Monotonicity in Asymmetric First-Price Auctions with Affiliation

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Abstract

I study monotonicity of equilibrium strategies in first-price auctions with asymmetric bidders, risk-aversion, affiliated types, and interdependent values. I prove that every mixed-strategy equilibrium is outcome equivalent to a monotone pure strategy equilibrium under the "priority rule" for breaking ties. This provides a missing link to establish uniqueness in Milgrom and Weber (1982)'s "general symmetric model". Non-monotone equilibria can exist under the "coin-flip rule" but they are distinguishable: all non-monotone equilibria have positive probability of ties whereas all monotone equilibria have zero probability of ties. This provides a justification for the standard empirical practice of restricting attention to monotone strategies.

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1 Introduction

A large and growing literature, both empirical and experimental, studies first-price auctions with the goal of testing whether observed behavior is consistent with equilibrium and/or of estimating the distribution of unobservables taking equilibrium behavior as given.[1] In symmetric settings, these papers focus on the symmetric monotone pure strategy equilibrium ("MPSE") demonstrated by Milgrom and Weber (1982). In asymmetric settings, they focus on MPSE guaranteed to exist by Reny and Zamir (2004). In either case, the possibility of non-monotone (including mixed strategy) equilibria is effectively ignored. But if such equilibria exist, then the conclusions of this literature would be jeopardized since they might be based on selecting the "wrong equilibrium." This paper raises an unnoticed problem with this literature and then proposes a solution.

The problem is that first-price auctions can possess non-monotone equilibria. In fact, I provide an example in which a non-monotone equilibrium Pareto dominates all monotone equilibria, i.e. all bidders *and* the auctioneer are better off in a non-monotone equilibrium than in every monotone equilibrium. No one has ever observed non-monotone bidding in first-price auctions in the field or in the laboratory, but this might simply be because the standard tools used to study auction data implicitly *assume* monotonicity. (When other bidders adopt monotone strategies, each bidder's first-order condition at bid-level b can be expressed simply in terms of others' types who bid b. The standard approach in the empirical literature is to use such first-order conditions for identification purposes, but this is not valid if others adopt non-monotone strategies.)

My solution to this problem is three-fold. First, I show that the existence of non-monotone equilibria hinges crucially on the *combination* of affiliated

[1] Hendricks, Pinkse, and Porter (2003) provide an overview of recent empirical work. For a survey of experimental work see Kagel and Levin (2002).

types and interdependent values. If bidders have (i) affiliated private values or (ii) interdependent values but independent types, then all equilibria must be MPSE. (See the discussion in Section 4.) Second, the existence of non-monotone equilibria hinges crucially on the details of the tie-breaking rule. Consider an alternative "priority rule" under which each bidder is assigned a priority before the bidding takes place and, in the event of a tie, the bidder with the highest priority wins. Given the priority rule, all equilibria are monotone. (See Theorem 2.) Third, when they do exist under the standard coin-flip rule, non-monotone equilibria are distinguishable from monotone equilibria. In particular, all non-monotone equilibria have a positive probability of ties whereas all monotone equilibria have zero probability of ties. (See Theorem 1.) In fact, as discussed in Section 4, *three* or more bidders must submit the same bid with positive probability in any non-monotone equilibrium. Thus, one may conclude that non-monotone equilibrium is not being played if such three-way ties do not occur.

Proving that all equilibria are monotone has the extra benefit of implying equilibrium uniqueness in those special cases in which others have proven uniqueness of MPSE. In particular, until now it has remained an open question whether there is a unique equilibrium in Milgrom and Weber (1982)'s classic "general symmetric model". Milgrom and Weber (1982)'s argument implies that this equilibrium is the unique symmetric MPSE and McAdams (2004) proves that there are no asymmetric MPSE, but the hardest problem is ruling out non-monotone equilibria. Thus, this paper provides the missing link for proving uniqueness in one of the most widely used first-price auction models.

The most closely related paper is Rodriguez (2000). He proves in the case of two bidders that all equilibria in the first-price auction must be monotone "within the support of winning bids". My result is stronger in that (i) there are $n \geq 2$ bidders and (ii) all equilibria must be monotone over the range of all bids, including losing bids. (Result (ii) eliminates the possibility that there might be some non-monotone equilibrium that "looks like" it has monotone strategies when we restrict attention to the winning bids but which is not outcome-equivalent to any monotone equilibrium. See Lemma 2.) Like Rodriguez, I allow for asymmetric bidders, risk-aversion, affiliated types, and/or interdependent values.

The rest of the paper is organized as follows. Section 2 lays out the model. Section 3 sketches the proof of the main result. Section 4 provides an example of a non-monotone equilibrium and related discussion. Section 5 concludes the paper with some remarks and directions for future work. Since the analysis is quite technical, most formal proofs are relegated to an Appendix.

2 Model

There are n asymmetric bidders and one object. The assumptions on information and payoffs are identical to those of Reny and Zamir (2004).

Information: Bidder types are one-dimensional with joint density $f(\mathbf{t})$ on the unit cube $[0,1]^n$. For each subset $I \subset \{1,...,n\}$, the conditional joint density will be denoted $f(\mathbf{t}_I|\mathbf{t}_{-I})$ where $\mathbf{t} \equiv (t_1,...,t_n)$, $\mathbf{t}_I \equiv (t_i : i \in I)$, and $-I \equiv \{1,...,n\}\backslash I$. (Bold notation will be used throughout the paper to refer to vectors of types, bids, and strategies.) Each bidder also receives a "randomization variable" τ_i, where $\boldsymbol{\tau} = (\tau_1,...,\tau_n)$ are i.i.d. uniform on $[0,1]$ and independent of types.

(A1) $f(\cdot)$ is measurable and positive on $[0,1]^n$

(A2) Bidder types $\mathbf{t}$ are affiliated, i.e. $f(\mathbf{t}' \vee \mathbf{t})f(\mathbf{t}' \wedge \mathbf{t}) \geq f(\mathbf{t}')f(\mathbf{t})$ for all type profiles $\mathbf{t}', \mathbf{t}$, where $\vee$ and $\wedge$ denote component-wise maximum and minimum, respectively

Affiliation is a powerful form of positive correlation that allows us to establish that certain conditional expectations are non-decreasing. (See Milgrom and Weber (1982) for more detailed discussion.)

Payoffs: Bidder i's utility upon losing is zero and upon winning with bid b has form $u_i(\mathbf{t}; b)$. I make the following assumptions on utility: for all i,

(A3) u_i measurable and continuous in b,

(A4) u_i strictly increasing in t_i, non-decreasing in t_j for all $j \neq i$, and strictly decreasing in b,

(A5) $u_i(\mathbf{t}; b') - u_i(\mathbf{t}; b)$ non-decreasing in $\mathbf{t}$ for all $b' > b$, and

(A6) $u(\mathbf{1}; p^h) < 0$ for some bid level $p^h < \infty$.

Bids: After learning t_i and τ_i, each bidder submits bid $b_i(t_i; \tau_i) \in OUT \cup \mathcal{P}$. Under a "continuum price grid", $\mathcal{P} = [0, \infty)$. Otherwise under a "general price grid", $\mathcal{P}$ is an arbitrary closed subset of the real line, e.g. finite. Bidding is voluntary: a bidder who chooses not to participate "bids" OUT. When bidders do not randomize, I will use $b_i(t_i)$ to denote type t_i's bid. Also, to avoid tedious technical complications, I will assume that each bid function $b_i(t_i; \tau_i)$ is measurable in both t_i, τ_i and that the correspondence $\{b : b = b_i(t_i; \tau_i) \text{ for some } \tau_i\}$ is piecewise continuous in t_i.

Outcomes: If all bidders bid OUT, then the auction is cancelled. Otherwise for a given profile of bids $\mathbf{b}$, the winner is a bidder from the set of highest bidders, $I(\mathbf{b}) \equiv \arg\max_{1 \leq i \leq n} b_i$, and pays its bid. I will consider two sorts of tie-breaking rules:

Definition 1 (Tie-breaking by coin-flip): Each member of $I(\mathbf{b})$ wins with probability $1/\#(I(\mathbf{b}))$.

Definition 2 (Tie-breaking by priority): The winner is whomever has the highest priority among the tying bidders. That is, $\arg\max_{i \in I(\mathbf{b})} \rho(i)$

wins, where ρ is a permutation of $\{1, ..., n\}$ ("priority ranking") that is known before the bidding takes place.

Ultimately, both rules may be thought of as ranking the bidders and awarding the object to the member of $I(\mathbf{b})$ with the highest rank. The difference is that under the coin-flip rule bidders are ranked after they submit their bids whereas in the priority rule they are ranked before they bid.

Definition 3 (Serious bid): $b > OUT$ is a *serious bid* for bidder i given others' strategies $\mathbf{b}_{-i}(\cdot;\cdot)$ if it is high enough to win outright (win without tying) with positive probability, i.e. $\Pr_{\mathbf{t}_{-i},\boldsymbol{\tau}_{-i}}\left(b > \max_{j\neq i} b_j(t_j;\tau_j)\right) > 0$.

Definition 4 (Outcome-Equivalent ("OE")): Two strategy profiles $\mathbf{b}'(\cdot;\cdot)$ and $\mathbf{b}(\cdot;\cdot)$ are *outcome-equivalent* if the high bid and the set of bidders submitting the high bid are the same with probability one:

$$\Pr\left(\max_i b_i'(t_i;\tau_i) = \max_i b_i(t_i;\tau_i)\right) = 1 \text{ and}$$
$$\Pr\left(\arg\max_i b_i'(t_i;\tau_i) = \arg\max_i b_i(t_i;\tau_i)\right) = 1$$

Equivalently, $\mathbf{b}'(\cdot;\cdot)$ and $\mathbf{b}(\cdot;\cdot)$ are OE iff $b_i'(t_i;\tau_i) = b_i(t_i;\tau_i)$ for all bidders i and a full measure set of those $(t_i;\tau_i)$ for which *either* $b_i'(t_i;\tau_i)$ or $b_i(t_i;\tau_i)$ is a serious bid.

Definition 5 (Equilibrium, Monotone): The profile $\mathbf{b}^*(\cdot;\cdot)$ is a *mixed strategy equilibrium* ("MSE") when, for each bidder i and all pairs (t_i,τ_i), $b_i^*(t_i,\tau_i)$ is a best response, i.e. maximizes bidder i's expected payoff conditional on t_i and others' strategies. The MSE $\mathbf{b}^*(\cdot;\cdot)$ is *monotone* iff $\inf b_i(t_i') \geq \sup b_i(t_i)$ for all $t_i' > t_i$. A monotone pure strategy equilibrium ("MPSE") is a monotone MSE in which $b_i^*(t_i;\tau_i') = b_i^*(t_i;\tau_i) \equiv b_i^*(t_i)$ for all t_i, τ', τ.

Notes: (a) Any monotone MSE involves mixing by at most countably many types and hence is outcome-equivalent to a MPSE. Thus, this paper's inquiry is ultimately directed at whether equilibria are monotone or non-monotone, not at whether they are pure or mixed. (b) The full-support assumption on the joint distribution of bidders' types is important to the results since extreme positive correlation of bidders' types can lead to existence of non-monotone equilibria. For a simple example, suppose that there are three bidders with private values that are always equal and distributed over $[0, 1]$. In addition to the MPSE in which $b_i(v_i) = v_i$ for each bidder i, mixed-strategy equilibria exist that are not outcome-equivalent to any MPSE. In one such equilibrium, $b_i(v_i) = v_i$ for $i = 1, 2$ but bidder 3 adopts a mixed strategy: $b_3(v_i; \tau_i) = 2\tau_i v_i$ when $\tau_i \in [0, 1/2]$ and $b_3(v_i; \tau_i) = v_i$ if $\tau_i \in [1/2, 1]$. No outcome-equivalent monotone strategy profile exists to this mixed strategy equilibrium, so certainly no outcome-equivalent MPSE exists.

3 Monotone Equilibria

Reny and Zamir (2004) prove that monotone pure strategy equilibrium ("MPSE") exists. The question addressed here is whether any mixed strategy equilibria ("MSE") exist that are not outcome-equivalent to MPSE.

Definition 6 (Zero probability of ties): A mixed-strategy equilibrium $\mathbf{b}^*(\cdot;\cdot)$ has zero probability of ties if $\Pr_{\mathbf{t},\boldsymbol{\tau}}\left(b_i^*(t_i, \tau_i) = b_j^*(t_j, \tau_j)\right) = 0$ for all pairs of bidders i, j.

Theorem 1: *Given tie-breaking by coin-flip and a continuum price grid, every MSE in the first-price auction with zero probability of ties is outcome-equivalent to some MPSE.*

Theorem 2: *Given tie-breaking by priority and a general price grid, every MSE in the first-price auction is outcome-equivalent to some MPSE.*

3.1 Proof sketch for Theorems 1,2

For a relatively simple illustration of what drives equilibria to be monotone, consider an equilibrium with n bidders and three simplifying properties: for

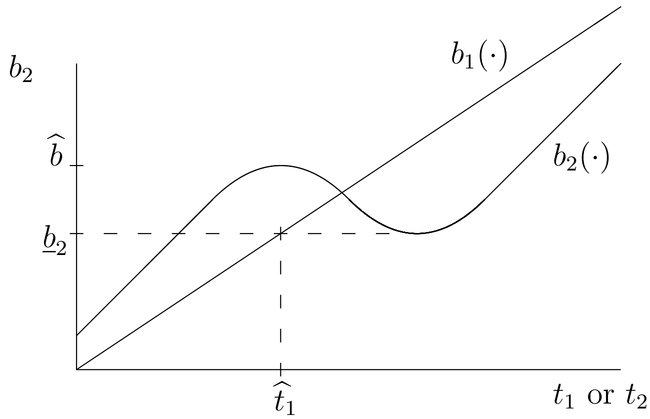

Figure 1: "Lowest trough" $\underline{b}_2$ of $b_2(\cdot)$ and bidder 1's threshold type $\widehat{t}_1$.

each i, (i) bidder i follows a pure strategy $b_i^*(\cdot)$, (ii) $\Pr(b_i^*(t_i) = b) = 0$ for all serious bids b and (iii) $\Pr(b_i^*(t_i) > \max_{j \neq i} b_j^*(t_j)) > 0$ for all $t_i > 0$. Property (i) rules out mixed strategies, (ii) implies that there are no atoms in equilibrium strategies, and (iii) implies that all bid types $t_i > 0$ make serious bids. Needless to say, these restrictions involve loss of generality and are not made in the Appendix proofs.

Presumption (ii) rules out atoms, so the details of the tie-breaking rule will be irrelevant in this proof sketch. For now, suffice it to say that the important difference between the priority rule and the coin-flip rule is that "ties are impossible" given the priority rule. More precisely, each bidder's probability of winning the object is always either 0% or 100% depending on others' bids, never 50%, 33%, etc.

Preparation: "lowest trough". The analysis will leverage a measure (the "lowest trough") of the extent to which a given strategy is non-monotone. Some other definitions will be useful as well. (The Appendix contains more general versions of some of these definitions, allowing for mixed strategies.)

Definition 7 (Decreasing/Increasing set): A subset $A \subset S \subset \mathcal{R}^k$

is *decreasing in* S if $x \in A$ implies $y \in A$ for all $y < x \in S$. Similarly, $A \subset S \subset \mathcal{R}^k$ is *increasing in* S when $S \backslash A$ is decreasing in S.

Definition 8 (Lowest trough): The *lowest trough* $\underline{b}_i(b_i(\cdot))$ (shorthand $\underline{b}_i$) of pure strategy $b_i(\cdot)$ is the supremum of all bid levels b such that $\{t_i : b_i(t_i) < x\}$ is decreasing in $[0,1]$ for all $x < b$. (The set of types that bid less than *or equal to* $\underline{b}_i$ need not be decreasing in $[0,1]$.) Similarly, let $\underline{b}_{-i} \equiv \min_{j \neq i} \underline{b}_j$.

Definition 9 (Threshold type): Bidder i has *threshold type* $\widehat{t_i}$ when $b_i(t_i) < \underline{b}_{-i}$ for all $t_i < \widehat{t_i}$ and $b_i(t_i) \geq \underline{b}_{-i}$ for all $t_i > \widehat{t_i}$.

The lowest trough $\underline{b}_i$ is well-defined since $\{t_i : b_i(t_i) < OUT\} = \emptyset$ is empty and hence trivially a decreasing set. Note that $b_i(\cdot)$ is monotone iff $\underline{b}_i = \infty$. Some bidders may not have a threshold type. Indeed, an important step in the analysis will be to *prove* that each bidder has one. See Lemma 1. (While threshold types play no explicit role in this proof sketch, they are central to the analysis in the Appendix.) Figure 1 provides a two-bidder illustration.

Preparation: affiliation tools. My workhorse is Theorem 23 from Milgrom and Weber (1982) regarding properties of expectations of affiliated random variables. Theorem 3 is a corollary of this powerful result.

Definition 10 (Lattice): $X \subset \mathcal{R}^k$ is a *lattice* when $(x_1^1, ..., x_n^1)$,$(x_1^2, ..., x_n^2) \in X$ implies $(\min\{x_1^1, x_1^2\}, ..., \min\{x_n^1, x_n^2\})$,$(\max\{x_1^1, x_1^2\}, ..., \max\{x_n^1, x_n^2\}) \in X$.

Theorem 3: *Let* $\mathbf{t} = (t_1, ..., t_n)$ *be affiliated and suppose that* $g_i : [0,1]^n \to \mathcal{R}$ *is non-decreasing in* $\mathbf{t}$ *and strictly increasing in* t_i. *Let* $X_{-i}, Y_{-i} \subset [0,1]^{n-1}$ *where* Y_{-i} *is a lattice and* X_{-i} *is decreasing in* Y_{-i}. *Then, for all fixed* $t_i \in [0,1]$ *and all* $t_i' > t_i$,

(a) $$E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in X_{-i}, t_i\right] \leq E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i\right] \leq E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i} \backslash X_{-i}, t_i\right]$$

(b) $$E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i\right] < E\left[g(t_i'; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i'\right]$$

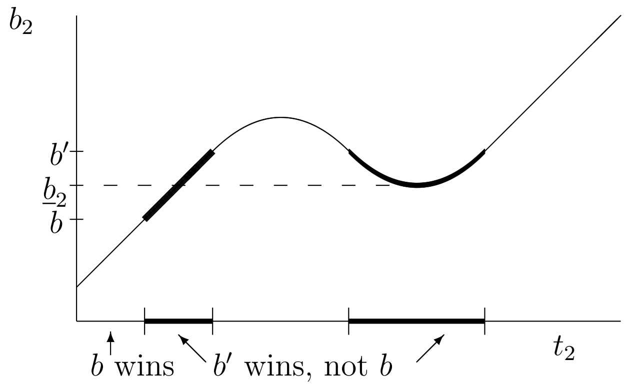

Figure 2: Bids b', b against non-monotone $b_2(\cdot)$

Proof. For all $t_i \in [0,1]$, $t_i \times X_{-i}$ is a decreasing set in the lattice $t_i \times Y_{-i}$. Thus, (a) follows immediately from Milgrom and Weber (1982)'s Theorem 23. Similarly, for all $t_i' > t_i$, $t_i \times Y_{-i}$ is a decreasing set in the lattice $\{t_i, t_i\} \times Y_{-i}$ so that $E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i\right] \leq E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i'\right]$. Since $g(\cdot)$ is strictly increasing in t_i, further, $E\left[g(t_i; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i'\right] < E\left[g(t_i'; \mathbf{t}_{-i}) | \mathbf{t}_{-i} \in Y_{-i}, t_i'\right]$. This proves (b). $\square$

"Preference reversals". We need to prove that there is no trough in any bidder's equilibrium bid function: $\underline{b}_i = \infty$ for all i. It suffices to show that the lowest trough in (say) bidder 1's equilibrium bid function is strictly higher than the lowest trough of someone else's equilibrium bid function: $\max_{i \neq 1} \underline{b}_i < \infty \Rightarrow \underline{b}_1 > \max_{i \neq 1} \underline{b}_i$. Without loss, consider bidder 1. Bidder 1 can only have a trough in his equilibrium bid function if he has a "preference reversal". That is to say, there exists bids $b' > b$ and types $t_1' > t_1$ such that the lower type t_1 prefers the higher bid b' while the higher type t_1' prefers the lower bid b. The heart of the analysis, then, is to determine when such a preference reversal is possible.

No preference reversal between (b, b') *when* $b \leq \underline{b}_{-1}$. Let $b \leq \underline{b}_{-1}$ and $b' > b$.

Consider the trade-offs associated with bidding b' versus b in terms of three events, as illustrated in Figures 2, 3.

$$\text{“}b'\text{ wins”} \equiv \{\mathbf{t}_{-1} : \max_{j \neq 1} b_j(t_j) < b'\}$$

$$\text{“}b\text{ wins”} \equiv \{\mathbf{t}_{-1} : \max_{j \neq 1} b_j(t_j) < b\}$$

$$\text{“}b'\text{ wins, not }b\text{”} \equiv \{\mathbf{t}_{-1} : \max_{j \neq 1} b_j(t_j) \in (b, b')\}$$

(i) If $\mathbf{t}_{-1} \in$ “b wins”, then bidder 1 wins regardless of whether it bids b' or b and so clearly prefers to win with the lower bid. (ii) If $\mathbf{t}_{-1} \in$ “b' wins, not b”, then bidding b' instead of b leads bidder 1 to win sometimes when b would have lost. (iii) If $\mathbf{t}_{-1} \notin$ “b' wins”, then both bids lose and bidder 1 is clearly indifferent between them.

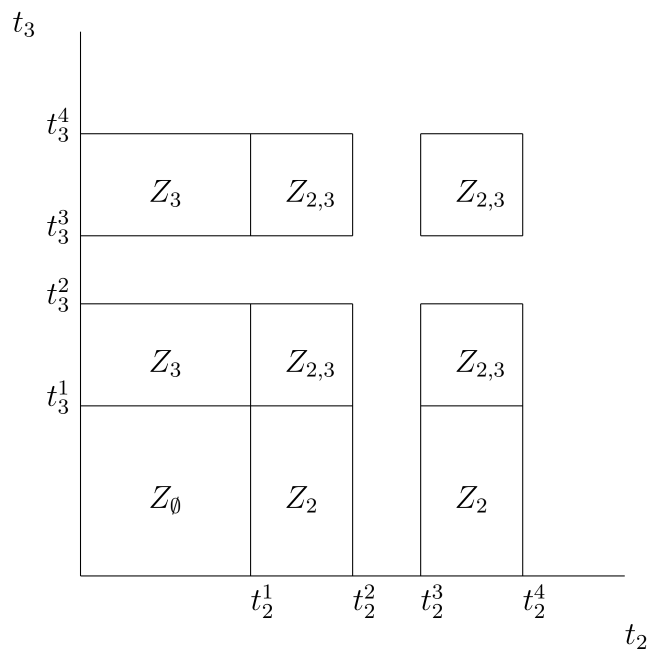

Figure 3: $Z_\emptyset =$ “b wins”. $Z_2 \cup Z_3 \cup Z_{2,3} =$ “b' wins, not b”

For the purposes of applying Theorem 3(a,b) later, note that “b wins” is decreasing in the lattice "b' wins". It will also be useful to partition "b' wins" into sets $\{Z_J : J \subset \{2, ..., n\}\}$ defined by $Z_J \equiv \{\mathbf{t}_{-1} : b_i(t_i) \in (b, b') \; \forall i \in J, b_i(t_i) < b \; \forall i \notin \{2, ..., n\} \backslash J\}$. ($Z_\emptyset$ is the event "b wins".) Note that Z_J is a lattice and that $Z_\emptyset$ is decreasing in the lattice $Z_\emptyset \cup Z_J$ for all $J \subset \{2, ..., n\}$. Figure 3 provides an example when $n = 3$ and bidders 2,3 follow a strategy with a single trough as in Figure 2 such that $b_j^*(t_j) \in [0, b)$ for all $t_j \in [0, t_j^1)$, $b_j^*(t_j) \in (b, b')$ for $t_j \in (t_j^1, t_j^2) \cup (t_j^3, t_j^4)$, and $b_j^*(t_j) > b'$ for $t_j \in (t_j^2, t_j^3) \cup (t_j^4, 1]$. ($Z_2, Z_3, Z_{2,3}$ are not connected in Figure 3.)

Suppose for the sake of contradiction that $\underline{b}_1 \leq \underline{b}_{-1} < \infty$. This means that there exists some pair of bids $b^h > b^l$ and pair of types $t_1^h > t_1^l$ such that (i) $b^l \leq \underline{b}_{-1}$, (ii) $b^h = b_1^*(t_1^l)$, and (iii) $b^l = b_1^*(t_1^h)$.[2] In particular, $EIU(t_1^l) \geq 0$ and $EIU(t_1^h) \leq 0$, where $EIU(t_1)$ is type t_1's expected incremental utility from bidding b^h rather than b^l:

$$\begin{aligned} EIU(t_1) &= \sum_{J \subseteq \{2,\ldots,n\}} \Pr(Z_J | t_1) \alpha(J, t_1) \\ &= \Pr(\text{"}b^h \text{ wins"} | t_1) E\left[\alpha(J(\mathbf{t}_{-1}), t_1) | t_1, \mathbf{t}_{-1} \in \text{"}b^h \text{ wins"}\right] \end{aligned}$$

where $J(\mathbf{t}_{-1}) = J$ iff $\mathbf{t}_{-1} \in Z_J$, and

$$\begin{aligned} Z_J &\equiv \left\{\mathbf{t}_{-1} : b_j(t_j) \in (b^l, b^h) \text{ for } j \in J, b_j(t_j) < b^l \text{ for } j \notin J\right\} \\ \alpha(\emptyset, t_1) &\equiv E[u_1(t_1, \mathbf{t}_{-1}, b^h) - u_1(t_1, \mathbf{t}_{-1}, b^l) | t_1, \mathbf{t}_{-1} \in Z_\emptyset] \\ \alpha(J, t_1) &\equiv E[u_1(t_1, \mathbf{t}_{-1}, b^h) | t_1, \mathbf{t}_{-1} \in Z_J] \text{ for } J \neq \emptyset \end{aligned}$$

By Theorem 3(a), $\alpha(J^2, t_1) \geq \alpha(J^1, t_1)$ whenever $J^2 \supset J^1$. Thus, $\alpha(J(\mathbf{t}_{-1}), t_1)$ is non-increasing in $\mathbf{t}_{-1}$. Similarly, Theorem 3(b) implies that $\alpha(J(\mathbf{t}_{-1}), t_1)$ is strictly increasing in t_1.

By presumption, type t_1^l at least weakly prefers bid b^h:

$$0 \leq EIU(t_1^l) \equiv \Pr(\text{"}b^h \text{ wins"} | t_1^l) E[\alpha(\mathbf{t}_{-1}, t_1^l) | t_1^l, \mathbf{t}_{-1} \in \text{"}b^h \text{ wins"}]$$

[2] Actually, matters are slightly more complex. It is possible that no bidder 1-type bids b^l but that there is a sequence of types $t_1^{h,\varepsilon} \nearrow t_1^h$ whose bids $b_1^*(t_1^{h,\varepsilon}) \searrow b^l$. See the Appendix.

But then Theorem 3(b) implies

$$E[\alpha(\mathbf{t}_{-1}, t_1^h)|t_1^h, \mathbf{t}_{-1} \in \text{“}b^h \text{ wins”}] > E[\alpha(\mathbf{t}_{-1}, t_1^l)|t_1^l, \mathbf{t}_{-1} \in \text{“}b^h \text{ wins”}]$$

so that

$$0 < \Pr(\text{“}b^h \text{ wins”}|t_1^h)E[\alpha(\mathbf{t}_{-1}, t_1^h)|t_1^h, \mathbf{t}_{-1} \in \text{“}b^h \text{ wins”}] \equiv EIU(t_1^h)$$

But then type t_1^h can not weakly prefer b^l. This contradiction finishes the proof given the extra assumptions made for this section. The proof in the Appendix follows the same outline; the major complications are that bidders may randomize, may tie with positive probability, and may have a positive measure of types that never win the object.

4 Ties and Non-Monotone Equilibria

In their work studying properties of monotone equilibria in asymmetric first-price auctions, Maskin and Riley (2000) (hereafter MR) dispense with the possibility of ties in a brief aside (pg. 453): “At most one bidder can bid b [a serious bid] with positive probability since, under any tie-breaking rule, at least one buyer would be strictly better off bidding slightly more than b.” The logic behind this result is very clear when there are just two bidders. Consider bidder 1's situation, supposing that bidder 2 follows a monotone strategy in which $b_2(t_2) < b$ for $t_2 < t_2'$ and $b_2(t_2) = b$ for $t_2 \in [t_2', t_2'']$.

1. Any type $\hat{t}_1$ that bids b must get positive expected payoff.

2. By Theorem 3(a),

$$E[u_1(t_1, t_2, b)|t_1 = \hat{t}_1, t_2 \in [t_2', t_2'']] \geq E[u_1(t_1, t_2, b)|t_1 = \hat{t}_1, t_2 \leq t_2'']$$

   In other words, when bidder 1 wins the object it is “good news” for him to learn that he tied.

3. Consequently, $E[u_1(\hat{t}_1, t_2, b)|t_2 \in [t_2', t_2'']] > 0$ and bidder 1 strictly prefers to bid slightly more given type $\hat{t}_1$ as long as doing so increases its chances of winning.

(This argument applies for all tie-breaking rules since some bidder must expect to be able to increase its chances of winning by bidding $b + \varepsilon$. Furthermore, it can be adapted to the case in which bidder 2 adopts a non-monotone strategy. For this reason, there can not be an equilibrium with positive probability of ties unless at least *three* bidders tie with positive probability.) Once there are several bidders, however, it is not necessarily "good news" to learn that one has tied with more bidders. For instance, when there are four or more other bidders, I may prefer to win with bid b in the event that I tie with two other bidders but prefer to lose if I tie with three other bidders, with the further twist that my probability of winning depends on the number of others that I tie with. This makes the issue of ruling out ties between three or more bidders quite complex.

In the end, I prove in Theorem 4 that MR's claim is correct in the context of the coin-flip rule. Given that others have adopted monotone strategies, your best response strategy will not lead you to tie with others with positive probability. (The more general claim vis-a-vis *all* tie-breaking rules remains to be verified and, indeed, may not be correct.)

THEOREM 4: *Given tie-breaking by coin-flip, ties occur with zero probability in every MPSE of the first-price auction.*

Proof. Follows from Theorem 5 in the Appendix. □

Theorems 1 and 4 leave open the possibility that there might be non-monotone equilibria with a positive probability of ties. Now I demonstrate such an equilibrium. It has the further property that all bidders and the auctioneer are better off in this equilibrium than in any monotone one. One might expect the presence of non-monotonicities to decrease the total surplus generated by the auction since higher types bidding lower than lower types leads to a more inefficient allocation. Another effect when some bidders adopt non-monotone strategies, however, is that others face a weaker "winner's curse": the expected value of the object conditional on winning with a low bid is higher if others submit even lower bids when they have relatively high types. This effect can increase total surplus if it leads to greater participation.

Example: Non-Monotone Equilibrium in the First-Price Auction

Three bidders $i = 1, 2, 3$ each have type $t_i \in \{L, H\}$.[3] These types $(t_1, t_2, t_3) = \mathbf{t}$ are affiliated and each bidder i's valuation $v_i(\mathbf{t})$ for the object is non-decreasing in $\mathbf{t}$. After receiving its type, each bidder chooses whether to participate and, if so, makes a bid $b_i \geq p^{min}$. The non-monotone equilibrium I exhibit will have the form

$$\begin{aligned} b_1^*(L) &= p^{min}, \quad b_1^*(H) = OUT \\ b_2^*(L) &= p^{min}, \quad b_2^*(H) = OUT \\ b_3^*(L) &= OUT, \quad b_3^*(H) = p^{min} \end{aligned}$$

Before fully specifying a concrete example it is helpful to examine the key preference reversal, how bidder 1 (say) could prefer bidding p^{min} to OUT upon receiving a low type but prefer OUT to p^{min} given a high type.

Ultimately, it must somehow be "bad news" for bidder 1 that she has received a higher type that she then prefers to lose for sure rather than sometimes win with bid p^{min}. To see how this could be the case, suppose for the sake of argument that (i) bidder 1 doesn't care about its own signal and gets high value $V_h > p^{min}$ from winning when $t_2 = H$ but a low value $V_l < p^{min}$ when $t_2 = L$, that (ii) bidders 1,3 have somewhat positively correlated types, and (iii) bidders 1,2 have independent types with (say) $\Pr(t_2 = L) =$

[3]Since all relevant bidder preferences are strict, one can easily modify the example presented here to fit within this paper's model, i.e. to settings in which $\mathbf{t}$ is drawn from an atomless distribution on $[0, 1]^3$ and in which bidder valuations are strictly increasing in own type, continuous, etc..

... with bid 100

t_2 \ t_3	L	H
H	100%	50%
L	50%	33%

... with bid $100 + \varepsilon$

t_2 \ t_3	L	H
H	100%	100%
L	100%	100%

Figure 4: Probability that bidder 1 wins ...

50%. (In the example, condition (iii) is not satisfied but is assumed here just to simplify the discussion.) Bidder 1's probability of winning with bid p^{min} is 100% if $(t_2, t_3) = (H, L)$, 50% if $(t_2, t_3) \in \{(L, L), (H, H)\}$, and 33% if $(t_2, t_3) = (L, H)$, as summarized in Figure 4. Conditional on bidder 3 having a low type, bidder 1 assesses conditional probability $\Pr(t_2 = H | t_3 = L,$ bidder 1 wins with bid $p^{min}) = 100/(100+50) = 67\%$ that the object has a high value to her. On the other hand, when bidder 3 has a high type, this conditional probability falls to $50/(50 + 33) = 60\%$. Thus, bidder 3's having a high type is bad news for bidder 1 vis-a-vis whether she wants to win with bid p^{min}. Since $t_3 = H$ is more likely when $t_1 = H$, having a high type herself is bad news for bidder 1 unless the direct benefit associated with having a higher type is large enough to offset this indirect effect.

Now to specifics. The joint probability of profile $(L, L, L) = a$, that of $(L, H, L) = (H, L, L) = b$, etc.. where $a, ..., f$ are determined by the probability ratios $b/a = c/b = d/c = 3/5$, $e/d = f/e = 2/3$, as depicted in Figure 5. $\mathbf{t}$ then is affiliated: as can be easily checked, a sufficient condition for affiliation is that $b/a \leq c/b \leq d/c \leq e/d \leq f/e$. Note that t_1, t_2, t_3 are independent when $b/a = c/b = d/c = e/d = f/e$, so this example is a relatively small departure from the independent case, for which one can easily

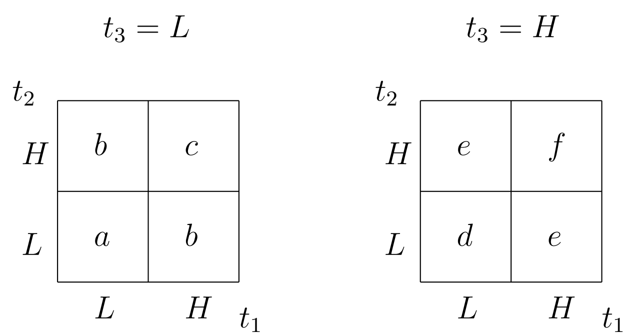

Figure 5: Joint densities in ratio $b/a = c/b = d/c = 3/5$, $e/d = f/e = 2/3$

prove that all equilibria must be monotone. (Consequently, while bidders' preferences are constructed to be strict, it is not surprising that some bidders turn out to be nearly indifferent between bids OUT, p^{min}.)

Let $p^{min} = 100$. Bidders' values for the object are non-decreasing in types. Bidders 1,2 are symmetric and care only about the other's type: $v_1(\mathbf{t}) = 135$ if $t_2 = H$ and $v_1(\mathbf{t}) = 59$ if $t_2 = L$, and vice versa for bidder 2. Bidder 3 strictly prefers to win with bid 100 only if $t_1 = t_2 = t_3 = H$. In particular, $v_3(\mathbf{t}) = 0$ if $t_3 = L$; otherwise, $v_3(\mathbf{t}) = 70$ if $(t_1, t_2) = (L, L)$, $v_3(\mathbf{t}) = 100$ if $(t_1, t_2) \in \{(L, H), (H, L)\}$, and $v_3(\mathbf{t}) = 200$ if $(t_1, t_2) = (H, H)$.

Claim 1: $\mathbf{b}^M(\cdot)$ *is a monotone equilibrium, where* $b_1^M(L) = b_1^M(H) = b_2^M(L) = b_2^M(H) = b_3^M(L) = OUT$ *and* $b_3^M(H) = 100$.

Proof. Bidders 1,2 are symmetric. Consider bidder 1. If he bids $b > 100$, he wins the object and gets negative expected surplus given either $t_1 = L$ since $(135 - 100)(b + e) + (59 - 100)(a + d) < 0$ or $t_1 = H$ since $(135 - 100)(c + f) + (59 - 100)(b + e) < 0$. Similarly, if he bids $b = 100$, his expected surplus conditional on winning is negative given either $t_1 = L$ since $(135-100)(b+e/2)+(59-100)(a+d/2) < 0$ or $t_1 = H$ since $(135-100)(c+f/2) + (59 - 100)(b + e/2) < 0$. Thus, his best response is always not to participate. Consider bidder 3. Her best response must be either 100 or OUT. Obviously $t_3 = L$ prefers OUT. The high type, however, gets positive expected surplus since $(200-100)f + (100-100)2e + (70-100)d > 0$. $\square$

CLAIM 2: *$\mathbf{b}^{NM}(\cdot)$ is also an equilibrium, where $b_1^{NM}(H) = b_2^{NM}(H) = b_3^{NM}(L) = OUT$ and $b_1^{NM}(L) = b_2^{NM}(L) = b_3^{NM}(H) = 100$. Bidders $1, 2$ adopt non-monotone strategies and each bidder always strictly prefers his bid over any other.*

Proof. We will first determine best responses for bidder 1. (Argument symmetric for bidder 2.) Given that bidders 2,3 never bid greater than 100, there are three possibilities: (a) OUT is his best response; (b) 100 is his best response; or (c) every bid in a neighborhood $(100, 100+\varepsilon)$ is preferred to both OUT and 100, in which case no best response exists. Bidding $100+\varepsilon$ allows bidder 1 to win the object with probability one regardless of others' types. In notation shorthand,

$$\begin{aligned} &\Pr(\text{“}100+\varepsilon \text{ wins”}|(L,L)) = \Pr(\text{“}100+\varepsilon \text{ wins”}|(L,H)) \\ &= \Pr(\text{“}100+\varepsilon \text{ wins”}|(H,L)) = \Pr(\text{“}100+\varepsilon \text{ wins”}|(H,H)) = 1 \end{aligned}$$

Bidding 100 leads him to win when $(t_2, t_3) = (H, L)$, tie with one other bidder when $(t_2, t_3) \in \{(H,H),(L,L)\}$, and tie with two bidders when $(t_2, t_3) = (L, H)$:

$$\begin{aligned} &\Pr(\text{“}100 \text{ wins”}|(L,L)) = 1/2, \Pr(\text{“}100 \text{ wins”}|(L,H)) = 1/3 \\ &\Pr(\text{“}100 \text{ wins”}|(H,L)) = 1, \Pr(\text{“}100 \text{ wins”}|(H,H)) = 1/2 \end{aligned}$$

After the null bid, of course, he never wins the object.

Null bid preferred to $100+\varepsilon$ by all types. When $t_1 = L$, the payoff to bidding $100+\varepsilon$ is negative as it is approximately: $(135-100)(b+e)+(59-100)(a+d) < 0$ since $35(\frac{3}{5}+\frac{4}{25}) - 41(1+\frac{6}{25}) = \frac{-606}{25}$. Similarly, when $t_1 = H$, the null bid is also preferred to $100+\varepsilon$: $(135-100)(c+f)+(59-100)(b+e) < 0$ since $35(\frac{9}{25}+\frac{8}{75}) - 41(\frac{3}{5}+\frac{4}{25}) = \frac{-1112}{75}$.

100 preferred to null bid by low type. When $t_1 = L$, bidding 100 yields positive payoff: $(135 - 100)(b + e/2) + (59 - 100)(a/2 + d/3) > 0$ since $35(\frac{3}{5} + \frac{2}{25}) - 41(\frac{1}{2} + \frac{2}{25}) = \frac{1}{50}$. Thus, bidder 1's best response is to bid 100 upon receiving a low type.

Null bid preferred to 100 by high type. When $t_1 = H$, bidding 100 yields negative payoff: $(135 - 100)(c + f/2) + (59 - 100)(b/2 + e/3) < 0$ since $35(\frac{9}{25} + \frac{4}{75}) - 41(\frac{3}{10} + \frac{4}{75}) = \frac{-1}{25}$. Thus, bidder 1's best response is to submit the null bid upon receiving a high type.

Finally, we determine best responses for bidder 3. Clearly she submits the null bid when $t_3 = L$. When $t_3 = H$, any bid $100 + \varepsilon$ leads her to always win for expected payoff approximately $(110-100)(e+e+f)+(85-100)d > 0$ since $10(\frac{2*4}{25} + \frac{8}{75}) - 15\frac{6}{25} = \frac{2}{3}$. To conclude the equilibrium verification, then, I need to show that she prefers to submit 100 over any bid $100+\varepsilon$. The payoff to 100 is $(110-100)(\frac{e}{2}+\frac{e}{2}+f)+(85-100)\frac{d}{3} = 10(\frac{4}{25}+\frac{8}{75})-15\frac{2}{25} = \frac{22}{15} > \frac{2}{3}$. □

Claim 3: $\mathbf{b}^{NM}(\cdot)$ *Pareto dominates* $\mathbf{b}^{M}(\cdot)$.

Proof. Bidders 1,2 are better off since they now get positive utility upon receiving low types. The auctioneer gets more revenue since now it sells the object whenever $t_1 = L$, $t_2 = L$, or $t_3 = H$. Bidder 3 is also better off since she still always wins the object for sure in the good event $(t_1, t_2) = (H, H)$ but now only with probability 1/3 in the bad event $(t_1, t_2) = (L, L)$. □

There are several essential aspects to this example: (1) bidder values are not private and (2) bidder types are not independent. Given either private values or independent types, one can easily prove that ties can not occur in equilibrium. (It suffices to show that each bidder's expected utility *conditional on tying* is increasing in own value. Given private values, this is obvious. Given independent signals, this follows from the fact that ex post utility is increasing in own value.) (3) Some bidders adopt non-monotone strategies *and* ties occur with positive probability. If bidders adopt monotone strategies, then

Theorem 4 shows that ties must occur with zero probability. Conversely, if ties occur with zero probability, Theorem 1 implies that all bidders must adopt monotone strategies. (4) Bidders are asymmetric. I have not managed to prove this, but I believe that no non-monotone equilibrium can exist given symmetric bidders. Some other aspects of the example may also be important: (a) some bidders' valuations are more sensitive to others' types than to their own and (b) ties occur exactly at the minimal permissible bid. (If $p^{min} = 100 - 2\varepsilon$ in the example below, bidder 1 would prefer to deviate with bid $100 - \varepsilon$ since then it would only win the object in the event that $t_2 = H, t_3 = L$.)

5 Concluding Remarks

When bidders have independent private values, it is well known that every equilibrium in the first-price auction must be outcome-equivalent to some monotone pure strategy equilibrium ("MPSE"). Once independence and/or private values is relaxed, however, existing theory is silent on whether non-monotone equilibria may exist. Yet in such models standard empirical identification approaches implicitly assume that monotone strategies are being played. In a model allowing for asymmetric bidders, risk aversion, affiliated types, and interdependent values, this paper provides the first theoretical justification for restricting attention to monotone strategies. For one thing, as discussed in the text, non-monotone equilibria can only exist in situations in which *both* independence and private values are relaxed.

Suppose instead that one wishes to study a model having both affiliated types and interdependent values. Given the coin-flip rule and a continuum price grid, one can reject the possibility of non-monotone equilibria as soon as one can reject the possibility that bidders tie with positive probability (Theorem 1). In most empirical applications, of course, bids must be made in discrete units so that ties can not be avoided. In such cases one can still rule out non-monotone equilibrium a priori if ties are broken using the priority rule (Theorem 2).

While my focus has been on asymmetric first-price auctions, all of the results hold as well for symmetric models. In particular, as discussed in the introduction, given existing results this paper proves that the symmetric MPSE in Milgrom and Weber (1982)'s "general symmetric model" is in fact its unique mixed strategy equilibrium. (More precisely, this is the unique equilibrium under the priority rule and the unique equilibrium having zero probability of ties under the coin-flip rule.)

Other tie-breaking rules than the coin-flip rule have been studied that involve selecting a random winner. For instance, in their proof of MPSE existence, Maskin and Riley (2000) make all tying bidders compete in a second-price auction with the coin-flip rule to break further ties. My proof approach does not extend to this tie-breaking rule nor does it apply to the second-price auction. Indeed, Reny and Zamir (2004) provide an example showing that all equilibria of the second-price auction may be non-monotone. If bidders were to adopt such non-monotone strategies in Maskin and Riley (2000)'s second bidding round, this might support non-monotone bidding in the first round as well.

Lastly, it is worth noting some assumptions of the model that interesting future work might attempt to relax:

One-dimensional types: Reny and Zamir (2004) provide a first-price auction example with multi-dimensional affiliated types in which all equilibria are non-monotone. In more specialized models that still allow for positively correlated types, however, it remains an open question whether some or all equilibria are monotone. In particular, existence of MPSE is unknown even in symmetric first-price auctions given multi-dimensional affiliated types.

Affiliated types: Affiliation is a very strong distributional assumption which has become widely used in the auction literature primarily for its analytical convenience. The strong results derived here rest in large part on affiliation. It remains an open question whether a weaker distributional assumption suffices even for existence of MPSE.

Independent payoff-irrelevant signals: In the analysis here, bidders receive independent payoff-irrelevant signals $\boldsymbol{\tau} = (\tau_1, ..., \tau_n)$. The conclusion that all mixed strategy equilibria are outcome-equivalent to MPSE implies, among other things, that bidders will never condition their bids on such signals. It remains an open question whether bidders will ever condition their bids on *correlated* payoff-irrelevant information.

Appendix

Proof of Theorem 1

The proof is divided into five parts. Most arguments to follow are framed in terms of bidder 1 but, of course, they apply to all bidders.

Part I. No Ties at b if Strategies Monotone up to b. I begin with several definitions, most of which are more general versions of definitions made in the text:

Definition 11 (Less-than sets $W_j(b), W_j^*(b)$): Type t_j belongs to the "sometimes less-than set" $W_j(b)$ when it bids weakly less than b with positive probability and belongs to the "always less-than set" $W_j^*(b)$ when it never bids more than b:

$$W_j(b) \equiv \left\{t_j : \Pr{}_{\tau_j} \left(b_j^*(t_j;\tau_j) \leq b\right) > 0\right\},$$
$$W_j^*(b) \equiv \left\{t_j : \Pr{}_{\tau_j} \left(b_j^*(t_j;\tau_j) \leq b\right) = 1\right\},$$

Definition 12 (Lowest trough $\underline{b}_j, \underline{b}_{-j}$): The "lowest trough" of bidder j's strategy, $\underline{b}_j(b_j^*(\cdot;\cdot))$ (shorthand $\underline{b}_j$), is the supremum of the set of bid levels b such that, for all $x < b$, there exists $\tilde{t}_j^x$ such that $W_j(x) \subset [0, \tilde{t}_j^x]$

and $W_j^*(x) \supset [0, \tilde{t}_j^x)$. (This condition is satisfied vacuously for $b = OUT$ so that $\underline{b}_j$ is well-defined.) Similarly, $\underline{b}_{-j} \equiv \min_{i \neq j} \underline{b}_i$ is the lowest lowest trough across all other bidders.

Definition 13 ("Monotone up to b"): Bidder j's strategy $b_j^*(\cdot;\cdot)$ will be said to be "monotone up to b" when $\underline{b}_j \geq b$.

Definition 14 ("No Ties at b"): There are "no ties at b" given strategy profile $\mathbf{b}(\cdot;\cdot)$ if $\Pr_{\mathbf{t};\boldsymbol{\tau}}(b = b_{j_1}(t_{j_1};\tau_{j_1}) = b_{j_2}(t_{j_2};\tau_{j_2})) = 0$ for all j_1, j_2. Similarly, there are "no ties" if there are no ties at b for all $b > OUT$.

Theorem 5 (No Ties at b if Monotone up to b): *Suppose that $b > OUT$ is a serious bid such that $b_j^*(\cdot;\cdot)$ is monotone up to b for all bidders j. Then there are no ties at b in the equilibrium $\mathbf{b}^*(\cdot;\cdot)$.*

Proof. Later in the Appendix. □

Note that Theorem 5 implies Theorem 4 since any monotone equilibrium is, by definition, monotone up to b for all bid levels b. In the example on page 15, ties occur with positive probability at $b = 100$. But $\underline{b}_1 = \underline{b}_2 = OUT$ so some strategies are not monotone up to 100.

Part II. Expected Payoffs Satisfy a Limited Strict Single-Crossing Property: The standard approach to showing that all of bidder 1's best response strategies are monotone given independent private values is to show that the expected incremental payoff from bidding higher satisfies strict single-crossing in own type. That is to say, if $t_1' > t_1$, $b' > b$, and type t_1 weakly prefers b' over b, then type t_1' must strictly prefer b' over b. Theorem 6 is the key result behind my proof, since it implies that expected incremental payoffs still satisfy strict single-crossing, though only with respect to a limited set of pairs of bids and types.

Theorem 6 (Limited Strict Single-Crossing): *For given bids $b < b'$, suppose that some type t_1 weakly prefers bid b' over both bids b, OUT. Furthermore, suppose that $\mathbf{b}^*(\cdot;\cdot)$ has no ties at b', that b' is a serious bid,* *and that $b_j^*(\cdot;\cdot)$ is monotone up to b for all bidders $j \neq 1$. Then every type $t_1' > t_1$ strictly prefers bid b' over both bids b, OUT.*

Proof. Later in the Appendix. $\square$

Part III. Decreasing Less-Than Sets for Minimal Winning Bid $\underline{R}$: Let $\mathbf{b}^*(\cdot;\cdot)$ be an equilibrium with no ties. A few definitions are useful for specifying which bids have a chance of winning or of winning outright (i.e. winning without tying) given others' strategies.

DEFINITION 15 (R_j, $\underline{R}_j$, $\underline{R}$): Define the closed convex hull of the support of bidder j's bid, $R_j \equiv cl\{b : \Pr(b < b_j^*(t_j;\tau_j)) \in (0,1)\}$. Let $\underline{R}_j \equiv \min R_j$, $\underline{R}_{-j} \equiv \max_{i\neq j} \underline{R}_j$, and $\underline{R} \equiv \max_j \underline{R}_j$.

LEMMA 1 (THRESHOLD TYPE $\hat{t}_1$): *In any equilibrium with no ties, threshold type $\widehat{t_1}$ exists such that (a) all types $t_1 > \widehat{t_1}$ get positive surplus and always bid strictly greater than $\underline{R}_{-1}$ while (b) all types $t_1 < \widehat{t_1}$ get zero surplus and always bid weakly less than $\underline{R}_{-1}$.*

Proof. Consider a serious bid $x = b_1^*(t_1, \tau_1)$, and define

$$\Pr\left(\text{“}x \text{ wins”}|\mathbf{t}_{-1}\right) \equiv \Pi_{j\neq 1} \Pr{}_{\tau_j}(b_j^*(t_j;\tau_j) \leq x)$$

(Recall that $\boldsymbol{\tau}$ are independent.) For this x, define a “derived joint density function” $f^x(\cdot)$:

$$f^x(\mathbf{t}) = \Pr\left(\text{“}x \text{ wins”}|\mathbf{t}_{-1}\right) f(\mathbf{t}) \text{ for all } \mathbf{t} \in [0,1]^n$$

(If a probability or expectation is not explicitly labelled otherwise, it is intended to be taken with respect to the original density f.) Note that $f^x(\cdot)$ is log-supermodular in $\mathbf{t}$ (LSPM) since both $f(\cdot)$ and $\Pr\left(\text{“}x \text{ wins”}|\mathbf{t}_{-1}\right)$ are LSPM. (f is LSPM since $\mathbf{t}$ are affiliated with respect to the original distribution; $\Pr\left(\text{“}x \text{ wins”}|\mathbf{t}_{-1}\right)$ is LSPM since it is a product of terms $\Pr_{\tau_j}(b_j^*(t_j;\tau_j) < x)$, each of which depends only on a one-dimensional variable t_j and hence is automatically log-supermodular.) Thus $\mathbf{t}$ are affiliated with respect to a new joint distribution having density $f^x(\cdot)$ over $[0,1]^n$ and a probability mass at (say) $(2,...,2)$ of $1-\int_{\mathbf{t}\in[0,1]^n} f^x(\mathbf{t})d\mathbf{t}$.

Bidder 1's expected payoff to bidding x can be easily expressed in terms of expectations taken with respect to $f^x(\cdot)$. First, given type t_1, bidder 1's expected payoff *conditional on winning outright* equals

$$E_{f^x}\left[(u_1(\mathbf{t},x))\,|t_1\right] = E_f\left[(u_1(\mathbf{t},x))\,\Pi_{j\neq 1}\Pr{}_{\tau_j}\left(b_j^*(t_j;\tau_j)<x\right)|t_1\right]$$

Since type t_1 bids x, no ties implies that at most one other bidder (call it h^*) bids exactly x with positive probability. If there is no such bidder, we may ignore ties at x; else let H denote the event in which bidder 1 would tie with bidder h^* by bidding b:

$$\begin{aligned} H \equiv & \{t_{h^*} : \Pr{}_{\tau_{h^*}}\left(b_{h^*}^*(t_{h^*};\tau_{h^*}) = x|t_1\right) > 0\} \times \\ & \times \prod_{j\neq 1,h^*}\left\{t_j : \Pr_{\tau_j}\left(b_j^*(t_j;\tau_j) < x|t_1\right) > 0\right\} \end{aligned}$$

Furthermore, bidder 1's incremental expected payoff (in the limit as $\varepsilon \to 0$) from bidding $x+\varepsilon$ versus x as well as between bidding x versus $x-\varepsilon$ must equal $V_H/2$, where

$$V_H = E_{f^x}\left[u_1(\mathbf{t},x)\Pr{}_{\tau_{h^*}}\left(b_{h^*}^*(t_{h^*},\tau_{h^*}) = x\middle|b_{h^*}^*(t_{h^*},\tau_{h^*})\leq x\right)\middle|t_1,\mathbf{t}_{-1}\in H\right]$$

Thus, x can only be preferred to both $x-\varepsilon$ and $x+\varepsilon$ if $V_H=0$. In other words, bidder 1's expected payoff conditional on tying must be zero. Since non-participation OUT gives guaranteed zero payoff, bidder 1's type t_1 must therefore get non-negative payoff *from winning outright* with bid x, i.e.

$$0 \leq E_{f^x}\left[u_1(\mathbf{t},x)|t_1\right] \tag{1}$$

Since $u_1(\cdot)$ is *strictly* increasing in t_1, Theorem 3(b) applied to (1) implies that $E_{g^x}\left[u_1(\mathbf{t})|t_1\right]$ is strictly increasing in t_1. Consequently, for all $t_1' > t_1$

bidder 1's payoff to bidding x in the event of winning outright is strictly positive. Similarly, since H is a lattice, types $t_1' > t_1$ must get non-negative expected utility conditional on tying, so that all together such types get positive expected payoff from bidding x. This implies, of course, that such types must always bid strictly greater than $\underline{R}_{-1}$: by definition, a bid less than or equal to $\underline{R}_{-1}$ never wins outright; and, in equilibrium with ties against at most one other bidder, one's expected payoff from winning by tying with bid $\underline{R}_{-1}$ must be zero.

Define $\widehat{t_1} \equiv \inf\{t_1 : \Pr_{\tau_1}\left(b_1^*(t_1;\tau_1) > \underline{R}_{-1}\right) > 0\}$. Since all types $t_1' > t_1$ *always* bid greater than $\underline{R}_{-1}$ if ever type t_1 bids greater than $\underline{R}_{-1}$, (a) all types $t_1 > \widehat{t_1}$ get positive surplus and always bid strictly greater than $\underline{R}_{-1}$ while (b) all types $t_1 < \widehat{t_1}$ get zero surplus and always bid weakly less than $\underline{R}_{-1}$. □

It will be useful later in the proof to observe that there is a bidder j^* for whom $\hat{t}_{j^*} = 0$. *Proof:* If $i = \arg\max_j \underline{R}_j$ then set $i = j^*$. Otherwise, suppose that all bidders in $\arg\max_j \underline{R}_j$ bid $\underline{R}$ with positive probability. In this case, they would tie with positive probability at $\underline{R}$, contradicting the assumption of no ties. So at least one of these bidders must bid $\underline{R}$ with zero probability. This bidder is our j^*.

Definition 16 (Bidder j^*): j^* is a bidder such that $j^* \in \arg\max_j \underline{R}_j$ and $\underline{t}_{j^*} = 0$.

Bidder j^* always wins outright with positive probability given any type $t_{j^*} > 0$. Furthermore, this lowest type $t_{j^*} = 0$ wins outright in the event that $t_j < \widehat{t_j}$ for all $j \neq j^*$. (This event may or may not have positive probability.)

Part IV. Strategies Monotone up to $\underline{R}$. Part III showed that all types $t_j < \widehat{t_j}$ lose with probability one while all types $t_j > \widehat{t_j}$ win outright with positive probability. Yet types $t_j < \widehat{t_j}$ are indifferent between all always-losing bids and hence could submit any always-losing bid (or mix over several such bids) in equilibrium. While these bids never win, it is conceivable that such bidding behavior might support others' equilibrium strategies. As it turns out, however, this is not the case. We may assume without loss that bidders adopt monotone strategies over the range of always-losing types.

LEMMA 2 (STRATEGIES MONOTONE UP TO $\underline{R}$): *Every mixed strategy equilibrium* $\mathbf{b}^*(\cdot;\cdot)$ *that has no ties at all bid levels has an outcome-equivalent equilibrium* $\tilde{\mathbf{b}}(\cdot;\cdot)$ *in which each bidder j's strategy is monotone up to* $\underline{R}$.

Proof. Consider any new strategy profile $\tilde{\mathbf{b}}(\cdot;\cdot)$ that satisfies three requirements: for all bidders j,

(i) $\tilde{b}_j(t_j;\tau_j) = b_j^*(t_j;\tau_j)$ for all $t_j > \widehat{t}_j$ and all τ: The sometimes-winning types bid the same as in the original equilibrium.

(ii) $\Pr\left(\tilde{b}_j(t_j;\tau_j) \geq \max_{i\neq j} \tilde{b}_i(t_i;\tau_i)\right) = 0$ for all $t_j < \widehat{t}_j$ and all τ: The always-losing types still always lose.

(iii) $\tilde{b}_j(t_j;\tau_j') = \tilde{b}_j(t_j;\tau_j) \equiv \tilde{b}_j(t_j)$ for all $t_j < \widehat{t}_j$, τ',τ, and $\tilde{b}_j(t_j) \leq \tilde{b}_j(t_j')$ for all $t_j < t_j' < \widehat{t}_j$: Monotone pure strategy over the range of always-losing types.

In the three cases below (a,b,c), I modify the original equilibrium strategies in *different* ways but in each case the new strategies satisfy conditions (i,ii,iii).

By (i,ii) the new strategy profile is outcome-equivalent to the original equilibrium. Each bidder's payoffs (and preferences) over the range of bids greater than $\underline{R}$, furthermore, remains the same. Since j^* always bids at least $\underline{R}$ in the original equilibrium, lastly, everyone else is indifferent between the null-bid and any bid less than $\underline{R}$. So, every bidder $j \neq j^*$ still finds her new strategy to be a best response and bidder j^*'s preferences amongst bids greater than $\underline{R}$ remain the same. The new strategy profile then is an equilibrium itself unless some *bidder* j^**-type* now prefers to bid weakly less than $\underline{R}$ who previously bid more than $\underline{R}$ or now prefers to bid strictly less than $\underline{R}$ who previously bid exactly $\underline{R}$.

There are three cases to consider:

(a) Some bidder $j \neq j^*$ bids exactly $\underline{R}$ with positive probability in the original equilibrium. But then to avoid ties j^* must bid strictly greater than $\underline{R}$ with probability one. Consider modified strategies $\mathbf{b}^0(\cdot;\cdot)$:

$$b_j^0(t_j;\tau_j) = b_j^*(t_j;\tau_j) \text{ for all } j, t_j > \hat{t}_j, \tau_j$$
$$b_j^0(t_j;\tau_j) = \underline{R} \text{ for all } j, t_j \leq \hat{t}_j, \tau_j$$

(Bidder j^*'s strategy is unchanged[4] and all other bidders' strategies remain the same for types $t_j > \hat{t}_j$.) Any bidder j^*-type certainly will not choose to bid strictly less than $\underline{R}$ now since that guarantees no chance of winning and hence zero payoff. Bidding exactly $\underline{R}$ might allow bidder j^* to win with positive probability, but only by tying with all of the other bidders. By the proof of Theorem 5, however, bidder j^* can only weakly prefer bidding $\underline{R}$ over both $\underline{R} + \varepsilon, OUT$ (for all small ε) if she gets zero expected utility from bidding $\underline{R}$, again no better than the payoff from her original equilibrium bid. Thus, $\mathbf{b}^0(\cdot;\cdot)$ is an equilibrium that is monotone up to $\underline{R}$.

(b) No bidder $j \neq j^*$ bids exactly $\underline{R}$ with positive probability and $\underline{R} = \underline{R}_{-j^*}$, i.e. some other bidder $j' \neq j^*$ also never bids less than $\underline{R}$. In this case, any strategy profile satisfying requirements (i,ii,iii) will be an outcome-equivalent equilibrium that is monotone up to $\underline{R}$: Bidder j^* has no chance of winning with a bid less than or equal to $\underline{R}$ so bidding behavior at and below $\underline{R}$ is irrelevant to bidder j^*'s best response. (For example, using strategies $b_j^{(\triangle,\beta)}(\cdot;\cdot)$ defined below would work.)

(c) No bidder $j \neq j^*$ bids exactly $\underline{R}$ with positive probability and $\underline{R} > \underline{R}_{-j^*}$. In this last and most difficult case, all bidder j^*-types win with positive probability in the original equilibrium. If $\underline{R} = p^{min}$, then all types $t_j \in [0, \hat{t}_j)$ must be bidding OUT for all $j \neq j^*$, and so $\mathbf{b}^*(\cdot;\cdot)$ is monotone up to $\underline{R}$

[4] To keep the proof relatively readable, I do not explicitly keep track of changes that occur on zero measure sets of types. All references to "no types" or "all types" should be understood as being made modulo a zero measure set.

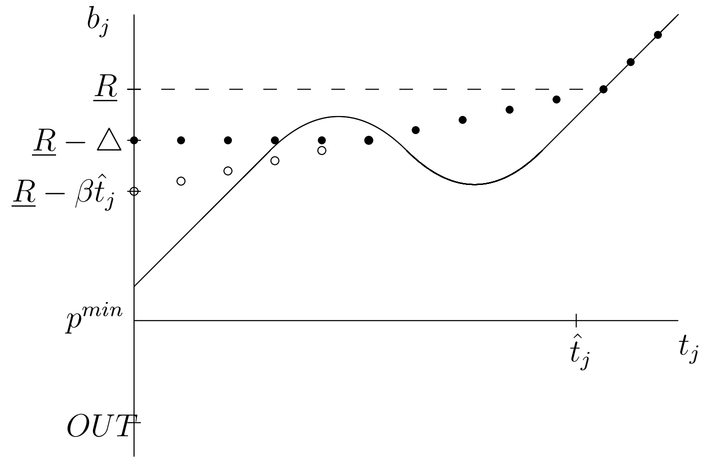

Figure 6: Graph of original equilibrium strategy traced by solid line; graph of new strategy $b_j^{(\triangle,\beta)}(\cdot)$ traced by filled circles

already. If $\underline{R} > p^{min}$, consider a family of modified strategies (indexed by $\triangle \in (0, \underline{R} - p^{min}]$ and $\beta \in (0, (\underline{R} - p^{min})/\hat{t}_j])$:

$$\begin{aligned} b_j^{(\triangle,\beta)}(t_j; \tau_j) &= b_j^*(t_j; \tau_j) \text{ for all } j, t_j > \hat{t}_j, \tau_j \\ b_j^{(\triangle,\beta)}(t_j) &= \underline{R} - \left(\hat{t}_j - t_j\right)\beta \text{ for all } j, t_j \in \left[\max\{0, \hat{t}_j - \triangle/\beta\}, \hat{t}_j\right] \\ &= \underline{R} - \triangle \text{ for all } j, t_j < \max\{0, \hat{t}_j - \triangle/\beta\} \end{aligned}$$

as illustrated in Figure 6. First, note that bidder j^*'s payoff to bidding exactly $\underline{R}$ remains the same as in the original equilibrium, so it remains for us only to rule out the possibility that some j^*-type who bid weakly greater than $\underline{R}$ before now prefers to bid strictly less than $\underline{R}$.

My claim is that, for small enough $(\triangle, \beta)$, no bidder j^*-type has incentive to bid strictly below $\underline{R}$. To see this note that, since $b_j^{(\triangle,\beta)}(t_j; \tau_j)$ is monotone up to $\underline{R}$ (for all $j \neq j^*$), Theorem 6 implies that *if* type $t_{j^*} = 0$ weakly prefers $b' > \underline{R}$ over $b \leq \underline{R}$ given these new strategies, then all types $t_{j^*} > 0$ must strictly prefer b' over b. Thus, it suffices for me to show that the lowest type $t_{j^*} = 0$ does not strictly prefer to bid strictly less than $\underline{R}$. The rest of this part of the proof focuses on this single type.

For given $(\triangle, \beta)$, all bids less than $p^{\triangle,\beta} \equiv \max\{\underline{R} - \triangle, \underline{R} - \hat{t}_j\beta\}$ lose for certain and so can not be strictly preferred by type $t_{j^*} = 0$ over its original equilibrium bid. Type $t_{j^*} = 0$'s utility from bidding $b \in [p^{\triangle,\beta}, \underline{R}]$ takes the form

$$\begin{aligned} \pi^\beta(b) &= P^\beta(b,0)V^\beta(b,b,0) \text{ where} \\ P^\beta(b^2,b^1) &\equiv \Pr\left(\max_{j\neq j^*} b_j^{\triangle,\beta}(t_j) \in [b^1,b^2] \Big| t_{j^*} = 0\right) \\ V^\beta(b',b^2,b^1) &\equiv E\left(u_{j^*}(\mathbf{t},b')\big| t_{j^*} = 0, \max_{j\neq j^*} b_j^{\triangle,\beta}(t_j) \in [b^1,b^2] \text{ for all } j \neq j^*\right) \end{aligned}$$

$P^\beta(b^2,b^1)$ is the probability that the highest bid by bidders $-j^*$ is in $[b^1,b^2]$. (For small enough β this doesn't depend on $\triangle$.) $P^\beta(b,0)$ is the probability that b is high enough to win. Similarly, $V^\beta(b',b^2,b^1)$ is type $t_{j^*} = 0$'s expected utility from paying b' for the object conditional on the highest bid by bidders $-j^*$ being in $[b^1,b^2]$. In particular, $V^\beta(b',b,OUT)$ is type $t_{j^*} = 0$'s expected utility from paying b' for the object conditional on b being high enough to win.

Note that, by design,

$$V^{\beta_1}(\underline{R},\underline{R}-\beta_1 x, OUT) = V^{\beta_2}(\underline{R},\underline{R}-\beta_2 x, OUT) \text{ for all } \beta_1,\beta_2 > 0, x \geq 0 \tag{2}$$

$$P^{\beta_1}(\underline{R}-\beta_1 x, OUT) = P^{\beta_2}(\underline{R}-\beta_2 x, OUT) \text{ for all } \beta_1,\beta_2 > 0, x \geq 0 \tag{3}$$

Before proceeding, I need to establish some smoothness properties of these functions. First, by **(A1)** the joint density $f(\cdot)$ is measurable so $P^\beta(b,OUT)$ is continuous in b at $\underline{R}$. Since $P^\beta(b,OUT)$ is also non-decreasing in the bid, the left-derivative $P_1^\beta(\underline{R},OUT) \geq 0$ is well-defined. Similarly, by **(A3)** utility is measurable in types so that $V^\beta(b',b,OUT)$ is continuous in b and utility is continuous in bid so that $V^\beta(b',b,OUT)$ is differentiable in b' (with $V_1^\beta(\underline{R},\underline{R},OUT) < 0$). Finally, since the strategies $b_j^{(\triangle,\beta)}(\cdot;\cdot)$ are monotone up to $\underline{R}$, $V^\beta(b',b,OUT)$ is non-decreasing in b as long as $b \leq \underline{R}$. Together with continuity, then, we have that the left-derivative $V_2^{\beta}(\underline{R}, \underline{R}, OUT) \geq 0$ is well-defined.

Next, using equations (2, 3), observe that

$$\begin{aligned} P_1^{\beta_2}(\underline{R}, OUT) &= \beta_1/\beta_2 P_1^{\beta_1}(\underline{R}, OUT) \text{ for all } \beta_1, \beta_2 \\ V_1^{\beta_2}(\underline{R}, \underline{R}, OUT) &= V_1^{\beta_1}(\underline{R}, \underline{R}, OUT) \text{ for all } \beta_1, \beta_2 \\ V_2^{\beta_2}(\underline{R}, \underline{R}, OUT) &= \beta_1/\beta_2 V_2^{\beta_1}(\underline{R}, \underline{R}, OUT) \text{ for all } \beta_1, \beta_2 \end{aligned}$$

(The smaller β, the more probability mass gets "stacked up" just below $\underline{R}$. This also affects the rate at which bidder j^*'s expected value from winning decreases as he lowers his bid, though not the rate at which lowering his bid decreases his expected payment.)

Figure 7 provides an illustration of type $t_{j^*} = 0$'s expected utility from submitting various bids both under the original equilibrium (solid line) as well as under strategies $\mathbf{b}^{(\triangle,\beta)}$ (trace of circles for $b < \underline{R}$; solid line for $b \geq \underline{R}$) in the most difficult case when j^* was originally indifferent between her equilibrium bid and some bid less than $\underline{R}$.[5] By construction all bids less than $p^{\triangle,\beta}$ lose for certain, get zero utility, and can not be strictly preferred to any equilibrium bid. What we need to show is that type $t_{j^*} = 0$'s utility is increasing in her bid in a neighborhood to the left of $\underline{R}$, i.e. that the (left-)derivative $\pi_1(\underline{R}) \geq 0$ where

$$\begin{aligned} (4) \qquad \pi_1(\underline{R}) \equiv & P_1^{\beta}(\underline{R}, OUT) V^{\beta}(\underline{R}, \underline{R}, OUT) \\ & + P^{\beta}(\underline{R}, OUT) \left(V_1^{\beta}(\underline{R}, \underline{R}, OUT) + V_2^{\beta}(\underline{R}, \underline{R}, OUT) \right) \end{aligned}$$

Given small enough $\triangle$, then, no bid less than $\underline{R}$ can be preferred to $\underline{R}$ (and hence to its original equilibrium bid).

Suppose first that $V^{\beta}(\underline{R}, \underline{R}, OUT) > 0$. Since $V_1^{\beta}(\underline{R}, \underline{R}, OUT)$ is bounded and $V_2^{\beta}(\underline{R}, \underline{R}, OUT) \geq 0$, we may choose β small enough that $P_1^{\beta}(\underline{R}, OUT)$ is large enough that $\pi_1(\underline{R}) > 0$.

[5] In the Figure, type $t_{j^*} = 0$ does not randomize and its equilibrium bid is strictly greater than $\underline{R}$. Neither feature is needed for the argument.

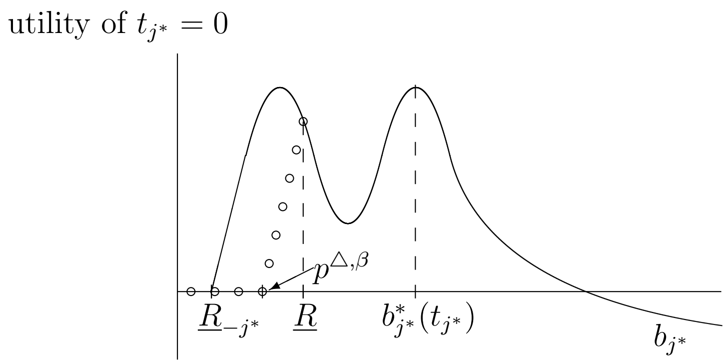

Figure 7: Graph of j^*'s utility from various bids; changes induced by strategy modifications traced by circles.

Suppose next that $V^\beta(\underline{R}, \underline{R}, OUT) = 0$. In this case, it suffices to show that $V_2^\beta(\underline{R}, \underline{R}, OUT) > 0$ since then we may choose β small enough that $V_2^\beta(\underline{R}, \underline{R}, OUT)$ is large enough that $\pi_1(\underline{R}) > 0$. For this purpose, observe that for any $b^1 < \underline{R}$,

$$P^\beta(\underline{R}, OUT)V^\beta(\underline{R}, \underline{R}, OUT) = P^\beta(b^1, OUT)V^\beta(\underline{R}, b^1, OUT) + P^\beta(\underline{R}, b^1)V^\beta(\underline{R}, \underline{R}, b^1)$$

or re-arranging,

$$V^\beta(\underline{R}, \underline{R}, OUT) - V^\beta(\underline{R}, b^1, OUT) = \frac{P^\beta(\underline{R}, b^1)\left(V^\beta(\underline{R}, \underline{R}, b^1) - V^\beta(\underline{R}, b^1, OUT)\right)}{P^\beta(\underline{R}, OUT)}$$

So, $V_2^\beta(\underline{R}, \underline{R}, OUT) = 0$ only if $V^\beta(\underline{R}, \underline{R}, \underline{R}) = V^\beta(\underline{R}, \underline{R}, OUT)$. By Theorem 3, however, $V^\beta(\underline{R}, \underline{R}, \underline{R}) = V^\beta(\underline{R}, \underline{R}, OUT)$ only if bidder j^*'s value for the object is constant in $\mathbf{t}_{-j^*}$ over the whole event $\left\{t_j \leq \widehat{t}_j \text{ for all } j \neq j^*\right\}$, i.e. only if bidder j^* has "private values" over this range of others' type profiles. In this case, however, to deter bidder j^* from wanting to deviate with a bid below $\underline{R}$, all we need to do is make sure that her probability of winning with any such bid is less than it was under the original equilibrium strategies. This can be done by making β sufficiently small. This completes the proof that any mixed-strategy equilibrium has an outcome-equivalent equilibrium that is monotone up to $\underline{R}$.

$\square$

Part V. Putting It All Together: By Lemma 2, we may assume that each bidder's strategy is monotone up to $\underline{R}$. To prove that every equilibrium must be monotone, it suffices to show that $\underline{b}_1 > \min_{j\neq 1} \underline{b}_j \equiv \underline{b}_{-1}$ whenever $\underline{b}_{-1} < \infty$. Suppose for the sake of contradiction that $\underline{R} \leq \underline{b}_1 \leq \underline{b}_{-1} < \infty$. This requires that, in equilibrium, there exists bid $b' > \underline{b}_{-1}$ and types $t_1'(\varepsilon), t_1$ such that type t_1 bids b' and each type $t_1'(\varepsilon) > t_1$ bids $b(\varepsilon) < \underline{b}_{-1} + \varepsilon$. First note that it must be that $b(\varepsilon) > \underline{b}_{-1}$ for all ε. Otherwise, since all bidders' strategies are monotone up to $\underline{b}_{-1}$ and by assumption there are no ties, Theorem 6 implies that all types greater than t_1 must *strictly* prefer b' over $b(\varepsilon)$ if type t_1 weakly prefers b' over $b(\varepsilon)$, a contradiction.

Without loss, also, we may assume that type t_1' exists such that $t_1'(\varepsilon) \in (t_1' - \varepsilon, t_1' + \varepsilon)$ for all small enough ε.[6] Define a sequence of functions

$$\begin{aligned}\gamma^{\varepsilon_k}(\mathbf{t}_{-1}, \tau_{-1}) &\equiv u_1\left(t_1'(\varepsilon_k), \mathbf{t}_{-1}, b(\varepsilon_k)\right) \text{ if } \max_{j\neq 1} b_j^*(t_j;\tau_j) < b(\varepsilon_k) \\ &\equiv 0 \text{ otherwise}\end{aligned}$$

γ^{ε_k} is bidder 1's ex post payoff from bidding $b(\varepsilon_k)$ when he has type $t_1'(\varepsilon_k)$. Note that γ^{ε_k} converges almost surely to γ^0 defined as

$$\begin{aligned}\gamma^{0}(\mathbf{t}_{-1}, \tau_{-1}) &\equiv u_1\left(t_1', \mathbf{t}_{-1}, \underline{b}_{-1}\right) \text{ if } \max_{j\neq 1} b_j^*(t_j;\tau_j) \leq \underline{b}_{-1} \\ &\equiv 0 \text{ otherwise}\end{aligned}$$

Consider 'hypothetical' bid $\underline{b}_{-1}^+$ defined by $\underline{b}_{-1} < \underline{b}_{-1}^+ < b'$ for all $b' > \underline{b}_{-1}$. γ^0 corresponds to bidder 1's ex post payoff from submitting bid $\underline{b}_{-1}^+$ given limiting type t_1'. Since utilities are bounded, we may apply the bounded convergence theorem to conclude that

$$\lim_{k\to\infty} E_{\mathbf{t}_{-1},\tau_{-1}}\left[\gamma^{\varepsilon_k}(\mathbf{t}_{-1}, \tau_{-1})\middle| t_1'\right] = E_{\mathbf{t}_{-1},\tau_{-1}}\left[\gamma^{0}(\mathbf{t}_{-1}, \tau_{-1})\middle| t_1'\right],$$

[6] Let t_1' be any accumulation point of a sequence $\{t_1'(\varepsilon_k)\}_{k=1}^{\infty}$ such that $\varepsilon_k \to 0$. Then we have that, for all ε, some type in $(t_1' - \varepsilon, t_1' + \varepsilon)$ bids $b(\varepsilon) \in \left(\underline{b}_{-1}, \underline{b}_{-1} + \varepsilon\right)$.

i.e. type t_1' must at least weakly prefer bidding $\underline{b}_{-1}^{+}$ to b'. On the other hand, by assumption type t_1 chose to bid b' rather than any bid in a neighborhood $(\underline{b}_{-1}, \underline{b}_{-1} + \partial)$, implying (by continuity of utility in the bids) that this lower type must weakly prefer b' over $\underline{b}_{-1}^{+}$. Finally, by Lemma 1, the set of $\mathbf{t}_{-1}$ profiles against which $\underline{b}_{-1}^{+}$ would win is a decreasing set and a lattice and that (by definition of this 'hypothetical' bid) no other bidder ever bids exactly $\underline{b}_{-1}^{+}$. Thus, Theorem 6 implies that any type greater than t_1 must strictly prefer b' to $\underline{b}_{-1}^{+}$, a contradiction. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2

Redefine the grid of permissible bids as follows: each bidder j submits a bid from the set $(OUT \cup \mathcal{P}) \times \rho(j)$ where $(OUT \cup \mathcal{P}) \times \{1, ..., n\}$ is endowed with the lexicographic order $(p', k') > (p, k)$ iff $p' > p$ or $p' = p, k' > k$. Now, define the lowest trough of each bidder's strategy in terms of this richer bid space and no two lowest troughs can be equal – by definition! All five parts of Theorem 1 proof go through with only minor changes (though several of the most challenging parts of the proof become irrelevant since ties at always-losing bid levels are no longer possible). $\square$

Proof of Theorem 5 (and hence Theorem 4)

Since each bidder's strategy is monotone up to b, there exists a threshold type $\widehat{t}_j^b$ such that $\Pr(b_j^*(t_j; \tau_j) < b) = 1$ for all $t_j < \widehat{t}_j^b$ and $\Pr(b_j^*(t_j; \tau_j) < b) = 0$ for all $t_j > \widehat{t}_j^b$. One can summarize this by saying that bidder j always bids less than b when $t_j \in [0, \widehat{t}_j^b)$ and bids *equal* to b with probability $\Pr_{\tau_j}\left(b_j^*(t_j; \tau_j) \leq b | t_j\right)$ when $t_j \in (\widehat{t}_j^b, 1]$. One may then characterize bidder 1's expected payoff from bidding b conditional on type t_1 (shorthand $\pi_1(b, t_1)$) as

$$\pi_1(b, t_1) \equiv \int_{[0,1]^{n-1}} \frac{1}{1 + \#\left(j \neq 1 : t_j > \widehat{t}_j^b\right)} \left(u_1(\mathbf{t}, b)\right) f^b(\mathbf{t}_{-1} | t_1) d\mathbf{t}_{-1}$$

(See page 24 for the definition of the induced joint density f^b.) Let $G(k)$ be the event in which bidder 1 ties with $k-1$ others at b:

$$G(k) \equiv \left\{\mathbf{t}_{-1} : \#\left(j \neq 1 : t_j > \widehat{t}_j^b\right) = k-1\right\}.$$

Then we may reformulate type t_1's expected payoff from bid b as

$$\pi_1(b, t_1) = \sum_{k=0}^{n-1} \frac{1}{k} \Pr{}_{f^b}\left(G(k)\right) E_{f^b}\left[u_1(\mathbf{t}, b) \middle| t_1, \mathbf{t}_{-1} \in G(k)\right]$$

If b is a best response for type t_1, then it must be he does not prefer to submit the null bid *OUT* nor to bid slightly more than b: $\pi_1(b, t_1) \geq \max\{0, \lim_{\partial \to 0} \pi_1(b + \partial, t_1)\}$, where

$$\lim_{d \to 0} \pi_1(b + \partial, t_1) = \sum_{k=1}^{n} \Pr{}_{f^b}\left(G(k)\right) E_{f^b}\left[u_1(\mathbf{t}, b) \middle| t_1, \mathbf{t}_{-1} \in G(k)\right]$$

In other words, in terms of further shorthand

$$A_k \equiv \Pr{}_{f^b}\left(G(k)\right) E_{f^b}\left[u_1(\mathbf{t}, b) \middle| t_1, \mathbf{t}_{-1} \in G(k)\right],$$

it must be that (i) $\sum_{k=1}^{n} \frac{1}{k} A_k \geq 0$ and (ii) $\sum_{k=2}^{n} \frac{k-1}{k} A_k \leq 0$.

Conditions (i, ii) imply that $E_{f^b}\left[u_1(\mathbf{t}, b) \middle| t_1\right] = \sum_{k=1}^{n} A_k = 0$. To see why, note first that $\cup_{l=1}^{k} G_l = B \cap [0,1]^n$ for some decreasing set B. (Each set in this union has the form, up to a zero measure set, of

$$G_k = \cup_{J \subset \{2,\ldots,n\} : \#(J) = \bar{k}} \Pi_{j \in J}\left(\widehat{t}_j^b, 1\right] \Pi_{j \notin J}\left[0, \widehat{t}_j^b\right)$$

So, if $\mathbf{t} \in G_k$ and $\mathbf{t}' < \mathbf{t}$, then $\mathbf{t}' \in G_{k'}$ for some $k' \leq k$.) Consequently, by Theorem 3(a),

$$E_{f^b}\left[u_1(\mathbf{t}, b) \middle| t_1\right] > 0 \Rightarrow \sum_{k=l}^{n} A_k > 0 \text{ for all } l = 1, ..., n$$

$$\Rightarrow \sum_{k=2}^{n} \frac{k-1}{k} A_k = \frac{1}{2} \sum_{k=2}^{n} A_k + \sum_{l=3}^{n} \left(\left(\frac{l-1}{l} - \frac{l-2}{l-1} \right) \sum_{k=l}^{n} A_k \right) > 0$$

Thus, $E_{f^b}\left[u_1(\mathbf{t},b)\middle|t_1\right] \leq 0$. On the other hand, $E_{f^b}\left[u_1(\mathbf{t},b)\middle|t_1\right] \geq 0$ since

$$E_{f^b}\left[u_1(\mathbf{t},b)\middle|t_1\right] < 0 \Rightarrow \sum_{k=1}^{l} A_k < 0 \text{ for all } l = 1, ..., n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} A_k = \frac{1}{n} \sum_{k=1}^{n} A_k + \sum_{l=1}^{n-1} \left(\left(\frac{1}{l} - \frac{1}{l+1} \right) \sum_{k=1}^{l} A_k \right) < 0$$

Finally, by Theorem 3(b) $E_{f^b}\left[u_1(\mathbf{t},b)\middle|t_1\right]$ is strictly increasing in t_1. We may conclude then that at most one bidder type t_1 bids b in equilibrium if all others' bids are monotone up to b. $\square$

Proof of Theorem 6

First, there are no ties at b by Theorem 5. Furthermore, since by assumption each bidder $j \neq 1$ adopts a strategy that is monotone up to b,the event "b wins" $\equiv \Pi_{j\neq 1} W_j(b)$ in which bidder 1 can sometimes win with bid b is decreasing and the same (up to a zero measure set) as the event $\Pi_{j\neq 1} W_j^*(b)$ in which bidder 1 always wins with bid b. Similarly, define "b' wins" $\equiv \Pi_{j\neq 1} W_j(b')$. Since "$b'$ wins" is a lattice,

$$E\left[u_1(\mathbf{t},b')\middle|t_1', \mathbf{t}_{-1} \in \text{"}b' \text{ wins"}\right] > E\left[u_1(\mathbf{t},b')\middle|t_1, \mathbf{t}_{-1} \in \text{"}b' \text{ wins"}\right] \geq 0$$

where the first inequality follows from Theorem 3(b) and the second from the assumption that type t_1 weakly prefers b' to OUT. Thus, type t_1' strictly prefers b' to OUT. Now we need to show that type t_1' strictly prefers b' to b. Suppose for the sake of contradiction that t_1' weakly prefers b to b'.

For every $J \subset \{2, ..., n\}$, define X^J to be the set of others' type profiles so that all bidders $j \in J$ submit a bid in (b, b') and all bidders $j \notin J$ bid less than b:

$$X^J \equiv \Pi_{j\in J} \left(W_j(b') \backslash W_j(b)\right) \Pi_{j\in\{2,\ldots,n\}\backslash J} W_j(b)$$

Each set X^J is a lattice, where $X^\emptyset =$ "b wins" and the union of all of these sets is "b' wins". Furthermore, for all $J \neq \emptyset$, $X^\emptyset \cup X^J$ is a lattice in which $X^\emptyset$ is decreasing.

Type t_1's incremental payoff to bidding b' versus b depends on whether bid b and/or b' win. Conditional on b winning, bidding b' leads to a (negative) incremental gain of $u_1(\mathbf{t}, b') - u_1(\mathbf{t}, b)$; conditional on b' winning and b losing, bidding b' leads to an incremental gain equal to the expected utility from winning and paying b'. Define

$$\begin{aligned} \phi_1(\mathbf{t}_{-1}) &\equiv u_1(\mathbf{t}, b') - u_1(\mathbf{t}, b) \text{ if } \mathbf{t}_{-1} \in X^{\emptyset} \\ &\equiv E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^J\right] \text{ whenever } \mathbf{t}_{-1} \in X^J \\ &\equiv 0 \text{ otherwise} \end{aligned}$$

(See page 24 for the definition of the density function $f^{b'}(\cdot)$.) $\phi_1(\mathbf{t}_{-1})$ is the ex post incremental payoff to type t_1'; $E_{f^{b'}}\left[\phi_1(\mathbf{t}_{-1}) | \widehat{t}_1\right] > 0$ iff bidder 1 would get positive expected incremental payoff from bidding b' over b *in a hypothetical situation* in which she got ex post payoffs like type t_1' but others' types were distributed as if she had type $\widehat{t}_1$. Consequently,

$$E_{f^{b'}}[\phi_1(\mathbf{t}_{-1}) | t_1'] \leq 0, \quad E_{f^{b'}}[\phi_1(\mathbf{t}_{-1}) | t_1] \geq 0$$

The first inequality states that type t_1' weakly prefers bid b over b'; the second inequality follows from the facts that type t_1 weakly prefers bid b' over b and that type t_1''s ex post incremental payoff to bidding b' over b is never less than type t_1's. On the other hand, Theorem 3(a) implies that $E_{f^{b'}}[\phi_1(\mathbf{t}_{-1}) | t_1'] > 0$ *if* $\phi_1(\cdot)$ is a non-decreasing function over the lattice "b' wins". In other words, to achieve a contradiction we need only show that, for all $J' \supset J \neq \emptyset$,

$$E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^{J'}\right] \geq E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^J\right] \tag{5}$$

$$E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^{J'}\right] \geq E_{f^{b'}}\left[u_1(\mathbf{t}, b') - u_1(\mathbf{t}, b) | t_1', \mathbf{t}_{-1} \in X^{\emptyset}\right] \tag{6}$$

Equation (5) follows immediately from Theorem 3(a) since X^J is a decreasing set in the lattice $X^J \cup X^{J'}$. For equation (6), note again by Theorem 3(a) that

$$E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^J\right] \geq E_{f^{b'}}\left[u_1(\mathbf{t}, b') | t_1', \mathbf{t}_{-1} \in X^{\emptyset}\right]$$

Thus it suffices to show that $E_{f^{b'}}\left[u_1(\mathbf{t},b)|t_1',\mathbf{t}_{-1}\in X^{\emptyset}\right]\geq 0$. But this follows directly from the presumption that type t_1' weakly prefers bid b over the null-bid. $\square$

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