

Efficient Investment in a Dynamic Auction Environment*

Brendan Daley[†] Michael Schwarz[‡] Konstantin Sonin[§]

November 2011

Abstract

We analyze an environment in which bidders' private values change over time due to both private investments and exogenous shocks. We demonstrate that a highly-decentralized mechanism achieves efficiency. The mechanism requires a stage of costly public announcements (i.e., signaling) to induce efficient investment. For this reason, an equilibrium selection issue arises, but can be handled by a minor modification in the spirit of virtual implementation.

*An earlier version of this paper circulated under the title “The Variable Value Environment: Auctions and Actions.” The authors are grateful to Rick Ericson, Drew Fudenberg, Giuseppe Lopomo, Eric Maskin, Paul Milgrom, Anna Mikoucheva, Eric Rasmusen, Mike Riordan, Ilya Segal, Thomas Sjostrom, Eric Stout, Jeroen Swinkels, Robert Townsend, and seminar participants at Harvard, MIT, and Columbia.

[†]Duke University, Fuqua School of Business, email: bd28@duke.edu

[‡]Yahoo! Labs, email: mschwarz@yahoo-inc.com

[§]New Economic School, email: ksonin@gmail.com

1 Introduction

A single, indivisible good will be available for use at some future date. If it were allocated today, each of its potential owners (call them *bidders*) would have some privately-known value for it. However, in the interim period each bidder’s value can fluctuate based on two factors:

- *Exogenous shocks.* Over time, bidders’ circumstances, information, and (therefore) values may change.
- *Investment in complimentary assets.* Interested bidders have the opportunity to invest in assets or technologies that will enhance their value of the available good. Of course, the bidder must be allocated the good for the investment decision to pay off.

For example, a commercial property will be available on a certain date. The value of the location to a given business will fluctuate based on exogenous market conditions. In addition, if a business were assured the location, it would engage in costly advertising for its “Grand Opening.” This investment is purely wasted if it does not receive the location. Or, a house will be available to move into on a future date. Before the house becomes available, bidders may have opportunities to buy other houses, essentially removing themselves from the market. We can consider “not buying another house” as an action that boosts a bidder’s value (where the cost of the action is the forgone surplus of not buying a different house).

This paper analyzes such a setting.¹ We take two perspectives, which we show are closely related. We first characterize an efficient mechanism for the sale of the object. The mechanism is extremely decentralized and induces a game worthy of study independent of any consideration of efficient mechanism design. We therefore further analyze the induced game, especially the issue of equilibrium multiplicity and selection.

We formally model the value-determination process as follows. First, each bidder receives a private signal, s_i , about her value. Next, each bidder decides whether or not to make an investment that increases her value for the object. The investment decision is binary and carries a cost of c . If the agent invests, her value for the object increases by $b > c$. After the investment decision, each bidder receives a second signal, t_i , about her value for the good. Bidder i ’s value is then $V(s_i, t_i) + a_i b$, where $a_i = 1$ if i invested and zero otherwise. This process takes place before the good is available for use.

¹Recent studies of environments with some similar features include Athey and Segal (2007), Battaglini (2005), Bergemann and Välimäki (2002, 2010), Calzolari and Pavan (2006), Cavallo, Parkes and Singh (2006), Crémer, Spiegel and Zheng (2009), Esö and Szentés (2003), Haile (2000, 2001, 2003), McAfee, Takacs and Vincent (1999), and Pavan, Segal and Toikka (2009). The main difference between these works and ours is that privately-informed bidders take actions that affect their values. This necessitates public messages within the mechanism, rather than messages sent only to the principal/designer/seller.

We demonstrate that the following is an efficient mechanism in this setting. After observing her first-period signal, s_i , but prior to making her investment decision, each bidder i makes a *public* announcement about her type and makes a corresponding payment. Investment behavior is then privately decided. After the realization of the final-period signals, a sealed-bid second-price auction is conducted with no reserve price. The payment scheme is designed such that truthful revelation is incentive compatible in the first stage (given the appropriate continuation play in the resulting subgame).

Notice that the mechanism is decentralized in the following two ways. First, the good is allocated via a standard second-price auction in the final stage—the previous decisions made by the bidders have no direct bearing on the allocation, nor on the payments in this stage. Second, investment decisions are private and, hence, non-contractible. Therefore, the incentive for a bidder to reveal her type, s_i , and make the prescribed payments in the first stage comes not from any action of the seller, but rather from influencing her competitors' investment behavior. This is why it is crucial that the announcement is publicly observable.

In addition, the game induced by our mechanism is of interest independent of considerations regarding efficient mechanism design. In the game, a second-price auction is to be held at some future date. Between now and then, bidders decide whether or not to make costly investment decisions and whether or not they wish to attempt to influence their competitors' behavior by dissipating resources as a signal of strength. We wish to understand this game for several reasons. First, it matches the obvious fact that auctions are *always* scheduled in advance. This is necessitated by the logistics of conducting an auction—the bidders must be told ahead of time where, when, and how to submit their bids. Second, it may be the case that the good in question will not be available for use for some time. Auctioning the good early may create inefficiency because each bidder's value for the good will change in the interim between an early auction and the date of actual availability. Third, it is rare to see more complicated dynamic mechanisms used in practice. Finally, it generalizes the study of auctions with costly entry by viewing the entry decision as a complimentary investment (in much the same way as *not buying* a substitute product is in the earlier example).²

On this front, our main result is the existence of an equilibrium in which each bidder has a strong incentive to immediately and credibly signal her initial value in order to discourage competitors from investing, thereby making the final auction stage more favorable toward her. Such behavior results in a complete revelation of initial values. *From this stage forward*, all investment decisions and the final allocation of the good are fully efficient. The only outlet

²Entry costs are investigated in various contexts by Milgrom (1981, 2004), Samuelson (1985), McAfee and McMillan (1987), Riordan and Sappington (1987), Levin and Smith (1994), Fullerton and McAfee (1999), McAfee, Quan and Vincent (2002), and Ye (2004, 2007).

for inefficiency is the potentially wasteful signaling activity. If an agent whose utility enters the social welfare calculation can benefit from the costly signaling activity of the bidders (the seller being the obvious choice for such an agent), then nothing is lost and the first-best societal welfare is attained. If such a capture is not possible, then the signaling activity will indeed be inefficient—sometimes so much so as to be more inefficient than if signaling were unavailable. This is common in models of signaling (see Riley (2001)), and is true of the standard (static) VCG mechanism as well.

To understand the efficient equilibrium in this game, let us reason via backward induction. In the auction stage, all bidders play their weakly-dominant strategy of bidding their value. This assures an efficient allocation of the good given the final values entering the auction stage. Given this behavior in the auction stage and each player i 's beliefs about \mathbf{s}_{-i} , we can solve for the equilibrium behavior in the investment stage. The structure of the investment decision (as well as the mild restrictions we impose on V and on the distribution of t_i given s_i), imply that, holding everything else constant, if investment is profitable for the player i when her type is s_i , it is also profitable for player i when her type is any $s'_i > s_i$. An important result is that *if \mathbf{s} were commonly known*, then there exists a unique symmetric equilibrium in monotone strategies where bidder i invests if and only if s_i is one of the $r(\mathbf{s})$ highest values. Further, this investment behavior maximizes social welfare (i.e., a social planner would designate this exact investment profile if she were so informed and empowered). This follows because the second-price auction allocates the good to the bidder with the highest value at the cost of the externality she creates by winning.

Holding fixed that bidders will play their weakly-dominant strategy of bidding their values in the final stage, the first and second stages comprise a complex signaling game (see Riley (2001) for an extensive survey of signaling games). Complex because, unlike a “standard” signaling game, the set of players is not clearly portioned into a set of *senders* and a set of *receivers*. Our setup is considerably more complicated in that we have multiple players each playing the role of sender and receiver to one another. That is, each player takes an action at every stage of the game. In the first stage she acts as a sender trying to influence the belief of her competitors. In the investment stage she acts as a receiver trying to optimize given her updated beliefs about her competition. Further, it is no longer true that bidders have unique best responses given their beliefs in the investment stage (i.e., when they are in the receiver role).

Even standard signaling games suffer from a well-known issue with equilibrium multiplicity. To obtain a sharp prediction in these models one must employ a strong equilibrium refinement—usually one based on restricting off-equilibrium-path beliefs (henceforth, *belief-based*) as pioneered by Banks and Sobel (1987) and Cho and Kreps (1987). Under commonly-

assumed “single-crossing” conditions, fully-revealing (or separating) equilibria usually withstand refinements, while equilibria involving pooling may fail depending on the strength of the refinement. We show that our fully-revealing equilibrium is indeed robust to belief-based refinements. However, the complexity of our setting allows equilibria involving pooling to sometimes survive any belief-based refinement as well.

This problem can be remedied by slightly altering the game/mechanism. Keeping everything else constant, award the bidder with highest announcement/payment in Stage 1 with a coupon worth ε off the price of the object if she is the eventual winner as well. This small change results in the fully-revealing equilibrium being the unique outcome consistent with symmetry and monotone investment strategies. As $\varepsilon \rightarrow 0$, the equilibrium converges to the efficient equilibrium we focus on throughout (again, that is efficient subject to the seller capturing the benefit of the first-stage payments). In this way, the ε -coupon serves as a non-belief-based refinement in the spirit of *virtual implementation* (Abreu and Sen, 1991).

The paper is structured as follows. In the next section we set up the model. We then present our efficient mechanism (Section 3). In Section 4, we study the related game more closely and consider the issue of equilibrium selection. We then offer concluding remarks. All proofs are found in the Appendix, and examples demonstrating that belief-based refinements cannot always eliminate full pooling equilibria are found in the Supplemental Material at the end.

2 The Setup

There is a single, indivisible asset to be allocated to one of N *ex ante* identical bidders. Each bidder’s value for the object is determined by both the private signals she receives as well as her investment decision. Specifically:

Period 1: Each agent i receives an independent signal s_i drawn according to a differentiable distribution function F on $[\underline{s}, \bar{s}]$.

Period 2: Each agent i then makes a private binary investment decision a_i : invest ($a_i = 1$) or not ($a_i = 0$). The investment decision carries an immediate cost of c and endows an additional benefit of $b > c$ if and only if player i is allocated the object.

Period 3: Each agent then receives a second private signal t_i drawn according to a differentiable distribution function $G(\cdot|s_i)$ on $[\underline{t}, \bar{t}]$. Signal t_i is drawn independently from a_i and s_j, a_j, t_j for all $j \neq i$. $G(\cdot|s_i)$ is differentiable in s_i

as well, and higher values of s_i make higher t_i values (weakly) more likely:
 If $s_i \geq s'_i$, then $G(\cdot|s_i)$ weakly first-order stochastically dominates $G(\cdot|s'_i)$
 [independence of t_i from s_i is a special case].

Agent i 's value for the object is given by $V(s_i, t_i) + a_i b$, where $V(\underline{s}, \underline{t}) \geq 0$. V is differentiable in both arguments and strictly (weakly) increasing in its first (second) argument. The utility of bidder i is

$$U_i = \mathbb{I}_i \cdot [V(s_i, t_i) + a_i b] - a_i c - p_i \quad (1)$$

where \mathbb{I}_i is the indicator function taking value 1 if i receives the object, and p_i is her total payment made within mechanism/game. Each bidder has an outside option worth zero, and all agents are expected-utility maximizers.

Efficiency

We measure efficiency by total surplus. There are N bidders and a seller (who we assume values the object at zero). The utility of each bidder is given by (1), and the utility of the seller is $\sum_{i=1}^N p_i$. Given an allocation and payments, the total surplus is

$$S = \sum_{i=1}^N U_i + p_i = V(s_{i^*}, t_{i^*}) + a_{i^*} b - \sum_{i=1}^N a_i c \quad (2)$$

where i^* represents the identity of the bidder receiving the object.³

3 An Efficient Mechanism

We begin by describing the general form of our efficient mechanism and providing a short discussion of its features. We then turn to analysis of the mechanism's specifics and substantiate our claims. We demonstrate that for any realization of \mathbf{s} , there exists a Bayes Nash Equilibrium of the game induced by our mechanism that maximizes expected surplus (where the expectation is taken over the realization of \mathbf{t}).

The forms of the mechanism and the equilibrium of interest are as follows (we use the term "stage" rather than "period" to distinguish the mechanism from the value-determination process):

Stage 1: After receiving her signal, s_i , each bidder i makes a public announcement, \hat{s}_i , and is assessed a payment, $h(\hat{s}_i, \hat{\mathbf{s}}_{-i})$, to the seller. In equilibrium, bidders report truthfully: $\hat{s}_i = s_i$.

³Notice that it is never efficient for the seller to retain the object.

Stage 2: After observing $\hat{\mathbf{s}}$, bidders make their private investment decisions. In equilibrium, they follow a cutoff rule $r(\hat{\mathbf{s}})$: $a_i = 1$ if and only if s_i is one of $r(\hat{\mathbf{s}})$ highest values.

Stage 3: After bidders observe their t -signals, a sealed-bid, second-price auction is held for the object. In equilibrium, bidders play their weakly-dominant strategy, bidding $V(s_i, t_i) + a_i b$.

Notice that the mechanism is decentralized in the following two ways. First, the good is allocated via a standard second-price auction in the final stage—the previous decisions made by the bidders have no direct bearing on the allocation nor on the payments in this stage. Second, because investment decisions are non-contractible, the incentive for a bidder to reveal her type s_i and make the prescribed payments in the first stage comes not from any action of the seller, but rather from influencing her competitors' investment behavior (by affecting the cutoff rule). Of course, in equilibrium her competitors must find it *optimal* to obey the cutoff rule, hence the bidder's incentive to choose $\hat{s}_i = s_i$ is more aptly described as a desire to affect the *beliefs* of her bidding competitors. This is why the observability of $\hat{\mathbf{s}}$ is crucial in the mechanism.

To argue that the mechanism is efficient, we will examine it stage-by-stage via backward induction. Along the way we will fill in the specifics of the r and h functions.

Stage 3: Efficient Allocation

In Stage 3, each bidder's value for the object is determined. Each bidder plays her weakly-dominant strategy of bidding her value, $V(s_i, t_i) + a_i b$, and the second-price auction allocates the object efficiently given these values. We are assured an efficient allocation *taking investment behavior as given*.

Stage 2: Full Revelation of First-Stage Signals Leads to Efficient Investment

Consider the subgame induced by the mechanism in which \mathbf{s} is common knowledge to all bidders. We first characterize the efficient investment behavior. We then demonstrate that, in this subgame, any symmetric equilibrium in monotone strategies (defined below) results in the efficient investments.

For any \mathbf{s} , let $\mathbf{a}^*(\mathbf{s})$ be an efficient profile of investment decisions. That is, $\mathbf{a}^*(\mathbf{s})$ maximizes the expected value of (2) given \mathbf{s} and that each bidder plays her weakly-dominant strategy of bidding her value in Stage 3. The following holds (all proofs are in the Appendix):

Cutoff Property: For any \mathbf{s} , $\mathbf{a}^*(\mathbf{s})$ is characterized by a cutoff, denoted $r^*(\mathbf{s})$, where bidder i invests ($a_i^*(\mathbf{s}) = 1$) if and only if s_i is one of the $r^*(\mathbf{s})$ highest values.⁴ Generically, the cutoff is unique.

The Efficient Rule is Incentive Compatible

We now demonstrate that the efficient cutoff rule is also incentive compatible. In Stage 3, a second-price auction is conducted. Each bidder follows her weakly-dominant strategy of bidding her value, $V(s_i, t_i) + a_i b$. Stepping back to the investment stage, each agent must simultaneously decide whether or not to invest *given that \mathbf{s} is common knowledge*. An investment strategy for bidder i is therefore a mapping from (s_i, \mathbf{s}_{-i}) to $\{0, 1\}$ specifying whether i invests or not given her signal s_i and all other bidders' signals \mathbf{s}_{-i} .⁵ An investment strategy is *monotone* if it is weakly increasing in s_i and weakly decreasing in \mathbf{s}_{-i} . We call an equilibrium *symmetric* if the following holds for almost all \mathbf{s} : letting \mathbf{a} and \mathbf{a}' denote the equilibrium investment profiles under \mathbf{s} and \mathbf{s}' respectively, if \mathbf{s}' is a permutation of \mathbf{s} , then \mathbf{a}' is the analogous permutation of \mathbf{a} .⁶

The following proposition establishes that any symmetric monotone equilibrium is characterized by efficient investment behavior.

Proposition 1 *In every symmetric equilibrium in monotone strategies, for any \mathbf{s} , investment decisions are characterized by an efficient cutoff $r^*(\mathbf{s})$, where bidder i invests ($a_i = 1$) if and only if s_i is one of the $r^*(\mathbf{s})$ highest values. Therefore, equilibrium play is generically unique.*

It is almost by definition that a symmetric monotone equilibrium will involve a cutoff rule. Only efficient cutoffs can be used in equilibrium because the second-price auction allocates the good to the bidder with highest value at the cost of the externality she creates by winning. Specifically, consider a bidder i 's investment decision taking the other players' decisions as given. There are three relevant events: 1) i wins regardless of her investment decision, 2) i loses regardless of her investment decision, and 3) i wins if and only if she invests. Clearly, if either (1) or (2) turn out to be true, then the difference in welfare from

⁴In the (non-generic) event that multiple bidders are tied for the $r^*(\mathbf{s})$ highest value (e.g., $\mathbf{s} = \{2, 1, 1\}$, and $r^*(\mathbf{s}) = 2$), the interpretation is that some, but not all, of the tied bidders invest in order to make the total number of bidders investing exactly $r^*(\mathbf{s})$ —because bidders are symmetric, any such profile simply permutes the utilities of the bidders and will be equally efficient.

⁵Allowing mixed strategies has no effect generically. However, it can introduce an inefficient equilibrium under the (non-generic) event described in footnote 4.

⁶The reason for the “almost all” caveat is found in footnote 4: in the (non-generic) event such that a subset of bidders with the equal s -values must coordinate their investments, symmetry is lost (though it could easily be restored by introducing a public randomization device, which we omit for simplicity).

investing or not is the same for player i as it is for total societal welfare because her decision does not affect the allocation. If case (3) turns out to be true, then player i 's utility increases from 0 to $V(s_i, t_i) + b - v - c$, where $v = \max_{j \neq i} V(s_j, t_j) + a_j b$. Total societal welfare changes from $v - \sum_{j \neq i} a_j c$ to $V(s_i, t_i) + (b - c) - \sum_{j \neq i} a_j c$. For both, the difference from investing is $V(s_i, t_i) + b - v - c$. Hence, each player i 's investment incentives are aligned with societal welfare state-by-state.

Proposition 1 hinges crucially on the use of the second-price auction to decide the final allocation. This illustrates that the second-price auction is a prudent choice in environments where investment opportunities arise prior to the allocation date and the designer cares about efficiency.⁷

We therefore specify that the mechanism sets its cutoff rule in Stage 2 equal to the efficient cutoff rule: $r(\cdot) = r^*(\cdot)$.⁸

Stage 1: Incentive-Compatible Full Revelation

Given our analysis to this point, our goal is to identify a function h on announcements $\hat{\mathbf{s}}$ that makes truthful revelation, followed by the continuation play specified above, incentive compatible. In fact, we will identify two. Under the first function, each bidder's payment may depend on the entire vector $\hat{\mathbf{s}}$, and truth-telling is *ex post* incentive compatible given the continuation play we have specified (where *ex post* here refers only to after the announcement $\hat{\mathbf{s}}$, but still prior to investment and the realization of \mathbf{t}). Under the second, each bidder's payment depends only on her own announcement \hat{s}_i , but truth-telling is not *ex post* incentive compatible. We have the familiar tradeoff between informational requirements and "robustness."

We will use the following notation. Fix a bidder i and \mathbf{s}_{-i} . Let $Q_i^k(\mathbf{s}_{-i}) \subset [\underline{s}, \bar{s}]$ be the set of values such that if $s_i \in Q_i^k(\mathbf{s}_{-i})$, then under the efficient cutoff rule, the bidders with the k highest values, *other than* s_i , invest.⁹ And let k^* satisfy $\underline{s} \in Q_i^{k^*}(\mathbf{s}_{-i})$, i.e., k^* is how many bidders, other than i , invest when $s_i = \underline{s}$. The number of other bidders investing is weakly decreasing in s_i , so $Q_i^{k^*}(\mathbf{s}_{-i}), Q_i^{k^*-1}(\mathbf{s}_{-i}), \dots$ is a partition of $[\underline{s}, \bar{s}]$ composed of intervals with boundaries denoted $\underline{s} = s_i^{k^*}(\mathbf{s}_{-i}) < s_i^{k^*-1}(\mathbf{s}_{-i}) < \dots$ ¹⁰ In other words, the number of other

⁷The following example demonstrates the necessity of the monotonicity restriction. Let $N = 2$ and $V(s, t) = s$. For a realization such that $0 < s_1 - s_2 < c$, it is an equilibrium for bidder 2 to invest and bidder 1 not to, even though the opposite is efficient.

⁸In the (non-generic) event that $r^*(\mathbf{s})$ is not unique, specify that $r(\mathbf{s}) = \max\{r^*(\mathbf{s})\}$. We will maintain this assumption, which has no bearing on efficiency, for the rest of the paper. Hence, future claims of "uniqueness" are technically subject to this caveat, but it is omitted for parsimony.

⁹Under the (non-generic) event described in footnote 4, the bidders coordinate their investment decisions by having the bidders with lower indices, i , invest.

¹⁰Because of footnote 8, in general, $s_i^k(\mathbf{s}_{-i}) \in Q_i^k(\mathbf{s}_{-i})$. The only exception is under the (non-generic)

bidders investing is a decreasing step function of s_i . Let \mathbf{a}_{-i}^k be the profile of investment decisions by bidders other than i such that the bidders with the k highest s -values invest and other do not. Let $\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i})$ be bidder i 's expected payoff (gross of any Stage 1 payments) given first-period signals \mathbf{s} , investment decisions \mathbf{a}_{-i} , and that i responds optimally given this information. Finally, Ξ denotes the candidate symmetric equilibrium:

Stage 1: Truthful revelation, $\hat{s}_i = s_i$

Stage 2: Following the efficient cutoff rule, $r^*(\cdot)$

Stage 3: Bidding one's value, $V(s_i, t_i) + a_i b$

We first establish an increasing differences property that will play a crucial role in designing the first-stage payment scheme.

Lemma 1 *For any \mathbf{s}_{-i} , $s'_i \geq s_i$, and $k' \leq k$,*

$$\pi_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k'}) - \pi_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) \geq \pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k'}) - \pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) \quad (3)$$

In words, the lemma establishes that the higher is the bidder's first-period signal, the more she benefits from less investment by competitors. With Lemma 1 in hand, we are ready to finish the construction of the mechanism. Theorem 1 specifies a payment rule where each bidder's payment depends on the entire vector of announcements $\hat{\mathbf{s}}$. For a given $\hat{\mathbf{s}}_{-i}$, we first define the payment required of i if she announces $\hat{s}_i \in Q_i^{k^*}(\hat{\mathbf{s}}_{-i})$. We then specify how much she is required to pay for announcements in $Q_i^{k^*-1}(\hat{\mathbf{s}}_{-i})$, in $Q_i^{k^*-2}(\hat{\mathbf{s}}_{-i})$, etc.

Theorem 1 *Under the following function, h , Ξ is both a Bayes Nash Equilibrium in the game induced by our mechanism and fully efficient. Further, truthful revelation in Stage 1 is ex post incentive compatible.*

$$\begin{aligned} \text{If } \hat{s}_i \in Q_i^{k^*}(\hat{\mathbf{s}}_{-i}): & \quad \text{then } h(\hat{s}_i, \hat{\mathbf{s}}_{-i}) = 0 \\ \text{If } \hat{s}_i \in Q_i^k(\hat{\mathbf{s}}_{-i}) \text{ for } k < k^*: & \quad \text{then } h(\hat{s}_i, \hat{\mathbf{s}}_{-i}) = h(s^0, \hat{\mathbf{s}}_{-i}) + \pi_i(s_i^k(\hat{\mathbf{s}}_{-i}), \hat{\mathbf{s}}_{-i}, \mathbf{a}_{-i}^k) - \pi_i(s_i^k(\hat{\mathbf{s}}_{-i}), \hat{\mathbf{s}}_{-i}, \mathbf{a}_{-i}^{k+1}) \\ & \quad \text{where } s^0 \in Q_i^{k+1}(\mathbf{s}_{-i}) \end{aligned}$$

Before turning to its particulars, let's start with a few simple observations that will help us understand the payment rule h . First, the higher other bidders believe s_i to be, the less of them invest. Second, regardless of s_i , bidder i is better off if less of the other bidders invest, but this benefit is increasing in s_i (Lemma 1). Putting these two together: bidder i

event described in footnote 4.

benefits from having her competitors believe that s_i is higher, and this benefit is increasing in the true value of s_i .

Under the h payment rule, an announcement of \underline{s} leads to a payment of zero regardless of the announcement of other bidders.¹¹ Now, suppose all bidders other than i announce truthfully. We need to consider how many other bidders will invest as a function of i 's announcement \hat{s}_i , given that they believe *all* announcements are truthful. If i announces $\hat{s}_i \in Q_i^{k^*}$, then she is saying she has a very low value, which prompts the most investment by others (k^* of them) and i 's payment is zero. If she announces $\hat{s}_i \in Q_i^{k^*-1}$, then her payment will increase, but she will have induced one less of her competitors to invest (only $k^* - 1$ will do so). In essence, she is “buying” less investment from her competitors. This is valuable regardless of s_i , but more valuable for higher realizations of s_i (Lemma 1).

In order for this behavior by her competitors to constitute best responses, it must be that i makes such an announcement if and only if $s_i \in Q_i^{k^*-1}$. Therefore, the payment required to garner this benefit must be exactly the one such that type $s_i^{k^*-1}$ is indifferent between doing so or not.¹² Lemma 1 then establishes that all types greater than $s_i^{k^*-1}$ find the benefit worth the cost, while types less than $s_i^{k^*-1}$ do not. The logic iterates as i considers higher announcements, making truthful reporting optimal for all values of s_i .

By taking the expectation over the (truthful) announcements of other bidders, we can construct a payment function that depends only on i 's own announcement, but is no longer *ex post* incentive compatible.

Theorem 2 *Under the following function, H, Ξ is both a Bayes Nash Equilibrium in the game induced by our mechanism and fully efficient.*

$$H(\hat{s}_i, \hat{\mathbf{s}}_{-i}) = H(\hat{s}_i) = E_{\mathbf{s}_{-i}}[h(\hat{s}_i, \mathbf{s}_{-i})]$$

where h is the function defined in Theorem 1.

The incentive compatibility of H is immediate: given that, for any \mathbf{s}_{-i} , reporting truthfully, $\hat{s}_i = s_i$, maximizes

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i})) - h(\hat{s}_i, \mathbf{s}_{-i})$$

when $\mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i})$ is the equilibrium response to the belief that $\mathbf{s} = (\hat{s}_i, \mathbf{s}_{-i})$ held by players

¹¹Hence the mechanism satisfies the participation constraint that all bidders receive non-negative expected utility. The same is true for the mechanism under the payment function H of Theorem 2.

¹²Any higher payment would dissuade types just above $s_i^{k^*-1}$ from the announcement, and any lower payment would induce the announcement from types just below $s_i^{k^*-1}$.

other than i , it also maximizes

$$E_{\mathbf{s}_{-i}}[\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i})) - h(\hat{s}_i, \mathbf{s}_{-i})] = E_{\mathbf{s}_{-i}}[\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i}))] - H(\hat{s}_i)$$

The last expression is precisely i 's expected utility in the mechanism, implying the mechanism is incentive compatible.

Equilibrium Selection

We have demonstrated that Ξ is an equilibrium in the game induced by our mechanism. However, other equilibria also exist. For instance, it is well known that the second-price auction has many equilibria involving weakly-dominated strategies. Additionally, taking the prescribed continuation play in Stage 3 and truthful revelation in Stage 1 as given, there are other behaviors in the investment stage consistent with equilibrium (see footnote 7). We take solace in that Ξ prescribes bidders to play their weakly-dominant strategies in Stage 3 and to invest according to the generically unique symmetric equilibrium in monotone strategies in the relevant subgame in Stage 2.

The main difficulty, however, comes from the interaction between Stages 1 and 2. Take as given that bidders will bid their final values in Stage 3. In the first stage, bidders attempt to influence the beliefs, and therefore investment decisions, of their opponents. This is essentially a signaling game (especially under the payment rule H in Theorem 2), although one that is much more complex than the “standard” signaling game. It is well known that signaling games suffer from a multitude of equilibria. Unfortunately, the relative complexity of our signaling game exacerbates the problem, as we discuss in the next section.¹³ However, we will offer an arbitrarily small alteration that ensures that the unique symmetric equilibrium, subject to an investment-monotonicity condition, is arbitrarily close to Ξ (see Theorem 3).

4 Further Study of the Signaling Game

In the previous section we designed an efficient mechanism to allocate an object in the environment outlined in Section 2. In addition to being efficient, the mechanism was highly decentralized and consisted of a standard second-price auction in the final stage. Now, a standard second-price auction may be held in environments outlined in Section 2, independent of our analysis on efficient mechanism design. For this reason, we turn to the study of

¹³This problem disappears if the environment allows the investment decision to be contractible.

the following game—be it the result of an efficiency-seeking designer or not.

Formally, the determination of values is just as in Section 2. In addition:

Stage 1: After receiving her signal s_i , each bidder i makes an announcement \hat{s}_i and can publicly dissipate any amount $x_i \in \mathbb{R}_+$.

Stage 2: After observing $\hat{\mathbf{s}}$ and \mathbf{x} , each bidder decides whether to invest or not.

Stage 3: After bidders observe their t -signals, a sealed-bid, second-price auction is held for the object.

Because the final round is a second-price auction, it makes no difference whether bidders observe the investment behavior of their opponents or not (if we restrict them to play their weakly-dominant strategy in the final subgame). Because we are taking both the investment opportunity and auction structure as fixed components of the game environment, we can think of auctions where entry is costly, be the cost paid to the auctioneer or not, as a particular case of what the investment opportunity is. [In the section on mechanism design, the investment opportunity was taken as a feature unaffected by the designer’s decisions and c did not accrue to the seller—entry costs are not a good candidate for such a feature].

Remark 1 (Costly Entry) *An auction with an entry decision, where bidders get private information both before and after their decision, is a particular case of our setup if we relax $V \geq 0$: Let $V(s_i, t_i) < 0 < V(s_i, t_i) + b$ for all (s_i, t_i) . The cost of investment, c , is the entry cost. If agent i decides not to “invest,” her value for the object is negative. It is equivalent to consider her as not participating in the auction.*

Equilibrium without Signaling

We now briefly touch on the case in which bidders do not signal their s -values to others via dissipation, which serves as a benchmark for comparison with the main signaling equilibrium of interest. The no-signaling case can be thought of in two ways: 1) signaling is not possible (i.e., the game differs by removing the option of announcement and dissipation), or 2) as a pooling equilibrium of the game as described: one in which every bidder chooses $x_i = 0$ regardless of s_i .¹⁴

Proposition 2 *There exists a unique symmetric equilibrium: In Stage 3, each bidder bids her value for the object: $V(s_i, t_i) + a_i b$. In Stage 2, bidder i invests according to a cutoff rule: $a_i = 1$ if and only if $s_i \geq s^*$, for uniquely-determined s^* .*

¹⁴The pooling equilibrium can be enforced by the off-path belief that any other choice of x_i leads all other bidders to believe that $s_i = \underline{s}$.

Two observations about efficiency are worth noting. First, the investment decisions generated by this equilibrium are, in general, inefficient.¹⁵ This is not surprising since, without signaling, the bidders lack the requisite public information about \mathbf{s} to coordinate their investment decisions.

Second, there often exist *asymmetric* equilibria that are more efficient *ex ante* than the unique symmetric equilibrium.¹⁶ The intuition for this is as follows. The symmetric equilibrium can suffer from over-investment when the realization of \mathbf{s} has many high components and under-investment when it has many low components. However, because of its cutoff structure, the symmetric equilibrium never suffers from what could be called “misallocation of investment”—that bidder i is investing when she should not be *and simultaneously* bidder j is not investing when she should be. Asymmetric equilibria can be constructed that trade off some of the former for the latter. Depending on the relative importance of each kind of inefficiency, the tradeoff may improve efficiency *ex ante*.¹⁷

Allowing the bidders to signal their s -values through dissipation will alleviate both of these. However, whether or not the signaling itself can be a source of inefficiency depends on whether or not it is beneficial to the seller (or any agent whose utility contributes to the social welfare measure)—see the final part of this section and Example 1.

A Signaling Equilibrium

The following is an equilibrium of this game:

Stage 1: Each bidder announces $\hat{s}_i = s_i$ and dissipates $x_i = H(s_i)$, where H is the function defined in Theorem 2.¹⁸

¹⁵Obviously, there are exceptions. For example, if b is very large, it can be efficient for all bidders to invest regardless of \mathbf{s} . In such examples $s^* = \underline{s}$, and the no-signaling equilibrium is therefore efficient.

¹⁶This implies that there must exist a positive measure of realizations of \mathbf{s} such that the asymmetric equilibrium is more efficient.

¹⁷A straightforward illustration of this is as follows. Let $N = 2$, s_i be distributed uniformly on $[0, 1]$, and $V(s, t) = s$, as in Example 1 (below) where we show that the total *ex ante* surplus in the symmetric equilibrium is $\frac{2}{3} + b(1 - \frac{c}{b})^2$. Suppose now that both c and $b - c$ are greater than $\frac{1}{2}$. The following is an asymmetric (non-signaling) equilibrium: $a_1 = 1$ and $a_2 = 0$ regardless of (s_1, s_2) . Hence, bidder 1 always wins and pays a price of s_2 . (It is sufficient to check that bidder 1 prefers to invest when $s_1 = 0$, and that bidder 2 prefers not to invest when $s_2 = 1$, which are guaranteed by the assumptions on c and b .) In this equilibrium the *ex ante* expected utilities are $b - c$, 0, and $\frac{1}{2}$ for bidders 1 and 2 and the seller respectively. So total surplus is $\frac{1}{2} + b - c$. Notice that this differs from the total possible surplus of $\frac{2}{3} + b - c$ by a constant (i.e., does not depend on b or c). This is precisely because there is always the efficient *level* of investment, just done by the wrong bidder half the time. In contrast, the level of inefficiency in the symmetric equilibrium depends on b, c because there is sometimes over- or under-investment. Heuristically, the asymmetric equilibrium is more efficient when the scale of the investment is large—for example, fixing any value of $\frac{c}{b}$ the asymmetric equilibrium is more efficient if b, c are sufficiently large.

¹⁸ H is not always an invertible function (it may be only weakly increasing). This is why we allow bidders to both make an announcement and to dissipate resources. However, if H is constant over a range, it is

Stage 2: Each bidder invests according the efficient cutoff rule $r^*(\cdot)$.

Stage 3: Each bidder bids $V(s_i, t_i) + a_i b$.

Given our analysis in Section 3, there is nothing new to verify. We call this a *signaling equilibrium* because each bidder credibly signals her type s_i by her costly choice of action $x_i = H(s_i)$.

The presence of the signaling equilibrium sheds some light on the maneuvering that bidders often make prior to an auction. For example, consider the sale of the Los Angeles license in the 1995 auction for broadband mobile-phone licenses. One bidder, Pacific Telephone, likely started with a higher private value than other bidders due to experience in California market and possible synergies between its wireline and wireless businesses. There were a number of important decisions (investments) that each bidder had to make before the auction for the Los Angeles license. These included forming alliances, making capital investments, and formulating strategies for other markets. It appears that Pacific Telephone signaled to other bidders (and would-be bidders) that it anticipated winning the Los Angeles license. Pacific Telephone made public statements like, “If somebody takes California away from us, they’ll never make any money.”¹⁹ To make these statement credible, Pacific Telephone made investments that were of little value without winning the license.²⁰ As a result, some potential bidders (including the industry giants such as Bell Atlantic, GTE, and MCI) were discouraged from participating in the auction—thus, failing to undertake a complimentary investment, in our interpretation. In fact, GTE and Bell Atlantic took actions that made them ineligible for the auction. As a result, revenues were quite low compared to initial estimates.²¹

Equilibrium Selection

As in virtually all signaling games, there are many equilibria in this game. Recall that in a “standard” signaling game the set of players is clearly partitioned into the set of senders (usually singleton) and the set of receivers. Each sender decides on her costly action then

because the announcement of any type over that range does not affect competitor behavior. Hence, we could dispense with type announcement, but it would complicate our notation for analysis of the investment stage.

¹⁹*Wall Street Journal*, October 31, 1994.

²⁰Some of the investments made by Pacific Telephone might be interpreted as actions and others as signals. Essentially, running a PR campaign aimed at signaling that Pacific Telephone is determined to win Los Angeles license can be interpreted as signaling. In contrast, making unobservable arrangements made to expedite creation of the wireless service in Southern California can be interpreted as an action.

²¹This is not the only possible interpretation of the example. Klemperer (2002) considers the history of this auction and suggests that the winner’s curse played an important role because the winner’s curse is particularly powerful in auctions where one bidder has an advantage. For a theoretical argument that uses this logic see Bulow, Huang and Klemperer (1999).

each receiver chooses his best response, which is usually assumed to be unique for any given belief held by the receiver. To obtain a sharp prediction in these models one must employ a strong equilibrium refinement. These refinements are usually “belief-based” in that they use specific criteria to restrict the beliefs the receivers may hold off the equilibrium path. Broadly put, by belief-based refinements we mean those that employ the following reasoning: type s had less incentive to undertake an observed deviation than other types did—therefore receivers lower their belief that the sender is type s based on the observed deviation. The refinements differ in their notions of “less incentive,” on which set of “other types” to use, and on how much to “lower their [the receivers’] belief.”²² By exploiting the differences between types, these refinements are primarily used to eliminate equilibria involving pooling. Because Ξ consists of full revelation, it survives any of these refinements.

Proposition 3 Ξ satisfies any belief-based refinement.

However, our setup is considerably more complicated in that we have multiple players, each playing the role of sender and receiver to one another. That is, each player takes an action at every stage of the game. In the first stage, she acts as a sender trying to influence the beliefs of her competitors. In the investment stage, she acts as a receiver trying to optimize given her updated beliefs about her competition. Further, it is no longer true that bidders have unique best responses given their beliefs in the investment stage (i.e., when they are in the receiver role). For these reasons, belief-based refinements do not have the same power in our setting. The Supplemental Material presents examples where belief-based refinements cannot eliminate the equilibrium where bidders pool at $x_i = 0$ and no information is revealed.

The ε -Coupon Mechanism

The examples in the Supplemental Material demonstrate that belief-based refinements are less effective in our context because there are situations in which a bidder gains nothing by costly signaling even if it endows her competitors with the most favorable possible beliefs about her type. The following minor change to the game/mechanism (in the spirit of *virtual implementation* (Abreu and Sen, 1991)) offers a very simple fix to this problem. Let Stages 2 and 3 remain the same.

Stage 1: After receiving their s -signals, bidders compete in an all-pay auction (with ties broken by lottery) for a coupon that entitles the winner to ε off the price in Stage 3 should she win the object. The vector of

²²Examples include the Intuitive Criterion, Divinity, D1, D2, and Never-a-Weak-Best-Response (see Banks and Sobel (1987), Cho and Kreps (1987), and Riley (2001) for definitions and implications in signaling games).

bids for the ε -coupon, denoted by \mathbf{x} , is publicly revealed before the investment stage.

The ε -coupon solves the problem encountered by belief-based refinements because, as long as it is possible for bidder i to win given s_i , the coupon has positive intrinsic value (in addition to any value that she receives from influencing her competitors' investment behavior). Further, because the coupon is only valuable conditional on winning, its worth is increasing in s_i . These two properties lead to full revelation in the first stage. Define x -monotonicity analogously to monotonicity in Section 3 with x_i, \mathbf{x}_{-i} substituted for s_i, \mathbf{s}_{-i} .

Theorem 3 *For $\varepsilon > 0$ small enough, there exists a unique symmetric equilibrium with x -monotone investment strategies. The equilibrium is fully separating and, as $\varepsilon \rightarrow 0$, the total equilibrium surplus converges to the efficient level.*

To provide an intuition for the result, let us argue that the profile under which no bidder makes a positive public bid for the coupon ($x_i = 0$ for all i) is not an x -monotone equilibrium when $\varepsilon > 0$. If $x_j = 0$ for all $j \neq i$, bidder i can deviate and win the coupon for an arbitrary small amount. The restriction to x -monotone investment strategies implies that this deviation will prompt her opponents to invest no more often. It is therefore strictly beneficial to undertake the deviation, breaking the equilibrium.

It is important to differentiate the implications of monotonicity, x -monotonicity, and typical belief-based refinements. Belief-based refinements restrict beliefs, while the monotonicity restriction from Section 3 is used to restrict strategies for *fixed* beliefs. This issue does not usually come up in signaling game because it is standard to assume that receivers have unique best responses given any belief that they hold. In Section 3, we saw this is not true in our setting. Finally, in this section we need to worry about how increasing x_i affects *both* the beliefs of i 's opponents *and* their investment response given those beliefs (of which there is not, in general, a unique best response). Hence, x -monotonicity, as opposed to a standard belief-based refinement, is needed to rule out that a deviation raises the beliefs a bidder's opponents hold about her type (in a way consistent with the belief-based refinement), but also (paradoxically) makes investment by her opponents more likely.²³

Costly Signaling and Efficiency Loss

In signaling games, it is common that senders' ability to credibly signal their types actually results in decreased societal welfare (Riley, 2001). For us, the essential question is: who

²³That is, x -monotonicity combines the spirit of both monotonicity and a belief-based refinement (most near to Divinity (Banks and Sobel, 1987)) in one notion.

benefits from the costly signaling? In the design of our mechanism in Section 3, we specified that first-stage payments were transferred to the seller and, hence, were no loss of efficiency. However, if we are taking the auction design as fixed, it may not include this feature. If first-stage signaling costs are purely dissipative, then, of course, they represent a drain on efficiency.²⁴ The following example illustrates that this drain can outweigh the benefit from coordinating investment behavior efficiently: total surplus would be enhanced if signaling was simply not permitted.

Example 1 *Let $N = 2$, s_i be distributed uniformly on $[0, 1]$ and $V(s, t) = s$. We will compare total surplus under three regimes: 1) no signaling, 2) dissipative signaling, and 3) signaling that benefits the seller. The ex ante total surplus is highest under (3), followed by (1), and lowest under (2).*

With No Signaling. *Without signaling, each bidder needs to decide whether to invest or not based only on s_i . It is easy to derive that the symmetric equilibrium is invest if and only if $s_i \geq s^* = \frac{c}{b}$ and that a bidder's expected surplus given s_i is $\frac{s_i^2}{2}$. Let $s^{(2)} \equiv \min\{s_1, s_2\}$. The seller's surplus is $s^{(2)}$ if either bidder does not invest, and $s^{(2)} + b$ if both do. Hence, his ex ante expected surplus is $E[s^{(2)}] + b(1 - \frac{c}{b})^2 = \frac{1}{3} + b(1 - \frac{c}{b})^2$.*

With Fully-Revealing Signaling. *After types are revealed by the signaling stage, the bidder with the higher s -value invests and the other does not. It can be shown that $H(s_i) = s_i(b - c)$ and, again, that a bidder's expected surplus given s_i is $\frac{s_i^2}{2}$. We can therefore compare total surpluses by looking only at the seller's surplus. If the bidders' signaling costs do not accrue to him, then his surplus is simply $s^{(2)}$, giving an expected surplus of $\frac{1}{3}$. If he does accrue the benefit from their signaling costs, this surplus is $s^{(2)} + (b - c)(s_1 + s_2)$, giving an expected surplus of $\frac{1}{3} + (b - c)(\frac{1}{2} + \frac{1}{2}) = \frac{1}{3} + (b - c)$.*

Surplus Ranking. *Notice that $b(1 - \frac{c}{b})^2 = (b - c)(1 - \frac{c}{b}) < b - c$, which establishes our claim. Comparing (1) and (2), the bidders destroy all of their benefit from efficient investment through their dissipative signaling. However, the seller gains from over-investment, but does not lose when there is under-investment. Hence, he prefers there to be no signaling if he cannot capture it as a transfer. We already knew that (3) would be fully efficient, explaining why it does better than (1) and (2).*

²⁴We thank an anonymous referee for encouraging us to expound on this issue.

5 Conclusion

We have studied an environment where an object is allocated after values have changed due to both exogenous factors and agents' decisions. We first provided a mechanism with several useful features: it is efficient, decentralized, and has minimal commitment requirements. We find the game induced by our mechanism to be so natural that we pursue a more in-depth study of its features, especially the issue of equilibrium selection.

One issue with our mechanism, however, is the lack of examples of real-world implementation. It is not common to have an initial round of heterogeneous payments to the seller (with bidders receiving nothing in return). This observation deserves response. First, while such intricate first-stage payments may not occur, auctioneers can and do charge entry fees. This can be seen as an implementation of a coarser (indeed, binary) first-stage payment/signaling function. This is different from considering entry fees as the investment if it is changed publicly and prior to the opportunity for further investment, though it illustrates that the two features are likely to be intimately related in the real world. One could extend our analysis by constraining the first-stage payment function and obtain an optimal entry fee. Second, the mechanism design perspective is not always a descriptive one. Many well-known solutions to mechanism design problems, VCG among them, are wanting as descriptive models of real-world mechanisms or contracts (Ausubel and Milgrom, 2005), but are nevertheless important to our economic understanding. Finally, we have drawn attention to a force—the necessity of bidders to signal their initial types if it is expected of them—that is present in dynamic settings and *could* be used in the design of mechanisms, even if it presently is not.

References

- Abreu, Dilip and Arunava Sen. 1991. "Virtual Implementation in Nash Equilibrium." *Econometrica* 59:997–1021.
- Athey, Susan and Ilya Segal. 2007. "An Efficient Dynamic Mechanism." Discussion Paper, Harvard University and Stanford University.
- Ausubel, Lawrence and Paul Milgrom. 2005. The Lovely but Lonely Vickery Auction. In *Combinatorial Auctions*. MIT Press.
- Banks, Jeffrey and Joel Sobel. 1987. "Equilibrium Selection in Signaling Games." *Econometrica* 55:647–662.
- Battaglini, Marco. 2005. "Long-Term Contracting With Markovian Consumers." *American Economic Review* 95:637–658.
- Bergemann, Dirk and Juuso Välimäki. 2002. "Information Acquisition and Efficient Mechanism Design." *Econometrica* 70:1007–1033.
- Bergemann, Dirk and Juuso Välimäki. 2010. "The Dynamic Pivot Mechanism." *Econometrica* 78:771–789.
- Bulow, Jeremy, Ming Huang and Paul Klemperer. 1999. "Toeholds and Takeovers." *Journal of Political Economy* 107:427–454.
- Calzolari, Giacomo and Alessandro Pavan. 2006. "Monopoly with Resale." *Rand Journal of Economics* 37:362–375.
- Cavallo, Ruggiero, David C. Parkes and Satinder Singh. 2006. Optimal Coordinated Planning Amongst Self-Interested Agents with Private State. In *Proceedings of the 22nd Conference on Uncertainty in Artificial Intelligence*. Cambridge.
- Cho, In Koo and David Kreps. 1987. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics* 102:179–221.
- Crémer, Jacques, Yossi Spiegel and Charles Zheng. 2009. "Auctions with Costly Information Acquisition." *Economic Theory* 38:41–72.
- Esö, Péter and Balázs Szentes. 2003. "The One Who Controls the Information Appropriates its Rents." Working Paper, <http://home.uchicago.edu/~szentes/adverse.pdf>.
- Fullerton, Richard L. and R. Preston McAfee. 1999. "Auctioning Entry into Tournaments." *Journal of Political Economy* 107:573–605.
- Haile, Philip. 2000. "Partial Pooling at the Reserve Price in Auctions with Resale Opportunities." *Games and Economic Behavior* 33:231–248.
- Haile, Philip. 2001. "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales." *American Economic Review* 92:399–427.

- Haile, Philip. 2003. "Auctions with Private Uncertainty and Resale Opportunities." *Journal of Economic Theory* 108:72–110.
- Klemperer, Paul. 2002. "What Really Matters in Auction Design." *Journal of Economic Perspectives* 16:169–189.
- Krishna, Vijay. 2002. *Auction Theory*. Academic Press, New York.
- Levin, Dan and James L. Smith. 1994. "Equilibrium in Auctions with Entry." *American Economic Review* 84:585–599.
- McAfee, R. Preston, Daniel Quan and Daniel Vincent. 2002. "How to Set Minimum Acceptable Bids, with Application to Real Estate Auctions." *Journal of Industrial Economics* L:391–416.
- McAfee, R. Preston and John McMillan. 1987. "Auctions with Entry." *Economic Letters* 23:343–347.
- McAfee, R. Preston, Wendy Takacs and Daniel Vincent. 1999. "Tarrifying Auctions." *Rand Journal of Economics* 30:158–179.
- Milgrom, Paul. 1981. "Rational Expectations, Information Acquisition, and Competitive Bidding." *Econometrica* 49:921–943.
- Milgrom, Paul. 2004. *Putting Auction Theory to Work*. Cambridge University Press.
- Pavan, Alessandro, Ilya Segal and Juuso Toikka. 2009. "Dynamic Mechanism Design." Working Paper, <http://faculty.wcas.northwestern.edu/~apa522/DMD--revised--final.pdf>.
- Riley, John. G. 2001. "Silver Signals: Twenty–Five Years of Screening and Signaling." *Journal of Economic Literature* 39:432–478.
- Riordan, Michael H. and David E. M. Sappington. 1987. "Awarding Monopoly Franchises." *American Economic Review* 77:375–387.
- Samuelson, William. 1985. "Competitive Bidding with Entry Costs." *Economic Letters* 17:53–57.
- Ye, Lixin. 2004. "Optimal Auctions with Endogenous Entry." *Contributions to Theoretical Economics* 4:1–27.
- Ye, Lixin. 2007. "Indicative Bidding and A Theory of Two–Stage Auctions." *Games and Economic Behavior* 58:181–207.

Appendix

We begin with a set of technical results that will be used in several instances. For any number (function, random variable) x , let $x^+ = \max\{x, 0\}$. A random variable X (first-order) *stochastically dominates* a random variable Y (denoted $X \succeq Y$) if and only if the cumulative density functions satisfy $F_X(z) \leq F_Y(z)$ for any $z \in \mathbf{R}$. An equivalent condition is that $E[h(X)] \geq E[h(Y)]$ for any increasing function h (e.g., Krishna (2002), Appendix B).

Lemma 2 *Let X and Y be random variables such that $X \succeq Y$.*

1. *For any random variable Z that is independent of X, Y , $\max\{X, Z\} \succeq \max\{Y, Z\}$.*
2. *For random variables W, Z , such that X, Z are independent, and Y, W are independent, if $W \succeq Z$, then $X - Z \succeq Y - W$.*
3. *For any constant $k \geq 0$, $E[(X + k)^+] - E[X^+] \geq E[(Y + k)^+] - E[Y^+]$.*
4. *For any random variable Z such that X, Y, Z are independent and any constant $k \geq 0$,*

$$E[\max\{X + k, Y, Z\}] \geq E[\max\{X, Y + k, Z\}]$$

Proof of Lemma 2.

(1)–(3) are straightforward calculations.

(4) For any x, y , $\max\{x, y\} = (x - y)^+ + y$. We start with the following identities:

$$\begin{aligned} \max\{X + k, Y, Z\} &= (X + k - \max\{Y, Z\})^+ + \max\{Y, Z\} \\ \max\{X, Y, Z\} &= (X - \max\{Y, Z\})^+ + \max\{Y, Z\} \end{aligned}$$

Then,

$$\begin{aligned} \max\{X + k, Y, Z\} - \max\{X, Y, Z\} &= (X + k - \max\{Y, Z\})^+ - (X - \max\{Y, Z\})^+ \\ \max\{X, Y + k, Z\} - \max\{X, Y, Z\} &= (Y + k - \max\{X, Z\})^+ - (Y - \max\{X, Z\})^+ \end{aligned}$$

From (1), we know that $\max\{X, Z\} \succeq \max\{Y, Z\}$. Then (2) implies that

$$X - \max\{Y, Z\} \succeq Y - \max\{X, Z\}$$

Using (3), and the linearity of the expectation operator, completes the proof. ■

Before proving the results from the body, we introduce the following definitions:

Definition 1 *For given vectors \mathbf{s} and \mathbf{a}_{-i} , let $G_i(\mathbf{s}, \mathbf{a}_{-i})$ be the change in total (expected) surplus when $a_i = 1$ compared to $a_i = 0$. Let $g_i(\mathbf{s}, \mathbf{a}_{-i})$ be the change in i 's expected utility when $a_i = 1$ compared to $a_i = 0$.*

Proof of Cutoff Property. Fix an arbitrary vector of signals \mathbf{s} . Consider two action vectors \mathbf{a} and \mathbf{a}' such that for two bidders j, k , $a_j = 1, a_k = 0, a'_j = 0, a'_k = 1$, and $a_i = a'_i$ for all $i \neq j, k$. We need to establish that if $s_j \geq s_k$, total expected surplus is weakly greater under \mathbf{a} than under \mathbf{a}' .

Notice that $\sum_i a_i = \sum_i a'_i$, therefore it is sufficient to demonstrate that

$$E[V(s_{i^*}, t_{i^*}) + a_{i^*} b | \mathbf{a}] \geq E[V(s_{i^*}, t_{i^*}) + a'_{i^*} b | \mathbf{a}']$$

Let $Z = \max_{i \neq j, k} V(s_i, t_i) + a_i b$. Under \mathbf{a} and \mathbf{a}' respectively

$$\begin{aligned} E[V(s_{i^*}, t_{i^*}) + a_{i^*} b | \mathbf{a}] &= E[\max\{V(s_j, t_j) + b, V(s_k, t_k), Z\}] \\ E[V(s_{i^*}, t_{i^*}) + a'_{i^*} b | \mathbf{a}'] &= E[\max\{V(s_j, t_j), V(s_k, t_k) + b, Z\}] \end{aligned}$$

Applying Lemma 2(4) establishes the claim.

To see that the cutoff will be unique generically, order the bidders such that $s_1 > s_2 > \dots > s_N$ (ties are non-generic) and let $X_j = V(s_j, t_j)$ and $Z_j = \max_{i \neq j} V(s_i, t_i) + a_i b$ given \mathbf{a}_{-j}^{j-1} (which is $a_i = 1$ if and only if $s_i > s_j$). Under this profile, the marginal contribution to surplus if $a_j = 1$ instead of $a_j = 0$ is

$$G_j(\mathbf{s}, \mathbf{a}_{-j}^{j-1}) = E[\max\{X_j + b, Z_j\}] - E[\max\{X_j, Z_j\}] - c \quad (4)$$

$$= E[(X_j + b - Z_j)^+] + E[Z_j] - (E[(X_j - Z_j)^+] + E[Z_j]) - c \quad (5)$$

$$= E[(X_j + b - Z_j)^+] - E[(X_j - Z_j)^+] - c \quad (6)$$

where the equivalence of (4) and (5) follows from the formula $\max\{x, y\} = (x - y)^+ + y$. Notice that if $s_j > s_k$, then $X_j \succeq X_k$ and $Z_k \succeq Z_j$. From Lemma 2(2), $X_j - Z_j \succeq X_k - Z_k$. Lemma 2(3) then implies that the marginal contribution to surplus from investing is lower for bidders with lower signals. The efficient cutoff is therefore determined by $\min\{s_j : G_j(\mathbf{s}, \mathbf{a}_{-j}^{j-1}) > 0\}$. (If there exists j such that $G_j(\mathbf{s}, \mathbf{a}_{-j}^{j-1}) = 0$, total expected surplus will be the same under $a_j = 0$ or $a_j = 1$. This is a non-generic event.) \blacksquare

Proof of Proposition 1. First, symmetry and monotonicity together imply a cutoff rule. To see this consider a generic vector of signals \mathbf{s} , and suppose that $a_j = 0$ and $a_k = 1$ despite $s_j > s_k$. Now consider \mathbf{s}' where $s'_j = s_k, s'_k = s_j$, and $s'_i = s_i$ for all $i \neq j, k$. By monotonicity, $a'_j \leq a_j$ and $a'_k \geq a_k$. Hence, $a'_j = 0$ and $a'_k = 1$. However, then \mathbf{a}' violates symmetry with \mathbf{a} since \mathbf{s}' is a permutation of \mathbf{s} . Hence, a cutoff rule is required.

To show that only efficient cutoff rules can be used in equilibrium we demonstrate that for each bidder i , fixing any \mathbf{a}_{-i} , the change to her own expected utility from investing is equal to the change in expected total surplus from investing. For an arbitrary \mathbf{a}_{-i} , let $Z_i = \max_{j \neq i} \{V(s_j, t_j) + ba_j\}$, and $X_i = V(s_i, t_i)$. By definition,

$$g_i(\mathbf{s}, \mathbf{a}_{-i}) = E[(X_i + b - Z_i)^+] - E[(X_i - Z_i)^+] - c$$

Using the formula $\max\{x, y\} = (x - y)^+ + y$, we get

$$\begin{aligned} G_i(\mathbf{s}, \mathbf{a}_{-i}) &= E[\max\{X_i + b, Z_i\}] - E[\max\{X_i, Z_i\}] - c \\ &= E[(X_i + b - Z_i)^+] + E[Z_i] - (E[(X_i - Z_i)^+] + E[Z_i]) - c \\ &= E[(X_i + b - Z_i)^+] - E[(X_i - Z_i)^+] - c = g_i(\mathbf{s}, \mathbf{a}_{-i}) \end{aligned}$$

Each bidder i increases utility by setting $a_i = 1$ if and only if doing so simultaneously increases total surplus. Therefore, using the efficient cutoff rule is an equilibrium. To see that there are no other equilibrium cutoff rules, order the bidders by decreasing s -values: $s_1 \geq s_2 \geq \dots \geq s_N$. If the bidders employed an inefficient cutoff rule with $j = \max\{i : a_i = 1\}$ and $k = \min\{i : a_i = 0\}$, then either $G_j(\mathbf{s}, \mathbf{a}_{-j}^{j-1}) < 0$ (and j would increase utility by switching to $a_j = 0$), or $G_k(\mathbf{s}, \mathbf{a}_{-k}^{k-1}) > 0$ (and k would increase utility by switching to $a_k = 1$), violating equilibrium. ■

Proof of Lemma 1. We first show that for any \mathbf{s}_{-i} , $\mathbf{a}_{-i} \geq \mathbf{a}'_{-i}$, and $s'_i \geq s_i$, if a_i and a'_i are i 's best responses to $\{s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}\}$ and $\{s'_i, \mathbf{s}_{-i}, \mathbf{a}'_{-i}\}$ respectively, then $a'_i \geq a_i$. To see this, let $Z_i = \max_{j \neq i} \{V(s_j, t_j) + ba_j\}$, $Z'_i = \max_{j \neq i} \{V(s_j, t_j) + ba'_j\}$, $X_i = V(s_i, t_i)$, and $X'_i = V(s'_i, t_i)$. Recall that

$$g_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}'_{-i}) = E[(X'_i + b - Z'_i)^+] - E[(X'_i - Z'_i)^+] - c \quad (7)$$

$$g_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}) = E[(X_i + b - Z_i)^+] - E[(X_i - Z_i)^+] - c \quad (8)$$

Given that $X'_i \succeq X_i$ and $Z_i \succeq Z'_i$, $(X'_i - Z'_i) \succeq (X_i - Z_i)$ by Lemma 2(2), and then $g_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}'_{-i}) \geq g_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i})$ by Lemma 2(3), establishing that $a'_i \geq a_i$.

Now, notice that

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}) = E[(X_i + a_i b - Z_i)^+] - a_i c$$

To establish the lemma, let us begin by precluding investment by player i . Then condition (3) is equivalent to

$$E[(X'_i - Z'_i)^+] - E[(X_i - Z'_i)^+] \geq E[(X'_i - Z_i)^+] - E[(X_i - Z_i)^+] \quad (9)$$

Define $Y_i = X'_i - X_i \geq 0$ (notice that X'_i and X_i are not independent, because both pertain to player i , but after two different realizations of the first signal, allowing us the freedom to create Y_i in this way). So $X'_i = X_i + Y_i$. Then condition (3) is now

$$E[(X_i + Y_i - Z'_i)^+] - E[(X_i - Z'_i)^+] \geq E[(X_i + Y_i - Z_i)^+] - E[(X_i - Z_i)^+]$$

Again we have $(X_i - Z'_i) \succeq (X_i - Z_i)$ by Lemma 2(2). Because Lemma 2(3) holds for every non-negative constant and $Y_i \geq 0$ always, the inequality holds.

The final step is verify that condition (3) holds when we allow for investment by player i . Given that a_i is increasing in s_i and decreasing in \mathbf{a}_{-i} , the results are straightforward to obtain. For instance, if $a_i = 1$ in all four scenarios, then the costs all cancel, and condition (3) becomes

$$E [(X_i + Y_i + b - Z'_i)^+] - E [(X_i + b - Z'_i)^+] \geq E [(X_i + Y_i + b - Z_i)^+] - E [(X_i + b - Z_i)^+]$$

The result still follows from Lemma 2(2,3) since $b \geq 0$. Or if $a_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}'_{-i}) = 1$ and $a_i = 0$ in the other three scenarios, then condition (3) becomes

$$E [(X_i + Y_i + b - Z'_i)^+] - c - E [(X_i - Z'_i)^+] \geq E [(X_i + Y_i - Z_i)^+] - E [(X_i - Z_i)^+]$$

However, if $a_i(s'_i, \mathbf{s}_{-i}, \mathbf{a}'_{-i}) = 1$, then, for player i to be best responding, it must be that

$$E [(X_i + Y_i + b - Z'_i)^+] - c \geq E [(X_i + Y_i - Z'_i)^+]$$

preserving the result. The other three cases are established analogously, completing the proof of the lemma. \blacksquare

Proof of Theorem 1. Fix an arbitrary bidder i and \mathbf{s}_{-i} , and suppose that 1) $\hat{\mathbf{s}}_{-i} = \mathbf{s}_{-i}$, and 2) in Stages 2 and 3 all bidders other than i play as outlined in Ξ under $\hat{\mathbf{s}} = \mathbf{s}$. It suffices to show that bidder i also wishes to report truthfully and follow the same continuation play (participation constraints are trivial given that any type gets weakly positive utility from reporting $\hat{s}_i = s_i$).

Now consider bidder i with first-period signal $s_i \in Q_i^k$ for arbitrary $k \leq k^*$ (since $\hat{\mathbf{s}}_{-i}$ is fixed, we suppress the dependence of Q_i^k, s_i^k on it). If i announces any $\hat{s}_i \in Q_i^k$, the response of other bidders, \mathbf{a}_{-i} , will be the same, so i is clearly indifferent between any such announcement and reporting truthfully. We only need to verify that i has no incentive to announce any $\hat{s}_i \in Q_i^{k'}, k' \neq k$.

Consider the deviation of i under-reporting with $\hat{s}_i \in Q_i^{k+1}$, (obviously if $s_i \in Q_i^{k^*}$ this is not feasible, so we needn't worry about this deviation). We now verify that this deviation results in a weakly lower payoff than reporting truthfully. Letting s^0 denote an arbitrary element of Q_i^{k+1} , reporting truthfully yields payoff

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) - h(s^0, \mathbf{s}_{-i}) - \pi_i(s_i^k, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) + \pi_i(s_i^k, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k+1}) \quad (10)$$

Under-reporting yields

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k+1}) - h(s^0, \mathbf{s}_{-i}) \quad (11)$$

Bidder i has no incentive to under-report if and only if (10) \geq (11). Notice that this is equivalent to

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) - \pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k+1}) \geq \pi_i(s_i^k, \mathbf{s}_{-i}, \mathbf{a}_{-i}^k) - \pi_i(s_i^k, \mathbf{s}_{-i}, \mathbf{a}_{-i}^{k+1})$$

which holds by Lemma 1, given that $s_i \geq s_i^k$ and $\mathbf{a}_{-i}^k \leq \mathbf{a}_{-i}^{k+1}$ (by definition). The same argument, with inequalities reversed, establishes that i does not wish to over-report with $\hat{s}_i \in Q_i^{k-1}$. Finally, that i does not wish to over- or under-report by a larger degree follows directly from transitivity of the increasing difference condition (3).

Hence, truthful reporting is incentive compatible, when continuation play is efficient. Therefore (using Proposition 1), together they constitute equilibrium play within the mechanism. ■

Proof of Theorem 2. Again, non-participation is weakly dominated by reporting $\hat{s}_i = \underline{s}$. From Theorem 1, for any \mathbf{s}_{-i} , reporting truthfully, $\hat{s}_i = s_i$, maximizes

$$\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i})) - h(\hat{s}_i, \mathbf{s}_{-i})$$

Therefore it also maximizes

$$\begin{aligned} E_{\mathbf{s}_{-i}}[\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i})) - h(\hat{s}_i, \mathbf{s}_{-i})] = \\ E_{\mathbf{s}_{-i}}[\pi_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i}(\hat{s}_i, \mathbf{s}_{-i}))] - H(\hat{s}_i, \mathbf{s}_{-i}) \end{aligned}$$

Hence, truthful reporting is incentive compatible when other bidders report truthfully and continuation play is efficient. Therefore (using Proposition 1), together they constitute equilibrium play within the mechanism. ■

Proof of Proposition 2. Let q_i denote an investment strategy for bidder i (i.e., it is a mapping from $[\underline{s}, \bar{s}]$ to $\{0, 1\}$ —distinguishing it from previous strategies that could depend on updated beliefs about other bidders' types). We first show that for *any* strategy profile of other bidders \mathbf{q}_{-i} , bidder i 's unique best response is a cutoff strategy $q_i^*(\mathbf{q}_{-i}) \equiv \{a_i = 1 \text{ if and only if } s_i \geq s_i^*(\mathbf{q}_{-i})\}$. The argument is similar to the one given in Lemma 1, except now we must take expectation over \mathbf{s}_{-i} , and correspondingly $\mathbf{a}_{-i} = \mathbf{q}_{-i}(\mathbf{s}_{-i})$, as well.

Let $\hat{g}_i(s_i, \mathbf{q}_{-i}) = E_{\mathbf{s}_{-i}}[g_i(s_i, \mathbf{s}_{-i}, \mathbf{a}_{-i})]$. Again, let $Z_i = \max_{j \neq i} \{V(s_j, t_j) + ba_j\}$, $X_i = V(s_i, t_i)$, and $X'_i = V(s'_i, t_i)$. Now, for $s'_i \geq s_i$,

$$\hat{g}_i(s'_i, \mathbf{q}_{-i}) = E[(X'_i + b - Z_i)^+] - E[(X'_i - Z_i)^+] - c \quad (12)$$

$$\hat{g}_i(s_i, \mathbf{q}_{-i}) = E[(X_i + b - Z_i)^+] - E[(X_i - Z_i)^+] - c \quad (13)$$

Given that $X'_i \succeq X_i$, then $(X'_i - Z_i) \succeq (X_i - Z_i)$ by Lemma 2(2). It follows that $\hat{g}_i(s'_i, \mathbf{q}_{-i}) \geq \hat{g}_i(s_i, \mathbf{q}_{-i})$ by Lemma 2(3), establishing that $a'_i \geq a_i$. So i 's best response is always a cutoff.

Finally, suppose that each bidder other than i employs the same cutoff s^0 , and let $s_i^*(s^0)$ be i 's optimal cutoff in response to this. The existence of a symmetric equilibrium, with cutoff s^0 , is therefore equivalent to $s_i^*(s^0) = s^0$. We now argue that exactly one such fixed point exists. It is sufficient to show that $s_i^*(\cdot)$ is a continuous, (weakly) decreasing function: continuity implies existence of the fixed point (*a la* Brouwer's fixed point theorem), and decreasing-ness implies

uniqueness. The continuity of the best response follows from the respective continuity of V , F , and G in both s and t . To see that $s_i^*(\cdot)$ is decreasing, use the same notation from (12) and (13), let $Z'_i = \max_{j \neq i} \{V(s_j, t_j) + ba'_j\}$, and let \mathbf{q}'_{-i} and \mathbf{q}_{-i} correspond to $s^{0'} \geq s^0$ respectively. Now, for any s_i ,

$$\begin{aligned}\hat{g}_i(s_i, \mathbf{q}'_{-i}) &= E[(X_i + b - Z'_i)^+] - E[(X_i - Z'_i)^+] - c \\ \hat{g}_i(s_i, \mathbf{q}_{-i}) &= E[(X_i + b - Z_i)^+] - E[(X_i - Z_i)^+] - c\end{aligned}$$

One last time, $Z_i \succeq Z'_i$ implies that $(X_i - Z'_i) \succeq (X_i - Z_i)$ by Lemma 2(2). It follows that $\hat{g}_i(s_i, \mathbf{q}'_{-i}) \geq \hat{g}_i(s_i, \mathbf{q}_{-i})$ by Lemma 2(3), establishing that $a'_i \geq a_i$. So i 's best-response cutoff function, $s_i^*(\cdot)$, is decreasing in her opponents' common cutoff s^0 . ■

Proof of Proposition 3. Given that V, F, G are all continuous, the schedule H is continuous. Because H is also non-decreasing with $H(\underline{s}) = 0$, the only off-path deviations for bidder i to consider are to dissipate an amount $x > H(\bar{s})$. Specify that doing so causes all other players to believe $s_i = \bar{s}$ with probability 1 and to behave as they would if i had dissipated $H(\bar{s})$, which leads to the same belief. Therefore such a deviation garners i the same continuation play from her opponents as simply imitating type \bar{s} , but she pays strictly more than she would by imitating \bar{s} . On-path incentive compatibility ensures that no type of i will find the deviation profitable. Finally, because \mathbf{a}_{-i} is weakly decreasing in the type that i 's opponents perceive her to be, the specified off-path belief is the most favorable possible one, and therefore satisfies any of the belief-based refinements. ■

Proof of Theorem 3. The outline of the proof is as follows. Restrict attention to symmetric equilibria. We first construct and verify a separating, x -monotone equilibrium by generalizing the arguments from Lemma 1, Theorem 1, and Theorem 2 to allow $\varepsilon > 0$. It is then immediately apparent that as $\varepsilon \rightarrow 0$, the total equilibrium surplus converges to the efficient level. We then demonstrate the uniqueness of the equilibrium in two steps: first showing that all non-separating equilibria fail x -monotonicity for $\varepsilon > 0$, and then that no other separating equilibrium exists.

Construction: The construction is a straightforward generalization of Theorems 1 and 2: we first construct a payment rule that is generically *ex post* incentive compatible (which we call h_ε), then use it to construct one that is *ex interim* incentive compatible, but where each bidder's payment depends only on her own announcement (which we call H_ε).²⁵ Unfortunately, this will require an investment in notation. Recall that when $\varepsilon = 0$, $s_i^k(\mathbf{s}_{-i})$ is the infimum of values of s_i such that in the efficient equilibrium of the subgame $\mathbf{a}_{-i} = \mathbf{a}_{-i}^k$, and k^* satisfies $s_i^{k^*}(\mathbf{s}_{-i}) = \underline{s}$. With $\varepsilon > 0$,

²⁵Notice that unlike the payment rule from Theorem 1, we only claim that h_ε is generically *ex post* incentive compatible. This is because our only interest in h_ε is in its use for constructing H_ε . Therefore, that truth-telling maximizes i 's expected utility (when other truth-tell as well) for almost all realizations of \mathbf{s} , and that i 's payoff in the mechanism according to h_ε is always bounded, are sufficient. This means we do not need to worry about the non-generic event described in footnote 4.

we will need to consider two variations of this. First, let $s_{0,i}^k(\mathbf{s}_{-i})$ be the infimum of values of s_i such that, in equilibrium, *if the coupon is awarded to* $\arg \max_{j \neq i} s_j$, then $\mathbf{a}_{-i} = \mathbf{a}_{-i}^k$. Second, let $s_{1,i}^k(\mathbf{s}_{-i})$ be the infimum of values of s_i such that, in equilibrium, *if the coupon is awarded to* i , then $\mathbf{a}_{-i} = \mathbf{a}_{-i}^k$. Also, for $y \in \{0, 1\}$ let $s_{y,i}^{k^*}(\mathbf{s}_{-i}) = \underline{s}$. So, by definition, if $\varepsilon = 0$, then $s_{0,i}^k = s_{1,i}^k = s_i^k$ for all k . In addition, let k_0^{**} (and analogously k_1^{**}) be the number of bidders, other than i , that invest when $s_i = \max_{j \neq i} s_j$ and the coupon is awarded to $\arg \max_{j \neq i} s_j$ (or awarded to i for k_1^{**}).

Consider the sequence

$$\underline{s} = s_{0,i}^{k_0^{**}}(\mathbf{s}_{-i}) \leq \dots \leq s_{0,i}^{k_0^{**}}(\mathbf{s}_{-i}) \leq \max_{j \neq i} s_j \leq s_{1,i}^{k_1^{**}-1}(\mathbf{s}_{-i}) \leq s_{1,i}^{k_1^{**}-2}(\mathbf{s}_{-i}) \leq \dots \quad (14)$$

which will play the same role as the sequence $s_i^{k_1^{**}}(\mathbf{s}_{-i}), s_i^{k_1^{**}-1}(\mathbf{s}_{-i}), \dots$ does in Theorem 1. To ease notation, denote the sequence given in (14) simply as $\sigma_0, \sigma_1, \sigma_2, \dots$. Finally, define $u_i(s_i, \mathbf{s}_{-i}, \hat{s}_i)$ to be i 's expected payoff when: 1) the realization of s -values is (s_i, \mathbf{s}_{-i}) , 2) the ε -coupon is awarded to i if $\hat{s}_i \geq \max_{j \neq i} s_j$ and to $\arg \max_{j \neq i} s_j$ otherwise,²⁶ 3) all bidders other than i invest according to the (efficient) equilibrium given the belief that $\mathbf{s} = (\hat{s}_i, \mathbf{s}_{-i})$ and the coupon allocation given by (2), and 4) i invests optimally given (1), (2) and (3).²⁷

We are now ready to define the payment rule h_ε as

$$\begin{aligned} h_\varepsilon(\hat{s}_i, \hat{\mathbf{s}}_{-i}) &= 0, \text{ if } \hat{s}_i < \sigma_1 \\ &= h_\varepsilon(\sigma_{n-1}, \hat{\mathbf{s}}_{-i}) + u_i(\sigma_n, \hat{\mathbf{s}}_{-i}, \sigma_n) - u_i(\sigma_n, \hat{\mathbf{s}}_{-i}, \sigma_{n-1}), \text{ if } \hat{s}_i \in [\sigma_n, \sigma_{n+1}), \text{ for } n \geq 1 \end{aligned}$$

As before, define a second payment rule $H_\varepsilon(\hat{s}_i) = E_{\mathbf{s}_{-i}}[h_\varepsilon(\hat{s}_i, \mathbf{s}_{-i})]$. A separating equilibrium is then given by $x^*(s_i) = H_\varepsilon(s_i)$, followed by efficient investment given the perfectly-revealed s -values and the allocation of the coupon, and each bidder bidding their (coupon-adjusted) value in the final stage.

Verification: The verification argument is analogous to that for Theorem 2: first provide an ‘‘increasing differences’’ lemma; then establish that if bidder i faced the payment rule h_ε and all other bidders truthfully revealed their types, then $\hat{s}_i = s_i$ is generically optimal; finally, note that the expectation operator preserves the optimality. We start with the lemma:

Lemma 3 *Fix $\varepsilon \geq 0$, \mathbf{s}_{-i} , and pairs: $s_i \leq s'_i$, and $\hat{s}_i \leq \hat{s}'_i$. Then,*

$$u_i(s'_i, \mathbf{s}_{-i}, \hat{s}'_i) - u_i(s'_i, \mathbf{s}_{-i}, \hat{s}_i) \geq u_i(s_i, \mathbf{s}_{-i}, \hat{s}'_i) - u_i(s_i, \mathbf{s}_{-i}, \hat{s}_i) \quad (15)$$

²⁶Notice that, because i is arbitrary, the specification when $s_i = \max_{j \neq i} s_j$ would cause a slight problem for using h_ε as a payment rule. However, because we are only interested in the *ex interim* rule H_ε , for any announcement \hat{s}_i by bidder i , the probability that $\hat{s}_i = \max_{j \neq i} s_j$ is zero, so the consideration is irrelevant.

²⁷Clearly, π_i and u_i are closely related. The only difference is that the third argument is in π_i is the investment decision undertaken by the i 's competitors, while in u_i it is their belief about her type. Clearly, the latter maps to the former via our equilibrium prescription for the investment stage, and the specification of u_i simplifies notation given that the sequence in (14) is more complex than is the comparable one in the absence of the coupon.

Proof. We follow the same outline as the proof of Lemma 1. First observe that an increase in the *perceived* value of s_i has two potential effects. One, it may cause i to switch from losing to winning the coupon. Two, it decreases the equilibrium investment response of i 's competitors, \mathbf{a}_{-i} (and this effect is strengthened if the first effect is realized). The first argument from the proof of Lemma 1 therefore easily extends to establish that, when i is best responding, a_i is increasing in both s_i and \hat{s}_i .

Notice that if bidder j wins the coupon, it is equivalent to simply having an ε higher final valuation of $V(s_j, t_j) + a_j b + \varepsilon$. Therefore, we need to consider each bidder's (potentially) "enhanced" value: $V_j + \mathbb{I}_j^C \varepsilon$, where \mathbb{I}_j^C is the indicator function for j winning the coupon. Let $Z_i = \max_{j \neq i} \left\{ V(s_j, t_j) + b a_j + \mathbb{I}_j^C \varepsilon | \hat{s}_i \right\}$, $Z'_i = \max_{j \neq i} \left\{ V(s_j, t_j) + b a_j + \mathbb{I}_j^C \varepsilon | \hat{s}'_i \right\}$, $X_i = V(s_i, t_i) + \mathbb{I}_i^C \varepsilon$, and $X'_i = V(s'_i, t_i) + \mathbb{I}_i^C \varepsilon$.

Again, notice that

$$u_i(s_i, \mathbf{s}_{-i}, \hat{s}_i) = E[(X_i + a_i b - Z_i)^+] - a_i c$$

To establish the lemma, let us begin by precluding investment by player i . Then condition (15) is equivalent to

$$E[(X'_i - Z'_i)^+] - E[(X_i - Z_i)^+] \geq E[(X'_i - Z_i)^+] - E[(X_i - Z_i)^+] \quad (16)$$

which (by design) looks identical to (9)—though when $\varepsilon > 0$, the four random variables are slightly different here from in Lemma 1. Nevertheless, it is still true that $X'_i \succeq X_i$ and $Z_i \succeq Z'_i$. So, we again define $Y_i = X'_i - X_i \geq 0$ and follow the same steps to reach the same conclusion: that (16) holds. The final step is to verify that condition (15) holds when we allow for investment by player i , and is identical to the argument given for this step in the proof of Lemma 1. ■

Now, fix a generic realization of \mathbf{s}_{-i} , and suppose that bidder i faced the payment rule h_ε and $\hat{\mathbf{s}}_{-i} = \mathbf{s}_{-i}$. The proof that $\hat{s}_i = s_i$ is optimal is identical to the proof of Theorem 1, with equations (10) and (11) replaced by (17) and (18) respectively. For $s_i \in [\sigma_n, \sigma_{n+1})$ the expected utility of truthfully reporting, and of under-reporting by one interval, respectively, are:

$$u_i(s_i, \mathbf{s}_{-i}, \sigma_n) - h_\varepsilon(\sigma_{n-1}, \mathbf{s}_{-i}) - u_i(\sigma_n, \mathbf{s}_{-i}, \sigma_n) + u_i(\sigma_n, \mathbf{s}_{-i}, \sigma_{n-1}) \quad (17)$$

$$u_i(s_i, \mathbf{s}_{-i}, \sigma_{n-1}) - h_\varepsilon(\sigma_{n-1}, \mathbf{s}_{-i}) \quad (18)$$

Having (17) \geq (18) is the same as

$$u_i(s_i, \mathbf{s}_{-i}, \sigma_n) - u_i(s_i, \mathbf{s}_{-i}, \sigma_{n-1}) \geq u_i(\sigma_n, \mathbf{s}_{-i}, \sigma_n) - u_i(\sigma_n, \mathbf{s}_{-i}, \sigma_{n-1}) \quad (19)$$

And, this is precisely what Lemma 3 demonstrates. The last step is to verify that the candidate equilibrium's specification that $x^*(s_i) = H_\varepsilon(s_i)$ is incentive compatible (given the prescribed continuation play). This is identical to the proof of Theorem 2, except with "almost all \mathbf{s}_{-i} " replacing "any \mathbf{s}_{-i} "—which makes no difference given that payoffs in the mechanism are always bounded.

To see that the equilibrium is x -monotone, we first argue that H_ε is strictly increasing. Clearly, h_ε is weakly increasing in \hat{s}_i , for any $\hat{\mathbf{s}}_{-i}$. Therefore, H_ε is also weakly increasing, and if $H_\varepsilon(s_i) = H_\varepsilon(s'_i)$ for $s_i < s'_i$, then it must be that $h_\varepsilon(s_i, \mathbf{s}_{-i}) = h_\varepsilon(s'_i, \mathbf{s}_{-i})$ for almost all values of \mathbf{s}_{-i} . But this is not possible when $\varepsilon > 0$ because there exists a positive measure of realizations of \mathbf{s}_{-i} such that $s_i < \max_{j \neq i} s_j < s'_i$ and h_ε prescribes different payments for the two types in such cases. Given that H_ε is strictly increasing, and the equilibrium adheres to the efficient cutoff rule, $r^*(\cdot)$, in the investment stage, it is x -monotone.

Efficiency: By construction, when $\varepsilon = 0$, the equilibrium constructed above is identical to the equilibrium of the efficient mechanism given in Theorem 2. Given that V, F, G are all continuous, for generic \mathbf{s} , the *ex interim* expected utility of each player (including the seller) in the equilibrium above converges to their *ex interim* expected utility in the equilibrium of Theorem 2 as $\varepsilon \rightarrow 0$. Therefore, the same is true for total surplus, giving the result since the equilibrium of Theorem 2 is efficient.

Uniqueness: There are two steps: 1) all x -monotone equilibrium must be separating, and 2) the x -monotone equilibrium constructed above is the unique separating equilibrium. First, for the purpose of contradiction, suppose that in equilibrium a positive measure of types pool on a single coupon bid of x_p . By symmetry, a type who is supposed to be in the pool has a positive probability of a tie for winning the coupon. If instead she deviated by bidding an infinitesimally small amount above x_p , she garners a discrete increase in her probability of winning the coupon. In addition, x -monotonicity implies that by increasing her bid, her opponents are no more likely to invest. Hence, the deviation is profitable, establishing that there can be no pooling in equilibrium.

Second, for uniqueness of the equilibrium separating function $x^*(\cdot)$, let $f_i(s_i, \hat{s}_i) \equiv E_{\mathbf{s}_{-i}}[u_i(s_i, \mathbf{s}_{-i}, \hat{s}_i)]$. Therefore, in any (symmetric) separating equilibrium with public bid function denoted $x^*(\cdot)$

$$E[U_i^* | s_i] = \max_{\hat{s}_i} f_i(s_i, \hat{s}_i) - x^*(\hat{s}_i) = f_i(s_i, s_i) - x^*(s_i) = E[U_i^* | s_i = \underline{s}] + \int_{\underline{s}}^{s_i} f_{i,1}(y, y) dy$$

where $f_{i,1}$ denotes the derivative of f_i with respect to its first argument. Clearly, $x^*(\underline{s}) = 0$, otherwise a bidder with type \underline{s} is paying a positive amount to induce the least-favorable belief about her type, and the most possible investment, among her opponents—so deviating to $x = 0$ would be an improvement. Therefore, $x^*(\cdot)$ is uniquely determined, completing the proof. ■

Supplemental Material

Examples: Pooling Can Satisfy Belief-Based Refinements

Example S1

Our first example illustrates that, given the complexity of the setup, belief-based refinements alone have less bite than in standard signaling games—necessitating restrictions (such as symmetry and monotonicity) on the investment-stage behavior in addition to refinements at the signaling stage.

Let $N = 2$, s_i be distributed uniform on $[0, 1]$, and $V(s, t) = s$. Also, define $k \equiv b - c$ and assume that $k < c$. In addition to Ξ , the following is also an equilibrium satisfying any belief-based refinement.

In the investment stage, bidder 2 invests “more often,” and bidder 1 “less often,” than they would in the efficient equilibrium. Because t is irrelevant, it must be that only one bidder is investing in equilibrium. In the efficient equilibrium bidder i should invest if $s_i > s_j$ (let bidder 1 invest if $s_1 = s_2$). Suppose bidder 2 is investing in some states she “shouldn’t” (where $s_2 < s_1$), so bidder 1 is not investing in these states. For this to hold in equilibrium it must be that no bidder wishes to deviate. If bidder 2 deviates to not investing, she gets zero (because $s_2 < s_1$). So we need: $s_2 + b - c - s_1 > 0$. For bidder 1 not to deviate to investing, we need: $s_1 + b - c - (s_2 + b) < 0$. Combined, the two constraints are simply: $s_1 - s_2 < \min\{b - c, c\} = k$. The following, for example, can be maintained in equilibrium: bidder 2 invests if and only if $s_2 > s_1 - \frac{k}{2}$.

Using standard techniques, we can find that the $x_i(s_i)$ payments in the first period to produce this investment behavior are:

$$x_1(s_1) = \begin{cases} 0 & \text{if } s_1 < \frac{k}{2} \\ \frac{3k}{2} (s_1 - \frac{k}{2}) & \text{if } s_1 \geq \frac{k}{2} \end{cases} \quad x_2(s_2) = \begin{cases} \frac{k}{2} s_2 & \text{if } s_2 < 1 - \frac{k}{2} \\ \frac{k}{2} (1 - \frac{k}{2}) & \text{if } s_2 \geq 1 - \frac{k}{2} \end{cases}$$

Notice that the equilibrium is not fully separating—bidder 1 does not distinguish her types lower than $\frac{k}{2}$ and bidder 2 does not distinguish her types above $1 - \frac{k}{2}$.

Finally, it is easy to verify that the equilibrium satisfies any belief-based refinement. These refinements only have bite when looking at off-path choices of x . For either bidder, the only off-path choice is some $x > x_i(1)$. However, for any type of bidder i , such a choice is strictly dominated by choosing $x_i(1)$. So, on-path incentive constraints guarantee satisfaction of any belief-based refinement.

Example S2

This example, and the next, show that belief-based refinements are still not powerful enough to always eliminate pooling, even when symmetry and monotonicity are imposed in the investment stage.

Fix all elements of the model except for c and let V be strictly increasing in both arguments. Then there exists two values $c_1 < c_2$ such that

- Full pooling at $(x_i, \hat{s}_i) = (0, \bar{s})$, followed by $a_i = 0$, for all i regardless of \mathbf{s} , satisfies any belief-based refinement whenever $c \geq c_1$.
- The efficient investment rule calls for $a_i = 0$ for all i , regardless of \mathbf{s} , if and only if $c \geq c_2$.

Therefore, for $c \in (c_1, c_2)$ an equilibrium involving full pooling satisfies any belief-based refinement and is also inefficient with regard to investments.

The first claim holds because, in this setting, investment is very costly. Suppose that if any bidder deviates to any positive x , all other bidders will ascribe to her the most favorable possible belief: that $s_i = \bar{s}$ with probability one. What does this gain her? Nothing—her opponents were already not investing. Its only worth signaling if it will affect one’s competitors’ behavior. So the deviation is not profitable no matter what belief it leads to.

The remainder of the example claims that this will not always be efficient. Consider a realization where $s_i = \bar{s}$ and $s_j = \underline{s}$ for all $j \neq i$. The minimum c such that it is inefficient for i to invest is larger than the minimum c needed to enforce non-investment when s -values are private information, delivering the result. That is, when $c \in (c_1, c_2)$, there will be a positive measure of realizations \mathbf{s} where it is efficient for some bidders to invest, but there will exist a pooling equilibrium in which no bidders invest.

A similar result obtains if investment is very inexpensive as well.

Example S3

Fix all elements of the model except for c and let V be strictly increasing in both arguments. Then there exists two values $c_1 < c_2$ such that

- Full pooling at $(x_i, \hat{s}_i) = (0, \bar{s})$, followed by $a_i = 1$, for all i regardless of \mathbf{s} , satisfies any belief-based refinement whenever $c \leq c_2$.
- The efficient investment rule calls for $a_i = 1$ for all i regardless of \mathbf{s} , if and only if $c \leq c_1$.

Therefore, for $c \in (c_1, c_2)$ an equilibrium involving full pooling satisfies any belief-based refinement and is also inefficient with regard to investments.

The logic here is similar. Even if deviating from the full pooling equilibrium leads to the most favorable belief, investment is inexpensive enough for all bidders to invest regardless of s_i anyway. When $c \in (c_1, c_2)$, there will be a positive measure of realizations \mathbf{s} where it is efficient for some bidders not to invest, but there will exist a pooling equilibrium in which all bidders invest. The result is strengthened as N increases because a unilateral deviation (even if it leads to the most favorable belief possible) has less impact on other bidders’ probability of winning.