**The value-added/turnover frontier**
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# The Value-Added/Turnover Frontier

*Insights to help the active manager judge the allowable level of turnover.*

Richard C. Grinold and Mark Stuckelman

Portfolio construction at its simplest balances a desire for increased performance against a fear of increased risk. This balancing act gets more complicated when we bring a third force, namely, transaction cost, to bear. Transaction costs are a snag; they reduce the gains of the skillful active manager and they complicate the analysis.

In this article we sprinkle a little sunshine amid the gloom, and insist that things are better than they appear. We've all heard about the extremely promising strategy that unfortunately requires 80% turnover per month. Such a policy should not be dismissed out of hand. We may be able to add considerable value with the strategy if we restrict turnover to 40%, 20%, or even 10% per month.

We shall build a simple framework for analyzing the effects of transaction costs and turnover for the active manager. Our purpose is to gain insight before we lose ourselves in a mass of details. In particular, we will:

- Help the active manager make an appropriate judgment of an allowable level of turnover.
- Evaluate ad hoc policies that seek to control transaction costs by limiting turnover.
- Provide a lower bound on the amount of value-added that can be gained with limited turnover.
- Clarify the link between transaction costs and turnover.
- Consider some important nuances when we depart from the assumptions of our simple model.

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Our main vehicle is a model that explores the relationship between value-added and turnover that we call the value-added/turnover frontier. Value-added is defined as risk-adjusted return where both the risk and return may be measured relative to a benchmark.[1] We find, not surprisingly, that more turnover allows for more value-added. We are also able to justify reasonable rules of thumb such as "for one-half the turnover you can get at least three-quarters of the value-added."

After we explore the theoretical model and its ramifications in the first section, we present sample calculations. Then we consider some of the finer points in measuring and managing transaction costs. Technical details are described in endnotes, and theoretical support is provided in the appendix.

It is certainly not a novel idea to consider the impact of transaction costs and turnover in portfolio selection. The articles by Rudd and Rosenberg [1979], Pogue [1970], and Schreiner [1980] are representative and give additional references. What is novel in this article is the focus on the trade-off between value-added and turnover. By combining the expected return and risk components of the object we are able to see the trade-off explicitly and gain insights that might otherwise be obscured.

## VALUE-ADDED AND TURNOVER

The model for value-added is simple and practical. For Portfolio P let $\alpha_P$ be the portfolio's alpha[2] and $\psi_P$ its active risk.[3] The value-added of Portfolio P, $VA_P$, is alpha less a penalty for active risk. That penalty is $\lambda\psi_P^2$ where the risk aversion parameter[4] $\lambda$ captures the manager's risk/reward trade-off.

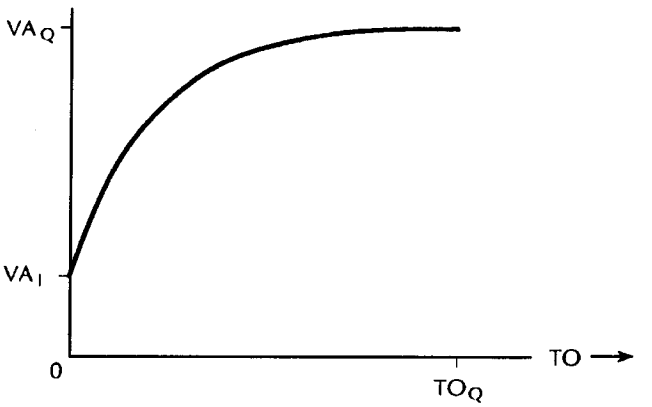
The formula for value-added is thus

$$VA_P = \alpha_P - \lambda\psi_P^2. \tag{1}$$

We can interpret value-added as a certainty-equivalent return. If $VA_P = 0.84\%$ (84 basis points), then the manager is indifferent[5] to holding Portfolio P or getting the benchmark return plus a certain 84 basis points of additional return.

The manager starts with an initial Portfolio I that has value-added $VA_I$. We will limit the portfolios that we can choose to a choice set C.[6] Portfolio Q is the portfolio in C with the highest possible value-added.[7] We will *assume* for the moment that our initial portfolio, I, is also in the choice set C, and later consider what happens when we relax that assumption.

**EXHIBIT 1**
**THE VALUE-ADDED/TURNOVER FRONTIER**




The increase in value-added that is possible as we move from I to Q is

$$\Delta VA_Q = VA_Q - VA_I. \tag{2}$$

We define turnover as one-half the sum of purchases and sales.[8] We will let $TO_P$ represent the amount of turnover needed to move from Portfolio I to Portfolio P. Thus the turnover required to move from I to Q is $TO_Q$.

If we restrict turnover to be less than $TO_Q$, we will be giving up some value-added to reduce cost. Let VA(TO) be the maximum amount of value-added if turnover is less than or equal to TO.[9] Exhibit 1 shows a typical situation.

VA(TO) increases from $VA_I$ to $VA_Q$. The concave shape of the curve indicates that we get a decreasing marginal return for each additional amount of turnover that we allow.[10]

### A Lower Bound

At the outset we assume that the choice set C is defined by linear equality constraints, such as specifying a level of cash (risk-free asset) in the portfolio or specifying the beta of the portfolio. We relax that restriction later.

With the assumption that the choice set is defined by equality constraints and our assumption that Portfolio I is in the choice set C, we can obtain a quadratic *lower* bound on potential value-added:[11]



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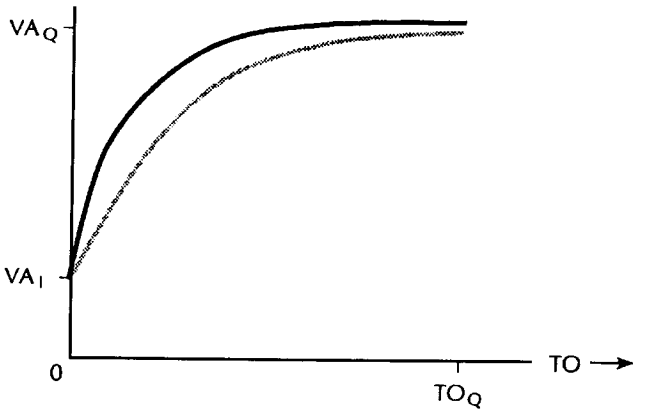
**EXHIBIT 2**
**A LOWER BOUND ON THE FRONTIER**




$$VA(TO) \geq VA_I + \Delta VA_Q\,[2[TO/TO_Q] - [TO/TO_Q]^2]. \quad (3)$$

The key to deriving this bound is noting that prorating a fraction $TO/TO_Q$t of the trades needed to move from the initial Portfolio I to the optimal Portfolio Q will give us a portfolio that is in the choice set and has turnover equal to TO.

### The Value of Scheduling the Trades

The difference between the actual value-added, VA(TO), and the bound is explained by judicious scheduling of the trades. We can skim off the most attractive opportunities.

For example, suppose there are only four assets. As we move from I to Q we purchase 10% in assets #1 and #2, and sell 10% of our holdings in assets #3 and #4. Turnover is 20%.

Suppose the alphas for the four assets are: 5%, 3%, -3%, and -5%, respectively. Then buying #1 and selling #4 has a bigger impact on alpha than swapping #2 for #3. If we have restricted turnover to 10%, then we could make an 8% trade of stock #1 for #4 and a 2% trade of #2 for #3 rather than doing 5% in both trades.

We have the situation shown in Exhibit 2. The solid line shows the frontier; the dotted line shows the lower bound. In particular, when $TO = 0.5TO_Q$, Equation (3) says

$$VA(0.5TO_Q) \geq VA_I + 0.75\Delta VA_Q. \quad (4)$$

This justifies our "rule of thumb" that you can get at least three-quarters of the (incremental) value-added with one-half the turnover.

### Transaction Costs

The simplest assumption we can make about transaction costs is to assume that round-trip costs are the same for all assets.[12] If TC is that level of costs, then the portfolio selection problem where transaction costs are considered is to choose a Portfolio P in the choice set C that will maximize

$$VA_P - TCTO_P. \quad (5)$$

We can solve this problem stating it graphically. Let SLOPE(TO) represent the slope of the value-added turnover frontier when the level of turnover is TO. Since the frontier is increasing and concave, SLOPE(TO) is positive and decreasing. The incremental gain from each additional amount of turnover is decreasing, so the slope of the frontier, SLOPE(TO), will decrease to zero as TO increases to $TO_Q$. The SLOPE(TO) represents the marginal gain in value-added by additional transactions, and TC represents the marginal cost of additional transactions.

The optimal level of turnover will occur where marginal cost equals marginal value-added, i.e., where SLOPE(TO*) = TC. As long as the transaction cost is positive and less than SLOPE(0), we can find a level of turnover, TO*, such that SLOPE(TO*) = TC. If TC > SLOPE(0), then it is not worthwhile to transact at all, and the best solution is to stick with Portfolio I.

The situation is shown in Exhibit 3.

### Implied Transaction Costs

The slope of the value-added/turnover frontier can be interpreted as a transaction cost. We can reverse the logic and link any level of turnover to a transaction cost; e.g., a turnover level of 20% corresponds to a round-trip transaction cost of 2.46%.

Transaction costs include observable components such as commissions and spreads as well as the unobservable market impact. Because managers cannot be sure they have a precise measure of transaction costs, they will often seek to control those costs by establishing an ad hoc policy such as "no more than 20% turnover per quarter." The insight that relates the slope of the value-added/turnover frontier to the level of transaction costs gives us an opportunity to analyze



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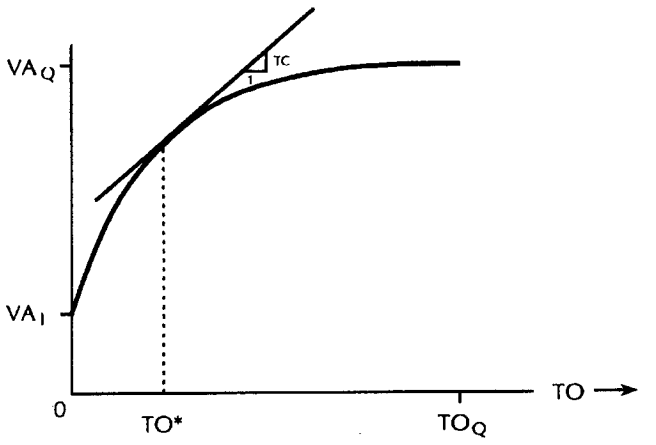
that cost control policy. One can fix the level of turnover at the required level $TO_R$, and then find the slope, $SLOPE(TO_R)$, of the frontier at $TO_R$.

Our ad hoc policy is consistent with an assumption that the general level of round-trip transaction costs is equal to $SLOPE(TO_R)$. If we have a notion that round-trip costs are around 2%, and we find that $SLOPE(TO_R)$ is about 4.5%, then something is awry.

We can make three possible adjustments to get things back in harmony: 1) increase our estimate of the round-trip costs from 2%; 2) increase the allowed level of turnover, $TO_R$, since we are giving up lots more marginal value-added (4.5%) than it is costing us to trade (2.0%); or 3) reduce our estimates of our ability to add value by scaling our alphas back toward zero.

A combination of these adjustments, i.e., a little give from all sides, would be fine as well. This type of analysis serves as a reality check on our policy and the overall investment process.

**EXHIBIT 3**
**MARGINAL VALUE-ADDED EQUALS TRANSACTION COSTS**




## SAMPLE CALCULATIONS

We did some calculations to determine the curve VA(TO), its slope SLOPE(TO), and the size of the scheduling benefit, i.e., how much VA(TO) is above its lower bound.

### The Setup

Our sample universe is the 100 stocks in the OEX. We generated alphas for these 100 stocks.[13] Then we centered and scaled the alphas so that 1) the capitalization-weighted average of the alphas for the OEX portfolio (the benchmark in this case) was zero, and 2) the optimal portfolio, Portfolio Q, would have an alpha of 3.2% and an active risk of 4% when we used a risk aversion parameter, $\lambda$ from Equation (1), of 0.1.

The initial portfolio was selected by randomly choosing twenty of the stocks and equal-weighting them. The initial portfolio has an alpha of 0.07% and an active risk of 5.29%. This would be typical when a manager takes over an existing account.

The playing field is laid out in Exhibit 4, where the alpha is on the vertical axis and active risk is the horizontal. Portfolios I (initial), B (benchmark), and Q (unrestricted optimal) are shown.

The straight line from B through Q sets the limit of the manager's opportunity set; i.e., the manager might prefer a portfolio with 5% alpha and 2% active risk, but that opportunity is not available. The curve $1.6\% = \alpha_P - 0.1\psi_P^2$ represents all $[\alpha_P, \psi_P]$ combinations that have a value-added equal to the 1.6% value-added of Portfolio Q. Portfolio Q is the only one of these in the opportunity set.

In our sample calculations we wanted to separate the effect of transaction costs and turnover from the effect of the constraints.

**EXHIBIT 4**
**THE OPPORTUNITY SET**

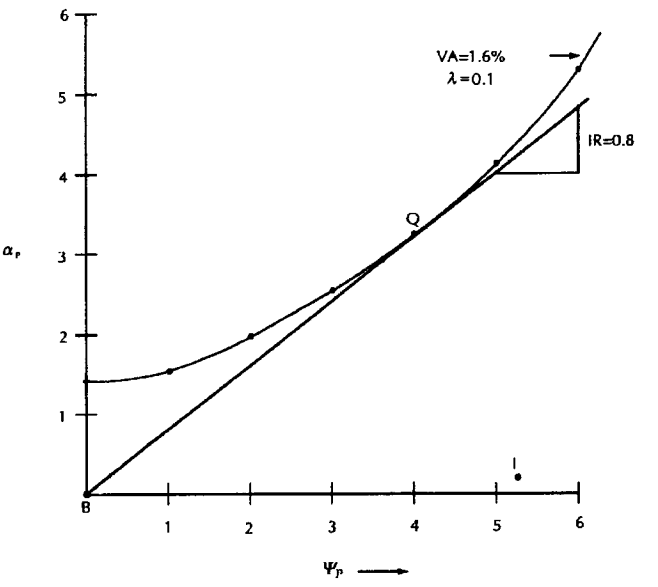





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**EXHIBIT 5**
**CHANGING OPPORTUNITY SETS FOR CASE 1**

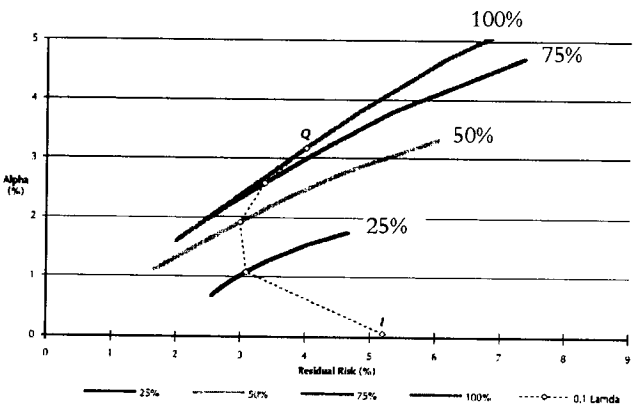
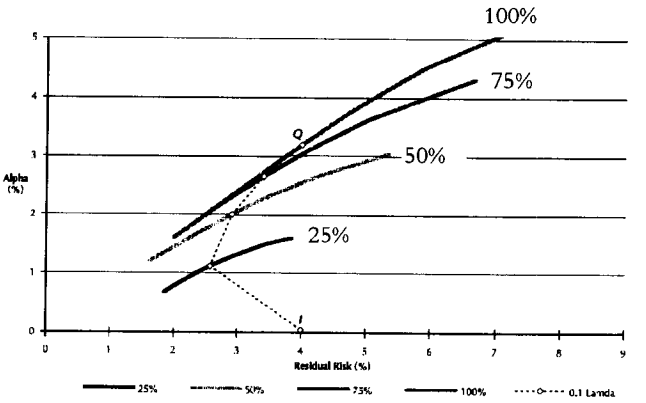



**EXHIBIT 6**
**CHANGING OPPORTUNITY SETS FOR CASE 2**




## Results

Restrictions on turnover limit opportunities. In Exhibit 5 we see the opportunity set shrink as we restrict turnover. The four curved lines represent the border of the opportunity set when turnover is $0.25TO_Q$, $0.5TO_Q$, $0.75TO_Q$, and $TO_Q$, respectively. The opportunity set grows rapidly with the first 25% and very slowly with the last 25%.

The path from I to Q in Exhibit 5 shows the sequence of optimal portfolios that will trace out the frontier VA(TO). This path is typical; i.e., if one draws a line from I to Q, one will find that the typical path will trace out a convex path to the left of that line.[14]

Exhibits 6 and 7 show two other cases. In our second and third cases we mixed the initial portfolio with the benchmark until we found a new initial portfolio with active risk equal to 4% (case #2) and 2% (case #3). The generic properties seen in Exhibit 5 carry over; the opportunity set grows rapidly at first and less rapidly later. In each of the cases the curved path shows the optimal portfolios as we ease the turnover restriction.

The calculations in the three cases are summarized in Exhibits 8, 9, and 10. We have turnover as a fraction of $TO_Q$ as the driving variable. We split the value-added into two parts, the lower bound and the

**EXHIBIT 7**
**CHANGING OPPORTUNITY SETS FOR CASE 3**

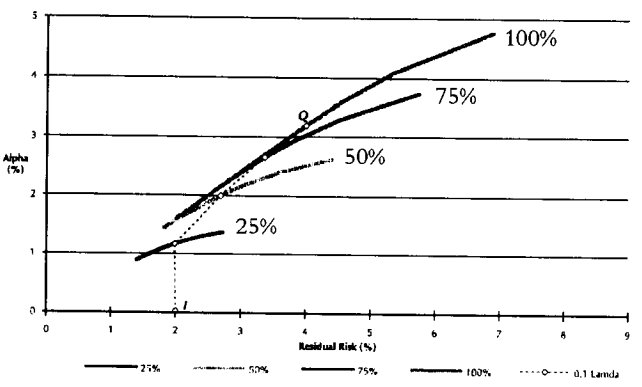



**EXHIBIT 8**
Case 1 Summary

| Percentage of $TO_Q$ | Value Added Lwr. Bnd. | Excess | Total | Fraction of $VA_Q$ | Implied Trans. Cost |
|---|---|---|---|---|---|
| 0.0% | -2.73 | 0.00 | -2.73 | 0.0% | |
| 10.0% | -1.91 | 0.87 | -1.03 | 39.2% | 8.66 |
| 20.0% | -1.17 | 1.00 | -0.18 | 59.0% | 5.12 |
| 30.0% | -0.52 | 0.88 | 0.36 | 71.4% | 3.40 |
| 40.0% | 0.04 | 0.70 | 0.74 | 80.2% | 2.50 |
| 50.0% | 0.52 | 0.51 | 1.03 | 86.8% | 1.90 |
| 60.0% | 0.91 | 0.34 | 1.24 | 91.8% | 1.43 |
| 70.0% | 1.21 | 0.19 | 1.40 | 95.5% | 1.02 |
| 80.0% | 1.43 | 0.09 | 1.51 | 98.0% | 0.67 |
| 90.0% | 1.56 | 0.02 | 1.58 | 99.5% | 0.33 |
| 100.0% | 1.60 | 0.00 | 1.60 | 100.0% | 0.00 |



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**EXHIBIT 9**
Case 2 Summary

| Percentage of $TO_Q$ | Value Added Lwr. Bnd. | Excess | Total | Fraction of $VA_Q$ | Implied Trans. Cost |
|---|---|---|---|---|---|
| 0.0% | -1.55 | 0.00 | -1.55 | 0.0% | |
| 10.0% | -0.95 | 0.59 | -0.37 | 37.6% | 6.55 |
| 20.0% | -0.42 | 0.65 | 0.24 | 56.7% | 4.08 |
| 30.0% | 0.06 | 0.58 | 0.63 | 69.3% | 2.87 |
| 40.0% | 0.47 | 0.46 | 0.93 | 78.6% | 2.19 |
| 50.0% | 0.81 | 0.34 | 1.15 | 85.8% | 1.68 |
| 60.0% | 1.10 | 0.23 | 1.32 | 91.2% | 1.24 |
| 70.0% | 1.32 | 0.13 | 1.45 | 95.1% | 0.90 |
| 80.0% | 1.47 | 0.06 | 1.53 | 97.9% | 0.58 |
| 90.0% | 1.57 | 0.01 | 1.58 | 99.5% | 0.28 |
| 100.0% | 1.60 | 0.00 | 1.60 | 100.0% | 0.00 |

**EXHIBIT 10**
Case 3 Summary

| Percentage of $TO_Q$ | Value Added Lwr. Bnd. | Excess | Total | Fraction of $VA_Q$ | Implied Trans. Cost |
|---|---|---|---|---|---|
| 0.0% | -0.37 | 0.00 | -0.37 | 0.0% | |
| 10.0% | 0.00 | 0.26 | 0.26 | 32.1% | 4.47 |
| 20.0% | 0.34 | 0.29 | 0.63 | 50.6% | 3.08 |
| 30.0% | 0.63 | 0.26 | 0.90 | 64.3% | 2.40 |
| 40.0% | 0.89 | 0.22 | 1.11 | 75.1% | 1.89 |
| 50.0% | 1.11 | 0.17 | 1.27 | 83.4% | 1.43 |
| 60.0% | 1.28 | 0.11 | 1.40 | 89.7% | 1.08 |
| 70.0% | 1.42 | 0.07 | 1.49 | 94.4% | 0.77 |
| 80.0% | 1.52 | 0.03 | 1.55 | 97.6% | 0.49 |
| 90.0% | 1.58 | 0.01 | 1.59 | 99.4% | 0.24 |
| 100.0% | 1.60 | 0.00 | 1.60 | 100.0% | 0.00 |

excess above the bound. The excess is the benefit we get by scheduling the best trades first.

The implied transaction costs are given in Exhibits 8, 9, and 10. We see that reasonable levels of round-trip costs (about 2%) do not call for large amounts of turnover, and that very low or high restrictions on turnover correspond to unrealistic levels of transaction costs.

Note also that the value of being able to pick off the best trades (the difference between the lower bound and the actual fraction of value-added) is largest when turnover is between 30% to 50% of the level required to move to Portfolio Q. We can also see that the rule of thumb of 75% of the value-added for 50% of the turnover seems to be quite conservative as we get at least 83% of the value-added in the three cases.[15]

## RECONSIDERATION OF THE ASSUMPTIONS AND OTHER DETAILS

We have made three assumptions: 1) the initial portfolio is in the choice set C; 2) the choice set is described by linear equalities; and 3) all round-trip transaction costs are the same. We will reconsider these in turn.

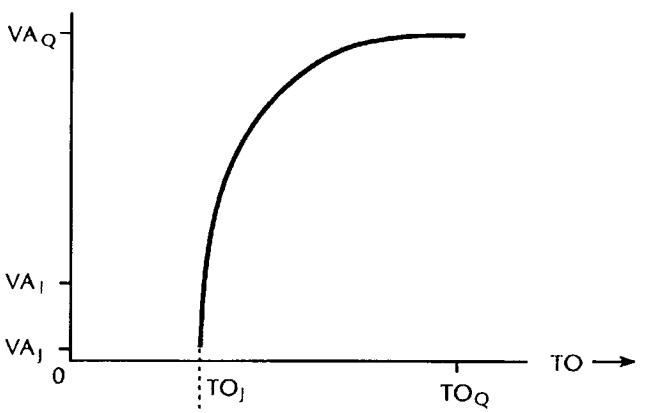
### Initial Portfolio is Not In the Choice Set

If the initial Portfolio I is not in the choice set, then we can think of the portfolio construction problem as being a two-step process. In Step 1, we find the Portfolio J in the choice set so that the turnover in moving from I to J is minimal. The value-added moving from I to J is not a consideration, so we may have $VA_I \geq VA_J$ or $VA_I < VA_J$.[16] The turnover required to move from I to J is $TO_J$. The lower bound result in Equation (3) will still apply, although we start from Portfolio J rather than Portfolio I.

We now have the situation shown in Exhibit 11. The presentation of Exhibit 11 illustrates some of the negative effects that constraints can have in portfolio selection. In general, constraints are either legal restrictions or ad hoc portfolio design parameters.

Typical legal restrictions are limitations on the

**EXHIBIT 11**
**INITIAL PORTFOLIO IS OUTSIDE THE CHOICE SET**



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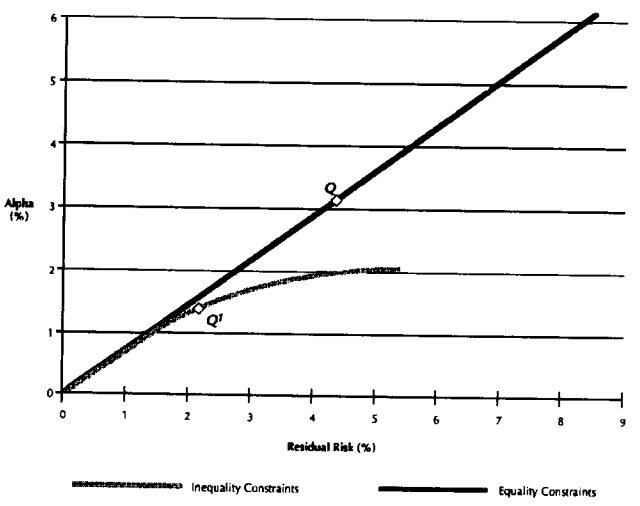
stocks that are available for purchase, a restriction on short sales, or a restriction on owning more than 5% of a company's outstanding equity. Portfolio design parameters are, in general, insurance against questionable inputs to the portfolio construction exercise. If we do not believe our estimates of return, risk, risk aversion, and transaction costs, then we may want to add constraints to limit the influence of the input information.

Design constraints may appear to be a reasonable idea, but they deal with the effects rather than the cause. Moreover, they can severely limit our freedom of action. Suppose we limit turnover to 10% per month. If the first 4% of the turnover is required to move the portfolio back into the choice set, then we only have 6% to take advantage of our new insights concerning the returns to the stocks.

### What Happens If There Are Inequality Constraints

If the choice set C is described by inequality constraints, such as a restriction on short-selling and upper limits on the individual asset holdings, then the analysis becomes more complicated. Exhibit 12 shows how the alpha, active risk possibilities, shrinks when we impose no short-selling restrictions on the data in Case 1. The portfolio that is optimal with no restrictions on turnover will, of course, differ. It is shown as Portfolio Q in Exhibit 12.[17]

**EXHIBIT 12**
**EFFECT OF INEQUALITY CONSTRAINTS ON THE OPPORTUNITY SET**
**ONE SAMPLE, 100 STOCKS**




Recall that the optimal portfolio with no restrictions on short-selling had an alpha of 3.2% and an active risk of 4%. For this example there is considerable loss in efficiency due to the lower limits on the asset's holdings.

The value-added/turnover frontier, VA(TO), will have the same increasing and concave slope that we see in Exhibit 1. There will be a quadratic lower bound on VA(TO), but a lower bound that is not as strong as we obtain in the case with only equality constraints.[18] You are not guaranteed three-quarters of the value-added for one-half the turnover. Nevertheless, in the cases we have solved to date we always find that we still get at least 75%.

### All Transaction Costs Are Not The Same

In part of our analysis above we made the assumption that all round-trip transaction costs are the same. It is *good news* for the portfolio manager if the transaction costs differ. Recall that the difference between the lower bound and the value-added/turnover frontier stemmed from our ability to schedule the trades and take the most value-enhancing trades first.

Our ability to discriminate adds value. When there is a difference in transaction costs among stocks, then our ability to discriminate is enhanced further.

To test this notion we went back to our three base cases. In each case we looked at the trades made when turnover was equal to one-half of the amount it took to move to Portfolio Q. This gives us a level of implied costs, $tc_0 = \text{SLOPE}(0.5TO_Q)$. We gave one-half (the first fifty) of the assets in our universe (the OEX) a round-trip cost of $tc_1 = 0.75\text{SLOPE}(0.5TO_Q)$, and we gave the second half a round-trip cost $tc_2$ adjusted so that the optimal trades with turnover restricted to $0.5TO_Q$ cost the same with the unequal costs $[tc_1, tc_2]$ as they did with the flat cost $tc_0$.

This setup makes it apparent that the unequal costs offer an opportunity. Just by sticking to the original schedule of trades we can do as well as we did with the flat costs, although we may be able to do better by taking more advantage of the lower-cost trades. The goal is to maximize value-added less costs as in (5).[19]

Exhibit 13 shows the effect of the unequal costs. For each case there are two rows: row 1 describes the optimal portfolio and cost under the



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equal cost $tc_0$, and row 2 the portfolio and costs under the unequal costs [$tc_1$, $tc_2$]. In all three cases we get to more or less the same portfolio alpha and active risk. The difference is in the cost of getting there.

We lower transaction costs by 0.37%, 0.24%, and 0.13% in the three cases, and almost all of this cost savings flows to the bottom line. This indicates the potential savings to a portfolio manager who has accurate measures of the cost of trading and who uses those measures as part of the portfolio rebalancing process.

### Transaction Costs and the Horizon

A portfolio optimization compares annualized alphas and annualized risk levels against the transaction cost that is paid at a given time. The transaction cost is like a capital investment, and the alpha is the annualized reward.

To put the transaction cost and the annualized measures of value-added on the same footing, it is necessary to amortize the transaction cost over the life of the investment. This is a tricky and circular problem that we are not going to solve here. The amortized transaction cost will depend on the life of the investment, and the life of the investment will depend on the amortized transaction cost.

We need to use some common sense to set up the appropriate rate of amortization. Consider two managers. Manager #1 would have 133% turnover per year if costs were ignored. That means that the ideal holding period for the manager is about nine months. We can think of this as saying that the manager's information about the asset's return has a nine-month horizon. For that manager, costs should be scaled up; replace TC with TC/0.75.

Manager #2 would have an ideal turnover of 50% per year (if costs were ignored). Manager #2 has an average holding period of two years, and should replace the cost TC with TC/2 to get an annual number.

Of course, once the costs come into play, a cost-conscious manager #1 will have less than 133% turnover per year, and manager #2 will have less than 50% turnover per year.

## SUMMARY

We have introduced a simple model of value-added and turnover. This model allows us to assemble several insights about turnover and related transaction cost. These are:

- The value-added/turnover frontier that shows the trade-offs between turnover and our ability to add value as measured in risk-adjusted expected return.
- The slope of the value-added/turnover frontier is an implied level of round-trip transaction cost.
- The rough rule of thumb that restricting turnover to one-half the level of turnover if there is no restriction on turnover will result in getting at least three-quarters of the value-added.
- The setting of a quadratic lower bound on the frontier.
- Attribution of the difference between the frontier and its lower bound to our ability to skim off the most valuable trades first.
- Demonstrating how to evaluate, or set, ad hoc policies for the control of turnover.
- Calculations that show how the opportunity set shrinks in the face of turnover constraints and the

**EXHIBIT 13**
Benefits of Different Transaction Costs

Case 1:
$VA_I = -2.73$; $TC_1 = 1.42$; $TC_2 = 2.26$; $TC_0 = 1.90$

| | Alpha | Active Risk | Trans. Cost | ΔVA Net | TO |
|---|---|---|---|---|---|
| Flat Cost | 1.92 | 2.99 | 1.24 | 2.52 | 0.65 |
| Different Cost | 1.99 | 3.10 | 0.87 | 2.89 | 0.66 |
| | | | | +0.37% | |

Case 2:
$VA_I = -1.58$; $TC_1 = 1.26$; $TC_2 = 2.04$; $TC_0 = 1.68$

| | Alpha | Active Risk | Trans. Cost | ΔVA Net | TO |
|---|---|---|---|---|---|
| Flat Cost | 2.00 | 2.92 | 0.98 | 1.72 | 0.58 |
| Different Cost | 2.05 | 3.01 | 0.74 | 1.96 | 0.59 |
| | | | | +0.24% | |

Case 3:
$VA_I = -0.37$; $TC_1 = 1.08$; $TC_2 = 1.82$; $TC_0 = 1.43$

| | Alpha | Active Risk | Trans. Cost | ΔVA Net | TO |
|---|---|---|---|---|---|
| Flat Cost | 2.00 | 2.69 | 0.71 | 0.94 | 0.50 |
| Different Cost | 2.02 | 2.75 | 0.58 | 1.07 | 0.51 |
| | | | | +0.13% | |



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path of an optimal solution as the turnover limit is raised.

- The observation that unequal transaction costs offer an additional opportunity for the shrewd manager. The ability to estimate transaction costs accurately is important to the bottom line.
- Constraints on portfolio construction that are safeguards against poor estimates of risk, expected return, or transaction costs can be counterproductive. They will cause unnecessary turnover, and they do not address the problem. They are a band-aid on an untreated wound. The portfolio manager can avoid the unnecessary turnover and improve the overall performance by making more accurate estimates of return, risk, and transaction costs.
- Finally, we must compare the transaction costs that are incurred at a given time with risk and hoped-for exceptional return that will occur over a span of time. This is similar to the problem incurred in capital budgeting where there are initial fixed costs and ending salvage costs, and the benefits are incurred in the interim.

## APPENDIX

### NOTATION

| | |
|---|---|
| $h$ | An N element vector of holdings in the risky assets |
| $C$ | The choice set: |
| | $C = [h \mid Ah \leq b]$ in the inequality case |
| | $C = [h \mid Ah = b]$ in the equality case |
| $h_P$ | The holdings in generic Portfolio P |
| $h_I$ | The initial portfolio. It is *assumed* that $h_I \in C$ |
| $h_Q$ | The optimal portfolio with no turnover restriction |
| $h_T$ | The trades; the difference between I and Q: $h_T = h_Q - h_I$ |
| $h_B$ | The benchmark portfolio |
| $\alpha$ | Alpha, an N element vector |
| $\alpha_P$ | The alpha of Portfolio P: $\alpha_P = \alpha' h_P$ holdings in Portfolio P |
| $\lambda$ | Active risk aversion |
| $V$ | The non-singular $N \times N$ covariance matrix |
| $\sigma_T^2$ | The variance of the trades; $\sigma_T^2 = h_T' V h_T$ |
| $\psi_P$ | Tracking error for Portfolio P: $\psi_P^2 = (h_P - h_B)' V (h_P - h_B)$ |
| $VA_P$ | Value-added for Portfolio P: $VA_P = \alpha_P - \lambda \psi_P^2$ |
| $\Delta VA_P$ | Incremental value-added; $\Delta VA_P = VA_P - VA_I$ |

### INEQUALITY CASE, $C = [h \mid Ah \leq b]$

Portfolio Q is optimal for the problem $\text{Max}[VA_P \mid h_P \in C]$. This means we can find non-negative Lagrange multipliers $\pi \geq 0$, such that

$$\alpha - 2\lambda V(h_Q - h_B) - A'\pi = 0,\ \pi \geq 0, \text{ and} \tag{1}$$

$$Ah_Q \leq b,\ \pi' A h_Q = \pi' b. \tag{2}$$

Since $h_I \in C$ we have $b - Ah_I \geq 0$ and $\pi \geq 0$, so

$$\pi'[b - Ah_I] = \kappa \geq 0, \text{ or } \pi' A h_I = \pi' b - \kappa. \tag{3}$$

If we premultiply (1) by $(h_I - h_Q)$ and use (2) and (3), we find that

$$-2\lambda(h_I - h_Q)' V (h_Q - h_B) = \alpha_Q - \alpha_I - \kappa. \tag{4}$$

Now consider the family of solutions as we move directly from I to Q.

$$h = h_I + \delta[h_Q - h_I] = h_Q + [1 - \delta][h_I - h_Q] = h_Q - [1 - \delta]\, h_T. \tag{5}$$

These solutions are all in C as long as $0 \leq \delta \leq 1$. The solutions have value-added

$$VA(\delta) = \alpha_Q + [1 - \delta][\alpha_I - \alpha_Q] - \lambda[\psi_Q^2 + 2[1 - \delta](h_I - h_Q)' V (h_Q - h_B) + [1 - \delta]^2 \sigma_T^2]. \tag{6}$$

If we use (4), then Equation (6) simplifies to

$$VA(\delta) = VA_Q - [1 - \delta]\kappa - \lambda[1 - \delta]^2 \sigma_T^2. \tag{7}$$

Since $VA(0) = VA_I$, $VA(1) = VA_Q$, and $\Delta VA_Q = VA_Q - VA_I$, we have

$$\kappa = \Delta VA_Q - \lambda \sigma_T^2 \geq 0. \tag{8}$$

Thus (7) simplifies further to

$$VA(\delta) = VA_I + \Delta VA_Q[2a\delta - b\delta^2], \tag{9}$$

where

$$a = [\lambda \sigma_T^2 + \kappa/2]/[\lambda \sigma_T^2 + \kappa] \leq 1, \text{ and} \tag{10}$$

$$b = [\lambda \sigma_T^2]/[\lambda \sigma_T^2 + \kappa] \leq a \leq 1. \tag{11}$$

The slope of $VA(\delta)$, Equation (9), is $2\Delta VA_Q[a - b\delta]$, which is positive for $0 \leq \delta < 1$, and decreases to $\kappa$ as $\delta$ approaches 1.

### EQUALITY CASE, $C = [h \mid Ah = b]$

The analysis is as before except $\pi$ in (1) is unrestricted in sign. Therefore, $\kappa = 0$ and, from (8), $\Delta VA_Q = \lambda \sigma_T^2$. Thus (9)



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simplifies to

$$VA(\delta) = VA_I + \Delta VA_Q[2\delta - \delta^2]. \quad (12)$$

## ENDNOTES

[1]This is a general case. If we take the benchmark to be all cash (risk-free), then we are left in the classic case of balancing expected excess return against risk.

[2]We split the return for each asset into a part that is correlated with the benchmark and an uncorrelated (residual) part. The alpha is measured as the expected residual return for each asset. If the alphas for each asset are zero, then the benchmark is a mean-variance-efficient portfolio.

[3]Let V be the $N \times N$ non-singular covariance matrix for the risky assets, and let $h_B$ be the holdings in the N risky assets of the benchmark portfolio. If the holdings in the risky assets for Portfolio P are $h_P$, then the active variance for Portfolio P is

$\psi_P^2 = (h_P - h_B)'V(h_P - h_B)$.

[4]A reasonable number for $\lambda$ is 0.1. That means that a tracking error of 3.16% results in a $1.00\% = 0.1(3.16)^2$ loss in expected return. When the benchmark is the risk-free asset, then value-added becomes the traditional risk versus expected return model.

[5]The indifference is a matter of preference. The 84 basis points for sure is not in the investors' opportunity set. While the investor is indifferent between Portfolio P and the certain 84 basis points, the portfolios selected have to come from the opportunity set.

[6]We are allowing for constraints that limit our choices such as full investment in risky assets, portfolio beta equals one, no short sales, etc. The choice set C is restricted to be closed and convex. We will consider two cases explicitly: where C is defined by equality constraints and where it is defined by inequality constraints.

[7]Portfolio $Q \in C$ and $VA_Q = \text{Max}[VA_P \mid P \in C]$.

[8]Let $h_{P,n}$ be Portfolio P's holding in asset n, and $[x]^+ = \max(0, x)$. The purchases moving from I to P are $pur_P = \Sigma_n [h_{P,n} - h_{I,n}]^+$, and the sales are $sal_P = \Sigma_n [h_{I,n} - h_{P,n}]^+$. The turnover is

$TO_P = [pur_P + sal_P]/2$.

[9]$VA(TO) = \text{Max}[VA_P \mid P \in C, TO_P \leq TO]$.

[10]The concavity of VA(TO) follows since the value-added function is concave in the holdings $h_P$, the turnover function is convex in $h_P$, and the choice set C is convex. The fact that VA(TO) is non-decreasing follows from common sense; i.e., a larger amount of turnover will let you do at least as well. The frontier will be made up of quadratic segments (piecewise quadratic) when the choice set is described by linear inequalities.

[11]See the appendix for justification.

[12]It is slightly more complicated. If our choice set C does not allow any change in the cash position, then we can just say that all round-trip costs are the same since total purchases and sales will be equal in that case. If the cash level can change, then we have to make the stronger assumption that all purchases and sales costs are the same.

[13]We produced 100 samples from the standard normal distribution.

[14]This has always been the case in the examples we have solved, and we would conjecture it must be the case, but we cannot provide proof.

[15]It is possible to make up a two-stock example where the lower bound on the frontier would be exact. Common sense indicates that the more stocks with a reasonable distribution of alphas, the more room to add value by plucking the best opportunities.

[16]In a formal sense we are defining $VA_P = \alpha_P - \lambda\psi_P^2$ if $P \in C$ and $VA_P = -\infty$ if $P \notin C$. This means $VA(TO) = -\infty$, if $TO < TO_J$.

[17]This point is generated using the same risk aversion of 0.1 as used in the case with no bounds on asset holdings. A risk aversion of 0.034 leads to a portfolio with an alpha of 2.14% and an active risk of 4%.

[18]See the appendix for justification.

[19]Let $tp_n$ be the purchase cost for asset n and $tq_n$ the sale costs. Let $[x]^+ = \max(0, x)$. The total costs are

$\Sigma_n tp_n[h_{P,n} - h_{I,n}]^+ + \Sigma_n tq_n[h_{I,n} - h_{P,n}]^+$

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