item inventory, since inventory of new is never used to satisfy excess demand of old. Next, we look at the effect of upward substitution. Just as in the case for TIS, upward substitution is profitable if and only if the upward substitution cost, α, is less than $m + p_2$, which is the cost if upward substitution is possible but does not take place in a given period.

Lemma 8 *On every sample path, $\mathcal{F} \succ \mathcal{D}$ and $\mathcal{U} \succ \mathcal{N}$, if and only if condition (4) holds. The same dominance relations hold when not all the customers accept upward substitution (i.e. $0 \leq \pi_U \leq 1$).*

Proof : We provide the proof for $\mathcal{F}$ vs. $\mathcal{D}$ only; the same idea applies to $\mathcal{U}$ vs. $\mathcal{N}$. Due to inventory recursions, we know $Y_{\mathcal{F}}^n = Y_{\mathcal{D}}^n$ for all n. $\mathcal{F}$ and $\mathcal{D}$ have the same cost for every period except for the periods where $\mathcal{F}$ uses upward substitution. In any period n, if $D_1^n < Y_{\mathcal{F}}^n$ and $D_2^n > S$, then upward substitution takes place in the amount of $us = \min\{L,\ K\}$ where $L = D_2^n - S$ and $K = Y_{\mathcal{F}}^n - D_1^n$. Let $K = Y_{\mathcal{F}}^n - D_1^n$. Let C_I^n denote the cost of policy I in period n, for $I = \mathcal{F}, \mathcal{D}, \mathcal{U}, \mathcal{N}$. Then, we have $C_{\mathcal{D}}^n = mK + p_2 L$, $C_{\mathcal{F}}^n = m(K - us) + p_2(L - us) + \alpha\ us$ and $C_{\mathcal{D}}^n - C_{\mathcal{F}}^n = (m + p_2 - \alpha)us$. Therefore $\mathcal{D}$ is costlier than $\mathcal{F}$ if and only if $\alpha < m + p_2$. ■

Next, we look at the (additional) value of downward substitution. Unlike TIS, the decision to use an excess new item in one period does *not* affect the number of new items in the next period under NIS. Therefore, the effect of substitution is easier to interpret.

Consider two consecutive periods; in the first downward substitution occurs. For each new item substituted for old, there is a cost γ. In the same period, the lost sales cost p_1 is saved, and one fewer item is carried to the next period, saving h. The greatest benefit that can be accrued from the downward substitution occurs if the demand for old items in the next period is low – one fewer item outdates, saving m. Thus, when

$$\gamma > m + h + p_1 \tag{9}$$

holds, downward substitution is *never* beneficial.

Now suppose, $\alpha > m + p_2$ holds (that is upward substitution is unprofitable, by Lemma 8). The worst thing that could happen after substitution is that an additional old item might be needed later, incurring a cost of p_1. If in this case downward substitution is profitable, then it is always beneficial. This holds under the following condition:

$$\gamma < h. \tag{10}$$

Finally, if upward substitution is viable (i.e. $\alpha < m + p_2$), in addition to the previous case (10), it may happen that in the next period an upward substitution is desired but cannot be enacted because of our previous period's downward substitution. This will save a substitution cost, α, but cost p_2. If even in this case substitution is profitable, then downward substitution is always beneficial, i.e. provided:

$$\gamma < \min\{h, h + \alpha - (p_2 - p_1)\}. \tag{11}$$

Formal statement of these results follow. Proofs are available in the Appendix.

Lemma 9 *($\mathcal{F}$ vs. $\mathcal{N}$) If conditions (4) and (10) hold, then $\mathcal{F} \succ \mathcal{N}$. If condition (4) fails to hold but condition (9) holds, then $\mathcal{N} \succ \mathcal{F}$.*

Lemma 10 *($\mathcal{F}$ vs. $\mathcal{U}$) If conditions (4) and (11) hold, then $\mathcal{F} \succ \mathcal{U}$. If conditions (4) and (9) hold, then $\mathcal{U} \succ \mathcal{F}$.*

Lemma 11 *($\mathcal{D}$ vs. $\mathcal{N}$) If condition (10) holds, then $\mathcal{D} \succ \mathcal{N}$. If condition (9) holds, then $\mathcal{N} \succ \mathcal{D}$.*

Lemma 12 *($\mathcal{D}$ vs. $\mathcal{U}$) If conditions (4) and (9) hold, then $\mathcal{U} \succ \mathcal{D}$. If condition (4) fails and condition (10) holds, then $\mathcal{D} \succ \mathcal{U}$. If condition (4) holds, there is no condition that guarantees $\mathcal{D} \succ \mathcal{U}$.*

Dominance	Condition(s) under TIS	Condition(s) under NIS
$\mathcal{F} \succ \mathcal{D}, \mathcal{U} \succ \mathcal{N}$	$\alpha < m + p_2$	$\alpha < m + p_2$
$\mathcal{D} \succ \mathcal{F}, \mathcal{N} \succ \mathcal{U}$	$\alpha > m + p_2$	$\alpha > m + p_2$
$\mathcal{F} \succ \mathcal{N}$	$\alpha < m + p_2$, $\gamma < \min\{m + 2p_1 - p_2, m + p_1 - \alpha, \frac{h+p_1}{2}\}$	$\alpha < m + p_2$, $\gamma < h$
$\mathcal{N} \succ \mathcal{F}$	Does Not Exist	$\alpha > m + p_2$, $\gamma > m + h + p_1$
$\mathcal{F} \succ \mathcal{U}$	$\alpha < m + p_2$, $\gamma < \min\{m + 2p_1 - p_2, m + p_1 - \alpha, \frac{h+p_1}{2}\}$	$\alpha < m + p_2$, $\gamma < \min\{h, h + \alpha - (p_2 - p_1)\}$
$\mathcal{U} \succ \mathcal{F}$	Does Not Exist	$\alpha < m + p_2$, $\gamma > m + h + p_1$
$\mathcal{D} \succ \mathcal{N}$	Does Not Exist	$\gamma < h$
$\mathcal{N} \succ \mathcal{D}$	Does Not Exist	$\gamma > m + h + p_1$
$\mathcal{U} \succ \mathcal{D}$	Does Not Exist	$\alpha < m + p_2$, $\gamma > m + h + p_1$
$\mathcal{D} \succ \mathcal{U}$	Does Not Exist	$\alpha > m + p_2$, $\gamma < h$

Table 1: Sufficient conditions on substitution costs that ensure dominance relations between substitution policies when inventory is replenished using TIS or NIS.

4.3 Summary of Cost Comparison Results

Our results are summarized in Table 1. Under NIS and TIS, the same condition on α is necessary and sufficient for upward substitution to result in lower costs ($\alpha < m+p_2$). Downward substitution is different: The benefit of downward substitution under general demand scenarios often cannot be established for TIS, but intuitive conditions on α and γ do exist under NIS. For instance, under NIS supplying a new item for an old one ($\mathcal{D}$ versus $\mathcal{N}$) is beneficial as long as the substitution cost is lower than the holding cost. However, when TIS is used, $\mathcal{D}$ and $\mathcal{N}$ can dominate each other based on the demand patterns, regardless of the costs.

As we have general, almost sure results, many of the bounds on the substitution costs are not tight. For instance under NIS, when γ satisfies $h < \gamma < m+h+p_1$, no dominance relation between $\mathcal{D}$ and $\mathcal{N}$ is provided. We use computational experiments to explore the behavior of the policies in such ranges in Section 6.

5 Comparison of Freshness of Inventory

Our focus so far has been on the economic benefit of substitution. However, a substitution policy's overall benefit cannot be assessed unless service issues are also taken into account. In this section we show that downward substitution leads to fresher inventory, which may help the supplier in a number of ways: These items have not lost their value through aging, customers are more likely to prefer them, and having newer items in stock decreases the risk of the obsolescence. Note that in Section 4 we showed that this does *not* necessarily lead to *cost-wise* superiority.

We first define what we mean by freshness: We look at the expected age of goods in stock – the greater the average remaining lifetime, the fresher the goods. For NIS, pairwise comparison of substitution policies with respect to freshness immediately follows from our earlier results.

Lemma 13 *For the same S, average age of inventory is lower (i.e. goods are fresher) under $\mathcal{F}$ and $\mathcal{D}$ than $\mathcal{U}$ and $\mathcal{N}$ when inventory is replenished using NIS.*

Proof : Follows from Proposition 4 (there are less old items due to downward substitution). ■

When inventory is replenished using TIS, we compare the average age of goods in stock for different substitution policies by studying the expected amount of new items in stock. For an arbitrary period n, we know we can have $X_F^n \geq X_N^n$ or $X_F^n < X_N^n$ based on our previous results; we also know $X_F^n = X_D^n$ and $X_N^n = X_U^n$ (almost surely) for all n. Below, we show in a time average sense that the inventory level of new items is greater, i.e. items are fresher under $\mathcal{F}$. The definition of a *pair* is the same as in Section 4.1. Outside the *pairs* it is obvious that there are more newer items under $\mathcal{F}$. The proposition below analyzes the situation inside a *pair*. (See the Appendix for the proof.)

Proposition 5 *Suppose inventory is replenished using TIS. Let periods $n-1$ and n constitute a* pair*. Then, in period $n-1$, (i) $D_1^{n-1} + D_2^{n-1} < S$, (ii) if there is substitution, it must be downward substitution, and (iii) $D_1^{n-1} < S - X_N^{n-1}$, i.e. the demand for old items must be less than the number of old items under policy $\mathcal{N}$.*

Based on Proposition 5, we can show that - for a *pair* in periods $n-1$ and n - the difference $X_F^{n-1} - X_N^{n-1} > 0$ is no less than $X_N^n - X_F^n > 0$. This leads to our next result:

Lemma 14 *For the same S, goods are fresher under $\mathcal{F}$ and $\mathcal{D}$ than $\mathcal{N}$ and $\mathcal{U}$, when inventory is replenished using TIS.*

Downward substitution - commonly used in practice - is clearly superior in this respect: Under both replenishment policies, it leads to fresher inventory. Note that the above results are true only for a *given* S. When substitution policies use different order-up-to levels (e.g. the supplier can choose the order-up-to level that minimizes total expected costs for each substitution policy), comparative analysis of the policies with respect to freshness of inventory is not straightforward. We use numerical examples (Section 6) to gain insights on this.

6 Computational Results

Below we report a series of experiments used to explore the performance of different substitution policies for different parameters. We discuss properties of the cost functions of different substitution policies, under both TIS and NIS in Section 6.1. We then study service levels and freshness in Section 6.2, and policy comparisons under proportional acceptance of substitute goods in Section 6.3. Throughout we focus on comparing the performance of different *substitution* policies; comparative analysis of the different inventory policies (TIS and NIS) is deferred to future work (Deniz et. al 2004). For given problem instances, we simulated the inventory system for 400,000 periods to compute average costs. The demand for old and new items are independent in all the examples.

6.1 Cost Properties of TIS and NIS

We first provide examples to discuss the structural properties of NIS and TIS.

Examples 1A&1B (Non-convexity/Multi-modality). In these examples common costs are h=1, p_1=3, p_2=9, α=7, γ=6 and the demand for new and old items are both discrete uniform distributed between 0 and 25. We vary m; $m = 5$ in 1A and $m = 2$ in 1B. Note that condition $\alpha < m + p_2$ is satisfied in both examples; therefore $\mathcal{F}$ has lower cost than $\mathcal{D}$, and $\mathcal{U}$ has lower cost than $\mathcal{N}$, under both TIS and NIS. However γ does not satisfy any of the sufficient conditions in Table 1 – there is no other TIS dominance relation among the substitution policies holding for all S, as we see in Figures 1 and 2 for 1A and 1B, respectively.

We also see that the expected cost function under TIS for policies $\mathcal{F}$ and $\mathcal{D}$ are non-convex in 1A; in 1B these functions become non-convex in the NIS case, and multi-modal in the TIS case. In addition, while $\mathcal{N}$ and $\mathcal{U}$ have higher optimal order-up-to levels compared to $\mathcal{F}$ and $\mathcal{D}$ under TIS, the opposite is true under NIS. Reducing the outdating cost m lowers the overall costs of all policies, but leaves the optimal S values relatively unchanged.

Overall, in both examples policies that do *not* downward substitute, i.e. $\mathcal{U}$ and $\mathcal{N}$ perform best, owing to the high substitution costs and $p_2 >> p_1$. The optimal costs of these policies are lower under NIS than TIS, although for any given S the expected cost of a substitution policy under NIS is not necessarily lower than that of TIS.

Example-2 (Making downward substitution more attractive). We now make old-item demand greater in magnitude than new item demand, and reduce substitution costs: Specifically, we choose h=1, m=2, p_1=3, p_2=9, α=3, γ=2. Again $\alpha < m + p_2$ is satisfied, while γ violates the

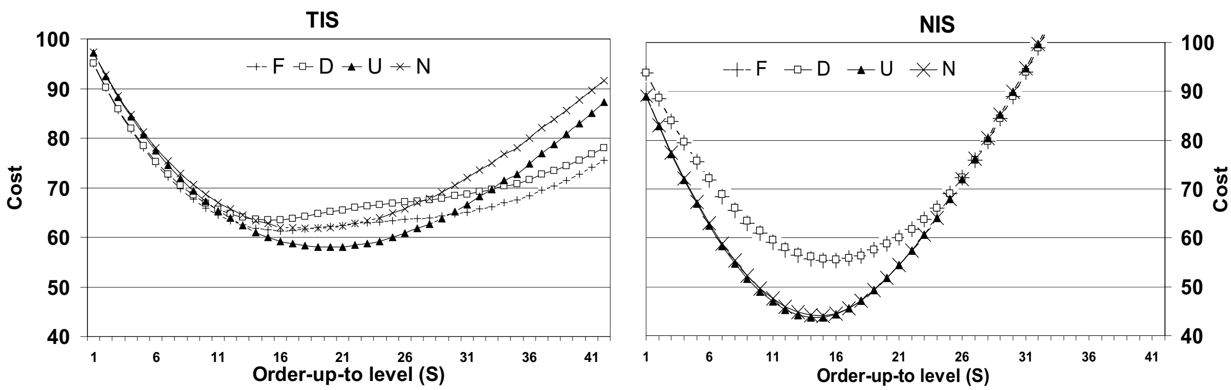

Figure 1: Expected costs of substitution policies in Example-1A.

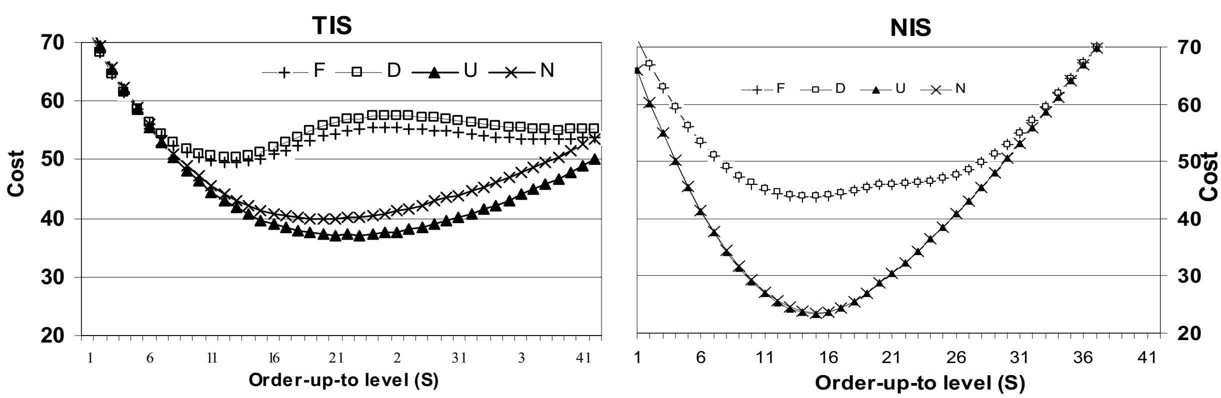

Figure 2: Expected costs of substitution policies in Example-1B.

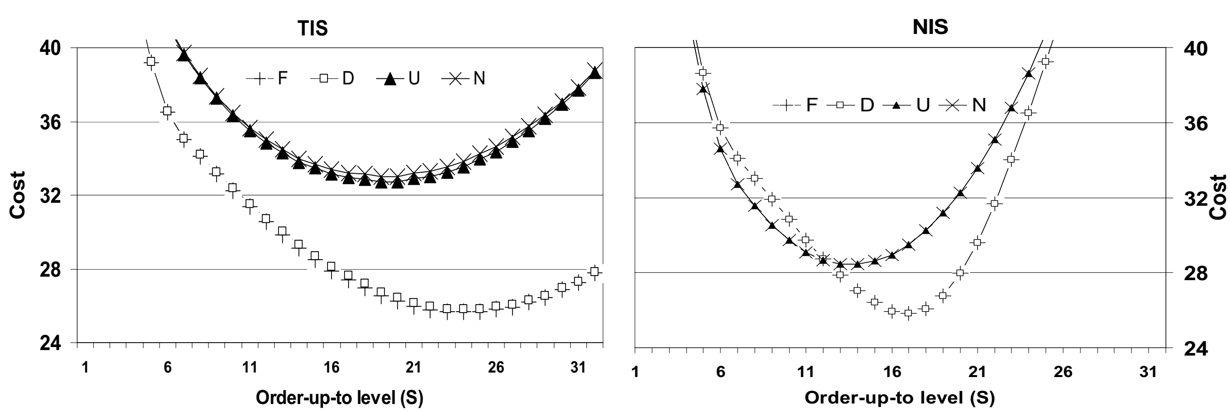

Figure 3: Expected costs of substitution policies in Example-2.

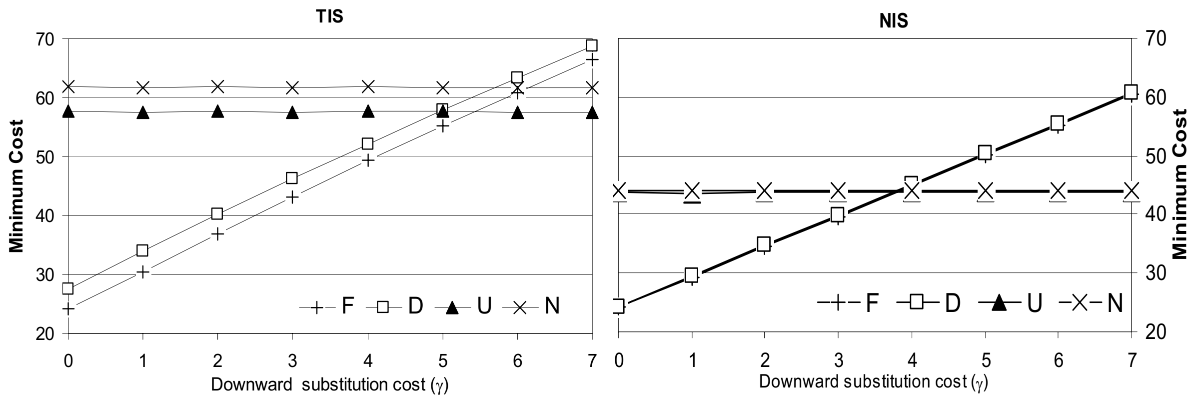

Figure 4: Minimum expected cost of policies as a function of downward substitution cost (γ) in Example-3.

sufficiency conditions in Table 1. The demand for new items is discrete uniform between 0 and 6, demand for old items is discrete uniform between 0 and 25. These changes make downward substitution more attractive – it is both less costly, and can occur more frequently.

The expected cost functions for Example-2 are presented in Figure 3. As we would expect, the lowest cost is achieved by policies that downward substitute, $\mathcal{F}$ and $\mathcal{D}$. In addition, given the increase in downward substitution, TIS now provides both lower cost than NIS, and greater robustness to choice of S. Our observations are consistent with the literature; TIS is effective when there is only one demand stream and demand is fulfilled in a FIFO fashion (i.e. downward substitution is practiced).

Example-3 (Effect of γ). We fix parameters h=1, m=5, p_1=3, p_2=9, α=7, and vary γ between zero and seven. The demand for both new and old items is discrete uniform distributed between 0 and 25. Note that upward substitution is beneficial since $\alpha < m + p_2$. Figure 4 shows the minimum expected cost of each policy as a function of γ; as γ increases, the cost of policies $\mathcal{F}$ and $\mathcal{D}$ increase almost linearly under both TIS and NIS.

From Table 1, under both NIS and TIS, $\gamma < 1$ guarantees $\mathcal{F} \succ \mathcal{N}$ and $\mathcal{F} \succ \mathcal{U}$. Under NIS, $\gamma > 9$ guarantees $\mathcal{N} \succ \mathcal{F}$ and $\mathcal{U} \succ \mathcal{F}$. Example-3 illustrates the behavior of substitution policies in the parameter range where our sufficient conditions do not hold. In this example, we observe a *critical* value of $\gamma = 4$ under NIS; for higher values of γ, $\mathcal{U}$ and $\mathcal{N}$ have lower expected costs. This critical value is between five and six for TIS. In either case, these values are in the mid-to-high portion of the parameter ranges identified in our sufficient conditions. Thus, while our analytical conditions of policies do not cover the entire γ range for dominance relations, they can provide guidance, especially when γ is closer to one sufficient condition than the other.

In all of the previous examples, when NIS is used to replenish inventory there is almost no

Policy	Type of demand	% lost		% substituted		Average Freshness	
		TIS	NIS	TIS	NIS	TIS	NIS
$\mathcal{F}$	new	7.0%	4.6%	2.3%	0.3%	1.85	1.82
	old	41.0%	34.6%	42.6%	35.8%		
$\mathcal{D}$	new	10.2%	5.0%	0.0%	0.0%	1.85	1.82
	old	44.9%	34.4%	40.4%	35.8%		
$\mathcal{U}$	new	13.9%	3.8%	3.1%	0.4%	1.70	1.71
	old	55.4%	50.9%	0.0%	0.0%		
$\mathcal{N}$	new	17.7%	4.1%	0.0%	0.0%	1.70	1.71
	old	57.2%	50.5%	0.0%	0.0%		

Table 2: Service levels and inventory age averaged across all experiments.

additional benefit to using upward substitution (over no substitution), but when TIS is used this difference is pronounced. This is due to the fact that *very little* upward substitution is taking place under NIS as compared to TIS, a fact borne out in the next section. This may be due to the fact that the optimal base-stock policy of NIS accounts directly for new item demands, reducing the need for upward substitution.

6.2 Service Levels and Freshness of Inventory

Next we study the policies in terms of service levels, conducting 32 experiments with the following cost parameters: $h = 1$, $m \in \{2, 5\}$, $p_1 \in \{1, 3\}$, $p_2 \in \{4, 9\}$, $\alpha \in \{3, 7\}$ and $\gamma \in \{2, 6\}$. Demand for new and old items is independent, and both are discrete uniform distributed between 0 and 25. For each cost parameter and demand distribution, we first compute the optimal order-up-to level S for each substitution and reordering policy. Then, using this order-up-to level, we compute average performance measures via simulation.

We study two aspects of service level: (i) the percentage of demand lost and (ii) the percentage of demand satisfied via substitution. Table 2 summarizes our results; the figures represent the averages across all 32 instances. Employing NIS as the replenishment policy increases the service level and decreases substitution: NIS not only tends to keep more inventory, but also better keeps the appropriate inventory of new items, reducing substitution. This effect is significant because the demand of old and new are both relatively high in this experiment.

In the same experiment, we also looked at freshness of inventory. We know that policies that use downward substitution have fresher inventory for a given S; however, when policies use different order-up-to levels (which is the case in our experiment), this result may no longer hold. We observed that $\mathcal{F}$ and $\mathcal{D}$ have fresher goods compared to $\mathcal{U}$ and $\mathcal{N}$ in 31 out of 32 of the experiments, when inventory is replenished by TIS. When NIS is used, this statistic falls to 29 out of 32. However, downward substitution is only slightly worse in these four instances; the average age of inventory of $\mathcal{F}$ and $\mathcal{D}$ is within 1% of $\mathcal{U}$ and $\mathcal{N}$. Average freshness numbers are also provided in Table 2.

6.3 Effect of customer behavior: Proportional acceptance of substitutes

Although our model is general enough to represent customer behavior where only a percentage of customers accept a substitute product, our analysis focused on four specific policies in which all customers accept substitution (either upward or downward). Here we present examples for which $0 \leq \pi_D, \pi_U \leq 1$; we call this general case the "proportional acceptance" model.

In all the examples, the demand is discrete uniform distributed between 0 and 25 for both new and old items. The examples below are analyzed for the case when TIS is used in replenishment. Our observations are similar for NIS, and hence are omitted.

Example-4 (Proportional Downward acceptance). In this example we use $h = 1$, $m = 2$, $p_1 = 3$, $p_2 = 4$, $\gamma = 6$, $\alpha = 3$. We study the effect of proportional acceptance of downward substitution by varying π_D, assuming $\pi_U = 1$ in this case. Thus, $\pi_D = 0$ represents policy $\mathcal{U}$ and $\pi_D = 1$ represents policy $\mathcal{F}$. The total cost as a function of π_D is presented in Figure 5. This graph shows how different components of the total cost change as we move from policy $\mathcal{U}$ to $\mathcal{F}$ as π_D increases.

In this particular example, the expected total cost increases almost linearly with π_D. As π_D increases, the amount of downward substitution increases (hence so too does its cost component) while the lost sales cost of old items decrease. Downward substitution under TIS increases the inventory turnover for new items; as the amount of new items used to satisfy the demand for old increases, TIS leads to higher number of newer items being replenished. Therefore, as π_D increases, the penalty for lost sales of new items also decreases. In this example, only a slight increase and decrease are observed in holding and outdating costs, respectively.

Example-5 (Proportional Upward acceptance). The costs are the same as in Example-1A. Here we look at the effect of acceptance proportion for upward substitution (π_U) and observe the changes as we move from policy $\mathcal{D}$ to $\mathcal{F}$. As we see in Figure 6, the effect of π_U on the costs is minimal; the total cost decreases slightly as π_U increases. This is not surprising because the difference between the lowest cost of $\mathcal{D}$ and $\mathcal{F}$ in Example-1A is very small. (This holds true for all the examples presented in Figures 1-3). There are examples where this difference may be slightly more (as in Example -4), but our main observation does not change: There is not a systematic effect of π_U on the cost components except the upward substitution cost. Thus in general results

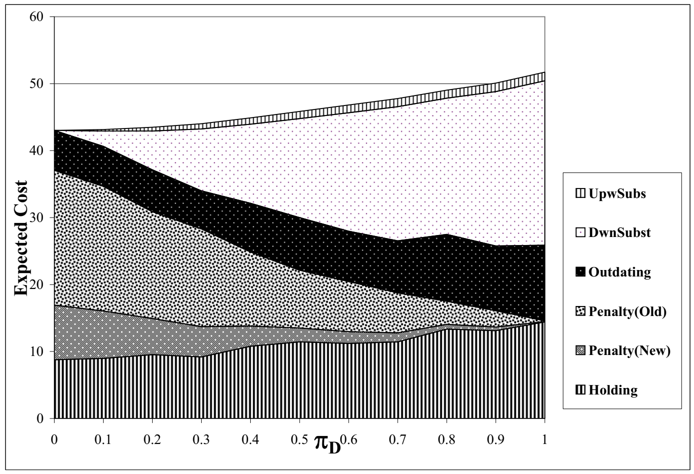

Figure 5: The effect of proportional acceptance of downward substitution on expected total cost in Example-4.

are more sensitive to π_D.

6.4 Summary of Computational Results

Based on the results of the numerical study presented above we conclude that downward substitution is an important lever for improving service levels and freshness of inventory; furthermore downward substitution has a more significant effect on the system performance than upward substitution (the value of π_D is more important than π_U). In contrast to what has appeared in the bulk of the literature, we see examples where NIS appears to be a more suitable replenishment policy with respect to costs, service levels and inventory freshness, especially when there is significant demand for new items and substitution costs are appreciable. We are also able to segment substitution policies by cost: The relative differences in expected costs of substitution policies are highest when we compare policies that use upward substitution to ones utilizing downward substitution. Hence, a supplier can significantly benefit from using $\mathcal{N}$ or $\mathcal{U}$ over $\mathcal{F}$ or $\mathcal{D}$, and vice versa.

7 Conclusion and Future Research Direction

In this study, we formalize and compare four different fulfillment policies for perishable goods, Full-Substitution ($\mathcal{F}$), Upward-Substitution ($\mathcal{U}$), Downward-Substitution ($\mathcal{D}$), and No-Substitution ($\mathcal{N}$) under two different base-stock type inventory replenishment policies, TIS and NIS. We show that substitution may or may not be beneficial with respect to operational costs, and provide conditions on cost parameters that characterize the regions in which different substitution strategies are most profitable, almost surely. We likewise show that downward substitution policies are always beneficial

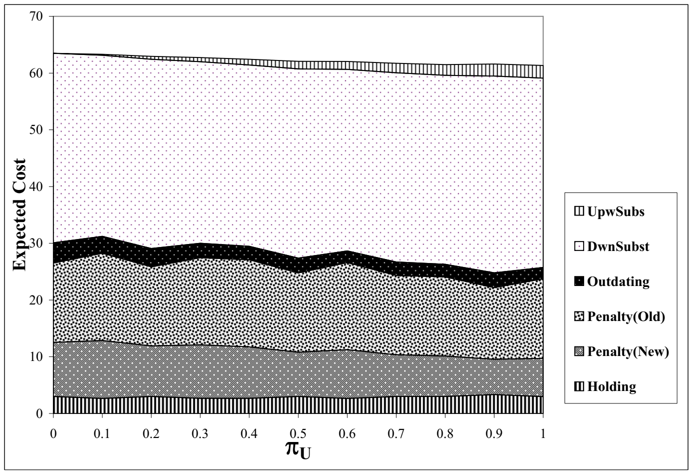

Figure 6: The effect of proportional acceptance of upward substitution on expected total cost in Example-5.

with respect to average freshness of products.

In light of our analysis, we can consider the substitution strategies of some of the examples in the introduction. Blood services utilize full substitution, owing to the high penalty costs (p_1 and p_2) of leaving customers (patients) unsatisfied, which would make equations (4), (8) and (11) true. Therefore, full substitution is justified under both NIS and TIS for blood banks. Furthermore, full substitution has the obvious advantage of leading to fresher inventory on the average - which is crucial in this setting. Similarly, fresh produce suppliers benefit from full substitution due to high lost sale costs and the low cost of substitution.

On the other hand, computer chip manufacturers typically cannot use upward substitution – if a customer needs a fast chip they often will not be satisfied with a slow one (i.e. α is high). They do practice downward substitution after fusing the chip – to control demand diversion (reducing γ). Fashion retailers do not have such a control; demand diversion may be a significant problem once substitutable products are made available. So to eliminate diversion, retailers may choose not to co-locate goods of different ages, offering no substitution. Finally, what about the local bagel shop? They put a high premium on serving customers the freshest bagels – or in our terms they have high p_2 and α costs (and only one demand stream – we are ignoring those who seek discounted day-old bagels). Thus upward substitution makes sense.

Our focus in this paper was on the effect of substitution given practical inventory policies. There are a number of possible future directions for research. A very natural one is studying different replenishment policies and comparing them to optimal; this is ongoing research (Deniz et al. 2004). Another challenge lies in the case where a supplier does not have to commit to a single substitution policy. Although such a policy is more difficult to implement (and to analyze), it provides more flexibility in satisfying the demand. Hence, investigation of static versus dynamic substitution decisions in fulfilling the demand warrants further attention. Likewise, evaluating substitution policies for a product that has a lifetime of more than two periods is intriguing; even for three-period lifetime case, substitution from new to old (in presence of products with medium age) may not be justified. In a related direction, using a multiple period lifetime with zero null demand for the first several periods of lifetime offers the possibility of modeling positive lead-times for perishable goods. Finally, adding customer behavior to the existing model (e.g a *probabilistic* acceptance model) or developing new models with customer gaming issues are also promising lines of future research.

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Appendix [3]

The proofs of the analytical results are given below.

A.1. Cost Comparison Results under TIS

To prove the results on TIS, we use the definition of a *pair* introduced in Section 4.1.1. C_I^n is the cost of substitution policy I in period n including holding, spoilage and penalty costs, X_I^n and Y_I^n are the amount of new and old items in stock, respectively, for policy I, $I = \mathcal{F}, \mathcal{N}, \mathcal{U}, \mathcal{D}$. We define a period as E (for equal) if $X_N^n = X_F^n$, and we call the periods between two E periods a *cycle* and J is the number of *pairs* in a cycle. If an E period follows another one, then it is called a *trivial cycle*. A *non-trivial cycle*, starts with an E period in which downward substitution takes place and ends with another E.

Proof of Proposition 2

Proof : Assume period n is an F period with $X_F^n = X_N^n + \Delta$ for some $\Delta > 0$. If demand for new items in this period is greater than or equal to the number of new items under $\mathcal{N}$ (i.e. $D_2^n \geq X_N^n$) then $\mathcal{N}$ will be at the inventory state $(S, 0)$ in the next period which implies that the period $n+1$ is not an F period. Therefore D_2^n must be less than X_N^n. That is, $\mathcal{F}$ has more than Δ unsold new items, which are available for substitution. If there is no need for the unsold new items ($D_1^n \leq S - X_N^n - \Delta$) then all of these items would be carried over to the next period. However this would make the period $n+1$ an N period because under policy $\mathcal{N}$, less new items would be carried. Hence if there are two consecutive F periods, then there must be downward substitution in the first one. ■

Proof of Proposition 3

Proof : Continuing from the proof of Proposition 2 let $D_1^n = S - X_N^n - \Delta + k_1$ and $D_2^n = X_N^n - k_2$ where $k_1, k_2 > 0$. Therefore $X_N^{n+1} = S - k_2$ and $X_F^{n+1} = S - (\Delta + k_2 - k_1)^+$ as $\mathcal{F}$ has $\Delta + k_2$ to substitute and k_1 is needed. Since period $n+1$ is an F period k_2 must be greater than $(\Delta + k_2 - k_1)^+$ thus $k_1 > \Delta$. (Note that $k_1 > \Delta$ implies demand for old items in period n must be more than old item inventory under $\mathcal{N}$, i.e. $D_1^n > S - X_N^n$.) The substitution amount is $\tilde{\Delta}_n = \min\{\Delta + k_2, k_1\}$ and $X_F^{n+1} - X_N^{n+1} = k_2 - (\Delta + k_2 - k_1)^+$ hence $X_F^{n+1} - X_N^{n+1} = \tilde{\Delta}_n - \Delta$ (i.e. the substitution amount minus Δ). ■

[3]Note to reviewers: Appendix is prepared to be an Online Addendum for the paper.

Proof of Lemma 2

We divide the proof of Lemma 2 into two Propositions.

Proposition 6 *If conditions (4) and (7) hold, then $\mathcal{F}$ has lower cost than $\mathcal{N}$ outside the* pairs.

Proof : We look at a series of F periods starting or ending either with an E period or a *pair*. Let period n be any F period in the series with $\Delta_n = X_F^n - X_N^n$. As was pointed out in the proof of Proposition 3 to have another F in period $n+1$, $D_2^n = X_N^n - k_2 < X_N^n$ and $D_1^n = S - X_N^n + k_1 > S - X_N^n$. Then $X_F^{n+1} = S - (k_2 - k_1)^+$ and $X_N^{n+1} = S - k_2$ (i.e. $\Delta_{n+1} = \min(k_1, k_2)$). Also:

$$C_N^n = hk_2 + p_1k_1, \quad C_F^n = h(k_2 - k_1)^+ + p_1(k_1 - k_2)^+ + \gamma(\Delta_n + \Delta_{n+1}),$$

$$C_N^n - C_F^n = (h + p_1)\Delta_{n+1} - \gamma(\Delta_n + \Delta_{n+1}).$$

Let the series start at period 1 and end at period $M-1$ (i.e. $\Delta_i > 0 \;\; \forall \; i = 1, ..., M-1$). In periods 0 and M there are 2 possibilities each:

- If period 0 is an E (i.e. $\Delta_0 = 0$), then:

$$C_N^0 - C_F^0 = (h + p_1)\Delta_1 - \gamma\Delta_1.$$

- If period 0 is an N (i.e. end of a *pair*), then: In this case $C_N^0 - C_F^0$ would be an amount carried over from the previous *pair* (the one that just ended in period 0). In other words an amount of $(h + p_1)\Delta_1 - \gamma\Delta_1$, which is saved from the previous *pair* (this is explained later in Proposition 7), will be used as $C_N^0 - C_F^0$. Thus,

$$C_N^0 - C_F^0 = (h + p_1)\Delta_1 - \gamma\Delta_1. \tag{12}$$

- If period M is an E (i.e. $\Delta_M = 0$) then:

$$C_N^{M-1} - C_F^{M-1} \geq -\gamma\Delta_{M-1}. \tag{13}$$

The proof of (13) is as follows; if period $M-1$ is F there are six demand scenarios that lead to an E period immediately after. We analyze these possible cases:

Case 1. $D_2^{M-1} = X_N^{M-1} - k_2$ and $D_1^{M-1} = S - X_N^{M-1} + k_1$ with $k_1, k_2 \geq 0$. These demands result in $X_F^M = S - (k_2 - k_1)^+$ and $X_N^M = S - k_2$. $\mathcal{F}$ is short of $\Delta_{M-1} + k_1$ old items while it has $\Delta_{M-1} + k_2$ unsold new items. Thus minimum of these is the downward substitution amount. Then we have $C_N^{M-1} = hk_2 + p_1k_1$ and $C_F^{M-1} = h(k_2 - k_1)^+ + p_1(k_1 - k_2)^+ + \gamma\min(\Delta_{M-1} + k_2, \Delta_{M-1} + k_1)$. In order for period M be an E, k_1 or k_2 (or both) must be 0; hence $C_N^{M-1} - C_F^{M-1} = -\gamma\Delta_{M-1}$.

Case 2. $D_2^{M-1} = X_N^{M-1} + k_2$ and $D_1 = S - X_N^{M-1} - \Delta_{M-1} + k_1$ with $0 \leq k_i < \Delta_{M-1}$ for $i \in \{1, 2\}$ so that $X_F^M = S - (\Delta_{M-1} - k_2 - k_1)^+$ and $X_N^M = S$. The downward substitution amount is $\Delta_{M-1} - k_2$ because in order for period M be an E, $k_1 + k_2 \geq \Delta_{M-1}$ must hold. Then we have $C_N^{M-1} = p_2 k_2 + m(\Delta_{M-1} - k_1)$ and $C_F^{M-1} = p_1(k_2 + k_1 - \Delta_{M-1}) + \gamma(\Delta_{M-1} - k_2)$; hence $C_N^{M-1} - C_F^{M-1} = (p_2 - p_1)k_2 + p_1(\Delta_{M-1} - k_1) + m(\Delta_{M-1} - k_1) - \gamma(\Delta_{M-1} - k_2) > -\gamma\Delta_{M-1}$.

Case 3. $D_2^{M-1} = X_N^{M-1} + k_2$ and $D_1^{M-1} = S - X_N^{M-1} + k_1$ with $k_1, k_2 \geq 0$ result in $X_F^M = X_N^M = S$. $\mathcal{F}$ is short of $\Delta_{M-1} + k_1$ old items while it has $\Delta_{M-1} - k_2$ unsold new items. Thus the downward substitution amount is $\Delta_{M-1} - k_2$. Then $C_N^{M-1} = p_2 k_2 + p_1 k_1$ and $C_F^{M-1} = p_1(k_1 + k_2) + \gamma(\Delta_{M-1} - k_2)$; hence $C_N^{M-1} - C_F^{M-1} = (p_2 - p_1)k_2 - \gamma(\Delta_{M-1} - k_2) > -\gamma\Delta_{M-1}$.

Case 4. $D_2^{M-1} = X_N^{M-1} + \Delta_{M-1} + k_2$ and $D_1^{M-1} = S - X_N^{M-1} - \Delta_{M-1} - k_1$ with $k_1, k_2 \geq 0$. $\mathcal{F}$ is short of k_2 old items while it has k_1 unsold old items. Then, there is an upward substitution in the amount of $\min(k_2, k_1)$ resulting in $X_F^M = X_N^M = S$. Thus $C_N^{M-1} = p_2(\Delta_{M-1} + k_2) + m(\Delta_{M-1} + k_1)$ and $C_F^{M-1} = p_2(k_2 - k_1)^+ + \alpha \min(k_1, k_2) + m(k_1 - k_2)^+$. If $k_1 < k_2$, then $C_N^{M-1} - C_F^{M-1} = (m + p_2)(\Delta_{M-1} + k_1) - \alpha k_1 > -\gamma\Delta_{M-1}$; otherwise $C_N^{M-1} - C_F^{M-1} = (m + p_2)(\Delta_{M-1} + k_2) - \alpha k_2 > -\gamma\Delta_{M-1}$ as $\alpha < m + p_2$.

Case 5. $D_2^{M-1} = X_N^{M-1} + \Delta_{M-1} + k_2$ and $D_1^{M-1} = S - X_N^{M-1} - \Delta_{M-1} + k_1$ with $\Delta_{M-1} > k_1 \geq 0,\ k_2 \geq 0$ result in $X_F^M = X_N^M = S$. No substitution takes place. Thus, $C_N^{M-1} = p_2(\Delta_{M-1} + k_2) + m(\Delta_{M-1} - k_1)$ and $C_F^{M-1} = p_2 k_2 + p_1 k_1$. Hence $C_N^{M-1} - C_F^{M-1} = m(\Delta_{M-1} - k_1) + p_2\Delta_{M-1} - p_1 k_1 > -\gamma\Delta_{M-1}$.

Case 6. $D_2^{M-1} = X_N^{M-1} + \Delta_{M-1} + k_2$ and $D_1^{M-1} = S - X_N^{M-1} + k_1$ with $k_1, k_2 \geq 0$ result in $X_F^M = X_N^M = S$. There is no substitution, and $C_N^{M-1} = p_2(\Delta_{M-1} + k_2) + p_1 k_1$ and $C_F^{M-1} = p_2 k_2 + p_1(\Delta_{M-1} + k_1)$; hence $C_N^{M-1} - C_F^{M-1} = (p_2 - p_1)\Delta_{M-1} > -\gamma\Delta_{M-1}$.

Therefore (13) is correct.

- If in period M a *pair* starts (i.e. the period M and $M+1$ are F and N respectively) then:

$$C_N^{M-1} - C_F^{M-1} = (h + p_1)\Delta_M - \gamma(\Delta_{M-1} + \Delta_M).$$

We reduce this amount by $(h + p_1 - \gamma)\Delta_M$; we save this $(h + p_1 - \gamma)\Delta_M$ as the "starting cost" of $\mathcal{N}$ for the forthcoming *pair* in period M and add it to the cost of $\mathcal{N}$ for the cost comparison in the *pairs* (Proposition 7). Therefore in this case:

$$C_N^{M-1} - C_F^{M-1} = -\gamma\Delta_{M-1}. \tag{14}$$

Thus we have the following cost structure:

$$
\begin{aligned}
C_N^0 - C_F^0 &= (h+p_1)\Delta_1 - \gamma\Delta_1, \\
C_N^1 - C_F^1 &= (h+p_1)\Delta_2 - \gamma(\Delta_1 + \Delta_2), \\
C_N^2 - C_F^2 &= (h+p_1)\Delta_3 - \gamma(\Delta_2 + \Delta_3), \\
&\dots \\
C_N^{M-2} - C_F^{M-2} &= (h+p_1)\Delta_{M-1} - \gamma(\Delta_{M-2} + \Delta_{M-1}), \\
C_N^{M-1} - C_F^{M-1} &\geq -\gamma\Delta_{M-1}.
\end{aligned}
$$

Hence,

$$\sum_{i=0}^{M-1} C_N^i - C_F^i \geq (h + p_1 - 2\gamma)\bar{\Delta}, \tag{15}$$

where

$$\bar{\Delta} = \sum_{i=1}^{M-1} \Delta_i.$$

Then left term in (15) is positive (i.e. $\mathcal{F}$ is less costly) as $\gamma < (h+p_1)/2$ is assumed. ■

Proposition 7 *If conditions (5) and (6) hold, then $\mathcal{F}$ has lower cost than $\mathcal{N}$ inside the* pairs.

Proof : Both $\mathcal{N}$ and $\mathcal{F}$ start at an E state at the beginning. Until the period in which downward substitution occurs both have same inventory state, they are E's. The period after the substitution takes place becomes an F period (i.e. $\mathcal{F}$ has more new items than $\mathcal{N}$). Due to this substitution $\mathcal{F}$ incurs a substitution cost γ but saves $h + p_1$ as a penalty cost is avoided by fulfilling a demand for old product and that item is sold instead of being held in inventory. Therefore, $\mathcal{F}$ incurs $h + p_1 - \gamma$ less cost than $\mathcal{N}$ per substitute item. There might be a number of F periods until an N period; using the accounting explained preceding equation (14), we say that at the beginning of a *pair*, $\mathcal{F}$ has a cost lower by an amount equal to the starting cost ($h + p_1 - \gamma$ per substitution).

Let the first *pair* start in period M with $X_F^M X_N^M + \Delta_M$ and $X_N^{M+1} = X_F^{M+1} + \Delta$. Intuitively this means under $\mathcal{F}$ more items (in the amount of Δ) have been aged and Δ less items perished at the end of period M resulting a cost difference of $(m - h)\Delta$ in favor of $\mathcal{F}$. (If these items did not perish then the Δ items would not have aged, they would have been substituted.)

We first prove that "the first part of the *pair* cost" until period $M+1$ is at least $(m+p_1-\gamma)\Delta$. To have an N period in period $M+1$ there are four possibilities for the demand in period M:

Case i. $D_2^M = X_N^M - k_2$ and $D_1^M = S - X_N^M - \Delta_M - k_1$ with $k_1, k_2 \geq 0$. No substitution takes place and the resulting inventory levels are $X_F^{M+1} = S - \Delta_M - k_2$ and $X_N^{M+1} = S - k_2$

(i.e. $\Delta = \Delta_M$). Then $C_N^M = hk_2 + m(\Delta_M + k_1)$ and $C_F^M = h(\Delta_M + k_2) + mk_1$. Hence $C_N^M - C_F^M = (m-h)\Delta_M$. After adding $(h + p_1 - \gamma)\Delta_M$ we have $(m + p_1 - \gamma)\Delta$ as the first part of the *pair* cost.

Case ii. $D_2^M = X_N^M - k_2$ and $D_1^M = S - X_N^M - \Delta_M + k_1$ with $\Delta_M > k_1 \geq 0,\ k_2 \geq 0$. There is downward substitution in the amount of k_1 so that $X_F^{M+1} = S - (\Delta_M + k_2 - k_1)$ and $X_N^{M+1} = S - k_2$ (i.e. $\Delta = \Delta_M - k_1$). Then $C_N^M = hk_2 + m(\Delta_M - k_1)$ and $C_F^M = h(\Delta_M + k_2 - k_1) + \gamma k_1$. Hence $C_N^M - C_F^M = (m-h)(\Delta_M - k_1) - \gamma k_1$. After adding $(h + p_1 - \gamma)\Delta_M$, we have $(m + p_1 - \gamma)\Delta + (h + p_1 - 2\gamma)k_1$ as the first part of the *pair* cost.

Case iii. $D_2^M = X_N^M + k_2$ and $D_1^M = S - X_N^M - \Delta_M - k_1$ with $\Delta_M > k_2 \geq 0,\ k_1 \geq 0$. There is no substitution then $X_F^{M+1} = S - (\Delta_M - k_2)$ and $X_N^{M+1} = S$ (i.e. $\Delta = \Delta_M - k_2$). Therefore $C_N^M = p_2k_2 + m(\Delta_M + k_1)$ and $C_F^M = h(\Delta_M - k_2) + mk_1$. Hence $C_N^M - C_F^M = (m-h)(\Delta_M - k_2) + (m+p_2)k_2$. After adding $(h+p_1-\gamma)\Delta_M$ we have $(m+p_1-\gamma)\Delta + (h+p_1-\gamma+m+p_2)k_2$ as the first part of the *pair* cost.

Case iv. $D_2^M = X_N^M + k_2$ and $D_1^M = S - X_N^M - \Delta_M + k_1$ with $0 \leq k_i < \Delta_M$ for $i \in \{1, 2\}$. Under $\mathcal{F}$ there are $\Delta_M - k_2$ unsold new items and k_1 more old items are needed and in order to have a *pair* $k_1 < \Delta_M - k_2$ must hold. Therefore $X_F^{M+1} = S - (\Delta_M - k_2 - k_1)$ and $X_N^{M+1} = S$ (i.e. $\Delta = \Delta_M - k_2 - k_1$). Then $C_N^M = p_2k_2 + m(\Delta_M - k_1)$ and $C_F^M = h(\Delta_M - k_2 - k_1) + \gamma k_1$. Hence $C_N^M - C_F^M = (m-h)(\Delta_M - k_2 - k_1) + (m+p_2)k_2 - \gamma k_1$. After adding $(h+p_1-\gamma)\Delta_M$ we have $(m + p_1 - \gamma)\Delta + (h + p_1 - \gamma + m + p_2)k_2 + (h + p_1 - 2\gamma)k_1$ as the first part of the *pair* cost.

In period $M+1$ if high demand for new items is seen this would benefit $\mathcal{N}$ since there is more new item under $\mathcal{N}$ in this period. We do a case-by-case analysis to show that this benefit is not greater than its cost, $(m+p_1-\gamma)\Delta$. The proof is by induction, that is we first show that the claim is true for $J = 1$ (J is number of *pairs* in the cycle) and 2, then assume it is true for $J = j$ and prove it stays correct for $J = j+1$. For $J = 1$ there are 3 possibilities. By examining these possible cases:

Case 1. $D_2^{M+1} = X_F^{M+1} + \Delta + k_2$ and $D_1^{M+1} = S - X_F^{M+1} + k_1$ where k_1 and $k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= p_1k_1 + p_2(\Delta + k_2), \quad C_N^{M+1} = p_1(\Delta + k_1) + p_2k_2 \\ C_F^{M+1} - C_N^{M+1} &= (p_2 - p_1)\Delta \leq (m + p_1 - \gamma)\Delta \end{aligned}$$

as we assume $p_2 - p_1 \leq m + p_1 - \gamma$.

Case 2. $D_2^{M+1} = X_F^{M+1} + \Delta + k_2$ and $D_1^{M+1} = S - X_F^{M+1} - \Delta + k_1$ where $\Delta \geq k_1 \geq 0$ and $k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= p_2(\Delta + k_2 - (\Delta - k_1)) + \alpha(\Delta - k_1) = p_2(k_2 + k_1) + \alpha(\Delta - k_1), \\ C_N^{M+1} &= p_2 k_2 + p_1 k_1 \\ C_F^{M+1} - C_N^{M+1} &= (p_2 - p_1)k_1 + \alpha(\Delta - k_1) \leq (m + p_1 - \gamma)\Delta \end{aligned}$$

as we assume $\alpha + \gamma \leq m + p_1$.

Case 3. $D_2^{M+1} = X_F^{M+1} + \Delta + k_2$ and $D_1^{M+1} = S - X_F^{M+1} - \Delta - k_1$ where k_1 and $k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= p_2(\Delta + k_2 - (\Delta + k_1))^+ + m(\Delta + k_1 - (\Delta + k_2))^+ \\ &= p_2(k_2 - k_1)^+ + m(k_1 - k_2)^+ + \alpha(\Delta + k_1) \\ C_N^{M+1} &= p_2 k_2 + m k_1 \end{aligned}$$

If $k_2 \geq k_1$, then $C_F^{M+1} - C_N^{M+1} = -(p_2 + m)k_1 + \alpha(\Delta + k_1) \leq (m + p_1 - \gamma)\Delta$.

If $k_2 < k_1$, then $C_F^{M+1} - C_N^{M+1} = -(p_2 + m)k_2 + \alpha(\Delta + k_2) < (m + p_1 - \gamma)\Delta$.

For all three cases above, in the next period $\mathcal{F}$ and $\mathcal{N}$ reach the same state, an E, as $D_2^{M+1} = X_F^{M+1} + \Delta + k_2$. If there are more F periods from period n until the cycle closure the claim is true due to Proposition 2.

For $J = 2$, there are three cases (of the original nine) with $D_2^{M+1} = X_F^{M+1} + k_2$ where $\Delta > k_2 \geq 0$, and the inventory state for $\mathcal{F}$ is $(S, 0)$ while the state for $\mathcal{N}$ is $(S - (\Delta - k_2), \Delta - k_2)$. This implies the period $M + 2$ is an F period with $\Delta_{M+2} = \Delta - k_2$.

Note that while proving for $J = 2$, in order to make a correct cost analysis, we subtract $(h + p_1 - \gamma)(\Delta_{M+2})$ from C_N^{M+1} for the first *pair*. We are "saving" this amount to use in the second *pair* in this cycle (either a series of Fs and then a *pair* or an immediate *pair*) as we mentioned in the proof of Proposition 6; see equation (12). This makes the $J = 2$ case identical to $J = 1$ case from that period on:

Case 4. $D_2^{M+1} = X_F^{M+1} + k_2$ and $D_1^{M+1} = S - X_F^{M+1} + k_1$ where $k_1 \geq 0$ and $\Delta > k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= p_2 k_2 + p_1 k_1, \quad C_N^{M+1} = h(\Delta - k_2) + p_1(\Delta + k_1) - (h + p_1 - \gamma)(\Delta - k_2) \\ C_F^{M+1} - C_N^{M+1} &= (p_2 - p_1)k_2 - \gamma(\Delta - k_2) < (m + p_1 - \gamma)\Delta \end{aligned}$$

Case 5. $D_2^{M+1} = X_F^{M+1} + k_2$ and $D_1^{M+1} = S - X_F^{M+1} - \Delta + k_1$ where $\Delta \geq k_1 \geq 0$ and $\Delta > k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= p_2[k_2 - (\Delta - k_1)]^+ + m[\Delta - k_1 - k_2]^+ + \alpha \ \min(k_2, \Delta - k_1) \\ C_N^{M+1} &= h(\Delta - k_2) + p_1 k_1 - (h + p_1 - \gamma)(\Delta - k_2). \end{aligned}$$

If $k_1 + k_2 \leq \Delta$, then

$$\begin{aligned} C_F^{M+1} - C_N^{M+1} &= m\Delta - mk_1 - mk_2 - h\Delta + hk_2 - p_1k_1 + \alpha k_2 \\ &\quad + (h + p_1 - \gamma)(\Delta - k_2) \\ &= (m + p_1)(\Delta - k_1 - k_2) + \alpha k_2 - \gamma(\Delta - k_2) \\ &= (m + p_1 - \gamma)(\Delta - k_2) + \alpha k_2 - (m + p_1)k_1 < (m + p_1 - \gamma)\Delta. \end{aligned}$$

If $k_1 + k_2 > \Delta$, then

$$\begin{aligned} C_F^{M+1} - C_N^{M+1} &= p_2k_2 + p_2k_1 - p_2\Delta - h\Delta + hk_2 - p_1k_1 + \alpha(\Delta - k_1) \\ &\quad + (h + p_1 - \gamma)(\Delta - k_2) \\ &= p_2(k_2 + k_1 - \Delta) + p_1(\Delta - k_2 - k_1) + h(-\Delta + k_2 + \Delta - k_2) \\ &\quad + \alpha(\Delta - k_1) - \gamma(\Delta - k_2) \\ &= (p_2 - p_1)(k_1 - (\Delta - k_2)) + \alpha(\Delta - k_1) - \gamma(\Delta - k_2) \quad < (m + p_1 - \gamma)\Delta. \end{aligned}$$

Case 6. $D_2^{M+1} = X_F^{M+1} + k_2$ and $D_1^{M+1} = S - X_F - \Delta - k_1$ where k_1 and $\Delta > k_2 \geq 0$:

$$\begin{aligned} C_F^{M+1} &= m(\Delta + k_1 - k_2) + \alpha k_2, \\ C_N^{M+1} &= h(\Delta - k_2) + mk_1 - (h + p_1 - \gamma)(\Delta - k_2) \\ C_F^{M+1} - C_N^{M+1} &= m(\Delta + k_1 - k_2) - h(\Delta - k_2) - mk_1 + (h + p_1 - \gamma)(\Delta - k_2) + \alpha k_2 \\ &= (m + p_1 - \gamma)(\Delta - k_2) + \alpha k_2 < (m + p_1 - \gamma)\Delta. \end{aligned}$$

We also have the cases where $D_2^{M+1} = X_F^{M+1} - k_2$ (i.e. new items are abundant) and there is no opportunity for the $\mathcal{N}$ policy to cash in on the extra new items that it has in period n. We also show that for these cases $C_F^{M+1} - C_N^{M+1} < (m + p_1 - \gamma)\Delta$ holds again, saving $(h + p_1 - \gamma)\Delta_{M+2}$:

Case 7. $D_2^{M+1} = X_F^{M+1} - k_2$ and $D_1^{M+1} = S - X_F^{M+1} + k_1$ where $k_1 \geq 0$ and $k_2 \geq 0$:

$X_F^{M+2} = S - (k_2 - k_1)^+$, $X_N^{M+2} = S - (\Delta + k_2)$ then $\Delta_{M+2} = \Delta + \min(k_1, k_2)$. Costs are as follows:

$$\begin{aligned} C_F^{M+1} &= h(k_2 - k_1)^+ + p_1(k_1 - k_2)^+ + \gamma \min(k_1, k_2), \\ C_N^{M+1} &= h(\Delta + k_2) + p_1(\Delta + k_1) - (h + p_1 - \gamma)\Delta_{M+2}. \\ C_F^{M+1} - C_N^{M+1} &= -\gamma\Delta \quad < (m + p_1 - \gamma)\Delta. \end{aligned}$$

Case 8. $D_2^{M+1} = X_F^{M+1} - k_2$ and $D_1^{M+1} = S - X_F^{M+1} - \Delta + k_1$ where $\Delta \geq k_1 \geq 0$ and $k_2 \geq 0$: $X_F^{M+2} = S - k_2$, $X_N^{M+2} = S - (\Delta + k_2)$ then $\Delta_{M+2} = \Delta$. Costs are follows:

$$\begin{aligned} C_F^{M+1} &= hk_2 + m(\Delta - k_1), \quad C_N^{M+1} = h(\Delta + k_2) + p_1 k_1 - (h + p_1 - \gamma)\Delta_{M+2}. \\ C_F^{M+1} - C_N^{M+1} &= (m + p_1)(\Delta - k_1) - \gamma\Delta, \quad \leq (m + p_1 - \gamma)\Delta. \end{aligned}$$

Case 9. $D_2^{M+1} = X_F^{M+1} - k_2$ and $D_1^{M+1} = S - X_F^{M+1} - \Delta - k_1$ where k_1 and $k_2 \geq 0$: $X_F^{M+2} = S - k_2$, $X_N^{M+2} = S - (\Delta + k_2)$ then $\Delta_{M+2} = \Delta$. Costs are follows:

$$\begin{aligned} C_F^{M+1} &= hk_2 + m(\Delta + k_1), \quad C_N^{M+1} = h(\Delta + k_2) + mk_1 - (h + p_1 - \gamma)\Delta_{M+2}. \\ C_F^{M+1} - C_N^{M+1} &= (m + p_1 - \gamma)\Delta. \end{aligned}$$

Hence $\mathcal{F}$ has lower cost than $\mathcal{N}$ in the *pairs* as well, and from then on we have a $J = 1$ case.

Assume the claim is true for $J = j$ and $\mathcal{F}$ has extra $(h + p_1 - \gamma)(\Delta - k_2)$ at the end of the last *pair* as in case $J = 2$. If the $(j+1)$st *pair* is immediately after the jth one; then $\mathcal{F}$ with the extra $(h + p_1 - \gamma)(\Delta - k_2)$ benefit, will be less costly if $m + 2p_1 > p_2 + \gamma$ holds. If there is no *pair* as an immediate successor of the jth *pair*, then there will be a number of F periods before the pair $j + 1$ and $\mathcal{F}$ will have a $h + p_1 - \gamma$ benefit before this next *pair* as discussed in Proposition 6. Then the proof is identical to the $J = 1$ case. Hence the claim is true. ■

Proof of Lemma 4

Proof : For cases in which there is no upward substitution, $\mathcal{U}$ is identical to $\mathcal{N}$ and we know that $\mathcal{F}$ is better than $\mathcal{N}$ under conditions (4) and (8). So we look at cycles where there is upward substitution.

For E periods, if there is any upward substitution, then $\mathcal{F}$ and $\mathcal{U}$ have the same costs and the next period is also an E period.

If there is upward substitution by $\mathcal{U}$ or $\mathcal{F}$ in an F period, then the next period is either a U period (i.e. there is a *pair*) or an E (specifically both policies reach the inventory state $(S, 0)$). Let $n-1$ be such an F period with $X_F^{n-1} = X_U^{n-1} + \Delta'$. To have an upward substitution, D_2^{n-1} must be greater than X_U^{n-1} which makes $X_U^n = S$. Let $\Delta = X_U^n - X_F^n$; then it must hold that $\Delta' > \Delta \geq 0$. This implies $D_2^{n-1} = X_U^{n-1} + \Delta' - \Delta$; and we also see that when $\Delta = 0$ then period n is E with state of $(S$,0).

We compare $\mathcal{U}$ with $\mathcal{N}$: $X_N^{n-1} = X_U^{n-1} + \Delta' - \Delta$ and in period n, both $\mathcal{U}$ and $\mathcal{N}$ are at state $(S, 0)$. The cost of having the Δ' difference is $(h + p_1 - \gamma)\Delta'$ more for $\mathcal{U}$ as compared to $\mathcal{F}$. As

$X_F^{n-1} - X_N^{n-1} = \Delta$, cost of $\mathcal{U}$ is greater than $\mathcal{N}$ by $(h + p_1 - \gamma)(\Delta' - \Delta)$ for the difference of new items because fewer items were substituted downward to get a net difference of Δ. $\mathcal{U}$ incurs the cost of upward substitution, which is α per item. Thus, if $D_1^{n-1} \leq S - X_U^{n-1} - (\Delta' - \Delta)$ then $\mathcal{U}$ is more costly than policy $\mathcal{N}$ by $(h + p_1 - \gamma)(\Delta' - \Delta) + \alpha(\Delta' - \Delta)$ until period n. Otherwise (i.e. if $D_1^{n-1} > S - X_U^{n-1} - (\Delta' - \Delta)$), $\mathcal{U}$ will have to substitute less than $\Delta' - \Delta$ and will incur p_2 for every new item demand that it cannot fulfill by substitution while $\mathcal{N}$ incurs p_1 as penalty costs. Therefore, for every state in $\mathcal{U}$, we can find a counterpart in $\mathcal{N}$ that has a lower cost than $\mathcal{U}$. As $\mathcal{N}$ has higher cost than $\mathcal{F}$, $\mathcal{U}$ is also more costly than $\mathcal{F}$. ■

A.2. Cost Comparison Results under NIS

Similar to what we did in Section 4, we define the *pairs*, and divide the horizon into different classes of periods. However, in this case a *pair* is defined based on the inventory level of *old* items: If the amount of old goods in stock in period n is higher under policy $\mathcal{F}$, then we call that period an F period. We call a period E if the amount of old items in stock for two policies is equal. We define *cycle*, *trivial cycle*, *non-trivial cycle*, and C_I^n for $I = \mathcal{F}, \mathcal{N}, \mathcal{U}, \mathcal{D}$ as previously.

Proof of Lemma 9

Proof : By Proposition 4, in any given period n $Y_N^n = Y_F^n + \Delta$ where $\Delta \geq 0$ is the downward substitution amount in period $n-1$. In an E period, $\Delta = 0$; otherwise it is N. $\mathcal{F}$ and $\mathcal{N}$ have the same cost in a trivial cycle. We look at a non-trivial cycle. For each substitution, $\mathcal{F}$ incurs a substitution cost γ but saves $h + p_1$; $\mathcal{F}$ incurs $h + p_1 - \gamma$ less cost than $\mathcal{N}$ per substitute item. In this N period (let this period be the nth period) there are three possibilities for $\mathcal{F}$:

i. No substitution takes place: Let the demand for old items be $D_1^n = Y_F^n + K$. As $Y_N^n = Y_F^n + \Delta$ we have the cost difference between the policies for the cycle given below. (Note that we do not include the costs that are incurred in cycle-ending E periods in our "cycle cost". We account for that cost in the subsequent cycle.)

$$\sum_{i=n-1}^{n} C_N^i - C_F^i = (h + p_1 - \gamma)\Delta + \begin{cases} -p_1\Delta & \text{if } K \geq \Delta, \quad D_2^n \geq S, \\ m(\Delta - K) - p_1 K & \text{if } \Delta > K \geq 0, \quad D_2^n \geq S, \\ m\Delta & \text{if } 0 > K \geq -X_F^n, \quad D_2^n \leq S. \end{cases}$$

The next period is an E and the cycle ends. Thus, for this case, the claim is true.

ii. Upward substitution takes place: To have upward substitution D_2^n must be greater than S and D_1^n must be less than Y_F^n, hence $D_2^n = S + L$ and $D_1^n = Y_F - K$ where $L, K > 0$. The cost difference between $\mathcal{F}$ and $\mathcal{N}$ for the cycle is as follows:

$$\sum_{i=n-1}^{n} C_N^i - C_F^i = (h + p_1 - \gamma + m)\Delta + (m + p_2 - \alpha)us$$

where $us = \min\{K, L\}$ is the upward substitution amount. As the next period is an E, the cycle ends. Therefore the claim is true for this case.

iii. Downward substitution takes place: A downward substitution in period n causes the period $n+1$ to be an N, therefore the cycle does not end in period $n+1$. There can be a number of Ns in a cycle, we define J as this number.

For $J = 2$, let $D_2^n = S - L$ and $D_1^n = Y_F^n + K$ where $S > L \geq 0$ and $K > 0$. Downward substitution takes place twice in this cycle, in periods $n-1$ and n. We compare the costs of $\mathcal{F}$ and $\mathcal{N}$ through the periods $n-1$, n and $n+1$. The cost difference between $\mathcal{N}$ and F for the first two periods of the cycle ($n-1$ and n) is as follows:

$$\sum_{i=n-1}^{n} C_N^i - C_F^i = (h + p_1 - \gamma)\Delta + (h + p_1 - \gamma)ds^n - p_1 K + m(\Delta - K)^+ + p_1(K - \Delta)^+$$

where $ds^n = \min\{K, L\}$ is the downward substitution amount in period n. Note that $\Delta = ds^{n-1}$. Thus:

$$\sum_{i=n-1}^{n} C_N^i - C_F^i = (h + p_1 - \gamma)\Delta + (h + p_1 - \gamma)ds^n - p_1 K + \begin{cases} m(\Delta - K) & \text{if } K < \Delta, \\ p_1(K - \Delta) & \text{if } K \geq \Delta. \end{cases}$$

$$= (h + p_1 - \gamma)ds^n + \begin{cases} (h - \gamma)\Delta + (m + p_1)(\Delta - K) & \text{if } K < \Delta, \\ (h - \gamma)\Delta & \text{if } K \geq \Delta. \end{cases}$$

However the $(h + p_1 - \gamma)ds^n$ part of the expression above will be accounted for the $n+1$st period's cost difference. The reasoning is as follows: The next period $(n+1)$ is an N. As there is no downward substitution in period $n+1$, the cycle ends in period $n+2$, which is an E. For the cost difference between $\mathcal{F}$ and $\mathcal{N}$ in period $n+1$ we refer to part **i** or **ii** of the proof as there is no downward substitution. Therefore for correct cost accounting $(h + p_1 - \gamma)ds^n$ is included in cost comparison in period $n+1$. Thus the claim is proved for $J = 2$.

The general $J = j$ case is depicted Table 3, and the following expression is the cost difference between $\mathcal{N}$ and $\mathcal{F}$ for the $J = j$ case:

$$\sum_{i=n-1}^{n+j-2} C_N^i - C_F^i =$$
$$\sum_{i=n-1}^{n+j-3} \left[(h + p_1 - \gamma)ds^i - p_1 K^{i+1} + m(ds^i - K^{i+1})^+ + p_1(K^{i+1} - ds^i)^+\right]$$
$$+ (h + p_1 - \gamma)ds^{n+j-2}.$$

Period	n-1	n	n+1	...	n+j-2	n+j-1	n+j
Down. Subs.	ds^{n-1}	ds^{n}	ds^{n+1}	...	ds^{n+j-2}	$ds^{n+j-1}=0$	ds^{n+j}
$X_N^i - X_F^i$	0	ds^{n-1}	ds^{n}	...	ds^{n+j-4}	ds^{n+j-2}	$ds^{n+j-1}=0$
Type	E	N	N	...	N	N	E

Table 3: A sequence of N periods under NIS.

where $ds^n = \min\{K^n, L^n\}$, $L^n = S - D_2^n$ and $K^n = D_1^n - Y_F^n$. Thus,

$$\begin{aligned}
\sum_{i=n-1}^{n+j-2} C_N^i - C_F^i &= (h+p_1-\gamma)ds^{n+j-2} + \sum_{i=n-1}^{n+j-3}(h+p_1-\gamma)ds^i - p_1 K^{i+1} \\
&\quad + \begin{cases} m(ds^i - K^{i+1}) & \text{if } K^{i+1} < ds^i, \\ p_1(K^{i+1} - ds^{i+1}) & \text{if } K^{i+1} \geq ds^i. \end{cases} \\
&= (h+p_1-\gamma)ds^{n+j-2} \\
&\quad + \sum_{i=n-1}^{n+j-3} \begin{cases} (h-\gamma)ds^i + (m+p_1)(ds^i - K^{i+1}) & \text{if } K^{i+1} < ds^i, \\ (h-\gamma)ds^i & \text{if } K^{i+1} \geq ds^i. \end{cases}
\end{aligned}$$

Similar to the $J=2$ case, $(h+p_1-\gamma)ds^{n+j-2}$ is included for the cost comparison in period $n+j-1$ referring part **i** or **ii**. Thus the claim is proved.

■

Proof of Lemma 10

Proof : First, as we assume upward substitution is sensible, $\alpha < m + p_2$. Comparing $\mathcal{F}$ and $\mathcal{U}$ is similar to the comparison of $\mathcal{F}$ and $\mathcal{N}$ as in any given period n $Y_N^n = Y_U^n = Y_F^n + \Delta$ for some $\Delta \geq 0$, by Proposition 4. Unless there is an upward substitution, $\mathcal{N}$ and $\mathcal{U}$ have the same cost, therefore part **i** (when $K \geq \Delta$) and **iii** of the proof of Lemma 9 provide the condition $\gamma < h$ that is needed for $\mathcal{F}$ to be less costly than $\mathcal{U}$; and the condition $\gamma > m + h + p_1$ so that $\mathcal{U}$ is less costly than $\mathcal{F}$. We analyze the remaining cases:

Upward substitution under both policies: $D_2^n = S + L$ and $D_1^n = Y_F^n - K$ leads to upward substitution under both $\mathcal{F}$ and $\mathcal{U}$, where $L > 0, Y_F^n > K > 0$.

$$C_U^n = (h+p_1-\gamma)\Delta + p_2(L - us_U) + m(\Delta + K - us_U) + \alpha\, us_U$$

where $us_U = \min\{\Delta + K, L\}$.

$$C_F^n = p_2(L - us_F) + m(K - us_F) + \alpha\, us_F$$

where $us_F = \min\{K, L\}$.

$$\begin{aligned} C_U^n - C_F^n &= (h + p_1 - \gamma)\Delta + m\Delta - (m + p_2 - \alpha)(us_U - us_F), \\ &= (m + h + p_1 - \gamma)(\Delta - (us_U - us_F)) \\ &\quad + (h - \gamma + \alpha - (p_2 - p_1))(us_U - us_F). \end{aligned} \tag{16}$$

(16) is positive if $\gamma < h + \alpha - (p_2 - p_1)$, as $\min\{\Delta + K, L\} \geq \min\{K, L\}$ and $\Delta \geq L - K$ (when $L > K$). (16) is negative if $\gamma > m + h + p_1$. Thus the claim is true for this case.

Upward substitution under $\mathcal{U}$, no substitution under $\mathcal{F}$: In this case $D_2^n = S + L$ and $D_1^n = Y_F^n + K$ with $L > 0, \Delta > K > 0$.

$$C_U^n = (h + p_1 - \gamma)\Delta + p_2(L - us) + m(\Delta - K - us) + \alpha\; us$$

where $us = \min\{\Delta - K, L\}$.

$$C_F^n = p_2 L + p_1 K.$$

$$\begin{aligned} C_U^n - C_F^n &= (h + p_1 - \gamma)\Delta + m(\Delta - K - us) + (\alpha - p_2)us - p_1 K, \\ &= (h - \gamma)\Delta + \begin{cases} (\alpha - p_2 + p_1)(\Delta - K) & \text{if } K + L > \Delta, \\ (m + p_1)(\Delta - K) - (m + p_2 - \alpha)L & \text{if } K + L \leq \Delta, \end{cases} \\ &= \begin{cases} (h - \gamma)K + [h - \gamma + \alpha - (p_2 - p_1)](\Delta - K) & \text{if } K + L > \Delta, \\ \\ (h - \gamma)(\Delta - L) + [h - \gamma + \alpha - (p_2 - p_1)]L & \\ \;+ (m + p_1)(\Delta - K - L) & \text{if } K + L \leq \Delta. \end{cases} \\ &= (h - \gamma)K + \\ &\quad \begin{cases} [h - \gamma + \alpha - (p_2 - p_1)](\Delta - K) & \text{if } K + L > \Delta, \\ (h - \gamma + m + p_1)(\Delta - L - K) + [h - \gamma + \alpha - (p_2 - p_1)]L & \text{if } K + L \leq \Delta. \end{cases} \end{aligned}$$

Thus if $\gamma < \min\{h, h + \alpha - (p_2 - p_1)\}$ $\mathcal{F}$ is less costly than $\mathcal{U}$. If $\gamma > h + m + p_1$ $\mathcal{U}$ is less costly than $\mathcal{F}$; as $h + m + p_1 > \max\{h, h + \alpha - (p_2 - p_1)\}$ when $m + p_2 > \alpha$. Thus the claim is proved.

■

Proof of Lemma 12

Proof : Comparison of $\mathcal{D}$ to $\mathcal{U}$ is identical to the comparison of $\mathcal{D}$ with $\mathcal{N}$ except when $\mathcal{U}$ does upward substitution. We analyze that case.

$Y_D^n + \Delta = Y_U^n$ by Proposition 4 and $D_2^n = S + L$, $D_1^n = Y_D^n + \Delta - K$, ($S \geq Y_D^n + \Delta > K > 0$ and $L > 0$) leads to upward substitution under $\mathcal{U}$ and no substitution under $\mathcal{D}$.

$$C_U^n = p_2(L - us) + m(K - us) + \alpha us,$$

where us is the upward substitution amount and is equal to $\min\{K, L\}$. Then

$$C_D^n = p_2 L + m(K - \Delta)^+ + p_1(\Delta - K)^+.$$

There are four demand scenarios:

Case 1. When $K \leq L$ and $K \leq \Delta$; $us = K$ and

$$\begin{aligned} C_U^n &= (h + p_1 - \gamma)\Delta + p_2(L - K) + \alpha K, \quad C_D^n = p_2 L + p_1(\Delta - K), \\ C_U^n - C_D^n &= (h - \gamma)\Delta + (\alpha - (p_2 - p_1))K, \\ &= (h - \gamma)(\Delta - K) + (h - \gamma + \alpha - (p_2 - p_1))K. \end{aligned}$$

Case 2. When $K \leq L$ and $K > \Delta$; $us = K$ and

$$\begin{aligned} C_U^n &= (h + p_1 - \gamma)\Delta + p_2(L - K) + \alpha K, \quad C_D^n = p_2 L + m(K - \Delta), \\ C_U^n - C_D^n &= (m + h + p_1 - \gamma)\Delta - (m + p_2 - \alpha)K, \\ &= (h - \gamma + \alpha - (p_2 - p_1))\Delta - (m + p_2 - \alpha)(K - \Delta). \end{aligned}$$

Case 3. When $K > L$ and $K \leq \Delta$; $us = L$ and

$$\begin{aligned} C_U^n &= (h + p_1 - \gamma)\Delta + m(K - L) + \alpha L, \quad C_D^n = p_2 L + p_1(\Delta - K), \\ C_U^n - C_D^n &= (h - \gamma)\Delta + (m + p_1)K - (m + p_2 - \alpha)L, \\ &= (h - \gamma)\Delta + (m + p_1)(K - L) + (\alpha - (p_2 - p_1))L. \end{aligned}$$

Case 4. When $K > L$ and $K > \Delta$; $us = L$ and

$$\begin{aligned} C_U^n &= (h + p_1 - \gamma)\Delta + m(K - L) + \alpha L, \quad C_D^n = p_2 L + m(K - \Delta), \\ C_U^n - C_D^n &= (m + h + p_1 - \gamma)\Delta - (m + p_2 - \alpha)L. \end{aligned}$$

Thus although the condition $\gamma < \min\{h, h + \alpha - (p_2 - p_1)\}$ is sufficient for $\mathcal{D}$ to be less costly than $\mathcal{U}$ in most of the demand scenarios (including no substitution and downward substitution cases), the analyses of cases 2 and 4 show that there is no condition that guarantees $\mathcal{D}$ to be less costly than $\mathcal{U}$ in general, unless $\alpha > m + p_2$, which in itself guarantees that upward substitution is an unattractive decision.

However, if $\gamma > h + m + p_1$, then $\mathcal{U}$ is guaranteed to be less costly than $\mathcal{D}$. (Note $\gamma > m + h + p_1$ implies $\gamma > h + \alpha - (p_2 - p_1)$ as $\alpha < m + p_2$.) ■

Proof of Lemma 11

Proof : Unless there is an upward substitution $\mathcal{D}$ and $\mathcal{F}$ have the same cost because $Y_F^n = Y_D^n = Y_N^n - \Delta$ for some $\Delta \geq 0$ by Proposition 4. Therefore parts **i** (no substitution) and **iii** (downward substitution) of the proof of Lemma 9 provide the conditions $\gamma < h$ that is required for $\mathcal{D}$ to be less costly than $\mathcal{N}$. We also need to analyze the following case (where there is no substitution): $D_2^n = S + L$ and $D_1^n = Y_D^n - K$ where $L > 0$ and $X_D^n > K > 0$. Then,

$$\begin{aligned} C_N^n &= (h + p_1 - \gamma)\Delta + p_2 L + m(\Delta + K), \quad C_D^n = p_2 L + mK, \\ C_N^n - C_D^n &= (m + h + p_1 - \gamma)\Delta. \end{aligned}$$

Thus the claim is true. ■

A.3. Results on Freshness of Inventory

Proof of Proposition 5

Proof : Inside a *pair*, we define $\Delta' = X_N^n - X_F^n > 0$ and $\Delta = X_F^{n-1} - X_N^{n-1} > 0$. To prove part (i), it is enough to observe: By definition of a *pair* $X_F^n = X_N^n - \Delta' \leq S - \Delta' < S$. But if $D_1^{n-1} + D_2^{n-1} \geq S$, then from the inventory recursions under TIS, we have $X_F^n = S$, which is contradicting. Part (ii) follows from the inventory recursions and substitution from old items to new items implies $X_F^n = S$. For part (iii), assume $D_1^{n-1} = S - X_N^{n-1} + k$ where $k > 0$. Since old item inventory under $\mathcal{F}$ in period $n-1$ is $S - X_N^{n-1} - \Delta$, there is excess demand of $\Delta + k$ for old items in period $n-1$. There must be enough new items to fulfill this shortage by downward substitution due to part (i). (Otherwise both $\mathcal{F}$ and $\mathcal{N}$ will be at inventory state $(S, 0)$.) Thus old item inventory in period n will be $[X_N^{n-1} - k - D_2^{n-1}]^+$ and $[X_N^{n-1} - D_2^{n-1}]^+$ for $\mathcal{F}$ and $\mathcal{N}$ respectively. This is a contradiction since in period n there must be less old items under $\mathcal{N}$ (i.e. there must be more new items under $\mathcal{N}$) by definition of a *pair*. Therefore the claim is true. ■

Proof of Lemma 14

Proof : Equalities follow from inventory recursions (i.e. $X_F^n = X_D^n$ and $X_U^n = X_N^n$ for all n). We show the inequality below.

Suppose there is a *pair* in periods $n-1$ and n. Inside a *pair*, we define $\Delta' = X_N^n - X_F^n > 0$ and $\Delta = X_F^{n-1} - X_N^{n-1} > 0$. There are two possible cases inside a *pair* due to Proposition 5.

Case 1. D_1^{n-1} and D_2^{n-1} are less than the inventories under policy $\mathcal{F}$; so no shortage or substitution takes place under $\mathcal{F}$. $D_2^{n-1} = X_N^{n-1} + \Delta - k$ $(S > k \geq 0)$ and $D_1^{n-1} = S - X_N^{n-1} - \Delta - l$

$(l \geq 0)$. Then,

$$X_F^n = S - (X_N^{n-1} + \Delta - X_N^{n-1} - \Delta - k) = S - k$$

and $X_N^n = S + \Delta - k$. Thus $\Delta^{'} = \Delta$.

Case 2. There is downward substitution and new item shortage for $\mathcal{N}$. Demands are: $D_2^{n-1} = X_N^{n-1} + \Delta - k$ $(S > k \geq 0)$ and $D_1^{n-1} = S - X_N^{n-1} - \Delta + l$, where $\Delta \geq l > 0$. Therefore,

$$X_F^n = S - (X_N^{n-1} + \Delta - X_N^{n-1} - \Delta + k - l)^+ = S - (k - l)^+,$$

the downward substitution amount is $\min\{k, l\}$, and $X_N^n = S + \Delta - k$. Thus if $k > l$, then $\Delta^{'} = S + \Delta - k - (S - k + l) = \Delta - l$. If $k \leq l$, then $\Delta^{'} = S + \Delta - k - S = \Delta - k$. Therefore $\Delta \geq \Delta^{'}$.

In both cases, we observe that $\Delta^{'} \leq \Delta$. This means on the average there are always more new items under $\mathcal{F}$, in the *pairs*. By definition $X_F^k \geq X_N^k$ for a period k that is outside the *pairs*. ■