Dynamic Assortment Customization with Limited Inventories

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We consider a retailer with limited inventories of identically priced, substitutable products. Customers arrive sequentially and the firm decides which subset of products to offer to each arriving customer depending on the customer's preferences, the inventory levels, and the remaining time in the season. We show that the optimal assortment policy is to offer all available products if the customer base is homogeneous with respect to their product preferences. However, with multiple customer segments characterized by different product preferences, it may be optimal to limit the choice set of some customers. That is, it may be optimal not to offer products with low inventory levels to some customer segments and reserve them for future customers (who may have a stronger preference for those products). In some settings, we prove that the optimal assortment policy is a threshold policy under which a product is offered to a customer segment if its inventory level is higher than a threshold value. The threshold levels are decreasing in time and increasing in the inventory levels of other products. For more general cases, we perform a numerical study and confirm that a threshold policy continues to be optimal. We also introduce an aggregation-based heuristic that computes an effective assortment customization policy. We find that the revenue impact of assortment customization can be significant, especially when customer heterogeneity is high and the starting inventory levels of the products are asymmetric. This demonstrates the use of assortment customization as another lever for revenue maximization in addition to pricing.
(*Key words: revenue management, retail assortment planning, inventory rationing, customer segmentation, mixed logit model*)

1. Introduction

Online retailing continues to be the retail industry's growth engine. In the last decade, online retailers have heavily invested in diverse technologies, such as sophisticated analytics and personalization tools. As a result, online retailers are able to send personalized promotions or display customized recommendations to individual customers based on their personal information and browsing and purchasing history, which can effectively enhance the customers' shopping experience and influence their purchasing activities. In contrast to traditional bricks-and-mortar stores, online retailers can not only easily collect personal information, but they can also record their customers' purchase histories and control the selection of products a customer is exposed to by strategically selecting the product variants displayed on the web page. For example, Amazon.com's web site, like many other online retailers' websites, requires a customer to log in using her/his account before making a purchase. This allows the company to track the customer's personal information and purchase history. Based on this data, the company can estimate the customer's preferences on styles, colors, and on other features of a product. An online retailer can use this information to customize the set of products made available to customers. In particular, websites typically display a selection of product options over multiple web pages. The products on the first few pages are likely to receive more attention. Therefore, the company may be able to control the selection of products offered to a customer by restricting the set of product variants displayed in the first few pages of the search results. In addition, the company can more directly exercise that control by selecting a product's "in-stock" or "out-of-stock" status information displayed to customers. In this paper, we argue that providing *customized assortments* to individual customers has the potential to increase overall sales – and therefore revenues – for online retailers. This also applies to online travel resellers such as Expedia.com, that purchase blocks of inventory (e.g., hotel rooms) to sell through their websites. These resellers may be able to maximize sales by choosing the right set of products displayed to each individual customer. The travel websites collect information about their customers' purchase histories and preferences, and have access to many products (e.g., hotels) with different inventory levels in the customer's desired price range.[1]

We consider a firm that sells multiple products in a retail category. There are limited inventories of the products to sell over a finite selling season. The selling prices of the products are all equal. The customer base is heterogeneous and characterized by multiple segments with different product preference distributions. These segments are essentially customer clusters with similar purchase histories as described in Linden et al. (2003). We use the Multinomial Logit framework to model the choice process of each customer segment. The retailer can identify the segment an arriving customer belongs to and customize the assortment to that particular customer without incurring additional cost. The customer then selects a product among those in the offered assortment or selects the no-purchase option. We formulate this as a dynamic assortment optimization problem

[1] In fact, Expedia.com is currently exploring the design of customized displays for its website (Expedia 2009).

in which the assortment decisions depend on the inventory levels, the current customer's preference distribution, and the distribution of preferences of future customers. In this setting, we find that the firm has the potential to increase revenues by strategically restricting the set of product options it makes available to customers, even when all products are in stock. In other words, the company may conceal a product short on inventory by showing it as out-of-stock, in anticipation of future sales to other customers who may have a stronger preference for this product (and who may therefore walk away if that product is not available).

We first characterize the optimal dynamic assortment policy for the case with a single customer segment and multiple product variants. In this case, it is optimal to offer the entire (in-stock) assortment to any arriving customer because all future customers have the same preference distribution as the current customer. In other words, we prove that the optimal assortment in each period is to offer all products with positive inventories.

In a setting with two products and two customer segments, we prove that the optimal assortment customization policy is a threshold-type policy. In particular, when both customer segments have opposite product preferences (i.e., each segment has a stronger preference for a different product) but would still consider buying the second product option, we show that each segment is always offered its most preferred product. In addition, a segment's less preferred product variant is offered only if that product's inventory level is above a certain threshold value. This threshold is decreasing in time. In other words, as time approaches the end of the selling season, it is more likely that both products will be offered to both customer segments. The threshold is also increasing in the inventory level of the segment's most preferred product. This implies, in particular, that both products are more likely to be offered if a customer's most preferred product variant has only a few remaining units in stock. We also find that as a product becomes relatively more popular (i.e., in terms of its utility) for any customer segment, the benefit of reserving this product for future customers increases – that is, the firm is more likely to ration this product to the customer segment with a relatively smaller preference for this product. We also consider another setting with two products in which one of the customer segments has exclusive interest in one particular product and would never buy the other variant. Customers in the other segment have both products in their consideration set. Many of the same results hold in this setting, although the second customer segment may not always be offered its most preferred product variant. We show that these results also extend to settings with two products and multiple customer segments. For the general case with multiple products and multiple customer segments, we introduce an aggregation-based heuristic that computes an effective assortment customization policy, recovering an average of around 64% of the gains achieved by using the optimal policy.

In a numerical study, we find that the threshold policy is optimal in more general settings with multiple products and multiple customer segments. This study also suggests that the revenue impact of assortment customization may be significant – up to a 12% increase in revenues in some examples – relative to a benchmark policy that does not consider customization and offers all products to all customers. In particular, we find that assortment customization is most valuable in settings with a highly heterogeneous customer base with overlapping product preferences and with asymmetric starting inventory levels. Assortment customization is contingent on the ability of a firm to segment its customer base. The results of the numerical study suggest that the revenue impact of customization can be large even with a small number of segments relative to the size of the product line. These findings demonstrate the revenue potential of dynamic assortment customization as a lever for revenue maximization. It is also important to note that assortment customization creates value for customers as well, because under this policy it is more likely that customers will find their most preferred products in stock.

Inventory rationing has been shown to be optimal in settings in which either selling prices or backordering costs are different across different customer groups. In contrast, inventory rationing in our model arises despite the fact that all products' selling prices are equal. In this case, inventory rationing is due to the heterogeneity in customer preferences.

The rest of the paper is organized as follows. Section 2 provides a review of the relevant literature. Section 3 describes the model formulation. Section 4 characterizes the optimal dynamic assortment policy for settings with two product variants. Section 5 presents a numerical study and proposes a heuristic to identify the impact of assortment customization and the characteristics of the optimal policy for the general case with multiple segments and multiple product variants. Section 6 concludes the paper. All proofs are provided in an appendix (we provide the key aspects of the proofs in Appendix A and the remaining details in Appendix B).

2. Literature Review

Our work is related to three streams of research. The first one is the literature on retail assortment planning, with papers focusing on assortment and inventory decisions for a single customer segment. Kök et al. (2008) provide a review of this literature. van Ryzin and Mahajan (1999) derive the optimal assortment policy for a category with homogenous products. Cachon et al. (2005) incorporate consumer search costs in a similar context. Kök and Xu (2010) study assortment and pricing decisions in retail categories with multiple subgroups of products. Smith and Agrawal (2000) discuss an optimization approach for the assortment selection and inventory management problems in a multi-item setting with demand substitution. Kök and Fisher (2007) describe a methodology for estimation of demand and substitution rates and for assortment optimization using data from a supermarket chain. Mahajan and van Ryzin (2001) and Honhon et al. (2008) optimize starting inventory levels for a model with dynamic customer substitution (i.e., customers choose from those products that are available at the time of their arrival). In our model, customers also engage in dynamic substitution, but the set of products displayed to each customer is a decision variable in our case. Caro and Gallien (2007), Saure and Zeevi (2009), and Ulu et al. (2010), study dynamic assortment selection with demand learning during a single selling season. Caro and Martinez-de-Albeniz (2009) find that renewing the assortment frequently can allow a firm to charge higher prices. The models in this stream of research do not consider multiple customer segments with different preferences, and therefore the assortment policy does not involve any form of customization. Kim et al. (2002) develop a methodology for estimating the product preferences of households and propose that web retailers such as Net Grocer and Peapod could offer customized assortments to each household – rather than the full assortment – to reduce the search cost of customers, which has been shown to negatively influence sales.

The second related stream of research includes work on choice-based network revenue management. Zhang and Cooper (2005) study revenue maximization in a setting where customers choose from a set of parallel flights. Talluri and van Ryzin (2004a) describe a framework for choice-based network revenue management models with multiple products (itineraries) and components (flight-legs), where product prices can be markedly different. The authors characterize the optimal assortment policy for settings in which all products share the same resource (aircraft capacity for one flight-leg). As in other revenue management papers with a single flight-leg, the optimal policy is a booking-limit based policy, under which some products with lower fare prices are not offered if the remaining capacity is low. In these models, the products command different prices but share the same resources. In contrast, each product variant has its own dedicated inventory in our setting.

For general choice-based network revenue management models, Gallego et al. (2004) and Liu and van Ryzin (2008) use a choice-based linear programming (CDLP) model to approximate the dynamic control problem. In addition, Liu and van Ryzin (2008) propose a dynamic programming decomposition heuristic and characterize the efficient sets which are used in the optimal policy. Zhang and Adelman (2009) use an affine function to approximate the value function of the dynamic program and Chen and Homem de Mello (2009) develop an approximation that consists of a sequence of two-stage stochastic programs with simple recourse. van Ryzin and Vulcano (2008a, b) study virtual nesting policies in a similar context where the demand process consists of a stochastic sequence of heterogeneous customers. Miranda Bront et al. (2009) show that the CDLP model of the assortment problem with multiple segments is NP-hard and propose a column generation algorithm. The above papers consider multiple segments characterized by different choice probabilities for the different products, and these products are sold at different prices (i.e., fare-route options). However, in these papers, the decision maker cannot observe the type of an arriving customer and therefore, at any given time, all customers are offered the same assortment (which is a list of fare class and route combinations). Hence, customization is not possible in those settings and the optimal assortment decisions are based on aggregate choice probabilities across segments, inventory levels, and price differences between products.

Cross-selling is commonly used to maximize revenues by online retailers. Netessine et al. (2006) study a retailer with limited inventories that dynamically selects product bundles to offer to each arriving customer (the bundles consist of the specific product requested by the customer plus a so-called packaging complement). For each arriving customer, the firm makes a decision regarding the packaging complement and the price of the bundle. The authors develop several heuristics and identify those that are effective in their setting. In a single product setting, Aydin and Ziya (2009) consider personalized dynamic pricing after receiving a signal about each customer's willingness-to-pay. Fudenberg and Villas-Boas (2006) provide a review of the literature on personalized pricing. Personalized pricing is also closely related to price discrimination, which has been studied extensively in the marketing literature. Price discrimination is achieved by offering a vertically differentiated product line (Mussa and Rosen 1978) and by offering product bundles (e.g., Fay and Xie 2008). Usually, in these settings, a static assortment is offered to all customers.

The ability of a company to limit the assortment to its customers is a form of inventory rationing. Therefore, our paper is also related to research on this topic. Ha (1997a) considers a single-item, make-to-stock production system with several demand classes (characterized by the different prices they are willing to pay) and lost sales, and demonstrates that the optimal policy is characterized by rationing levels for each demand class. Ha (1997b) studies a similar system with two demand classes and backordering. de Vericourt et al. (2002) extend this model to a setting with multiple demand classes characterized by different backorder penalty costs.

3. Model Formulation

We consider a retailer that sells a set of product variants within a retail category over a finite selling season. The retailer decides an assortment to offer to each arriving customer from a set of horizontally differentiated products, denoted by $\mathcal{N} = \{1, \cdots, N\}$. Each product variant may represent a combination of features that does not change the functionality of the product, such as a color/size/design combination in a clothing category. Other examples include music CDs, DVDs, variations of similar products from different brands, and online travel services. The selling season has T periods. There is no replenishment during the season. The initial amount of stock for all products is denoted by an N-dimensional vector $\mathbf{y}_0 = (y_{01}, \cdots, y_{0N})$ and $\mathbf{y}$ denotes a generic vector of inventory levels. This model is applicable, for example, to short-life-cycle products with long procurement lead times and to end-of-season sales for seasonal products.

There are M customer segments characterized by different product preferences. We model customer preferences using a Mixed Logit model. That is, each customer belongs to a segment with a certain probability and the choice process of all customers in a segment follows a specific Multinomial Logit (MNL) model.[2] Given an assortment $S \in \mathcal{N}$, the utility derived from choosing product $i \in S$ for a customer in segment m is $u_{mi} + \xi_{mi}$, where u_{mi} is the expected utility derived

[2]Mixed Logit models are quite flexible in modeling demand elasticities and do not suffer from some of the main criticisms of the Multinomial Logit Model, such as the IIA (independence of irrelevant alternatives) assumption.

from product i and ξ_{mi} is a random variable representing the heterogeneity of utilities across customers in the same segment. In addition, customers can always choose to not purchase any product, receiving a utility $u_{m0}+\xi_{m0}$. Each customer chooses the product that offers the maximum utility. We assume that ξ_{mi} are i.i.d. random variables following a Gumbel distribution with mean zero and variance $\pi^2/6$. The probability of a customer choosing product i that arises from this utility maximization problem is given by

$$q_{mi}(S) = \frac{\theta_{mi}}{\sum_{j \in S} \theta_{mj} + \theta_{m0}}, \; i \in S \cup \{0\}$$

where $\theta_{mi} = e^{u_{mi}}$. See Anderson et al. (1992) for more details on the MNL model and Kök et al. (2008) for a comparison of the MNL model with other demand models. We refer to θ_{mi} as a segment m customer's preference for product i and we let Θ_m denote the preference vector $(\theta_{m1}, \theta_{m2}, \cdots, \theta_{mN})$. We assume, without loss of generality, that all segments have the same preference for the no-purchase option, i.e. $\theta_{m0} = \theta_0 > 0$. There are in total M different preference vectors, one for each of the M customer segments. We assume that the retailer knows the preference vector for each customer segment. The retailer is able to estimate these preferences based on the customers' purchase history. Mixed Logit models are commonly used by retailers and marketing firms to identify multiple latent customer segments in the customer base and to estimate purchasing behavior for each segment (see, e.g., Gupta and Chintagunta 1994 and Wedel and Kamakura 1998). A common approach in online retailing is collaborative filtering (Linden et al. 2003), which measures similarity of customers to each other based on past history and infers preferences of an arriving customer for the category of interest.

We consider a Poisson arrival process and assume that at most one customer arrives in each period. The sequence of events in each period is as follows: At the beginning of the period, a consumer arrives with probability λ and the arriving customer belongs to segment m with probability ρ_m. (Clearly, $\sum_{i=1}^{M} \rho_i = 1$.) Because of the identification process that takes place during the log-in, the retailer has perfect information on the customer's segment upon arrival. The retailer offers an assortment (subset of the available products) to the customer. Next, the customer makes a purchasing decision according to the choice process and the revenue is received if a product is sold.

To isolate the effect of customer heterogeneity, we focus on a model with identical prices for all products, denoted by p. In a setting with non-identical prices, there is a clear incentive for rationing (e.g., not offering a lower-priced product at certain levels of inventory) even with a homogeneous customer base. In practice, the price of a product is usually the same for all variants within categories of horizontally differentiated products, supporting the assumption of identical prices in our context. We also assume that a product's price is the same for all customers, that is, we do not consider dynamic customized pricing of individual products. It is a matter of debate whether customized pricing is legal with respect to antitrust laws (Ramasastry 2005). As an example, in 2000, Amazon.com acknowledged that it had presented different prices to different customers in its DVD store for experimentation purposes, but denied that it did so on the basis of any past purchasing behavior at Amazon.

Finally, without loss of generality, we assume that the salvage value for unsold units at the end of the season is zero. We use the following notation throughout the paper. We let e_i denote the i^{th} unit vector. In addition, we let $\overline{S}(\mathbf{y})$ be the set of products with positive inventory and denote the cardinality of set S as $|S|$.

3.1 The Dynamic Assortment Optimization Problem

Define the value function in period t as $V_t(\mathbf{y}|m)$, given the vector of inventories $\mathbf{y}$ and that the customer arriving in this period is in segment m. Taking expectation across all customer segments, the value function at the beginning of period t is given by

$$V_t(\mathbf{y}) = \sum_{m \in \mathcal{M}} \rho_m V_t(\mathbf{y}|m).$$

Provided that the current arriving customer is in segment m, the goal is to select an assortment to maximize revenues for the current period and for the rest of the season. Therefore, the optimality equation is given by

$$V_t(\mathbf{y}|m) = \max_{S \subset \overline{S}(\mathbf{y})} \left[\sum_{i \in S} \lambda q_{mi}(S)(p + V_{t+1}(\mathbf{y} - e_i)) + \lambda q_{m0}(S) V_{t+1}(\mathbf{y}) \right] + (1-\lambda) V_{t+1}(\mathbf{y}). \quad (1)$$

The term inside the brackets is the value function assuming that the arriving customer is in segment m (an arrival occurs with probability λ). For a given choice of assortment S, this term accounts for the probability of selling one unit of variant i in S, earning a revenue of p in this period, plus the revenue-to-go function in period $t+1$ evaluated at the current inventory level minus the unit sold in period t. The term also accounts for the possibility that the customer does not make a purchase, in which case all inventories are left to the next period and the revenue is the profit-to-go function in period $t+1$ evaluated at the current vector of inventory levels. We maximize this term over all possible subsets of variants with positive inventories. The last term is the future value function if no customer arrives (with probability $1-\lambda$). Let $S^*_{it}(\mathbf{y})$ denote the optimal assortment for a segment i customer at time t given inventory levels $\mathbf{y}$. The total optimal revenue over the selling season is given by $V_1(\mathbf{y}_0)$. The terminal condition of this dynamic program is the value function in period T. Because there are no more customers beyond the last period, the optimal policy is to offer all products with positive inventory to any arriving customer regardless of the segment the customer belongs to. Therefore, $V_T(\mathbf{y}|m) = \sum_{i\in\overline{S}(\mathbf{y})} \lambda q_{mi}(\overline{S}(\mathbf{y}))p$.

This formulation leads to a dynamic program with an N-dimensional state space and a large action space (for each segment, there are $2^{|\overline{S}(\mathbf{y})|}$ possible assortments that can be offered). Thus, the above dynamic assortment optimization problem is intractable for large N and M. In the next section, we provide general properties of the optimal solution to this dynamic program and characterize the optimal policy in some settings.

4. The Optimal Assortment Policy

In this section, we turn our attention to the optimal dynamic assortment policy. We first discuss properties of the optimal policy for the general problem, and then proceed to a full characterization of the optimal policy in some specific settings.

For $i \in \overline{S}(\mathbf{y})$, let us define $\Delta^i_{t+1}(\mathbf{y}) = V_{t+1}(\mathbf{y}) - V_{t+1}(\mathbf{y}-e_i)$ as the marginal expected revenue generated by the y_i-th unit of inventory of product i in period $t+1$. Clearly, $\Delta^i_{t+1}(\mathbf{y})$ is nonnegative and cannot exceed the price p, i.e., $p \geq \Delta^i_{t+1}(\mathbf{y}) \geq 0$. In addition, $\Delta^i_{t+1}(\mathbf{y}) = 0$ when $y_i \geq T-t+1$, that is, when there is enough inventory to satisfy demand throughout the remaining selling season. We can then rewrite (1), the optimality equation, as

$$V_t(\mathbf{y}|m) = \max_{S\subset\overline{S}(\mathbf{y})} \left\{ \sum_{i\in S} \lambda q_{mi}(S)(p - \Delta^i_{t+1}(\mathbf{y})) \right\} + V_{t+1}(\mathbf{y}). \tag{2}$$

If product i is offered and sold in period t, then the revenue consists of the price p minus the lost opportunity revenue of selling this unit after period t, given by $\Delta^i_{t+1}(\mathbf{y})$. Therefore, we denote the effective marginal price of product i in period t as $p^i_t(\mathbf{y}) = p - \Delta^i_{t+1}(\mathbf{y})$ if $i \in \overline{S}(\mathbf{y})$ and $p^i_t(\mathbf{y}) = 0$

if $i \in \mathcal{N} \setminus \overline{S}(\mathbf{y})$. Consider an ordering of the products in period t given inventory levels $\mathbf{y}$ so that $p_t^{i_1}(\mathbf{y}) \geq \cdots \geq p_t^{i_N}(\mathbf{y})$. Note that the indices $(i_j)_{j=1}^N$ depend on the inventory vector $\mathbf{y}$ and the period t. Based on this ordering, we define a set consisting of the products with the largest effective marginal prices, given by $A_k(\mathbf{y}) = \{i_1, \cdots, i_k\}$. Then, if the value function $V_{t+1}(\cdot)$ is known for each inventory level, the problem in (2) reduces to a one-period assortment optimization problem. This one-period problem is similar to that in Talluri and van Ryzin (2004b). We now show that their result extends to the setting when the assortment can be customized for each customer segment. We also provide additional properties of the optimal assortment.

Proposition 1 *Given inventory levels* $\mathbf{y}$ *in period* t*, the optimal assortment for a segment* m *customer is given by* $S^*_{mt}(\mathbf{y}) \in \{A_1(\mathbf{y}), \cdots, A_N(\mathbf{y})\}$*. Furthermore,* $S^*_{mt}(\mathbf{y}) \subset S^*_{rt}(\mathbf{y})$ *if the following holds:* $\sum_{k=1}^{l} \theta_{m,i_k} > \sum_{k=1}^{l} \theta_{r,i_k}$ *for all* $l = 1, .., N$ *and* $\theta_{m,i_k} \leq \theta_{r,i_k}$ *for all* $k = 2, .., N$.

This result shows that the optimal assortment for each customer segment is restricted to one of N possible sets $A_i(\mathbf{y})$. In particular, $i < |\overline{S}(\mathbf{y})|$ indicates an assortment specifically customized for that particular customer segment. The characterization of these sets requires the computation of the effective marginal prices $\left(p_t^i(\mathbf{y})\right)_{i \in \mathcal{N}}$, which themselves depend on the value function $V_{t+1}(\cdot)$. These marginal prices are easy to compute in some special cases. For example, when the inventory level of product i is large relative to the remaining time-horizon, say $y_i \geq T - t + 1$, product i's marginal effective price is p and, therefore, it is optimal to offer this product to all segments. In general, the computation of the future value function and the effective marginal prices at time t requires solving the dynamic program up to that period. Nevertheless, Proposition 1 provides a useful tool for determining the structure of the optimal assortment policy and will be used in many of the subsequent results. In particular, this result implies that there is a product – product i_1 – that will be offered to any customer arriving in period t.

The result also suggests that, in any period, the optimal assortment sets offered to different customer segments have a nested structure. That is, the optimal assortment sets can be partially ordered based on the customer segments' preference vectors and the resulting marginal effective prices in that period (which, in turn, depend on the inventory levels). The proposition provides a necessary condition on the preference vectors that implies the nested structure of the optimal assortment sets. The first set of conditions is equivalent to the majorization order (see van

Ryzin and Mahajan 1999), which measures and compares the dispersion of preferences for any two customer segments. These conditions imply that a segment that has a relatively stronger preference for products with high effective marginal prices would be offered a relatively smaller assortment. If a product i has a high effective marginal price at time t, $p_t^i(\mathbf{y}) = p - \Delta_{t+1}^i(\mathbf{y})$, then $\Delta_{t+1}^i(\mathbf{y})$ must be small, implying that there is a relatively large amount of inventory for that product. Therefore, if a segment of customers has a strong preference for the products with high effective marginal prices (relative to other customers), then their assortment should be mostly restricted to those products to force them to buy products with high inventory levels (instead of allowing them to potentially buy a product with a lower effective marginal price and therefore a higher marginal value of inventory). In contrast, other customers would be offered a relatively larger assortment consisting of those same products (because they have high inventory levels) plus other products according to their segment's overall preferences.

We characterize the optimal assortment policy in a setting with a single customer segment and multiple product variants in Section 4.1 and then proceed to the characterization of the optimal policy in a setting with two product variants in Section 4.2.

4.1 Single Customer Segment and Multiple Product Variants

Let us consider the case with a single customer segment, i.e., $M = 1$. This case represents, in particular, a situation in which a company is unable to obtain or use its customers' purchase history to segment the market according to the customers' preferences. Clearly, in this setting, customization is not possible. Therefore, the optimal policy in this setting serves as a benchmark to assess the value of customization (in terms of profit gains) when segmentation is possible. We first present results on the value function and marginal expected revenue.

Proposition 2 *Given inventory levels* $\mathbf{y}$ *in period* t*,*

(i) $\Delta_t^i(\mathbf{y})$ *is decreasing in* y_j, $j \in \mathcal{N}$.

(ii) If $y_i = y_j \geq 1$ *and* $\theta_{1i} \geq \theta_{1j}$, *then* $\Delta_t^i(\mathbf{y}) \geq \Delta_t^j(\mathbf{y})$.

(iii) Both $V_t(\mathbf{y})$ *and* $\Delta_t^i(\mathbf{y})$ *are decreasing in* t.

The first property in Proposition 2 implies that the value function is concave in y_i and submodular in $\mathbf{y}$. In other words, the marginal expected profit of a product decreases as the inventory level of any product increases because a higher inventory level lowers the probability that any existing unit will be sold over the remaining time in the selling season. The second property shows that, when both products have the same inventory levels, the product with a higher expected utility has a higher marginal value than the product with a lower expected utility. The last part indicates that both the value function and the marginal expected profit decrease as the remaining time until the end of the selling season gets shorter, because the probability that a unit will be sold decreases with time t. Based on these results, we characterize the optimal assortment in this setting.

Theorem 1 *Consider a setting with a single customer segment and N products. The optimal assortment in each period is to offer all products with positive inventory.*

Theorem 1 shows that, in this setting, it is optimal to follow the myopic policy, i.e., offer all products in each period, thus maximizing the probability of selling any one product in that period. Because all product variants have identical prices, optimizing the profit over the selling season is equivalent to maximizing total sales. In addition, because customers are homogeneous, all future customers have the same preferences as the current arriving customer in expectation, so the firm cannot benefit from reserving a product in anticipation of future sales.

Note that the result does not directly follow from the properties in Proposition 2. The proof relies on inductively showing that the value-to-go function for any assortment that leaves out an available product can be improved by adding that product to the assortment.

4.2 Two Product Variants and Multiple Customer Segments

We now consider settings with two product variants. Most of the results in this section are derived for the case of two customer segments, although we later generalize some of these results for the case of multiple customer segments. We first focus on settings with two customer segments.

The preference vectors for customer segments 1 and 2 are given by $(\theta_{11}, \theta_{12})$ and $(\theta_{21}, \theta_{22})$, respectively. There are multiple possible scenarios to consider in this setting given by the possible orderings of the preference values for both customer segments. We first consider a setting in which each segment has a stronger preference for a different product. Specifically, we assume that $\theta_{11} \geq \theta_{12}$ and $\theta_{22} \geq \theta_{21}$, that is, segment 1 [segment 2] customers have a stronger preference for product 1 [product 2]. Moreover, we assume that $\theta_{11} \geq \theta_{21}$ and $\theta_{22} \geq \theta_{12}$.

In this setting, if both products are available, then segment 1 customers are more likely to purchase product 1, while segment 2 customers are more likely to purchase product 2. In addition, when only product i is offered, a customer from segment i purchases that product with a higher probability than a customer from the other segment. Therefore, these two customer segments are distinctively different in terms of their product preferences. The following result characterizes the optimal assortment policy.

Theorem 2 *The optimal assortment policy in period t is as follows.*

If $\overline{S}_t(\mathbf{y}) = \{k\}$, i.e., only product k is available, then it is optimal to offer that product to any arriving customer. If $\overline{S}_t(\mathbf{y}) = \{1,2\}$, then:

(i) Product i is always in the optimal assortment set for segment i, i.e., $i \in S^*_{it}(\mathbf{y})$.

(ii) For a segment i customer, there exists a threshold level $y^*_{jt}(y_i)$ such that:

If $y_j \geq y^*_{jt}(y_i)$, then $S^*_{it}(\mathbf{y}) = \{1,2\}$; If $y_j < y^*_{jt}(y_i)$, then $S^*_{it}(\mathbf{y}) = \{i\}$.

(iii) $y^*_{it}(y_j)$ is increasing in y_j.

(iv) The threshold levels are decreasing in t.

Sketch of the Proof: We prove the following five properties for each period t using an induction argument that makes use of these five properties in period $t+1$.

P1(t, i, y_1, y_2): $\Delta^i_t(y_1, y_2)$ is decreasing in y_j, $j \in \{1,2\}$.

P2(t, i, y_1, y_2): $\Delta^i_t(y_1, y_2)$ is decreasing in t.

P3(t, i, y_1, y_2): $p - \Delta^i_t(y_1, y_2) \geq \frac{\theta_{ij}}{\theta_{ij}+\theta_0}(p - \Delta^j_t(y_1, y_2))$ for $j \neq i$.

P4(t, i, y_1, y_2): $\Delta^j_t(\mathbf{y}) - \Delta^j_t(\mathbf{y} + \mathbf{e}_i) \geq \frac{\theta_{ii}}{\theta_{ii}+\theta_0}(\Delta^i_t(\mathbf{y}) - \Delta^i_t(\mathbf{y} + \mathbf{e}_i))$ for $j \neq i$.

P5(t, i, y_1, y_2): If $p - \Delta^i_t(\mathbf{y}) \geq \frac{\theta_{jj}}{\theta_{jj}+\theta_0}(p - \Delta^j_t(\mathbf{y}))$, then $p - \Delta^i_t(\mathbf{y} - \mathbf{e}_j) \geq \frac{\theta_{jj}}{\theta_{jj}+\theta_0}(p - \Delta^j_t(\mathbf{y} - \mathbf{e}_j))$ for $j \neq i$.

P1 implies that the expected marginal revenue of a unit of inventory is decreasing in any product's inventory level, implying concavity in y_i and submodularity in $\mathbf{y}$. P2 implies that the expected marginal revenues are decreasing in time. P3 implies part (i) of the theorem, i.e., that the assortment offered to a segment always contains that segment's most preferred product. P4 is an inequality that implies the threshold policy in part (ii), which can be expressed as follows: If it is optimal to offer product j to segment i at inventory level $\mathbf{y}$, then it is also optimal to offer product j to segment i when there is one more unit of inventory of product j, i.e., if $p - \Delta_t^j(\mathbf{y}) \geq \frac{\theta_{ii}}{\theta_{ii}+\theta_0}(p - \Delta_t^i(\mathbf{y}))$ then $p - \Delta_t^j(\mathbf{y}+\mathbf{e}_i) \geq \frac{\theta_{ii}}{\theta_{ii}+\theta_0}(p - \Delta_t^i(\mathbf{y}+\mathbf{e}_i))$. P5 implies that if it is optimal to offer product i to segment j at inventory level $\mathbf{y}$, then it is also optimal to offer this product to segment j when the vector of inventory levels is $\mathbf{y} - \mathbf{e}_j$, that is, $y_i \geq y_i^*(y_j - 1)$. This property implies part (iii) of the theorem. The proof of each property requires comparing the alternative assortments in the current period given the customer choice probabilities and the continuation value functions based on the implementation of the optimal assortment policy in future periods. This results in a large number of cases to be checked (corresponding to multiple future outcomes), and the properties do not necessarily hold in all cases. Properties 3-5 are used to rule out some cases, and we prove that the remaining cases satisfy the five properties. ■

The result shows that, in this setting, each customer segment is always offered its most preferred product in the optimal dynamic assortment policy. Furthermore, a segment i customer is also offered product j if the inventory level of that product is large enough. Otherwise, the customer is only offered product i. This policy is characterized by a set of threshold levels y_{it}^* and y_{jt}^* in each period t. Because the products prices are equal, it is optimal to sell as many units of either product as possible throughout the selling season. Then, for an arriving customer of segment i, if the inventory level of product j is low, it may be optimal to reserve that inventory for future arriving customers of segment j, since those customers are more likely to leave without purchasing any product if product j is not available. On the other hand, if the inventory level of product j is large enough, then it is optimal to offer both product variants as future sales to segment j customers are not highly compromised. Thus, the optimal policy involves inventory rationing. This result is driven by the heterogeneity in customer preferences, suggesting another reason for rationing inventory that is not related to differences in prices. We next present an example to further clarify the intuition behind this result.

Example 1 Consider the second to last period before the end of the horizon, i.e., period $t = T-1$, and let $\mathbf{y} = (y_1 = 1, y_2 = 2)$. Suppose that a customer from segment 2 arrives in period $T-1$ and consider the realization of her preferences for the three options – purchase product 1, purchase product 2, or not purchase any product. If the customer's highest realized preference is either to purchase product 2 or the no-purchase option, then not offering product 1 has no effect on current or future revenues. Now, consider the case where the customer's first (realized) choice is product 1. If the offered assortment is $S_{2,T-1} = \{1, 2\}$, then the customer buys product 1, yielding a revenue of $p + V_T(0, 2)$. If the offered assortment is $S_{2,T-1} = \{2\}$, then the customer will either buy product 2 (with probability $\theta_{22}/(\theta_0 + \theta_{22})$), yielding a revenue of $p + V_T(1, 1)$, or will choose the no-purchase option, yielding a revenue of $V_T(1, 2)$. Note that $V_T(1, 2) = V_T(1, 1)$ and $V_T(0, 2) = V_T(0, 1)$, because the firm can sell at most one unit in the last period. Hence, the assortment $S_{2,T-1} = \{1, 2\}$ yields a revenue of $p + V_T(0, 1)$ and $S_{2,T-1} = \{2\}$ yields a revenue of $p(\theta_{22}/(\theta_0 + \theta_{22})) + V_T(1, 1)$. With $S_{2,T-1} = \{2\}$, the firm loses some revenue in the current period, but it gains $V_T(1, 1) - V_T(0, 1)$ in the last period. Thus, depending on the choice probabilities and the size of the segments, $S_{2,T-1} = \{2\}$ could be the optimal solution – implying rationing product 1. This will be the case, for example, when θ_{22} is high relative to θ_0 or when segment 1 is relatively large (ρ_1 is close to one). In such cases, it is optimal to direct the segment 2 customer to buy product 2, reserving product 1 for the last period. ∎

To conclude the characterization of the optimal policy, we discuss the monotonicity properties of the threshold levels stated in Theorem 2. Part (*iii*) implies that, given inventory levels y_i and y_j, if assortment $\{i, j\}$ is offered to segment j, then it is also optimal to offer the full assortment to segment j for lower inventory levels of product j. Part (*iv*) suggests that towards the end of the selling season, there is more incentive to offer both variants to any arriving customer.

Figure 1 below illustrates these properties. Figure 1(a) shows the threshold values for both customer segments in a given period t, each as a function of the inventory level of the segment's most preferred product. These thresholds divide the inventory (y_1, y_2) space in three regions. In the upper-left region, segment 1 is offered both products, but it is optimal to ration product 1 to an arriving customer of segment 2. The reverse situation arises in the lower-right region. In the center region of the graph, any arriving customer is offered both products. As shown in Theorem 2(iii), both threshold levels are increasing. Thus, for a given amount of inventory of product 2, it is more likely to offer that product to an arriving customer of segment 1 when the inventory level of product 1 is low. In this case, it may be optimal to re-direct some segment 1 customers to purchase product 2 so as to reduce the likelihood of running out of stock of product 1 in the future. On the other hand, when the inventory level of product 1 is high, it may be optimal to offer only that product to an arriving customer of segment 1, therefore increasing the demand for that product. Figure 1(b) exhibits the threshold for product 2 as a function of the inventory level of product 1 in period t, for various values of t. The threshold levels decrease as time increases, indicating that less rationing occurs as time approaches the end of the selling season. This occurs because there is less concern about future sales and therefore it is optimal to sell as many units as possible.

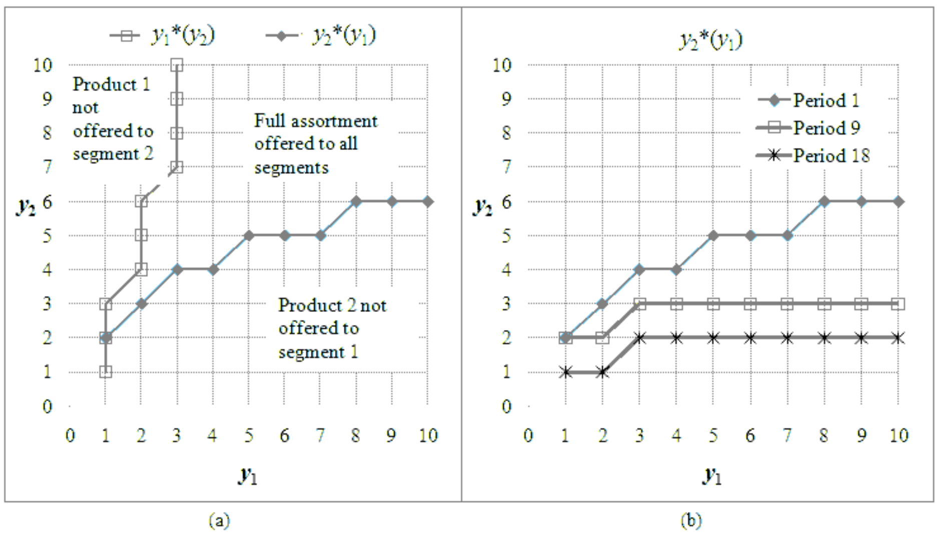

Figure 1: Threshold values characterizing the optimal assortment policy. Parameters: $p = 1, \rho_1 = 0.5, \Theta_1 = (20, 10, 6), \Theta_2 = (20, 2, 10)$. For (a), $t = 18$.

Next, we demonstrate the potential impact of assortment customization on revenue with numerical examples. We measure the impact on revenue of employing the optimal policy by calculating the percentage revenue increase due to assortment customization relative to a benchmark policy, under which all available products are offered to any arriving customer. Figure 2 below shows the percentage increase in revenue due to assortment customization as a function of the inventory levels. Consider the case with symmetric demands in Figure 2(a). When inventory levels of both products are large, the retailer is less likely to run out of stock, so there is no need for rationing inventory and therefore the percentage revenue impact is small. When inventory levels of both products are low relative to the time horizon (low values of y_1 and y_2 in the graph), there is enough time to sell all units, so both products are likely to be offered in the optimal solution, and again the revenue impact is small. The revenue impact is more significant in the areas where the inventory level of one product is relatively high while the inventory level of the other product is relatively low (a maximum of 1.74% in this example). These are the cases in which strategically selecting the assortment to offer customers is most important. In the example in Figure 2(b), demand for product 1 is higher (because both customer segments have a strong preference for that product). Thus, customization is important when the inventory level for product 1 is relatively low. In that example, the maximum revenue gain is 7.1%. That is, depending on the inventory levels, the revenue impact of assortment customization can be substantial. The products' inventory levels may become unbalanced at any point during the selling season (relative to their expected demands) due to the stochasticity of the demand process, even if the initial inventory levels were selected optimally. This may be particularly the case in settings with high demand uncertainty, such as in fashion and apparel products, where it is difficult to predict the popular products in advance of the selling season. In any case, because retailers usually operate with low profit margins, even a small percentage increase in revenue can have a significant impact on profit.

We further expand the example displayed in Figure 2 by varying one problem parameter at a time. The revenue impact of employing the optimal policy is large when it involves a significant level of assortment customization. In turn, this occurs when the threshold levels are relatively high. Consider the threshold level for segment 1 in any given period t, $y_{2t}^*(y_1)$. Recall that, for a segment 1 customer, it is optimal to offer product 2 if $y_2 \geq y_{2t}^*(y_1)$. We find that the threshold level $y_{2t}^*(y_1)$ for a fixed inventory level y_1 decreases with ρ_1. That is, as the proportion of customers from segment 1 increases, offering both products to segment 1 customers becomes more attractive as this has the potential of increasing sales. Furthermore, as ρ_1 approaches 1, rationing is less likely because most customers belong to one segment (and therefore the setting resembles the one discussed in Section 4.1). We also find that the threshold level $y_2^*(y_1)$ increases with $\theta_{22} - \theta_{21}$, a measure of segment 2's preference for product 2 over product 1. When this difference increases, a segment 2 customer is more likely to purchase product 2. Thus, reserving product 2 for future arriving customers of segment 2 is more likely to lead to higher sales. In addition, as either θ_{12} or θ_{22} increases (one customer segment has a higher relative preference for product 2), demand for product 2 becomes stochastically larger. Then, it becomes more desirable to reserve units of product 2 for customers of segment 2. That is, the threshold level $y_{2t}^*(y_1)$ increases with θ_{12} and

θ_{22}. However, this threshold decreases as either θ_{11} or θ_{21} increases (and, therefore, the probability of selling product 1 increases). In this case, making both products available to segment 1 customers is less likely to have a severe impact on product 2's future inventory levels, and therefore on overall sales. Finally, as the utility of the no-purchase option increases, i.e., when customers are more likely to leave without purchasing any product, total demand decreases and therefore both the threshold levels and the revenue impact of customization are lower.

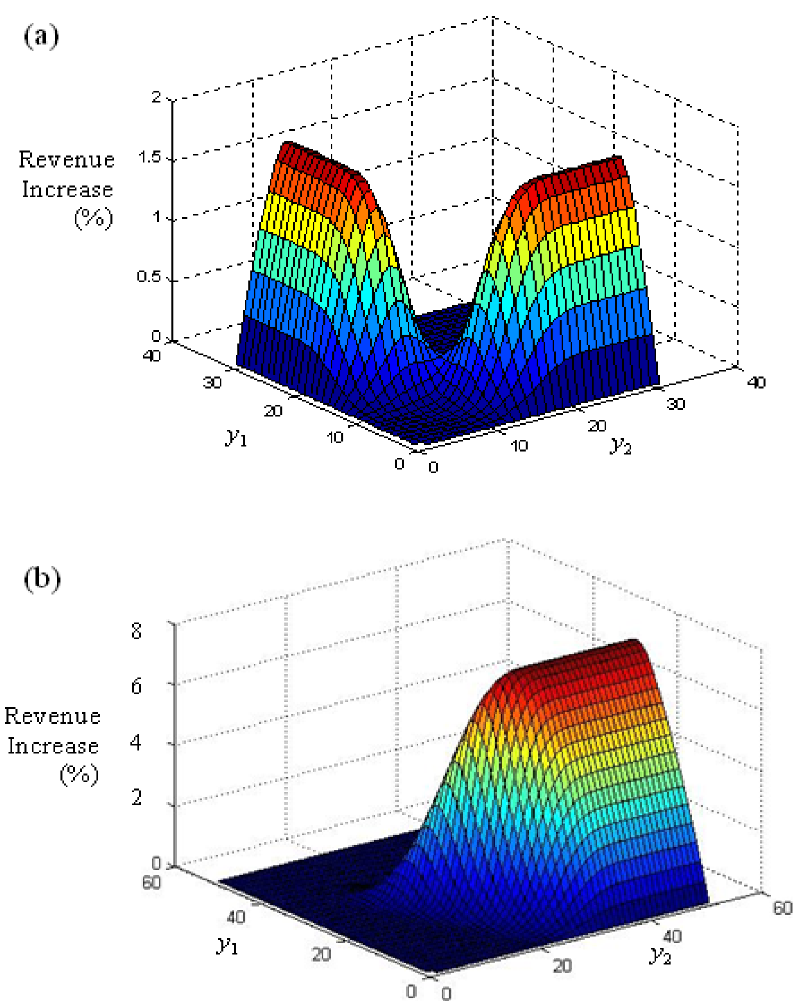

Figure 2: Percentage revenue increase due to assortment customization relative to the benchmark policy that offers all available products to all customers. Parameters: $p = 1, \boldsymbol{\rho} = (0.5, 0.5)$,
(a) $\Theta_1 = (1, 5), \Theta_2 = (5, 1), \theta_0 = 2, T = 30$.
(b) $\Theta_1 = (10, 2), \Theta_2 = (10, 10), \theta_0 = 2, T = 50$.

In the setting analyzed above, in which each customer segment has a distinctive preference for a different product, we show that it is optimal to offer each segment its most preferred product regardless of the inventory levels.[3] This result may not always hold, even in settings with two products and two customer segments. The following simple setting illustrates that it may be optimal to ration a segment's most preferred product even to customers from that segment. Suppose that segment 1 customers are only interested in one of the two product variants, say product 1. If this product is not available, segment 1 customers will not make a purchase. On the other hand, segment 2 customers may purchase either product. Specifically, assume that $\theta_{11} > 0$ and $\theta_{12} = 0$, while $\theta_{21} > 0$ and $\theta_{22} > 0$. In this setting, an arriving segment 1 customer is always offered the assortment $\{1\}$, the only product in their consideration set, if this product is available. The following result characterizes the optimal assortment policy for an arriving customer of segment 2.

Proposition 3 *The optimal assortment policy in period t is as follows.*
If $\overline{S}_t(\mathbf{y}) = \{k\}$, i.e., only product k is available, then it is optimal to offer that product to any arriving customer. If $\overline{S}_t(\mathbf{y}) = \{1, 2\}$, then:
(i) $S^*_{1t}(\mathbf{y})$ equals $\{1\}$ or $\{1, 2\}$, indistinctively, for any $\mathbf{y}$; Product 2 is always included in $S^*_{2t}(\mathbf{y})$.
(ii) For a segment 2 customer, there exists a threshold level $y^*_{1t}(y_2)$ such that:
If $y_1 \geq y^*_{1t}(y_2)$, then $S^*_{2t}(\mathbf{y}) = \{1, 2\}$; If $y_1 < y^*_{1t}(y_2)$, then $S^*_{2t}(\mathbf{y}) = \{2\}$.

For an arriving customer from segment 2, the possible assortment sets in the optimal solution are either $\{1, 2\}$ or $\{2\}$, regardless of which product is the most preferred by that segment. Whether or not product 1 will be rationed to segment 2 is determined by a threshold. If the inventory level for product 1 is low, then that product is reserved for segment 1 customers, who would not make a purchase if product 1 is out of stock, therefore rationing product 1 to segment 2 customers. Some of those customers may be willing to purchase product 2 instead, increasing the total expected number of units sold throughout the selling season. So, rationing may apply to a segment's most preferred product – this is driven by the relative strength of product preferences for each customer segment (and not simply by the order of their preferences).

Consider now a setting with M customer segments. The following result establishes details of the optimal policy for the case of M customer segments, under some specific conditions on the

[3] Note that this does not mean that each *individual* customer is always offered its most preferred product. Due to the randomness in the customers' preferences, it is possible that a segment 1 customer actually prefers product 2 over product 1. The optimal assortment would still include product 1, but it may not include product 2.

preference parameters.

Proposition 4 *Consider a setting with two products and M customer segments. Let $\theta_{\min,i} = \min_{k=1,...,M}\{\theta_{ki}\}$, for $i = 1, 2$. Assume that segments can be classified into two types: Segment k is of type i if $\theta_{ki} \geq \theta_{k,3-i} = \theta_{\min,3-i}$, for $i = 1, 2$ and $k = 1, ..., M$. Then, an arriving customer of segment k, which is itself of type i, is always offered product i. Moreover, this customer is also offered product $3-i$ if the inventory level of that product is larger than or equal to a threshold level (that depends on and is increasing in θ_{ki}).*

The conditions in the proposition hold if there are two types of segments, one type with a stronger preference for product 1 and all equal preferences for product 2, and the reverse for the other type. For example, $\Theta_1 = (10, 1), \Theta_2 = (9, 1)$, both of type 1, and $\Theta_3 = (5, 8), \Theta_4 = (5, 12)$, both of type 2. This result will be helpful for the development of the heuristic introduced in Section 5.

5. General Case

In this section, we consider the general problem with N products and M customer segments. As mentioned before, the computation time for this dynamic programming problem grows exponentially with the number of products. We perform a numerical study to investigate whether the properties of the optimal policy derived in Section 4 hold for the general case and to identify the settings in which the impact of assortment customization is most significant. We then introduce a heuristic to compute an assortment customization policy for the general case.

The study consists of 60 cases – all with four customer segments, four product variants, $p = 1$, $\lambda = 1$, and $\theta_0 = 2$. The parameters λ, θ_0, and T, are interchangeable in the sense that values of $\lambda < 1$ and/or $\theta_0 > 2$ lead to the same results as relatively lower values of T. We therefore fix the values of λ and θ_0, and consider five different values for T, $T = 10, 20, 30, 40, 50$. For each case, we evaluate the optimal profit for T^4 different starting inventory levels in the set $\{\mathbf{y}_0 : 1 \leq y_{i0} \leq T, i = 1, 2, 3, 4\}$.[4] The preference vectors are given in Table 1: G1 represents a setting with distinctively different preferences across customer segments; the same applies for the first three customer segments under G2, while the fourth segment is somewhat indifferent

[4]When the inventory level of a product is larger than T, there is enough inventory of that product to satisfy all demands, regardless of their segment. Therefore, that product is always in the optimal assortment.

between products; in G3 and G4, customers in each segment have roughly equal preferences for two and three of the products, respectively; We also consider three possible vectors of segment sizes: $\boldsymbol{\rho}^1 = (0.25, 0.25, 0.25, 0.25)$, $\boldsymbol{\rho}^2 = (0.1, 0.1, 0.4, 0.4)$, $\boldsymbol{\rho}^3 = (0.1, 0.1, 0.1, 0.7)$.

Group	Θ_1	Θ_2	Θ_3	Θ_4
G1	$(10, 1, 1, 1)$	$(1, 10, 1, 1)$	$(1, 1, 10, 1)$	$(1, 1, 1, 10)$
G2	$(10, 1, 1, 1)$	$(1, 10, 1, 1)$	$(1, 1, 10, 1)$	$(9, 9, 9, 10)$
G3	$(10, 9, 1, 1)$	$(1, 10, 9, 1)$	$(1, 1, 10, 9)$	$(9, 1, 1, 10)$
G4	$(10, 9, 8, 1)$	$(1, 10, 9, 8)$	$(8, 1, 10, 9)$	$(9, 8, 1, 10)$

Table 1: Preference vectors of customer segments in the numerical study.

We find that the optimal policy is determined by a set of inventory threshold levels for each segment across all cases and for all periods and inventory levels. Recall that in the two-product two-segment case with reverse preferences, each segment is always offered its most preferred product. As discussed in Section 4.2, this may not always be the case. In fact, we find that this property may not hold in a setting with more than two products even when segments have reversed preferences. Consider, for example, the case with segment sizes given by $\boldsymbol{\rho}^1$ and preference vectors given by G3. Suppose that $t = T - 1$ and $\mathbf{y} = (1, 1, 1, 0)$. The optimal assortment for segment 1 is given by $S_{1t}(\mathbf{y}) = \{2\}$, hence segment 1 is not offered its most preferred product. A segment 1 customer is likely to make a purchase from this set. If the customer was also offered product 1, it would purchase that product with a high probability, therefore possibly creating a critical shortage for a segment 4 customer in the next period, whose other most preferred product is already out-of-stock.

In order to evaluate the revenue impact of assortment customization, we compare the revenue under the optimal dynamic assortment policy for a given set of inventory levels with the revenue obtained from the benchmark policy, under which all available products are offered to all customers. We make this comparison by considering three different subsets of initial inventory levels, each of which satisfying a given load factor condition. More specifically, we compare the revenue impact of customization over the set of initial inventory levels that satisfy an equation of the type $\sum_{i=1}^{4} y_{i0} = \alpha T$, for three different values α. (Recall that there are a total of T^4 possible initial inventory levels.) In particular, we consider the cases of $\alpha = 1$, $\alpha = 10/9$, and $\alpha = 3/2$, corresponding to a 100% load factor, a 90% load factor, and a 66% load factor, respectively. Table 2 presents a summary of these comparisons for each group of parameters for settings with $T = 30$ and $T = 50$. The patterns of performance are similar for $T = 10$, $T = 20$, and $T = 40$. The average of maximum gaps across all initial inventory levels is 3.76% for $T = 10$ and 5% for $T = 20$, while it is 5.62% for $T = 30$, 6.13% for $T = 40$, and 6.25% for $T = 50$.

Preferences	Segment Sizes	% Revenue increase due to assortment customization							
		$T = 30$				$T = 50$			
		Max	Average			Max	Average		
			$\alpha = 1$	$\alpha = 10/9$	$\alpha = 3/2$		$\alpha = 1$	$\alpha = 10/9$	$\alpha = 3/2$
G1	$\boldsymbol{\rho}^1$	1.81	1.02	0.94	0.64	2.02	1.14	1.04	0.71
	$\boldsymbol{\rho}^2$	2.39	0.74	0.70	0.50	2.78	0.83	0.77	0.56
	$\boldsymbol{\rho}^3$	2.59	0.50	0.47	0.35	3.21	0.56	0.52	0.39
G2	$\boldsymbol{\rho}^1$	6.42	2.85	2.54	1.39	6.98	3.13	2.70	1.50
	$\boldsymbol{\rho}^2$	8.09	3.13	2.78	1.51	8.88	3.43	2.96	1.64
	$\boldsymbol{\rho}^3$	7.20	2.26	2.00	1.04	8.09	2.53	2.17	1.15
G3	$\boldsymbol{\rho}^1$	6.65	2.79	2.36	0.83	7.40	3.09	2.50	0.90
	$\boldsymbol{\rho}^2$	6.83	2.37	2.06	0.95	7.56	2.61	2.18	1.02
	$\boldsymbol{\rho}^3$	4.25	1.75	1.51	0.73	4.87	1.92	1.59	0.79
G4	$\boldsymbol{\rho}^1$	6.55	2.17	1.72	0.28	7.30	2.36	1.76	0.30
	$\boldsymbol{\rho}^2$	8.21	2.23	1.77	0.34	8.96	2.42	1.80	0.36
	$\boldsymbol{\rho}^3$	6.50	1.93	1.51	0.44	6.97	2.09	1.54	0.47

Table 2: Impact of assortment customization. The maximum is calculated over all possible inventory values, while the average is calculated over all $\mathbf{y}$ such that $\Sigma_i y_i = \alpha T$.

We make the following observations from Table 2. First, the impact of assortment customization is larger in cases in which segments are distinctively different but have some overlapping preferences (as in groups G2, G3 and G4). The value of customization is relatively low in G1 because the interaction between the segments is minimal (each segment has a strong preference for a different product and very low preferences for all other products). In the extreme case, if each segment had a single product in its consideration set, then offering all products or just their favorite products would lead to the same revenue performance. In group G2, segment 4's preferences overlap with all segments and this increases the value of customization, which will affect mostly the assortments offered to segment 4 customers. As noted from the table, a setting involving the parameters in G2 and $\boldsymbol{\rho}^2$ leads to a maximum gap of 8.88%. This case parallels the example discussed in Figure 2(b), but here $N = 4$ and $M = 4$. Groups G3 and G4 also provide cases with large revenue impact from customization. In those cases, the segments are also tightly connected (each product is strongly preferred by at least two segments), but at the same time the segments are distinctively different from each other.

In general, the impact of assortment customization is lower for a less heterogeneous population (i.e., where one segment conforms a large fraction of the population, such as in $\boldsymbol{\rho}^3$). This effect is less pronounced in G1 – because each customer segment has a strong preference for a single product, a large concentration of the market in one particular segment makes customization more valuable (by strictly rationing that segment's most preferred product to the other segments. Recall that in the extreme case with a single customer segment, there is no value to customization as it is always optimal to offer all products. However, the concentration of customers in fewer segments may lead to an increased value of assortment customization, if those segments have overlapping product preferences. This can be seen for G3, comparing the settings with $\boldsymbol{\rho}^2$ and $\boldsymbol{\rho}^3$. In the case of $\boldsymbol{\rho}^2$, segments 3 and 4 are both of fairly large size and they both have a high preference for product 4 – in this case, assortment customization can have a significant impact for certain inventory values. In the case of $\boldsymbol{\rho}^3$, however, a large part of the population belongs to a single segment (namely, segment 4), and the value of customization is smaller.

To expand on this issue, we have run a set of numerical experiments with $N = 4$ and $M = 2$, where the preference of each of the two segments is the average of the preferences of segments 1-2 and segments 3-4, respectively, in the original problem setting with $M = 4$. We consider two vectors of segment sizes that result from the combination of the respective segments in the original setting, i.e., $\boldsymbol{\rho}^1_{M=2} = (0.5, 0.5)$ and $\boldsymbol{\rho}^2_{M=2} = (0.2, 0.8)$. The percentage revenue increase was substantial in these settings as well, e.g., under $M = 2$ and $\boldsymbol{\rho}^1_{M=2}$, the maximum gaps are 4.38%, 7.71%, 7.49%, and 1.93%, for G1, G2, G3, and G4, respectively.[5] This finding suggests that assortment customization may be valuable even with a small number of segments relative to the total number of products. That is, even a rather coarse segmentation of the market may be conducive to profit gains through assortment customization.

We further analyze the data to identify those inventory values that generate high revenue increases under assortment customization. We find the impact to be larger in cases where the inventory levels of highly preferred products are somewhat low and the inventory levels of other products are relatively high. As discussed for the case of two products and two customer segments, if all inventory levels are low, then there is not much potential for customization as all products will be rapidly stocked out. If the inventory levels are too high, then there is no need for rationing.

[5]Combining segments 1-2 and segments 3-4 under G4 leads to two somewhat similar customer segments, in terms of their product preferences. Therefore, in this case, the percentage revenue gain from assortment customization is relatively small.

These observations suggest that the inventory-to-demand ratios may provide valuable information about whether or not it may be optimal to ration a product. We investigate this issue with a set of numerical examples. For a given period t, consider the prevailing inventory level y_i for product i and the expected demand for this product across all customer segments assuming all products are offered throughout the rest of the selling season, which we denote as $E[D_i]$. For a large set of inventory levels (a total of 4,320 inventory vectors) in the study with four products and four customer segments, we calculate the ratios $r_i = y_i/E[D_i]$ for each product i and count the number of customer segments that have access to that product under those inventory levels. We find that in 99.2% [resp., 97.6%] of the instances in which $r_i > 2$ [resp., $r_i > 1.5$] for a given product i, this product is offered to all customer segments regardless of the inventory levels of the other products. This indicates that when the inventory level of a product is at least 50% larger than the expected demand for this product for the rest of the selling season, it is safe to offer this product to all customer segments without much loss in profitability. In the other extreme, we find that 99.5% [resp., 96.6%] of the instances in which $r_i < 0.4$ [resp., $r_i < 0.5$] for a given product i, this product is rationed to most customer segments (at least 2 in this study) and this, again, is regardless of the inventory levels of all other products. This suggests that when the inventory level of a product is roughly half the expected demand for that product in the remaining selling season, it may be optimal to ration the product to all segments with a relatively low preference for this product.

Based on the observations in the preceding paragraph, and using some of the results for the case of two products derived in Section 4, we now introduce a heuristic to compute an assortment customization policy for the general case with N products and M customer segments.

Heuristic

Step 1. Select a ratio value r_0.

Step 2. Given inventory levels y_i and expected demands until the end of the horizon $E[D_i]$, compute the ratios $r_i = y_i/E[D_i]$ for each product i. If $r_i \geq r_0$ for all $i = 1, \cdots, N$, then use the policy that offers all products to all arriving customers. Otherwise, let $Z_1 = \{j : r_j < r_0\}$ and $Z_2 = \{j : r_j \geq r_0\}$.

Step 3. Create two aggregated products: Product $Z1$, with inventory level $\tilde{y}_1 = \sum_{j \in Z_1} y_j$; Product $Z2$, with inventory level $\tilde{y}_1 = \sum_{j \in Z_2} y_j$. Compute segment k's preference vector for the aggregated products: $\tilde{\theta}_{k1} = \sum_{j \in Z_1} \theta_{kj}$ and $\tilde{\theta}_{k2} = \sum_{j \in Z_2} \theta_{kj}$.

Step 4. Compute the optimal policy for this setting with two products and M customer segments.

Step 5. Implement the solution in the original setting with N products and M customer segments, as follows: If product $Z1$ [product $Z2$] is rationed to segment k in the solution computed in Step 4 in period t, then ration all products in Z_1 [in Z_2] to an arriving customer of segment k in period t.

Essentially, the heuristic aggregates all variants with low levels of inventory in one product and all variants with ample capacity in another product. The computation of the optimal policy for the case of two products in Step 4 is significantly more manageable than for more than two products (due to the dimensionality of the dynamic program). Moreover, for many parameter values, we can fully characterize the optimal policy. We have tested the heuristic with 40 different inventory vectors and $T = 30$, for each of the 12 parameter combinations described earlier in this section. We have also conducted another set of experiments with six products and six customer segments. In this set of experiments, we consider four groups of preference parameters and three sets of ρ-vectors, all of which are natural extensions of those presented for the case of four products and four customer segments (see Appendix A for a detailed description of this numerical study). For this study, we set $T = 15$ and consider 18 different initial inventory vectors for each parameter combination. The results of the numerical studies suggest that the heuristic performs very well. In a few instances, particularly when the gains from customization are relatively small, the heuristic may result in a policy that performs worse than just offering all products. However, in those cases, the optimal policy, the policy that suggests offering all products, and the heuristic, all result in very similar profit values. We therefore examine the performance of the heuristic for those settings in which the gains from assortment customization (relative to offering all products) is larger than 0.5%. For those cases, the heuristic recovers a large portion of the gains from assortment customization. In the cases with four products and four customer segments, the heuristic recovers an average of 66% of the gains achieved with the optimal policy (whereas the median is 92% and the 75^{th} percentile is 98%). In the cases with six products and six customer segments, the average and maximum gains from assortment customization, relative to offering all products, are 2.99% and 12.54%, respectively. The heuristic recovers an average of 61% of the gains achieved with the optimal policy (with a median of 90% and a 75^{th} percentile value of 98%).

6. Conclusion

We consider a retailer with limited inventories of substitutable products in a category with identical prices. The retailer faces a heterogeneous customer base, consisting of multiple segments that are characterized by different preferences for the products. We consider settings in which the retailer can identify the segment of each arriving customer and can therefore offer a customized assortment based on that segment's preferences. This is the case, for example, for online retailers, as they usually have information about a customer and its product preferences based on past purchasing and browsing history. We assume that the retailer does not have the opportunity to replenish inventory as in the case of seasonal products and products with short life-cycles. We formulate this problem as a dynamic assortment customization problem where the assortment decision depends on the inventory levels of the products, the time remaining in the season, the preference distribution of the arriving customer and of the anticipated future customers.

With a homogeneous customer base, and therefore a single customer segment, it is optimal to offer all available products to any arriving customer. In contrast, with a heterogeneous customer base, it may not be optimal to offer all product variants to all customers, even if the products' prices are equal. Specifically, for a setting with two products and two segments, we show that it is optimal to follow a threshold policy under which it is optimal to offer a product to a customer segment only if the inventory level of that product is higher than a threshold level. Otherwise, the product is not offered to that segment, at the cost of losing some potential sales, in order to reserve those units for customers who have a stronger preference for that product. This is a form of inventory rationing – restricting access of products to some customer segments. We show that the threshold levels are increasing in the other product's inventory level and decreasing in time, making customization (and rationing) less likely as the system runs out of inventory or time to sell the products. Inventory rationing has been studied extensively in inventory and revenue management contexts and it generally arises due to price and cost differences between customer groups. Our model demonstrates that the heterogeneity in the customer preferences is another reason for rationing.

In a numerical study of general systems, we find that the impact of assortment customization on revenues may be significant (as much as 9% in some examples with four products and 12% with six products) relative to a policy that involves no assortment customization, offering all available products to all customers. We find that the optimal assortment policy continues to be of the threshold-type in the general case. We also propose an effective heuristic to compute an assortment customization policy for general systems. In the numerical study, we identify the settings under which assortment customization is more valuable. As expected, the value of assortment customization increases with the heterogeneity of the customer base, provided that the segments have some overlapping preferences, and when the inventory levels of highly preferred products are relatively low. Hence, online retailers have an opportunity to significantly increase end-of-season revenues by understanding the composition of their customer base and by strategically customizing their product offerings to individual customers. Finally, our results demonstrate the potential of using customized assortment sets as another lever for revenue maximization, in addition to dynamic pricing.

In this paper, we considered the assortment offered to a customer as a decision variable and assumed identical prices to isolate the effect of heterogeneous customer preferences. However, our results indicating the need for inventory rationing (effectively directing some customers to specific products) due to differences in customer preferences would also apply in more general settings. Consider the case of two products with non-identical prices $p_1 > p_2$. In this case, our results (based on the heterogeneity in customer preferences) suggest that there could be an incentive to ration either product depending on their inventory levels. In the absence of heterogeneity in customer preferences, the difference in prices would support rationing product 2 and offering product 1 to all customers. This effect will work against the incentive to ration product 1 due to heterogeneous preferences and amplify the need to ration product 2. The combination of these two forces will determine the optimal assortment policy.

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Technical Appendix
Dynamic Assortment Customization with Limited Inventories

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Appendix A

This appendix contains the proofs of the results in the paper. Some of the proofs involve verifying the result for multiple cases. For those proofs, the results are shown for a few representative cases here, while Appendix B contains the proofs of all remaining cases.

Proposition 1

Proof: We first prove that $S^*_{mt} \in \{A_1(\mathbf{y}), \cdots, A_N(\mathbf{y})\}$. The optimality equation in (2) can be rewritten as

$$V_t(\mathbf{y}|m) = \max_{S \subset \overline{S}(\mathbf{y})} \left\{ \sum_{i \in S} \lambda q_{mi}(S) p^i_t(\mathbf{y}) \right\} + V_{t+1}(\mathbf{y}). \tag{3}$$

For a given vector of inventory levels $\mathbf{y}$, assume, without loss of generality, that $p^{i_1}_t(\mathbf{y}) \geq \cdots \geq p^{i_N}_t(\mathbf{y})$ and $\overline{S}(\mathbf{y}) = \mathcal{N}$. Note that these effective marginal prices are the same for all customer segments. We prove the result by contradiction. Suppose that the arriving customer in period t belongs to segment m. If S^*_{mt} is not one of the sets in $\{A_1(\mathbf{y}), \cdots, A_N(\mathbf{y})\}$, then there exist i_k and i_l such that $i_k \in S^*_{mt}$, $i_l \in \mathcal{N} \setminus S^*_{mt}$, and $p^{i_l}_t(\mathbf{y}) > p^{i_k}_t(\mathbf{y})$. Because $i_k \in S^*_{mt}$, we have $p^{i_k}_t(\mathbf{y}) \geq \sum_{j \in S^*_{mt}} q_{mj}(S^*_{mt}) p^{i_j}_t(\mathbf{y})$. (If this inequality did not hold, then $(1 - q_{mi_k}(S^*_{mt})) p^{i_k}_t(\mathbf{y}) < \sum_{j \in S^*_{mt} \setminus \{i_k\}} q_{mj}(S^*_{mt}) p^{i_j}_t(\mathbf{y})$, which implies $p^{i_k}_t(\mathbf{y}) < \sum_{j \in S^*_{mt} \setminus \{i_k\}} q_{mj}(S^*_{mt} \setminus \{i_k\}) p^{i_j}_t(\mathbf{y})$. This, in turn, implies that the assortment $S^*_{mt} \setminus \{i_k\}$ leads to a higher profit than that of S^*_{mt}, resulting in a contradiction.) Thus, $p^{i_l}_t(\mathbf{y}) > p^{i_k}_t(\mathbf{y}) \geq \sum_{j \in S^*_{mt}} q_{mj}(S^*_{mt}) p^{i_j}_t(\mathbf{y}) = a/b$, where $a = \sum_{j \in S^*_{mt}} \theta_{mj} p^{i_j}_t(\mathbf{y})$ and $b = \sum_{j \in S^*_{mt}} \theta_{mj} + \theta_{m0}$. Denote θ and x as the preference and effective marginal price of the new product, respectively. Because $(a + \theta x)/(b + \theta)$ is larger than a/b for any positive θ if and only if $x \geq a/b$, and $p^{i_l}_t(\mathbf{y}) > \sum_{j \in S^*_{mt}} q_{mj}(S^*_{mt}) p^{i_j}_t(\mathbf{y})$, we have

$$\sum_{j \in S^*_{mt} \cup \{i_l\}} q_{mj}(S^*_{mt} \cup \{i_l\}) p^{i_j}_t(\mathbf{y}) > \sum_{j \in S^*_{mt}} q_{mj}(S^*_{mt}) p^{i_j}_t(\mathbf{y}).$$

This implies that assortment $S^*_{mt} \cup \{i_l\}$ is strictly better than S^*_{mt}, which is a contradiction.

We next prove that $S^*_{mt}(\mathbf{y}) \subset S^*_{rt}(\mathbf{y})$ if $\sum_{k=1}^{l} \theta_{m,i_k} > \sum_{k=1}^{l} \theta_{r,i_k}$ for all $l = 1, .., N$ and $\theta_{m,i_k} \leq \theta_{r,i_k}$ for all $k = 2, .., N$. Under these conditions, for any set A_l, we have $\theta_{m,i_1}(\sum_{k=1}^{l} \theta_{r,i_k} + \theta_0) p^{i_1}_t > \theta_{r,i_1}(\sum_{k=1}^{l} \theta_{m,i_k} + \theta_0) p^{i_1}_t$ and $\theta_{m,i_j}(\sum_{k=1}^{l} \theta_{r,i_k} + \theta_0) p^{i_j}_t < \theta_{r,i_j}(\sum_{k=1}^{l} \theta_{m,i_k} + \theta_{r0}) p^{i_j}_t$ for any $j = 2, \cdots, l$. Then,

$$\Gamma_1 = \sum_{j=2}^{l} \left(\theta_{m,i_j} (\sum_{k=1}^{l} \theta_{r,i_k} + \theta_0) - \theta_{r,i_j} (\sum_{k=1}^{l} \theta_{m,i_k} + \theta_0) \right) p^{i_j}_t >$$

$$\sum_{j=2}^{l}\left(\theta_{m,i_j}(\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0)-\theta_{r,i_j}(\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0)\right)p_t^{i_1}=\Gamma_2. \tag{4}$$

Based on this inequality, we have

$$\begin{aligned}
&(\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0)(\sum_{k=1}^{l}\theta_{m,i_k}p_t^{i_k})-(\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0)(\sum_{k=1}^{l}\theta_{r,i_k}p_t^{i_k})\\
&=\left(\theta_{m,i_1}(\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0)-\theta_{r,i_1}(\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0)\right)p_t^{i_1}+\Gamma_1\\
&>\left(\theta_{m,i_1}(\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0)-\theta_{r,i_1}(\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0)\right)p_t^{i_1}+\Gamma_2\\
&=\left((\sum_{k=1}^{l}\theta_{m,i_k})(\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0)-(\sum_{k=1}^{l}\theta_{r,i_k})(\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0)\right)p_t^{i_1}\\
&=\left((\sum_{k=1}^{l}\theta_{m,i_k})\theta_0-(\sum_{k=1}^{l}\theta_{r,i_k})\theta_0\right)p_t^{i_1}>0.
\end{aligned}$$

Thus,

$$\frac{\sum_{k=1}^{l}\theta_{m,i_k}p_t^{i_k}}{\sum_{k=1}^{l}\theta_{m,i_k}+\theta_0}>\frac{\sum_{k=1}^{l}\theta_{r,i_k}p_t^{i_k}}{\sum_{k=1}^{l}\theta_{r,i_k}+\theta_0}. \tag{5}$$

Recall that if $p_t^{i_l}$ is greater than the left-hand side of the inequality in (5), then it is optimal to include product $l+1$ in the assortment for segment m customers. The inequality in (5) implies that if it is optimal to include product $l+1$ in the assortment for segment m customers, then it is also optimal to include that product for segment r customers. Because the inequality holds for any l, we have $S^*_{mt}(\mathbf{y})\subset S^*_{rt}(\mathbf{y})$. ■

Parts (i) and (ii) of Proposition 2 and Theorem 1 follow from these technical properties:

Q1(t, i, j, y): $V_t(\mathbf{y}-e_j)-V_t(\mathbf{y}-e_i-e_j)-V_t(\mathbf{y})+V_t(\mathbf{y}-e_i)\geq 0$, $i,j\in\mathcal{N}$.

Q2(t, i, j, y): $V_t(\mathbf{y}-e_i)\leq V_t(\mathbf{y}-e_j)$ where $y_i=y_j\geq 1$ and $\theta_{1i}\geq\theta_{1j}$.

Q3(t, i, y): For any $S\subset\mathcal{N}\setminus\{i\}$, $p-\Delta_t^i(\mathbf{y})\geq\frac{\sum_{j\in S}\theta_{1j}(p-\Delta_t^j(\mathbf{y}))}{\sum_{j\in S}\theta_{1j}+\theta_0}$. In other words,

$$\theta_0 p+(\sum_{j\in S}\theta_{1j}+\theta_0)V_t(\mathbf{y}-e_i)-\sum_{j\in S}\theta_{1j}V_t(\mathbf{y}-e_j)-\theta_0 V_t(\mathbf{y})\geq 0$$

Q1 and Q2 lead to (i) and (ii) of Proposition 2, respectively. Q3 implies that the optimal assortment in period $t-1$ is to offer all available products, which is the result in Theorem 1. We prove these three properties together by induction. Because the optimal assortment in period T is to offer all available products, it is easy to verify that the results hold for period T. We then assume that these properties hold in period $t+1$, and prove that they continue to hold in period t. Note that Q3 in period $t+1$ implies that the optimal assortment in period t is to offer all available products. Finally, we prove Q1 and Q2 within the proof of Proposition 2 and Q3 within the proof of Theorem 1. Then, the entire proof is completed by induction. Without loss of generality, we assume that $\overline{S}_t(\mathbf{y}) = \mathcal{N}$. Finally, we assume that $\lambda = 1$. The results follow similarly for the case with $\lambda < 1$.

Proposition 2

Proof:

Parts (i) and (ii): We begin with the proof of Q1. There are in total seven cases to consider. If $i = j$, there are three cases: (1) $y_i = 1$; (2) $y_i = 2$; (3) $y_i > 2$. If $i \neq j$, there are four cases: (4) $y_i \geq 2$ and $y_j \geq 2$; (5) $y_i = 1$ and $y_j \geq 2$; (6) $y_i \geq 2$ and $y_j = 1$; (7) $y_i = 1$ and $y_j = 1$. Cases 1, 3, and 4 can be proved easily by induction. Cases 5 and 6 are similar. Here, we prove cases 2 and 5, and the proof of case 7 is provided in Appendix B. We use the following notation throughout this proof: $\Lambda_{\mathcal{S}} = \sum_{k\in\mathcal{S}} \theta_{1k} + \theta_0$, $e_{i,j} = e_i + e_j$, and $e_{i,j,k} = e_i + e_j + e_k$.

We begin with case 2. Because $y_i = 2$, the inventory vector $\mathbf{y} - e_{i,i}$ includes no units of product i in stock. Then,

$$\begin{aligned}
-V_t(y - e_{i,i}) &= -\frac{\sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k} p}{\Lambda_{\mathcal{N}\setminus\{i\}}} - \frac{\sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k} V_{t+1}(y - e_{i,i,k}) + \theta_0 V_{t+1}(y - e_{i,i})}{\Lambda_{\mathcal{N}\setminus\{i\}}} \\
&= -\frac{\sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k} p}{\Lambda_{\mathcal{N}\setminus\{i\}}} - \frac{\sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k} V_{t+1}(y - e_{i,i,k}) + \theta_0 V_{t+1}(y - e_{i,i})}{\Lambda_{\mathcal{N}\setminus\{i\}}} \\
&\quad - \frac{\sum_{k\in N\setminus\{i\}} \theta_{1k} V_{t+1}(y - e_{i,i,k}) + \theta_0 V_{t+1}(y - e_{i,i})}{\Lambda_{\mathcal{N}}} \\
&\quad + \frac{\sum_{k\in N\setminus\{i\}} \theta_{1k} V_{t+1}(y - e_{i,i,k}) + \theta_0 V_{t+1}(y - e_{i,i})}{\Lambda_{\mathcal{N}}},
\end{aligned}$$

where we have added and subtracted the same term in the second equality. Then,

$$\begin{gathered}
V_t(\mathbf{y} - e_i) - V_t(\mathbf{y} - e_{i,i}) - V_t(\mathbf{y}) + V_t(\mathbf{y} - e_i) = \\
\frac{\theta_0 \theta_{1i} p}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})} + \frac{\sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k}(2V_{t+1}(\mathbf{y} - e_i - e_k) - V_{t+1}(\mathbf{y} - e_{i,i,k}) - V_{t+1}(\mathbf{y} - e_k))}{\Lambda_{\mathcal{N}}} \\
+ \frac{\theta_0(2V_{t+1}(\mathbf{y} - e_i) - V_{t+1}(\mathbf{y} - e_{i,i}) - V_{t+1}(\mathbf{y}))}{\Lambda_{\mathcal{N}}} + \frac{\theta_{1i}(2V_{t+1}(\mathbf{y} - e_{i,i}) - V_{t+1}(\mathbf{y} - e_i))}{\Lambda_{\mathcal{N}}} \\
- \frac{\theta_{1i} \sum_{k\in\mathcal{N}\setminus\{i\}} \theta_{1k} V_{t+1}(\mathbf{y} - e_{i,i,k})}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})} - \frac{\theta_{1i}\theta_0 V_{t+1}(\mathbf{y} - e_{i,i})}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}.
\end{gathered} \tag{6}$$

The second and third terms on the right side of (6) are positive due to $Q1(t+1, i, i, \mathbf{y} - e_k)$ and $Q1(t+1, i, i, \mathbf{y})$. Thus, we have

$$(6) \geq \frac{\theta_0 \theta_{1i} p}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})} + \frac{\theta_{1i}(2V_{t+1}(\mathbf{y} - e_{i,i}) - V_{t+1}(\mathbf{y} - e_i))}{\Lambda_{\mathcal{N}}}$$

$$-\frac{\theta_{1i}\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,i,k})}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}-\frac{\theta_{1i}\theta_0V_{t+1}(\mathbf{y}-e_{i,i})}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}. \tag{7}$$

We rewrite the second term in (7) as follows:

$$\begin{aligned}\frac{\theta_{1i}(2V_{t+1}(\mathbf{y}-e_{i,i})-V_{t+1}(\mathbf{y}-e_i))}{\Lambda_{\mathcal{N}}} &= \frac{(\Lambda_{\mathcal{N}\setminus\{i\}})\theta_{1i}V_{t+1}(\mathbf{y}-e_{i,i})-\theta_{1i}\theta_0V_{t+1}(\mathbf{y}-e_i)}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}\\ &\quad+\frac{\theta_{1i}\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{i,i})-V_{t+1}(\mathbf{y}-e_i))}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}\\ &\quad+\frac{\theta_{1i}\theta_0(V_{t+1}(\mathbf{y}-e_{i,i})}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}\\ &\stackrel{def}{=} A_1+A_2+A_3\end{aligned}$$

We now rewrite (7) by combining A_1 with the first term in the equation and canceling the last term with A_3:

$$\begin{aligned}(7)&=\frac{\theta_{1i}}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}[\theta_0p+(\Lambda_{\mathcal{N}\setminus\{i\}})V_{t+1}(\mathbf{y}-e_{i,i})-\theta_0V_{t+1}(\mathbf{y}-e_i)]\\ &+\frac{\theta_{1i}\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{i,i})-V_{t+1}(\mathbf{y}-e_i)-V_{t+1}(\mathbf{y}-e_{i,i,k}))}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}.\end{aligned} \tag{8}$$

Following the inductive argument from Q3$(t+1,i,\mathbf{y}-e_i)$, we have

$$\theta_0p+(\Lambda_{\mathcal{N}\setminus\{i\}})V_{t+1}(\mathbf{y}-e_{i,i})-\theta_0V_{t+1}(\mathbf{y}-e_i)\geq\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,k}).$$

Using this inequality in the first term of equation (8), we obtain

$$(8)\geq\frac{\theta_{1i}\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{i,k})+V_{t+1}(\mathbf{y}-e_{i,i})-V_{t+1}(\mathbf{y}-e_i)-V_{t+1}(\mathbf{y}-e_{i,i,k}))}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{i\}})}.$$

This expression is positive due to Q1$(t+1,i,k,\mathbf{y}-e_i)$, completing the proof of Q1$(t,i,j,\mathbf{y})$ for case 2.

Next, we prove case 5. Because $y_i=1$, the vectors $\mathbf{y}-e_{i,j}$ and $\mathbf{y}-e_i$ do not include product i in stock. Then,

$$\begin{aligned}-V_t(y-e_{i,j}) &= -\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}p}{\Lambda_{\mathcal{N}\setminus\{i\}}}-\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,j,k})+\theta_0V_{t+1}(y-e_{i,j})}{\Lambda_{\mathcal{N}\setminus\{i\}}}\\ &\stackrel{def}{=} -B_1-B_2.\end{aligned}$$

$$\begin{aligned}V_t(y-e_i) &= \frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}p}{\Lambda_{\mathcal{N}\setminus\{i\}}}+\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,k})+\theta_0V_{t+1}(y-e_i)}{\Lambda_{\mathcal{N}\setminus\{i\}}}\\ &\stackrel{def}{=} C_1+C_2.\end{aligned}$$

We rewrite B_2 and C_2 as follows:

$$\begin{aligned}
-B_2 &= -\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,j,k})+\theta_0V_{t+1}(y-e_{i,j})}{\Lambda_{\mathcal{N}\setminus\{i\}}} \\
&\quad -\frac{\sum_{k\in N\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,j,k})+\theta_0V_{t+1}(y-e_{i,j})}{\Lambda_{\mathcal{N}}} \\
&\quad +\frac{\sum_{k\in N\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,j,k})+\theta_0V_{t+1}(y-e_{i,j})}{\Lambda_{\mathcal{N}}} \\
&\stackrel{def}{=} -B_2-D_1+D_1.
\end{aligned}$$

$$\begin{aligned}
C_2 &= \frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,k})+\theta_0V_{t+1}(y-e_{i})}{\Lambda_{\mathcal{N}\setminus\{i\}}} \\
&\quad +\frac{\sum_{k\in N\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,k})+\theta_0V_{t+1}(y-e_{i})}{\Lambda_{\mathcal{N}}} \\
&\quad -\frac{\sum_{k\in N\setminus\{i\}}\theta_{1k}V_{t+1}(y-e_{i,k})+\theta_0V_{t+1}(y-e_{i})}{\Lambda_{\mathcal{N}}} \\
&\stackrel{def}{=} C_2+E_1-E_1.
\end{aligned}$$

The expression in Q1 is given by

$$V_t(\mathbf{y}-e_j)-V_t(\mathbf{y}-e_{i,j})-V_t(\mathbf{y})+V_t(\mathbf{y}-e_i)=$$

$$V_t(\mathbf{y}-e_j)-V_t(\mathbf{y})-V_t(\mathbf{y}-e_{i,j})+V_t(\mathbf{y}-e_i)=$$

$$\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{j,k})-V_{t+1}(\mathbf{y}-e_k))}{\Lambda_{\mathcal{N}}}+\frac{\theta_{1i}(V_{t+1}(\mathbf{y}-e_{j,i})-V_{t+1}(\mathbf{y}-e_i))}{\Lambda_{\mathcal{N}}}$$
$$-B_1-B_2-D_1+D_1+C_1+C_2+E_1-E_1. \tag{9}$$

We combine the first term in (9) with $-D_1$ and E_1, obtaining the first two terms in the equation below. Furthermore, $C_1=B_1$ and the remaining terms in (9) add up to the third term of the equation below.

$$(9)=\frac{\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{j,k})-V_{t+1}(\mathbf{y}-e_{i,j,k})-V_{t+1}(\mathbf{y}-e_k)+V_{t+1}(\mathbf{y}-e_{i,k}))}{\Lambda_{\mathcal{N}}}$$
$$+\frac{\theta_0(V_{t+1}(\mathbf{y}-e_j)-V_{t+1}(\mathbf{y}-e_{i,j})-V_{t+1}(\mathbf{y})+V_{t+1}(\mathbf{y}-e_i))}{\Lambda_{\mathcal{N}}}$$
$$\frac{\theta_{1i}\sum_{k\in\mathcal{N}\setminus\{i\}}\theta_{1k}(V_{t+1}(\mathbf{y}-e_{j,i})-V_{t+1}(\mathbf{y}-e_i)-V_{t+1}(\mathbf{y}-e_{i,j,k})+V_{t+1}(\mathbf{y}-e_{i,k}))}{(\Lambda_{\mathcal{N}\setminus\{i\}})(\Lambda_{\mathcal{N}})}. \tag{10}$$

Because of $Q1(t+1,i,j,\mathbf{y}-e_k)$, $Q1(t+1,i,j,\mathbf{y})$, and $Q1(t+1,j,k,\mathbf{y}-e_i)$, all three terms in (10) are positive. Then, $(10)\geq 0$. This completes the proof of $Q1$ in period t.

Next, we prove $Q2$ in period t. We focus on the case where $y_i=y_j=1$. The proof of the case where $y_i=y_j>1$ is similar. For $y_i=y_j=1$, we have

$$V_t(\mathbf{y}-e_j)-V_t(\mathbf{y}-e_i)=\frac{(\theta_{1i}-\theta_{1j})\theta_0 p}{(\Lambda_{\mathcal{N}\setminus\{j\}})(\Lambda_{\mathcal{N}\setminus\{i\}})}$$

$$+\frac{\sum_{k\in\mathcal{N}\cup\{0\}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_{\mathcal{N}\setminus\{j\}}}-\frac{\sum_{k\in\mathcal{N}\cup\{0\}\setminus\{i\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,k})}{\Lambda_{\mathcal{N}\setminus\{i\}}}$$

$$=\frac{(\theta_{1i}-\theta_{1j})[\theta_0 p+(\Lambda_{\mathcal{N}\setminus\{i,j\}})V_{t+1}(\mathbf{y}-e_{i,j})-\sum_{k\in\mathcal{N}\cup\{0\}\setminus\{i,j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,k})]}{(\Lambda_{\mathcal{N}\setminus\{j\}})(\Lambda_{\mathcal{N}\setminus\{i\}})}$$

$$+\frac{\sum_{k\in\mathcal{N}\cup\{0\}\setminus\{i,j\}}\theta_{1k}[V_{t+1}(\mathbf{y}-e_{j,k})-V_{t+1}(\mathbf{y}-e_{i,k})]}{\Lambda_{\mathcal{N}\setminus\{j\}}}.$$

This is positive due to Q3$(t+1,j,\mathbf{y}-e_i)$ and Q2$(t+1,i,j,\mathbf{y}-e_k)$, completing the proof of $Q2$ in period t.

Finally, we move to part (iii) of this proposition. Because there is one more period to sell products starting from period t compared to period $t+1$, we can easily prove that $V_t(\mathbf{y})$ and $\Delta_t^i(\mathbf{y})$ are decreasing in time by using a sample path argument. ■

Theorem 1

Proof: We prove Q3 in period t, which implies the result in the theorem. Define $S'=\{j : j\in S, y_j=1\}$. There are two cases to consider: (1) $y_i>1$; (2) $y_i=1$. We prove case (1) here and leave the proof of case (2) to Appendix B. In case (1), if $S'\neq\emptyset$, then for any $j\in S'$,

$$V_t(\mathbf{y}-e_j)=\frac{(\Lambda_{\mathcal{N}\setminus\{j\}}-\theta_0)p}{\Lambda_{\mathcal{N}\setminus\{j\}}}+\frac{\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_{\mathcal{N}\setminus\{j\}}}$$

and for any $m\in S\setminus\{S'\}$,

$$V_t(\mathbf{y}-e_m)=\frac{(\Lambda_{\mathcal{N}}-\theta_0)p}{\Lambda_{\mathcal{N}}}+\frac{\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{m,k})}{\Lambda_{\mathcal{N}}}.$$

Then,

$$\theta_0 p+(\Lambda_S)V_t(\mathbf{y}-e_i)-\sum_{j\in S}\theta_{1j}V_t(\mathbf{y}-e_j)-\theta_0V_t(\mathbf{y})$$

$$=\theta_0 p+(\Lambda_S)V_t(\mathbf{y}-e_i)-\sum_{j\in S'}\theta_{1j}V_t(\mathbf{y}-e_j)-\sum_{j\in S\setminus S'}\theta_{1j}V_t(\mathbf{y}-e_j)-\theta_0V_t(\mathbf{y})$$

$$=\theta_0 p+\sum_{j\in S'}\frac{\theta_{1j}^2\theta_0 p}{(\Lambda_{\mathcal{N}})(\Lambda_{\mathcal{N}\setminus\{j\}})}+\frac{(\Lambda_S)(\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,k}))}{\Lambda_{\mathcal{N}}}$$

$$+\frac{(\Lambda_S)\theta_0V_{t+1}(\mathbf{y}-e_i)}{\Lambda_{\mathcal{N}}}-\sum_{j\in S\setminus S'}\theta_{1j}\frac{\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_{\mathcal{N}}}$$

$$-\sum_{j\in S'}\theta_{1j}\frac{\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_{\mathcal{N}\setminus\{j\}}}-\sum_{j\in S\setminus S'}\theta_{1j}\frac{\theta_0V_{t+1}(\mathbf{y}-e_j)}{\Lambda_{\mathcal{N}}}$$

$$-\sum_{j\in S'}\theta_{1j}\frac{\theta_0V_{t+1}(\mathbf{y}-e_j)}{\Lambda_{\mathcal{N}\setminus\{j\}}}-\frac{\theta_0\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_k)}{\Lambda_{\mathcal{N}}}-\frac{\theta_0^2V_{t+1}(\mathbf{y})}{\Lambda_{\mathcal{N}}}. \quad (11)$$

We next rewrite the third term in (11):

$$\begin{aligned}\frac{(\Lambda_S)(\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{i,k}))}{\Lambda_\mathcal{N}} &= \sum_{j\in S'}\frac{\theta_{1j}^2(\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}+\theta_0)V_{t+1}(\mathbf{y}-e_{j,i})}{(\Lambda_\mathcal{N})(\Lambda_{\mathcal{N}\setminus\{j\}})} \\ &\quad+\frac{\sum_{k\in S'}\theta_{1k}(\sum_{j\in S\setminus\{k\}}\theta_{1j}+\theta_0)V_{t+1}(\mathbf{y}-e_{i,k})}{\Lambda_\mathcal{N}} \\ &\quad+\frac{\sum_{k\in\mathcal{N}\setminus S'}\theta_{1k}(\Lambda_S)V_{t+1}(\mathbf{y}-e_{i,k})}{\Lambda_\mathcal{N}} \\ &\stackrel{def}{=} X_1+X_2+X_3.\end{aligned}$$

Because of Q3$(t+1,i,\mathbf{y}-e_j)$,

$$\begin{aligned}\sum_{j\in S'}\frac{\theta_{1j}^2\theta_0 p}{(\Lambda_\mathcal{N})(\Lambda_{\mathcal{N}\setminus\{j\}})}+X_1 &= \sum_{j\in S'}\frac{\theta_{1j}^2}{(\Lambda_\mathcal{N})(\Lambda_{\mathcal{N}\setminus\{j\}})}\left[\theta_0 p+\left(\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}+\theta_0\right)V_{t+1}(\mathbf{y}-e_{j,i})\right] \\ &\geq \sum_{j\in S'}\frac{\theta_{1j}^2}{(\Lambda_\mathcal{N})(\Lambda_{\mathcal{N}\setminus\{j\}})}\left[\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})+\theta_0V_{t+1}(\mathbf{y}-e_j)\right].\end{aligned} \tag{12}$$

In addition, by summing up the terms on the right side of inequality (12) and the sixth and eighth terms in (11) together, we obtain

$$\begin{aligned}&\sum_{j\in S'}\frac{\theta_{1j}^2}{(\Lambda_\mathcal{N})(\Lambda_{\mathcal{N}\setminus\{j\}})}\left[\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})+\theta_0V_{t+1}(\mathbf{y}-e_j)\right] \\ &-\sum_{j\in S'}\frac{\theta_{1j}}{(\Lambda_{\mathcal{N}\setminus\{j\}})}\left[\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})+\theta_0V_{t+1}(\mathbf{y}-e_j)\right] \\ &=-\sum_{j\in S'}\frac{\theta_{1j}}{\Lambda_\mathcal{N}}\left[\sum_{k\in\mathcal{N}\setminus\{j\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})+\theta_0V_{t+1}(\mathbf{y}-e_j)\right]\stackrel{def}{=}-Y_1.\end{aligned}$$

We add up $-Y_1$ and the fifth and seventh terms in (11), to obtain

$$-Y_1-\sum_{j\in S\setminus S'}\theta_{1j}\frac{\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_\mathcal{N}}-\sum_{j\in S\setminus S'}\theta_{1j}\frac{\theta_0V_{t+1}(\mathbf{y}-e_j)}{\Lambda_\mathcal{N}}=$$

$$-\frac{\sum_{k\in S'}\theta_{1k}\sum_{j\in S\setminus\{k\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_\mathcal{N}}-\frac{\sum_{k\in\mathcal{N}\setminus S'}\theta_{1k}\sum_{j\in S}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_\mathcal{N}}-\sum_{j\in S}\theta_{1j}\frac{\theta_0V_{t+1}(\mathbf{y}-e_j)}{\Lambda_\mathcal{N}}$$

Therefore,

$$\begin{aligned}(11)\geq \theta_0 p+X_2+X_3-\frac{\sum_{k\in S'}\theta_{1k}\sum_{j\in S\setminus\{k\}}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_\mathcal{N}} \\ -\frac{\sum_{k\in\mathcal{N}\setminus S'}\theta_{1k}\sum_{j\in S}\theta_{1k}V_{t+1}(\mathbf{y}-e_{j,k})}{\Lambda_\mathcal{N}}-\sum_{j\in S}\theta_{1j}\frac{\theta_0V_{t+1}(\mathbf{y}-e_j)}{\Lambda_\mathcal{N}}\end{aligned}$$

$$+\frac{(\Lambda_S)\theta_0 V_{t+1}(\mathbf{y}-e_i)}{\Lambda_{\mathcal{N}}} - \frac{\theta_0\sum_{k\in\mathcal{N}}\theta_{1k}V_{t+1}(\mathbf{y}-e_k)}{\Lambda_{\mathcal{N}}} - \frac{\theta_0^2 V_{t+1}(\mathbf{y})}{\Lambda_{\mathcal{N}}}$$

$$= \frac{\sum_{k\in S'}\theta_{1k}[\theta_0 p + (\sum_{j\in S\setminus\{k\}}\theta_{1j}+\theta_0)V_{t+1}(\mathbf{y}-e_{i,k}) - \sum_{j\in S\setminus\{k\}}\theta_{1j}V_{t+1}(\mathbf{y}-e_{k,j}) - \theta_0 V_{t+1}(\mathbf{y}-e_k)]}{\Lambda_{\mathcal{N}}}$$

$$+\frac{\sum_{k\in\mathcal{N}\setminus S'}\theta_{1k}[\theta_0 p + (\Lambda_S)V_{t+1}(\mathbf{y}-e_{i,k}) - \sum_{j\in S}\theta_{1j}V_{t+1}(\mathbf{y}-e_{k,j}) - \theta_0 V_{t+1}(\mathbf{y}-e_k)]}{\Lambda_{\mathcal{N}}}$$

$$+\frac{\theta_0[\theta_0 p + (\Lambda_S)V_{t+1}(\mathbf{y}-e_i) - \sum_{j\in S}\theta_{1j}V_{t+1}(\mathbf{y}-e_j) - \theta_0 V_{t+1}(\mathbf{y})]}{\Lambda_{\mathcal{N}}}. \tag{13}$$

The right-hand side of the inequality in (13) is positive due to Q3$(t+1, i, \mathbf{y}-e_k)$ and Q3$(t+1, i, \mathbf{y})$. The case in which $S' = \emptyset$ can be proved similarly. This completes the proof of the three properties by induction. ■

Theorem 2 follows from the technical properties described below. Note that these results are presented for segment 2 customers only – we can similarly prove the results for segment 1. For simplicity, define $\Lambda^m_{\{ij0\}} = \theta_{mi} + \theta_{mj} + \theta_0$ and $\Lambda^m_{\{i0\}} = \theta_{mi} + \theta_0$, where m denotes the customer segment, and i and j represent the products. Throughout the proof, we will frequently use the fact that, for a given vector of inventory levels (y_1, y_2), the set $\{1, 2\}$ is preferred over the set $\{2\}$ [resp., $\{1\}$] for a customer of segment 2 [resp., 1] at time t if and only if $p - \Delta^1_{t+1}(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_{t+1}(y_1, y_2)\right)$ [resp., $p - \Delta^2_{t+1}(y_1, y_2) \geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}\left(p - \Delta^1_{t+1}(y_1, y_2)\right)$].

P1(t, 2, y_1, y_2): $\Delta^2_t(y_1, y_2)$ is decreasing in y_i, $i \in \{1, 2\}$.

P2(t, 2, y_1, y_2): $\Delta^2_t(y_1, y_2)$ is decreasing in t.

P3(t, 2, y_1, y_2): $p - \Delta^2_t(y_1, y_2) \geq \frac{\theta_{21}}{\theta_{21}+\theta_0}(p - \Delta^1_t(y_1, y_2))$. In other words,

$$\theta_0 p + (\theta_{21} + \theta_0)V_t(y_1, y_2 - 1) \geq \theta_0 V_t(y_1, y_2) + \theta_{21}V_t(y_1 - 1, y_2).$$

P4(t, 2, y_1, y_2): If $p - \Delta^1_t(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1, y_2)\right)$, then $p - \Delta^1_t(y_1+1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1 + 1, y_2)\right)$.
We actually prove a stronger version of this statement:

$$\Delta^1_t(y_1, y_2) - \Delta^1_t(y_1 + 1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}(\Delta^2_t(y_1, y_2) - \Delta^2_t(y_1 + 1, y_2)).$$

In other words,

$$(\theta_{22}+2\theta_0)V_t(y_1, y_2)+\theta_{22}V_t(y_1, y_2-1) \geq (\Lambda^2_{\{20\}})V_t(y_1-1, y_2)+\theta_{22}V_t(y_1+1, y_2-1)+\theta_0 V_t(y_1+1, y_2).$$

P5(t, 2, y_1, y_2): If $p - \Delta^1_t(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1, y_2)\right)$, then $p - \Delta^1_t(y_1, y_2-1) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1, y_2 - 1)\right)$.
In other words, if $\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1 - 1, y_2) \geq \theta_0 V_t(y_1, y_2) + \theta_{22}V_t(y_1, y_2 - 1)$, then $\theta_0 p + (\theta_{22} + \theta_0)V_t(y_1 - 1, y_2 - 1) \geq \theta_0 V_t(y_1, y_2 - 1) + \theta_{22}V_t(y_1, y_2 - 2)$.

P3 implies that product 2 is always in the optimal assortment of segment 2 and P4 implies the optimality of the threshold policy. P5 indicates that the threshold value $y^*_{it}(y_j)$ is increasing in y_j. These results are proved together by induction. First, because the optimal assortment in period T is to offer all available products, it is easy to verify that the results hold for period T. We then assume that these five properties hold in period $t+1$, and show that they also hold in period t. Note that if P3 and P4 hold in period $t+1$, then this implies that the optimal assortment policy in period t is a threshold policy. Note also that because $V_t(y_1, y_2)$ is a linear combination of $V_t(y_1, y_2|m)$ for $m = 1$ and $m = 2$, we show that the results hold for $V_t(y_1, y_2|m)$ for a given $m = 1, 2$, which then implies that they hold for $V_t(y_1, y_2)$. Without loss of generality, the proof focuses on the case with $\lambda = 1$. The results can be similarly proof for $\lambda < 1$. Finally, we focus on the case in which $\overline{S}(\mathbf{y}) = \{1, 2\}$. Otherwise, if only one product is available, then it is always optimal to offer that product to any arriving customer.

Theorem 2

Proof: We start with P1. There are two inequalities we have to demonstrate: $V_t(y_1, y_2) - V_t(y_1 - 1, y_2) - V_t(y_1, y_2+1) + V_t(y_1-1, y_2+1) \geq 0$ and $V_t(y_1, y_2) - V_t(y_1, y_2-1) - V_t(y_1+1, y_2) + V_t(y_1+1, y_2-1) \geq 0$. We prove the first inequality, and the second one can be proved similarly. For the first inequality, we further consider two cases: (i) $p - \Delta^1_{t+1}(y_1, y_2+1) \geq p - \Delta^2_{t+1}(y_1, y_2+1)$ and (ii) $p - \Delta^1_{t+1}(y_1, y_2+1) < p - \Delta^2_{t+1}(y_1, y_2+1)$. We focus on (i) and the proof for case (ii) follows a similar argument. The inequality in (i) implies that $S^*_{2t}(y_1, y_2+1) = \{1, 2\}$. In addition, since by induction P5 holds in period $t+1$, we have that $S^*_{2t}(y_1, y_2) = \{1, 2\}$ (a similar argument follows in the other cases). The optimal assortment sets for both customer segments in period t can therefore be summarized as follows:

	(y_1, y_2)	(y_1-1, y_2)	(y_1, y_2+1)	(y_1-1, y_2+1)
S^*_{1t}	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$ or $\{2\}$	$\{1,2\}$	$\{1,2\}$ or $\{2\}$

As shown in this table, there are several scenarios to consider, depending on the possible optimal assortment sets under the various inventory levels. We consider one specific scenario here (the most challenging one in terms of its proof), and the rest are proved in Appendix B:

	(y_1, y_2)	(y_1-1, y_2)	(y_1, y_2+1)	(y_1-1, y_2+1)
S^*_{1t}	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
S^*_{2t}	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$

For a segment 2 customer, we have

$$V_t(y_1, y_2|2) - V_t(y_1-1, y_2|2) - V_t(y_1, y_2+1|2) + V_t(y_1-1, y_2+1|2)$$

$$= [V_t(y_1, y_2|2) - V_t(y_1, y_2+1|2)] + [V_t(y_1-1, y_2+1|2) - V_t(y_1-1, y_2|2)]$$

$$= \frac{1}{\Lambda^2_{\{120\}}}[\theta_{21}V_{t+1}(y_1-1, y_2) - \theta_{21}V_{t+1}(y_1-1, y_2+1) + \theta_{22}V_{t+1}(y_1, y_2-1) - \theta_{22}V_{t+1}(y_1, y_2)$$

$$+\theta_0 V_{t+1}(y_1,y_2)-\theta_0 V_{t+1}(y_1,y_2+1)]+\frac{1}{\Lambda^2_{\{20\}}}[\theta_{22}V_{t+1}(y_1-1,y_2)-\theta_{22}V_{t+1}(y_1-1,y_2-1)$$

$$+\theta_0 V_{t+1}(y_1-1,y_2+1)-\theta_0 V_{t+1}(y_1-1,y_2)]$$

$$=\frac{-1}{\Lambda^2_{\{120\}}}\left[\theta_{21}\Delta^2_{t+1}(y_1-1,y_2+1)+\theta_{22}\Delta^2_{t+1}(y_1,y_2)+\theta_0\Delta^2_{t+1}(y_1,y_2+1)\right]$$

$$+\frac{1}{\Lambda^2_{\{20\}}}\left[\theta_{22}\Delta^2_{t+1}(y_1-1,y_2)+\theta_0\Delta^2_{t+1}(y_1-1,y_2+1)\right]. \tag{14}$$

Following the inductive argument for $\mathrm{P1}(t+1,2,y_1-1,y_2+1)$ and $\mathrm{P1}(t+1,2,y_1,y_2)$, we have that $\Delta^2_{t+1}(y_1-1,y_2)\geq\Delta^2_{t+1}(y_1-1,y_2+1)\geq\Delta^2_{t+1}(y_1,y_2+1)$ and $\Delta^2_{t+1}(y_1-1,y_2)\geq\Delta^2_{t+1}(y_1,y_2)$, respectively. Then, (14) ≥ 0.

For a segment 1 customer, because $S^*_{1t}(y_1,y_2)=\{1\}$, we have that

$$\begin{aligned}V_t(y_1,y_2|1) &= \frac{1}{\Lambda^1_{\{10\}}}[\theta_{11}p+\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)]\\ &\quad-\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_{12}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2)]\\ &\quad+\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_{12}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2)].\end{aligned}$$

It follows that

$$V_t(y_1,y_2|1)-V_t(y_1-1,y_2|1)-V_t(y_1,y_2+1|1)+V_t(y_1-1,y_2+1|1)$$

$$=\frac{1}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)]-\frac{\theta_{12}\theta_0 p}{(\Lambda^1_{\{10\}})(\Lambda^1_{\{120\}})}$$

$$-\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_{12}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2)]$$

$$+\frac{\theta_{11}}{\Lambda^1_{\{120\}}}[V_{t+1}(y_1-1,y_2)-V_{t+1}(y_1-2,y_2)-V_{t+1}(y_1-1,y_2+1)+V_{t+1}(y_1-2,y_2+1))]$$

$$+\frac{\theta_{12}}{\Lambda^1_{\{120\}}}[V_{t+1}(y_1,y_2-1)-V_{t+1}(y_1-1,y_2-1)-V_{t+1}(y_1,y_2)+V_{t+1}(y_1-1,y_2)]$$

$$+\frac{\theta_0}{\Lambda^1_{\{120\}}}[V_{t+1}(y_1,y_2)-V_{t+1}(y_1-1,y_2)-V_{t+1}(y_1,y_2+1)+V_{t+1}(y_1-1,y_2+1)]. \tag{15}$$

Because $S^*_{1t}=\{1\}$, we have that $p-\Delta^2_{t+1}(y_1,y_2)<\frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2)]$. This, in turn, is equivalent to $-\theta_0 p-(\Lambda^1_{\{10\}})V_{t+1}(y_1,y_2-1)>-\theta_{11}V_{t+1}(y_1-1,y_2)-\theta_0 V_{t+1}(y_1,y_2)$. Applying this inequality to the first three terms in (15), we obtain

$$\frac{1}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)]-\frac{\theta_{12}\theta_0 p}{(\Lambda^1_{\{10\}})(\Lambda^1_{\{120\}})}$$

$$-\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_{12}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2)]$$

$$\geq \frac{1}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)] - \frac{\theta_{12}(\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2))}{(\Lambda^1_{\{10\}})(\Lambda^1_{\{120\}})}$$
$$-\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)] = 0$$

The last three terms of (15) are positive due to P1$(t+1,2,y_1-1,y_2+1)$, P1$(t+1,2,y_1,y_2)$, and P1$(t+1,2,y_1,y_2+1)$, respectively. Therefore, (15) ≥ 0.

We now turn attention to P2. As for P1, there are two cases to consider: $p-\Delta^1_{t+1}(y_1,y_2) \geq p-\Delta^2_{t+1}(y_1,y_2)$ and $p-\Delta^1_{t+1}(y_1,y_2) < p-\Delta^2_{t+1}(y_1,y_2)$. We show the proof for the first case, and the second case can be proved similarly. The first inequality implies that product 1 is offered to both customer segments. This and the inductive argument imply the possible optimal assortment sets for both customer segments in period t:

	(y_1,y_2)	(y_1,y_2-1)
S^*_{1t}	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$

To prove the result, it is sufficient to show that $V_t(y_1,y_2|m) - V_t(y_1,y_2-1|m) \geq V_{t+1}(y_1,y_2) - V_{t+1}(y_1,y_2-1)$ for $m=1,2$. For $m=2$, because it is optimal to offer both products at time t, the proof follows by noting that $V_t(y_1,y_2|2)-V_t(y_1,y_2-1|2)-V_{t+1}(y_1,y_2)+V_{t+1}(y_1,y_2-1) = \frac{\theta_{21}}{\Lambda^2_{\{120\}}}\left(\Delta^2_{t+1}(y_1-1,y_2)-\Delta^2_{t+1}(y_1,y_2)\right)+\frac{\theta_{22}}{\Lambda^2_{\{120\}}}\left(\Delta^2_{t+1}(y_1,y_2-1)-\Delta^2_{t+1}(y_1,y_2)\right)$. The last term is non-negative using P1$(t+1,2,y_1,y_2)$. Following a similar argument, the inequality can be shown for $m=1$ when $S^*_{1t}(y_1,y_2)=\{1,2\}$ and $S^*_{1t}(y_1,y_2-1)=\{1,2\}$, or when $S^*_{1t}(y_1,y_2)=\{1\}$ and $S^*_{1t}(y_1,y_2-1)=\{1\}$. The cases in which the optimal sets are different for (y_1,y_2) and (y_1,y_2-1) require a different argument. We focus on the case $S^*_{1t}(y_1,y_2)=\{1,2\}$ and $S^*_{1t}(y_1,y_2-1)=\{1\}$. Using the knowledge about the optimal assortment sets for each vector of inventory levels, we can write

$$V_t(y_1,y_2|1)-V_t(y_1,y_2-1|1)-V_{t+1}(y_1,y_2)+V_{t+1}(y_1,y_2-1)$$
$$= \frac{\theta_{11}p+\theta_{12}p+\theta_{11}V_{t+1}(y_1-1,y_2)+\theta_{12}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2)}{\Lambda^1_{\{120\}}}$$
$$-\frac{\theta_{11}p+\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2-1)}{\Lambda^1_{\{10\}}} - V_{t+1}(y_1,y_2)+V_{t+1}(y_1,y_2-1)$$
$$= \frac{\theta_{12}}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})}[\theta_0 p + (\Lambda^1_{\{10\}})V_{t+1}(y_1,y_2-1)] + \frac{\theta_{11}}{\Lambda^1_{\{120\}}}V_{t+1}(y_1-1,y_2)$$
$$-\frac{\theta_{11}}{\Lambda^1_{\{10\}}}V_{t+1}(y_1-1,y_2-1) - \frac{\theta_{11}+\theta_{12}}{\Lambda^1_{\{120\}}}V_{t+1}(y_1,y_2) + \frac{\theta_{11}}{\Lambda^1_{\{10\}}}V_{t+1}(y_1,y_2-1)$$
$$\geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[V_{t+1}(y_1-1,y_2)+V_{t+1}(y_1,y_2-1)-V_{t+1}(y_1-1,y_2-1)-V_{t+1}(y_1,y_2)]$$
$$= \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[\Delta^2_{t+1}(y_1-1,y_2)-\Delta^2_{t+1}(y_1,y_2)] \geq 0.$$

The first inequality follows from the fact that $\theta_0 p+(\theta_{11}+\theta_0)V_{t+1}(y_1,y_2-1) \geq \theta_{11}V_{t+1}(y_1-1,y_2)+\theta_0 V_{t+1}(y_1,y_2)$ in the first term, which in turn follows from $p-\Delta^2_{t+1}(y_1,y_2) \geq \frac{\theta_{11}}{\theta_{11}+\theta_0}[p-\Delta^1_{t+1}(y_1,y_2)]$ (this inequality holds because $S^*_{1t}(y_1,y_2)=\{1,2\}$). The last inequality is due to P1$(t+1,2,y_1,y_2)$.

We next prove P3. If $V_t(y_1,y_2-1) \geq V_t(y_1-1,y_2)$, then $\theta_0 p+(\theta_{21}+\theta_0)V_t(y_1,y_2-1) \geq \theta_0 p+(\theta_{21}+\theta_0)V_t(y_1-1,y_2) \geq \theta_0 V_t(y_1,y_2)+\theta_{21}V_t(y_1-1,y_2)$ because $p+V_t(y_1-1,y_2) \geq V_t(y_1,y_2)$. We then consider the case in which $V_t(y_1,y_2-1) < V_t(y_1-1,y_2)$. For this, we again consider two possible scenarios: $p-\Delta^1_{t+1}(y_1,y_2) \geq p-\Delta^2_{t+1}(y_1,y_2)$ and $p-\Delta^1_{t+1}(y_1,y_2) < p-\Delta^2_{t+1}(y_1,y_2)$. We show the result for the first scenario, and the second one can be proved similarly. For the case under consideration, the optimal assortment sets under the various inventory levels are given by one of the combinations shown in the table below:

	(y_1,y_2)	(y_1-1,y_2)	(y_1,y_2-1)
S^*_{1t}	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$ or $\{2\}$	$\{1,2\}$

We further consider two possible cases: (a) $p-\Delta^2_{t+1}(y_1,y_2) \geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2)]$ and (b) $p-\Delta^2_{t+1}(y_1,y_2) < \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2)]$. Under case (a) and inventory levels (y_1,y_2), the set $\{1,2\}$ is preferred over the set $\{1\}$ for a customer of segment 1 in period t. The same holds for inventory levels (y_1-1,y_2) from P5$(t+1,1,y_1,y_2)$. Then, when $y_1,y_2 \geq 2$, the optimal assortment sets are:

	(y_1,y_2)	(y_1-1,y_2)	(y_1,y_2-1)
S^*_{1t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$ or $\{2\}$	$\{1,2\}$

When $p-\Delta^2_{t+1}(y_1,y_2-1) \geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2-1)]$ and $p-\Delta^1_{t+1}(y_1-1,y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}[p-\Delta^2_{t+1}(y_1-1,y_2)]$, the optimal assortment sets are:

	(y_1,y_2)	(y_1-1,y_2)	(y_1,y_2-1)
S^*_{1t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

and the result follows easily by induction. Instead, if $p-\Delta^2_{t+1}(y_1,y_2-1) < \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2-1)]$ and $p-\Delta^1_{t+1}(y_1-1,y_2) < \frac{\theta_{22}}{\Lambda^2_{\{20\}}}[p-\Delta^2_{t+1}(y_1-1,y_2)]$, the optimal assortment sets are:

	(y_1,y_2)	(y_1-1,y_2)	(y_1,y_2-1)
S^*_{1t}	$\{1,2\}$	$\{1,2\}$	$\{1\}$
S^*_{2t}	$\{1,2\}$	$\{2\}$	$\{1,2\}$

For a segment 1 customer, because $S^*_{1t}(y_1,y_2-1)=\{1\}$,

$$
\begin{aligned}
V_t(y_1,y_2-1|1) \quad = \quad & \frac{\theta_{11}p+\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2-1)}{\Lambda^1_{\{10\}}} \\
& -\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_{12}V_{t+1}(y_1,y_2-2)+\theta_0 V_{t+1}(y_1,y_2-1)] \\
& +\frac{1}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_{12}V_{t+1}(y_1,y_2-2)+\theta_0 V_{t+1}(y_1,y_2-1)]
\end{aligned}
$$

(where we are adding and subtracting the same term). Then, for customer segment 1, we have

$$\theta_0 p + (\theta_0+\theta_{21})V_t(y_1,y_2-1|1) - \theta_0 V_t(y_1,y_2|1) - \theta_{21}V_t(y_1-1,y_2|1)$$

$$= \frac{(\theta_0+\theta_{21})\theta_{11}p}{\Lambda^1_{\{10\}}} - \frac{\theta_{21}(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}} - \frac{\theta_0(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}}$$

$$+\frac{\theta_0+\theta_{21}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2-1)]$$

$$-\frac{\theta_0+\theta_{21}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_{12}V_{t+1}(y_1,y_2-2)+\theta_0 V_{t+1}(y_1,y_2-1)]$$

$$+\frac{\theta_{11}}{\Lambda^1_{\{120\}}}[\theta_0 p+(\theta_0+\theta_{21})V_{t+1}(y_1-1,y_2-1)-\theta_0 V_{t+1}(y_1-1,y_2)-\theta_{21}V_{t+1}(y_1-2,y_2)]$$

$$+\frac{\theta_{12}}{\Lambda^1_{\{120\}}}[\theta_0 p+(\theta_0+\theta_{21})V_{t+1}(y_1,y_2-2)-\theta_0 V_{t+1}(y_1,y_2-1)-\theta_{21}V_{t+1}(y_1-1,y_2-1)]$$

$$+\frac{\theta_0}{\Lambda^1_{\{120\}}}[\theta_0 p+(\theta_0+\theta_{21})V_{t+1}(y_1,y_2-1)-\theta_0 V_{t+1}(y_1,y_2)-\theta_{21}V_{t+1}(y_1-1,y_2)]. \tag{16}$$

The last three terms are positive due to P3$(t+1,2,y_1-1,y_2)$, P3$(t+1,2,y_1,y_2-1)$, and P3$(t+1,2,y_1,y_2)$, respectively. Therefore,

$$(16) \geq \frac{(\theta_0+\theta_{21})\theta_{11}p}{\Lambda^1_{\{10\}}} - \frac{\theta_{21}(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}} - \frac{\theta_0(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}}$$

$$+\frac{\theta_0+\theta_{21}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1,y_2-1)]$$

$$-\frac{\theta_0+\theta_{21}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-1,y_2-1)+\theta_{12}V_{t+1}(y_1,y_2-2)+\theta_0 V_{t+1}(y_1,y_2-1)]$$

$$= \frac{(\theta_0+\theta_{21})\theta_{11}p}{\Lambda^1_{\{10\}}} - \frac{\theta_{21}(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}} - \frac{\theta_0(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}}$$

$$+\frac{(\theta_0+\theta_{21})\theta_{12}}{(\Lambda^1_{\{10\}})(\Lambda^1_{\{120\}})}[\theta_{11}V_{t+1}(y_1-1,y_2-1)-(\Lambda^1_{\{10\}})V_{t+1}(y_1,y_2-2)+\theta_0 V_{t+1}(y_1,y_2-1)]$$

$$> \frac{(\theta_0+\theta_{21})\theta_{11}p}{\Lambda^1_{\{10\}}} - \frac{\theta_{21}(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}} - \frac{\theta_0(\theta_{11}+\theta_{12})p}{\Lambda^1_{\{120\}}} + \frac{(\theta_0+\theta_{21})\theta_{12}\theta_0 p}{(\Lambda^1_{\{10\}})(\Lambda^1_{\{120\}})} = 0.$$

The last inequality holds because $-(\Lambda^1_{\{10\}})V_{t+1}(y_1,y_2-2) > \theta_0 p - \theta_0 V_{t+1}(y_1,y_2-1) - \theta_{11}V_{t+1}(y_1-1,y_2-1)$, which itself follows from $p-\Delta^2_{t+1}(y_1,y_2-1) < \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2-1)]$.

For a segment 2 customer, we have

$$\theta_0 p + (\theta_0+\theta_{21})V_t(y_1,y_2-1|2) - \theta_0 V_t(y_1,y_2|2) - \theta_{21}V_t(y_1-1,y_2|2)$$

$$= \theta_0 p + \frac{\theta_{21}^2\theta_0 p}{(\Lambda^2_{\{120\}})(\Lambda^2_{\{20\}})} + \frac{(\theta_0+\theta_{21})}{\Lambda^2_{\{120\}}}[\theta_{21}V_{t+1}(y_1-1,y_2-1)$$

$$+\theta_{22}V_{t+1}(y_1, y_2-2) + \theta_0 V_{t+1}(y_1, y_2-1)] - \frac{\theta_0}{\Lambda^2_{\{120\}}}[\theta_{21}V_{t+1}(y_1-1, y_2)$$

$$+\theta_{22}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1, y_2)] - \frac{\theta_{21}}{\Lambda^2_{\{20\}}}[\theta_{22}V_{t+1}(y_1-1, y_2-1)$$

$$+\theta_0 V_{t+1}(y_1-1, y_2)]$$

$$= \frac{\theta_{21}\theta_0 p}{\Lambda^2_{\{20\}}} + \frac{(\theta_0+\theta_{21})\theta_{21}V_{t+1}(y_1-1, y_2-1)}{\Lambda^2_{\{120\}}}$$

$$\frac{\theta_{22}}{\Lambda^2_{\{120\}}}[\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-2)] + \frac{\theta_0}{\Lambda^2_{\{120\}}}[\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-1)]$$

$$-\frac{\theta_0}{\Lambda^2_{\{120\}}}[\theta_{21}V_{t+1}(y_1-1, y_2) + \theta_{22}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1, y_2)]$$

$$-\frac{\theta_{21}}{\Lambda^2_{\{20\}}}[\theta_{22}V_{t+1}(y_1-1, y_2-1) + \theta_0 V_{t+1}(y_1-1, y_2)]. \tag{17}$$

Using inductively P3$(t+1, 2, y_1, y_2-1)$, which is equivalent to $\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-2) \geq \theta_0 V_{t+1}(y_1, y_2-1) + \theta_{21}V_{t+1}(y_1-1, y_2-1)$, and P3$(t+1, 2, y_1, y_2)$, which is equivalent to $\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-1) \geq \theta_0 V_{t+1}(y_1, y_2) + \theta_{21}V_{t+1}(y_1-1, y_2)$, we have

$$\frac{\theta_{22}}{\Lambda^2_{\{120\}}}[\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-2)] \geq \frac{\theta_{22}}{\Lambda^2_{\{120\}}}[\theta_0 V_{t+1}(y_1, y_2-1) + \theta_{21}V_{t+1}(y_1-1, y_2-1)]$$

$$\frac{\theta_0}{\Lambda^2_{\{120\}}}[\theta_0 p + (\theta_0+\theta_{21})V_{t+1}(y_1, y_2-1)] \geq \frac{\theta_0}{\Lambda^2_{\{120\}}}[\theta_0 V_{t+1}(y_1, y_2) + \theta_{21}V_{t+1}(y_1-1, y_2)]$$

Applying these two inequalities to the third and fourth terms after the last equality in (17), we have

$$(17) \geq \frac{\theta_{21}\theta_0 p}{\Lambda^2_{\{20\}}} + \frac{\theta_{21}\theta_0}{\Lambda^2_{\{20\}}}(V_{t+1}(y_1-1, y_2-1) - V_{t+1}(y_1-1, y_2)) \geq 0.$$

In sum, $\theta_0 p + (\theta_0+\theta_{21})V_t(y_1, y_2-1|m) - \theta_0 V_t(y_1, y_2|m) - \theta_{21}V_t(y_1-1, y_2|m) \geq 0$ for $m = 1, 2$, which implies P3$(t, 2, y_1, y_2)$. The proofs of all other cases not considered here are provided in Appendix B.

We now proceed with P4. Here again, we consider two cases: $p - \Delta^1_{t+1}(y_1, y_2) \geq p - \Delta^2_{t+1}(y_1, y_2)$ and $p - \Delta^1_{t+1}(y_1, y_2) < p - \Delta^2_{t+1}(y_1, y_2)$. As before, we the first case and the second one follows similarly. In this setting, and using induction, the optimal assortment sets under the various inventory levels can be summarized as follows:

	(y_1, y_2)	(y_1, y_2-1)	(y_1-1, y_2)	(y_1+1, y_2-1)	(y_1+1, y_2)
S^*_{1t}	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$ or $\{2\}$	$\{1,2\}$	$\{1,2\}$

We further consider two scenarios: (a) $p - \Delta^2_{t+1}(y_1, y_2) \geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p - \Delta^1_{t+1}(y_1, y_2)]$ and (b) $p - \Delta^2_{t+1}(y_1, y_2) < \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p - \Delta^1_{t+1}(y_1, y_2)]$, and focus attention on (a). When $y_1, y_2 > 1$, the possible optimal assortment sets are

	(y_1, y_2)	(y_1, y_2-1)	(y_1-1, y_2)	(y_1+1, y_2-1)	(y_1+1, y_2)
S^*_{1t}	$\{1,2\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$ or $\{2\}$	$\{1,2\}$	$\{1,2\}$

We prove one specific case here and all the other cases are proved in Appendix B. Specifically, we consider the optimal assortment sets given by:

	(y_1, y_2)	(y_1, y_2-1)	(y_1-1, y_2)	(y_1+1, y_2-1)	(y_1+1, y_2)
S^*_{1t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1\}$	$\{1,2\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$

For a segment 1 customer, because $S^*_{1t}(y_1+1, y_2-1) = \{1\}$, we have

$$\begin{aligned} -\theta_{22}V_t(y_1+1, y_2-1|1) \quad = \quad & -\frac{\theta_{22}}{\Lambda^1_{\{10\}}}[\theta_{11}p + \theta_{11}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1+1, y_2-1)] \\ & +\frac{\theta_{22}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1+1, y_2-1)] \\ & -\frac{\theta_{22}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1+1, y_2-1)] \qquad (18) \end{aligned}$$

We denote (18) as $A(y_1+1, y_2-1|1)$, since this equation will be used in the other cases provided in Appendix B. Using (18) and the optimal assortment sets under the other inventory levels, we have

$$(\theta_{22}+2\theta_0)V_t(y_1, y_2|1)+\theta_{22}V_t(y_1, y_2-1|1)-(\Lambda^2_{\{20\}})V_t(y_1-1, y_2|1)-\theta_{22}V_t(y_1+1, y_2-1|1)-\theta_0 V_t(y_1+1, y_2|1)$$

$$\begin{aligned} &= \frac{\theta_{11}}{\Lambda^1_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1-1, y_2) + \theta_{22}V_{t+1}(y_1-1, y_2-1) \\ &-(\Lambda^2_{\{20\}})V_{t+1}(y_1-2, y_2) - \theta_{22}V_{t+1}(y_1, y_2-1) - \theta_0 V_{t+1}(y_1, y_2)] \\ &+\frac{\theta_0}{\Lambda^1_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1, y_2) + \theta_{22}V_{t+1}(y_1, y_2-1) \\ &-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1, y_2) - \theta_{22}V_{t+1}(y_1+1, y_2-1) - \theta_0 V_{t+1}(y_1+1, y_2)] \\ &+\frac{\theta_{12}\theta_0\theta_{22}p}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})} + \frac{\theta_{12}}{\Lambda^1_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1, y_2-1) + \theta_{22}V_{t+1}(y_1, y_2-2) \\ &-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1, y_2-1) - \theta_0 V_{t+1}(y_1+1, y_2-1)] + \frac{\theta_{22}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1, y_2-1) \\ &+\theta_0 V_{t+1}(y_1+1, y_2-1)] - \frac{\theta_{22}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1, y_2-1) + \theta_0 V_{t+1}(y_1+1, y_2-1)]. \qquad (19) \end{aligned}$$

Due to P4$(t+1, 2, y_1-1, y_2)$ and P4$(t+1, 2, y_1, y_2)$, the first two terms in (19) are non-negative. Then,

$$(19) \geq \frac{\theta_{12}\theta_0\theta_{22}p}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})} + \frac{\theta_{12}}{\Lambda^1_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1, y_2-1) + \theta_{22}V_{t+1}(y_1, y_2-2)$$

$$-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1,y_2-1)-\theta_0 V_{t+1}(y_1+1,y_2-1)]+\frac{\theta_{22}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1,y_2-1)$$

$$+\theta_0 V_{t+1}(y_1+1,y_2-1)]-\frac{\theta_{22}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1+1,y_2-1)]$$

$$=\frac{\theta_{12}\theta_{22}}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})}[\theta_0 p+\Lambda^1_{\{10\}}V_{t+1}(y_1,y_2-2)]+\frac{\theta_{12}}{\Lambda^1_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1,y_2-1)$$

$$-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1,y_2-1)-\theta_0 V_{t+1}(y_1+1,y_2-1)]+\frac{\theta_{22}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1,y_2-1)$$

$$+\theta_0 V_{t+1}(y_1+1,y_2-1)]-\frac{\theta_{22}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1,y_2-1)+\theta_0 V_{t+1}(y_1+1,y_2-1)] \qquad (20)$$

Using the assumption that $p-\Delta^2_{t+1}(y_1,y_2-1)\geq \frac{\theta_{11}}{\Lambda^1_{\{10\}}}[p-\Delta^1_{t+1}(y_1,y_2-1)]$, which is equivalent to $\theta_0 p+(\Lambda^1_{\{10\}})V_{t+1}(y_1,y_2-2)\geq \theta_0 V_{t+1}V_{t+1}(y_1,y_2-1)+\theta_{11}V_{t+1}(y_1-1,y_2-1)$, in the first term above, we obtain

$$(20)\geq \frac{\theta_{12}\theta_0(\theta_{22}+\theta_{11}+\theta_0)}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})}[2V_{t+1}(y_1,y_2-1)-V_{t+1}(y_1-1,y_2-1)-V_{t+1}(y_1+1,y_2-1)]\geq 0$$

The last inequality follows from P1$(t+1,1,y_1,y_2-1)$.

For a segment 2 customer, because $S^*_{2t}(y_1-1,y_2)=\{2\}$,

$$\begin{aligned}-(\Lambda^2_{\{20\}})V_t(y_1-1,y_2|2) &= -\theta_{22}p-\theta_{22}V_{t+1}(y_1-1,y_2-1)-\theta_0 V_{t+1}(y_1-1,y_2)\\ &\quad +\frac{\Lambda^2_{\{20\}}}{\Lambda^2_{\{120\}}}[\theta_{22}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1-1,y_2)]\\ &\quad -\frac{\Lambda^2_{\{20\}}}{\Lambda^2_{\{120\}}}[\theta_{22}V_{t+1}(y_1-1,y_2-1)+\theta_0 V_{t+1}(y_1-1,y_2)]\end{aligned}$$

Then, using this equation and the optimal assortment sets under the other inventory levels for customers of segment 2, we have

$$(\theta_{22}+2\theta_0)V_t(y_1,y_2|2)+\theta_{22}V_t(y_1,y_2-1|2)-(\Lambda^2_{\{20\}})V_t(y_1-1,y_2|2)-\theta_{22}V_t(y_1+1,y_2-1|2)-\theta_0 V_t(y_1+1,y_2|2)$$

$$=\frac{\theta_{22}}{\Lambda^2_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1,y_2-1)+\theta_{22}V_{t+1}(y_1,y_2-2)$$

$$-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1,y_2-1)-\theta_{22}V_{t+1}(y_1+1,y_2-2)-\theta_0 V_{t+1}(y_1+1,y_2-1)]$$

$$+\frac{\theta_0}{\Lambda^2_{\{120\}}}[(\theta_{22}+2\theta_0)V_{t+1}(y_1,y_2)+\theta_{22}V_{t+1}(y_1,y_2-1)$$

$$-(\Lambda^2_{\{20\}})V_{t+1}(y_1-1,y_2)-\theta_{22}V_{t+1}(y_1+,y_2-1)-\theta_0 V_{t+1}(y_1+1,y_2)]$$

$$+\frac{\theta_{21}\theta_0 p}{\Lambda^2_{\{120\}}}+\frac{\theta_{21}}{\Lambda^2_{\{120\}}}[(\Lambda^2_{\{20\}})V_{t+1}(y_1-1,y_2)-\theta_{22}V_{t+1}(y_1,y_2-1)-\theta_0 V_{t+1}(y_1,y_2)]. \qquad (21)$$

The first three terms are positive due to P4$(t+1, 2, y_1, y_2-1)$ and P4$(t+1, 2, y_1, y_2)$. Then,

$$(21) \geq \frac{\theta_{21}\theta_0 p}{\Lambda^2_{\{120\}}} + \frac{\theta_{21}}{\Lambda^2_{\{120\}}}[(\Lambda^2_{\{20\}})V_{t+1}(y_1-1, y_2) - \theta_{22}V_{t+1}(y_1, y_2-1) - \theta_0 V_{t+1}(y_1, y_2)] \geq 0$$

The last inequality follows from $p - \Delta^1_{t+1}(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}[p - \Delta^2_{t+1}(y_1, y_2)]$. This completes the proof of P4 for this case.

We now proceed to the proof of P5. If $V_t(y_1-1, y_2-1) \geq V_t(y_1, y_2-2)$, then $\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1-1, y_2-1) \geq \theta_0 V_t(y_1, y_2-1) + \theta_{22}V_t(y_1, y_2-2)$ because $p + V_t(y_1-1, y_2-1) \geq V_t(y_1, y_2-1)$. Therefore, we focus on the case in which $V_t(y_1-1, y_2-1) < V_t(y_1, y_2-2)$. We further consider two scenarios: $\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1-2, y_2-1) \geq \theta_0 V_t(y_1-1, y_2-1) + \theta_{22}V_t(y_1-1, y_2-2)$ and $\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1-2, y_2-1) < \theta_0 V_t(y_1-1, y_2-1) + \theta_{22}V_t(y_1-1, y_2-2)$. We consider the first scenario. Under that assumption, when $y_1 > 1$ and $y_2 > 2$, the possible optimal assortment sets for the various inventory levels are

	(y_1-1, y_2-1)	(y_1, y_2-2)	(y_1, y_2-1)
S^*_{1t}	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$	$\{1,2\}$ or $\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

If $\theta_0 p + (\Lambda^1_{\{10\}})V_t(y_1-1, y_2-2) \geq \theta_0 V_t(y_1-1, y_2-1) + \theta_{11}V_t(y_1-2, y_2-1)$ and $\theta_0 p + (\Lambda^1_{\{10\}})V_t(y_1, y_2-3) \geq \theta_0 V_t(y_1, y_2-2) + \theta_{11}V_t(y_1-1, y_2-2)$, then the optimal assortment for any customer type under either of the inventory levels (y_1-1, y_2-1), (y_1, y_2-2), or (y_1, y_2-1), is the set $\{1,2\}$. In this case, it is easy to prove that P5 holds for each customer segment. We therefore consider the case in which the optimal assortment sets are given by

	(y_1-1, y_2-1)	(y_1, y_2-2)	(y_1, y_2-1)
S^*_{1t}	$\{1,2\}$	$\{1\}$	$\{1\}$
S^*_{2t}	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

It is similarly straightforward to prove the result for a customer in segment 2. For a segment 1 customer,

$$\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1-1, y_2-1|1) - \theta_0 V_t(y_1, y_2-1|1) - \theta_{22}V_t(y_1, y_2-2|1)$$

$$= \theta_0 p + (\Lambda^2_{\{20\}})\frac{\theta_0\theta_{12}p}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})} + \frac{\Lambda^2_{\{20\}}}{\Lambda^1_{\{120\}}}[\theta_{11}V_{t+1}(y_1-2, y_2-1) + \theta_{12}V_{t+1}(y_1-1, y_2-2) + \theta_0 V_{t+1}(y_1-1, y_2-1)]$$

$$-\frac{\theta_0}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1, y_2-1) + \theta_0 V_{t+1}(y_1, y_2-1)] - \frac{\theta_{22}}{\Lambda^1_{\{10\}}}[\theta_{11}V_{t+1}(y_1-1, y_2-2) + \theta_0 V_{t+1}(y_1, y_2-2)]$$

$$= (\Lambda^2_{\{20\}})\frac{\theta_{12}}{(\Lambda^1_{\{120\}})(\Lambda^1_{\{10\}})}[\theta_0 p - \theta_{11}V_{t+1}(y_1-2, y_2-1) + (\Lambda^1_{\{10\}})V_{t+1}(y_1-1, y_2-2) - \theta_0 V_{t+1}(y_1-1, y_2-1)]$$

$$+\frac{\theta_{11}}{\Lambda^1_{\{10\}}}[\theta_0 p + (\Lambda^2_{\{20\}})V_{t+1}(y_1-2, y_2-1) - \theta_0 V_{t+1}(y_1-1, y_2-1) - \theta_{22}V_{t+1}(y_1-1, y_2-2)]$$

$$+\frac{\theta_0}{\Lambda^1_{\{10\}}}[\theta_0 p + (\Lambda^2_{\{20\}})V_{t+1}(y_1-1, y_2-1) - \theta_0 V_{t+1}(y_1, y_2-1) - \theta_{22}V_{t+1}(y_1, y_2-2)] \geq 0$$

The first term is positive because $S^*_{1t}(y_1-1, y_2-1) = \{1\}$. The second and third terms are positive because $S^*_{2t}(y_1-1, y_2-1) = \{1,2\}$ and $S^*_{2t}(y_1, y_2-1) = \{1,2\}$, respectively. This completes the proof for this case. The proofs of the remaining cases are provided in Appendix B.

We now prove part (iv) of Theorem 2, which states that the threshold values are decreasing in time t. We again consider the case of segment 2 customers. Assume that the optimal assortment is given by $\{1,2\}$ in period t under the inventory level (y_1, y_2), i.e., $\theta_0(p - \Delta^1_{t+1}(y_1, y_2)) + \theta_{22}(\Delta^2_{t+1}(y_1, y_2) - \Delta^1_{t+1}(y_1, y_2)) \geq 0$. It suffices to prove that the optimal assortment is still given by $\{1,2\}$ in period $t+1$ under the same inventory level, i.e., $\theta_0(p - \Delta^1_{t+2}(y_1, y_2)) + \theta_{22}(\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2)) \geq 0$. If $\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2) \geq 0$, then $\theta_0(p - \Delta^1_{t+2}(y_1, y_2)) + \theta_{22}(\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2)) \geq 0$ and the proof is complete. If $\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2) < 0$, then the marginal expected revenue of product 1 is larger than that of product 2 in period $t+2$, i.e., $V_{t+2}(y_1, y_2-1) - V_{t+2}(y_1-1, y_2) > 0$. Similar to the proof of P2, we can show that $V_{t+1}(y_1, y_2-1) - V_{t+1}(y_1-1, y_2) \geq V_{t+2}(y_1, y_2-1) - V_{t+2}(y_1-1, y_2)$ by induction, which implies that $\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2) \geq \Delta^2_{t+1}(y_1, y_2) - \Delta^1_{t+1}(y_1, y_2)$. Moreover, from P2, we have that $\Delta^i_{t+1}(y_1, y_2) > \Delta^i_{t+2}(y_1, y_2)$. Therefore, $\theta_0(p - \Delta^1_{t+2}(y_1, y_2)) + \theta_{22}(\Delta^2_{t+2}(y_1, y_2) - \Delta^1_{t+2}(y_1, y_2)) \geq \theta_0(p - \Delta^1_{t+1}(y_1, y_2)) + \theta_{22}(\Delta^2_{t+1}(y_1, y_2) - \Delta^1_{t+1}(y_1, y_2)) \geq 0$, concluding the proof. ■

Proposition 3

Proof: It is easy to prove that the optimal assortment for an arriving customer of segment 1 is $\{1\}$ by sample path analysis. For the remaining results, we can prove them similarly as Theorem 2. Similarly, we have five properties for this setting as well.

P1'(t, j, y_1, y_2): $\Delta^j_t(y_1, y_2)$ is decreasing in y_i, $i \in \{1,2\}$ and $j \in \{1,2\}$.

P2'(t, j, y_1, y_2): $\Delta^j_t(y_1, y_2)$ is decreasing in t for any $j \in \{1,2\}$.

P3'(t, 2, y_1, y_2): $p - \Delta^2_t(y_1, y_2) \geq \frac{\theta_{21}}{\theta_{21}+\theta_0}(p - \Delta^1_t(y_1, y_2))$. In other words, $\theta_0 p + (\theta_0 + \theta_{21})V_t(y_1, y_2 - 1) \geq \theta_0 V_t(y_1, y_2) + \theta_{21} V_t(y_1-1, y_2)$. The optimal assortment for segment 1 customers is always to offer product 1 only.

P4'(t, 2, y_1, y_2):

$$\Delta^1_t(y_1, y_2) - \Delta^1_t(y_1+1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}(\Delta^2_t(y_1, y_2) - \Delta^2_t(y_1+1, y_2)).$$

In other words,

$$(\theta_{22}+2\theta_0)V_t(y_1, y_2) + \theta_{22}V_t(y_1, y_2-1) \geq (\Lambda^2_{\{20\}})V_t(y_1-1, y_2) + \theta_{22}V_t(y_1+1, y_2-1) + \theta_0 V_t(y_1+1, y_2)$$

P5'(t, 2, y_1, y_2): If $p - \Delta^1_t(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1, y_2)\right)$, then $p - \Delta^1_t(y_1, y_2-1) \geq \frac{\theta_{22}}{\Lambda^2_{\{20\}}}\left(p - \Delta^2_t(y_1, y_2-1)\right)$. In other words, If $\theta_0 p + (\Lambda^2_{\{20\}})V_t(y_1-1, y_2) \geq \theta_0 V_t(y_1, y_2) + \theta_{22}V_t(y_1, y_2-1)$, then $\theta_0 p + (\theta_{22}+\theta_0)V_t(y_1-1, y_2-1) \geq \theta_0 V_t(y_1, y_2-1) + \theta_{22}V_t(y_1, y_2-2)$.

The only difference between P1-P5 and P1'-P5' is in P3'. First of all, it is easy to show that the optimal assortment for segment 1 customers is always to offer product 1 only. Secondly, compared to the proof of P3, P3' simplify the proof by limiting possible optimal assortments for segment 1 customers. For segment 2 customers, the proofs are similar.

Proposition 4

Proof: Without loss of generality, we prove the results for segment i which is assumed to be of type 1, i.e., $\theta_{i2} \geq \theta_{i1} = \theta_{min,1}$. The properties R1-R5 below for segment i are sufficient to establish the optimality threshold policy.

R1(t, y_1, y_2): $\Delta_t^j(y_1, y_2)$ is decreasing in y_i, $i \in \{1, 2\}$ and $j \in \{1, 2, \cdots, M\}$.

R2(t, y_1, y_2): $\Delta_t^2(y_1, y_2)$ is decreasing in t.

R3(t, y_1, y_2): $p - \Delta_t^2(y_1, y_2) \geq \frac{\theta_{min,1}}{\theta_{min,1}+\theta_0}(p - \Delta_t^1(y_1, y_2))$ and $p - \Delta_t^1(y_1, y_2) \geq \frac{\theta_{min,2}}{\theta_{min,2}+\theta_0}(p - \Delta_t^2(y_1, y_2))$.

R4(t, i, y_1, y_2):

$$\Delta_t^1(y_1, y_2) - \Delta_t^1(y_1 + 1, y_2) \geq \frac{\theta_{i2}}{\theta_{i2} + \theta_0}(\Delta_t^2(y_1, y_2) - \Delta_t^2(y_1 + 1, y_2)).$$

In other words,

$$(\theta_{i2}+2\theta_0)V_t(y_1, y_2)+\theta_{i2}V_t(y_1, y_2-1) \geq (\theta_{i2}+\theta_0)V_t(y_1-1, y_2)+\theta_{i2}V_t(y_1+1, y_2-1)+\theta_0 V_t(y_1+1, y_2)$$

R5(t, i, y_1, y_2): If $p - \Delta_t^1(y_1, y_2) \geq \frac{\theta_{i2}}{\theta_{i2}+\theta_0}\left(p - \Delta_t^2(y_1, y_2)\right)$, then $p - \Delta_t^1(y_1, y_2 - 1) \geq \frac{\theta_{i2}}{\theta_{i2}+\theta_0}\left(p - \Delta_t^2(y_1, y_2 - 1)\right)$.

The proof of these five properties is similar to that of Theorem 2. We next prove that the threshold level is increasing in the preference of product 1 for segments in type 1. Consider two customer segments with the minimum utility of product 1, i.e., $\theta_{i1} = \theta_{j1} = \theta_{min,1}$ and $\theta_{i2} > \theta_{j2} > \theta_{min,1}$. R4 implies that product 2 is always offered to these segments and product 1 is offered only if its inventory level is larger than a threshold. If $p - \Delta_t^1(y_1, y_2) \geq \frac{\theta_{i2}}{\theta_{i2}+\theta_0}\left(p - \Delta_t^2(y_1, y_2)\right)$, then $p - \Delta_t^1(y_1, y_2) \geq \frac{\theta_{j2}}{\theta_{j2}+\theta_0}\left(p - \Delta_t^2(y_1, y_2)\right)$, because $\frac{\theta_{i2}}{\theta_{i2}+\theta_0} > \frac{\theta_{j2}}{\theta_{j2}+\theta_0}$. Hence, the threshold level of product 1 for segment i is greater than that in segment j.

Numerical Study for $N = 6$ and $M = 6$

To evaluate the performance of the aggregation-based heuristic in Section 5, we have tested a set of experiments, all with six products, six customer segments, $p = 1$, $\lambda = 1$, and $T = 15$. The preference vectors are given in Table 3: G1 represents a setting with distinctively different preferences across customer segments; the same applies for the first five customer segments under G2, while the last segment is somewhat indifferent between products; in G3 and G4, customers in each segment have roughly equal preferences for two and three of the products, respectively.

Group	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6
G1	$(10, 1, 1, 1, 1, 1)$	$(1, 10, 1, 1, 1, 1)$	$(1, 1, 10, 1, 1, 1)$	$(1, 1, 1, 10, 1, 1)$	$(1, 1, 1, 1, 10, 1)$	$(1, 1, 1, 1, 1, 10)$
G2	$(10, 1, 1, 1, 1, 1)$	$(1, 10, 1, 1, 1, 1)$	$(1, 1, 10, 1, 1, 1)$	$(1, 1, 1, 10, 1, 1)$	$(1, 1, 1, 1, 10, 1)$	$(9, 9, 9, 9, 9, 10)$
G3	$(10, 9, 1, 1, 1, 1)$	$(1, 10, 9, 1, 1, 1)$	$(1, 1, 10, 9, 1, 1)$	$(1, 1, 1, 10, 9, 1)$	$(1, 1, 1, 1, 10, 9)$	$(9, 1, 1, 1, 1, 10)$
G4	$(10, 9, 8, 1, 1, 1)$	$(1, 10, 9, 8, 1, 1)$	$(1, 1, 10, 9, 8, 1)$	$(1, 1, 1, 10, 9, 8)$	$(8, 1, 1, 1, 10, 9)$	$(9, 8, 1, 1, 1, 10)$

Table 3: Preference vectors of customer segments in the numerical study with six customer segments and six product variants.

We also consider three possible vectors of segment sizes: $\boldsymbol{\rho}^1 = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$, $\boldsymbol{\rho}^2 = (1/12, 1/12, 1/12, 3/12, 3/12, 3/12)$, $\boldsymbol{\rho}^3 = (1/12, 1/12, 1/12, 1/12, 1/12, 7/12)$. For this study, we consider 18 different initial inventory vectors for each parameter combination:

$$\mathbf{y} = \{(15, 1, 1, 1, 1, 1), (1, 15, 1, 1, 1, 1), (1, 1, 15, 1, 1, 1), (1, 1, 1, 15, 1, 1), (1, 1, 1, 1, 15, 1), (1, 1, 1, 1, 1, 15), (8, 8, 1, 1, 1, 1), (1, 1, 1, 8, 8, 1), (1, 1, 1, 1, 8, 8), (8, 1, 8, 1, 1, 1), (8, 1, 1, 8, 1, 1), (8, 1, 1, 1, 1, 8), (5, 5, 5, 1, 1, 1), (1, 1, 1, 5, 5, 5), (5, 1, 5, 1, 1, 5), (5, 1, 1, 5, 5, 1), (5, 1, 1, 1, 5, 5), (1, 1, 5, 1, 5, 5)\}.$$

The study consists of 216 cases.