Uniqueness in Symmetric First-Price Auctions with Affiliation

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Abstract

The first-price auction has a unique monotone pure strategy equilibrium when there are n symmetric risk-averse bidders having affiliated types and interdependent values.

Key words: First-price auction, uniqueness, affiliation, interdependent values, all-pay auction.

1 Introduction

A growing empirical literature studies symmetric first-price auctions in which bidders do not have independent private values but rather affiliated types and interdependent values (also known as 'common values'). [1] The standard practice in this literature is to assume that bidders play the symmetric monotone pure strategy equilibrium (MPSE) described by Milgrom and Weber [15].

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[1] See e.g. Hendricks, Pinkse, and Porter [6]. For a survey of experimental work, see Kagel [7].

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Unfortunately, theory has provided no justification for focusing on this symmetric equilibrium, even though the possibility of *asymmetric* equilibria in symmetric auctions is a real concern. In models with independent private values, the symmetric war of attrition has multiple equilibria, including a continuum of asymmetric equilibria [Nalebuff and Riley [16]]. Once the private values assumption is relaxed, the second-price auction and the open ascending-price auction also are well-known to have a continuum of asymmetric equilibria (in undominated strategies) [Milgrom [13], Bikhchandani and Riley [4]].[2] These asymmetric equilibria have a natural 'winner's curse' intuition. When some bidders are more aggressive, this decreases other bidders' expected value conditional on winning with any given bid, leading these other bidders to bid less aggressively, and vice versa. This intuition would seem to apply to the first-price auction as well, suggesting that symmetric first-price auctions might also have asymmetric equilibria.

This paper proves that symmetric first-price auctions with affiliated types and interdependent values do not have asymmetric MPSE. More precisely, a unique MPSE exists and this equilibrium is in symmetric strategies. Could still other *mixed* strategy equilibria exist, not in monotone pure strategies? This possibility has been ruled out in some important special cases, but not in general.[3] See Athey and Haile [2], especially Theorem 2.1(ii), for a survey

[2] The symmetric private value second-price auction is also well-known to have many asymmetric equilibria in weakly dominated strategies. See Blume and Heidhues [5].

[3] If (asymmetric) bidders have private values *or* independent signals, McAdams "Monotonicity in Asymmetric First-Price Auctions with Affiliation", mimeo (2003), proves that any mixed strategy equilibrium is outcome-equivalent to a MPSE, i.e. bidding strategies are identical to those in a MPSE except possibly for subsets of types whose equilibrium bids win with probability zero.

of the best available results on uniqueness of mixed strategy equilibrium.

Lebrun [9,10], Maskin and Riley [12], and Bajari [3] prove uniqueness of mixed strategy equilibrium given independent private values and any number of asymmetric bidders.[4] Given two asymmetric bidders having affiliated types and interdependent values, Lizzeri and Persico [11] (LP) proves uniqueness of MPSE. This paper differs from LP by allowing for more than two bidders while requiring symmetry.

The rest of the paper is organized as follows. Section 2 lays out the model and assumptions. Section 3 then proves the main result on uniqueness of MPSE given symmetric bidders. Section 4 concludes with an extension to all-pay auctions.

2 Model and preliminaries

Information: Bidder types are one-dimensional random variables having joint density $f(\mathbf{t})$ on the unit cube $[0,1]^n$. For each subset $I \subset \{1,...,n\}$, the conditional joint density will be denoted $f(\mathbf{t}_I|\mathbf{t}_{-I})$ where $\mathbf{t} \equiv (t_1,...,t_n)$, $\mathbf{t}_I \equiv (t_i : i \in I)$, and $-I \equiv \{1,...,n\}\backslash I$. (Bold notation will be used throughout the paper to refer to vectors of types, bids, and strategies.)

(A1) Bidder types are affiliated, i.e. $f(\mathbf{t}' \vee \mathbf{t})f(\mathbf{t}' \wedge \mathbf{t}) \geq f(\mathbf{t}')f(\mathbf{t})$ for all type profiles $\mathbf{t}', \mathbf{t}$ where $\mathbf{t}' \vee \mathbf{t}, \mathbf{t}' \wedge \mathbf{t}$ are their component-wise maximum and minimum, respectively.

[4] Lebrun [10] proves uniqueness under the weakest distributional assumptions.

Affiliation is a powerful form of positive correlation; see Milgrom and Weber [15] for more detailed discussion.

(A2) There exists $f_{high}, f_{low} > 0$ such that $f(\mathbf{t}) \in [f_{low}, f_{high}]$ for all $\mathbf{t}$.

(A3) f is continuously differentiable on $[0,1]^n$.

Bids and Payoffs: After learning its type, each bidder submits a bid $b_i \in OUT \cup [r, \infty)$ where r is the reserve price. Bidding is voluntary: a bidder who chooses not to participate 'bids' OUT. If all bidders bid OUT, then the auction is cancelled. Otherwise the highest bidder wins the object, with ties broken by a coin-flip: if k bidders each submit the highest bid, then each wins the object with probability $1/k$. Bidder i's utility upon losing is zero and upon winning with bid b has form $u_i(t_i; \mathbf{t}_{-i}; b)$. I make the following assumptions on utility: for all i,

(A4) u_i is twice continuously differentiable.

(A5) u_i is strictly increasing in t_i, non-decreasing in t_j for all $j \neq i$, and strictly decreasing in b

(A6) $\frac{\partial u_i}{\partial b}$ is non-decreasing in $\mathbf{t}$ and non-increasing in b

(A7) $u_i(1; \mathbf{1}; b^h) < 0$ for some bid level $b^h < \infty$

An important special case arises when utility takes the form $u_i(v_i(t_i; \mathbf{t}_{-i}) - b)$, where v_i is strictly increasing in t_i, non-decreasing in $\mathbf{t}_{-i}$, and so on. In this case, **(A6)** is satisfied when $u_i'' \leq 0$. Thus, the model is consistent with any sort of risk-aversion.

The model is a special case of Reny and Zamir [17]. The most important additional restriction imposed here is symmetry.

(A8) (i) Density f is symmetric in all types. (ii) Each bidder's utility is symmetric in others' types. (iii) Different bidders' utility depends on own type and others' types in the same way. [5]

Strategies: In a monotone pure strategy, type t_i bids $b_i(t_i)$, where $b_i(t_i') \geq b_i(t_i)$ for all $t_i' > t_i$. (By definition, the non-participation 'bid' $OUT < b$ for all other bids $b \geq r$.) A monotone pure strategy equilibrium (MPSE) is a Nash equilibrium in monotone pure strategies, i.e. for all i, $b_i(t_i)$ is a best response for bidder i for a full measure set of types $t_i \in [0,1]$. [6]

All assumptions presented in this section, including **(A1)-(A8)**, are maintained throughout the entire analysis.

Preliminaries and useful shorthand

Let $\mathbf{b}(\cdot)$ be a given MPSE. ($b_i(\cdot)$ will always refer to an *equilibrium* strategy.)

Several basic results are gathered together here and proven in the Appendix. First, some results derived from Milgrom and Weber [15] ('MW') Theorem 23, based on our assumption that $\mathbf{t}$ are affiliated. (A is a *decreasing subset of* X when $x \in A, y \leq x \in X$ implies $y \in A$.)

Lemma 1 *(a) Let* X *be a sublattice of* $[0,1]^n$ *and let* A *be a decreasing subset*

[5] Formally, for all b, $i,j,k \in \{1,...,n\}$, $t,t' \in [0,1]$, and $\mathbf{t}_{-ij}, \mathbf{t}_{-jk} \in [0,1]^{n-2}$, (i) $f(t_i = t, t_j = t', \mathbf{t}_{-ij}) = f(t_i = t', t_j = t, \mathbf{t}_{-ij})$, (ii) $u_i(t_i; t_j = t, t_k = t', \mathbf{t}_{-ijk}; b) = u_i(t_i; t_j = t', t_k = t, \mathbf{t}_{-ijk}; b)$, and (iii) $u_i(t_i = t; t_j = t', \mathbf{t}_{-ij}; b) = u_j(t_j = t; t_i = t', \mathbf{t}_{-ij}; b)$.

[6] In principle, some bidder types might fail to have a best response. A side-implication of the proof, however, is that every type of every bidder has a best response in every MPSE.

of X. Then

$$E\left[u_i(t_i;\mathbf{t}_{-i};b)|\mathbf{t}\in A\right]\leq E\left[u_i(t_i;\mathbf{t}_{-i};b)|\mathbf{t}\in X\right]\leq E\left[u_i(t_i;\mathbf{t}_{-i};b)|\mathbf{t}\in X\backslash A\right]$$

for all i. (b) $U_i(t_i,b)$ is strictly increasing in t_i for all $b\geq r$.

Lemma 2 (No ties) *Suppose that* $\Pr\left(b\geq \max_i b_i(t_i)\right)>0$ *for some $b\geq r$. Then* $\Pr\left(b_i(t_i)=b_j(t_j)=b\right)=0$ *for all i,j.*

Lemma 3 (No atoms) *Suppose that* $\Pr\left(b>\max_i b_i(t_i)\right)>0$ *for some $b>r$. Then* $\Pr(b_i(t_i)=b)=0$ *for all i.*

Limits of bids: Define $b_i(t-)\equiv\lim_{\varepsilon\to 0}b_i(t-\varepsilon)$ for all $t\in(0,1]$ and $b_i(0-)\equiv b_i(0)$. Define $b_i(t+)\equiv\lim_{\varepsilon\to 0}b_i(t+\varepsilon)$ for all $t\in[0,1)$ and $b_i(1+)\equiv b_i(1)$.

Lowest bids: $\underline{b}_i\equiv b_i(0+)$ is bidder i's 'lowest bid'. $\underline{b}_I\equiv\underline{b}_I$ is the *highest* lowest bid among bidders in I, with shorthand $\underline{b}\equiv\underline{b}_{1,\ldots,n}$.

Several useful facts about lowest bids follow immediately from the definition (proof omitted): for all $I\subset\{1,...,n\}$, (a) $b>\underline{b}_I$ iff $\Pr(b>\max_{i\in I}b_i(t_i))>0$, (b) $b<\underline{b}_I$ implies $\Pr(b\geq\max_{i\in I}b_i(t_i))=0$, and (c) $b=\underline{b}_I$ implies $\Pr(b\geq\max_{i\in I}b_i(t_i))=\Pr(b=\max_{i\in I}b_i(t_i))$. By (a,b), bidder i will win with positive probability whenever he bids more than $\underline{b}_{-i}$ and win with zero probability whenever he bids less than $\underline{b}_{-i}$. By (c), bidder i may win after bidding exactly $\underline{b}_{-i}$, but only by tying with other(s).

Inverse bid function: For all $b\geq r$, define $\phi_i(b)\equiv\inf\{t_i:b_i(t_i)\geq b\}$ and $\phi_i(b+)=\lim_{\varepsilon\to 0}\phi_i(b+\varepsilon)$. By Lemma 3, $\phi_i(b+)=\phi_i(b)$ for all $b>\underline{b}$.

Winning event: Define $W_i(b)\equiv\times_{j\neq i}[0,\phi_j(b)]\equiv[\mathbf{0},\boldsymbol{\phi}_{-i}(b)]$. $W_i(b)$ is the event (up to a zero measure set) in which all bidders $j \neq i$ bid strictly less than b. As long as $b > r$, there are no atoms at bid-level b by Lemma 3, and $W_i(b)$ is the event in which bidder i would win with bid b.

Winning probability: $P_i(t_i, b) \equiv \Pr_{\mathbf{t}_{-i}|t_i}(W_i(b)|t_i)$

Winning expected utility: $U_i(t_i, b) \equiv E\left[u_i(t_i; \mathbf{t}_{-i}; b)|t_i, \mathbf{t}_{-i} \in W_i(b)\right]$

Interim expected payoff: $\Pi_i(t_i, b)$ is bidder i's expected utility (or 'payoff') from bidding b conditional on own type t_i and others' equilibrium strategies. For all $b > \max\{\underline{b}, r\}$,[7]

$$\Pi_i(t_i, b) \equiv P_i(t_i, b)U_i(t_i, b) = \int_{\mathbf{t}_{-i} \in W_i(b)} u_i(t_i; \mathbf{t}_{-i}; b) f(\mathbf{t}_{-i}|t_i) \mathrm{d}\mathbf{t}_{-i}$$

$\Pi_i(t_i, b_i(t_i)) = \sup_b \Pi_i(t_i, b)$ for all types t_i having a best response.

Lemma 4 *(a) For all i, $\sup_b \Pi_i(t_i, b)$ is continuous in t_i. (b) For all i, $b_i(t_i-)$, $b_i(t_i+)$ is a best response for type t_i when $b_i(t_i-), b_i(t_i+) > \max\{\underline{b}, r\}$, respectively.*

Lemma 5 *$P_i(t_i, b(t_i)) > 0$ implies $\sup_b \Pi_i(t_i', b) > 0$ for all $t_i' > t_i$.*

Highest bids: Let $\bar{b}_i \equiv b_i(1-)$ be bidder i's 'highest bid', and $\bar{b} \equiv \max_i \bar{b}_i$.

Lemma 6 Either $\bar{b} = OUT$ or $\bar{b}_i > \max\{\underline{b}, r\}$ *and* $\Pi_i(1, \bar{b}_i) = \sup_b \Pi_i(1, b)$ *for all i.*

[7] See equation (7) in the Appendix for a general formulation of payoffs allowing for atoms.

3 Uniqueness

Reny and Zamir [17] guarantees existence of monotone pure strategy equilibrium ('MPSE') in a more general model allowing for bidder asymmetry.

Theorem 1 *There is a unique MPSE in the symmetric first-price auction, up to the bids made by a zero measure set of types.*

The rest of the paper proves Theorem 1, via several intermediate 'claims'. More technical parts of the proof are relegated to the Appendix.

Symmetry at the highest types

Recall our notation for highest bids, $b_i(1-) = \overline{b}_i$ and $\overline{b} = \max_i \overline{b}_i$.

Claim 1 Either $b_i(1-) = OUT$ *for all* i or $b_i(1-) = \overline{b} > \max\{\underline{b}, r\}$ *for all* i.

Proof of Claim 1. By Lemma 6, either $\overline{b}_i = OUT$ for all i or $\overline{b}_i > \max\{\underline{b}, r\}$ for all i. We need to prove that $\overline{b}_i = \overline{b}$ for all i given that $\overline{b}_i > \max\{\underline{b}, r\}$ for all i.

Without loss, suppose for the sake of contradiction that $\overline{b}_1 = \overline{b}$ and $\overline{b}_2 < \overline{b}$. By Lemma 3, there are no atoms at bid-level $\overline{b}_i$ for any i. By Lemma 6, $\Pi_i(1, \overline{b}_i) = \sup_b \Pi_i(1, b)$ for all i. In particular, $\Pi_1(1, \overline{b}_1) \geq \Pi_1(1, \overline{b}_2)$ and $\Pi_2(1, \overline{b}_2) \geq \Pi_2(1, \overline{b}_1)$.

Since there are no atoms at $\overline{b}_1$, each bidder would win with probability one if he were to bid $\overline{b}_1$. Thus, bidder 2 must get the same expected utility as bidder

1 given type $t = 1$ when bidding $\bar{b}_1$:

$$\begin{aligned}\Pi_2(1, \bar{b}_1) &= \Pr(W_2(1, \bar{b}_1)|t_2 = 1)E[u_2(1; \mathbf{t}_{-2}; \bar{b}_1)|t_2 = 1, \mathbf{t}_{-2} \in W_2(1, \bar{b}_1)] \\ &= \Pr(W_1(1, \bar{b}_1)|t_1 = 1)E[u_1(1; \mathbf{t}_{-1}; \bar{b}_1)|t_1 = 1, \mathbf{t}_{-1} \in W_1(1, \bar{b}_1)] \\ &= \Pi_1(1, \bar{b}_1)\end{aligned}$$

On the other hand, if bidder 1 were to bid $\bar{b}_2$, he would win more frequently than bidder 2 does with the same bid, and have a weakly higher expected utility when winning:

$$\begin{aligned}\Pi_1(1, \bar{b}_2) &= \Pr\left(W_1(\bar{b}_2)|t_1 = 1\right) E\left[u_1(t_1; \mathbf{t}_{-1}; \bar{b}_2)|t_1 = 1, W_1(\bar{b}_2)\right] \\ &\geq \Pr\left(W_1(\bar{b}_2)|t_1 = 1\right) E\left[u_2(t_2; \mathbf{t}_{-2}; \bar{b}_2)|t_2 = 1, W_2(\bar{b}_2)\right] \qquad (1) \\ &> \Pr\left(W_2(\bar{b}_2)|t_2 = 1\right) E\left[u_2(t_2; \mathbf{t}_{-2}; \bar{b}_2)|t_2 = 1, W_2(\bar{b}_2)\right] = \Pi_2(1, \bar{b}_2) \qquad (2)\end{aligned}$$

(1) follows from symmetry of bidders 1,2 combined with Lemma 1(a). (Set $X \equiv W_1(\bar{b}_2) = [0, 1] \times [\mathbf{0}, \boldsymbol{\phi}_{-12}(\bar{b}_2)]$ and $A \equiv W_2(\bar{b}_2) = [0, \phi_1(\bar{b}_2)] \times [\mathbf{0}, \boldsymbol{\phi}_{-12}(\bar{b}_2)]$.) (2) uses the fact that bidder 1 is strictly more likely to win with bid $\bar{b}_2$ than bidder 2. ($\phi_1(\bar{b}_2) < \phi_2(\bar{b}_2) = 1$ by presumption since $\bar{b}_1 > \bar{b}_2$.) Thus, $\Pi_1(1, \bar{b}_2) > \Pi_2(1, \bar{b}_2) \geq \Pi_2(1, \bar{b}_1) = \Pi_1(1, \bar{b}_1) = \sup_b \Pi_1(1, b)$, a contradiction.

Intuition for Claim 1. The crucial step was to show that $\bar{b}_1 > \bar{b}_2$ implies $\Pi_1(1, \bar{b}_2) > \Pi_2(1, \bar{b}_2)$. There are two reasons for this. First, bidder 2 bids less than $\bar{b}_2$ with probability one while bidder 1 bids more than $\bar{b}_2$ with positive probability. Consequently, conditional on bidding $\bar{b}_2$, bidder 1 is *strictly* more likely to win the object than bidder 2. Second, conditional on bidding $\bar{b}_2$ and winning, bidder 1 faces 'winner's curse' to a lesser degree than bidder 2. This is because bidder 1 wins the object regardless of bidder 2's type, whereas bidder 2 only wins when bidder 1 has a relatively low type.

Continuous differentiability near where symmetric

Claim 2 *Suppose* $b_i(\hat{t}+) = \hat{b} > \max\{\underline{b}, r\}$ *for all* i*. Then* $b_i(\hat{t}-) = \hat{b}$ *for all* i*.*

Claim 3 *Suppose* $b_i(\hat{t}+) = \hat{b}$ *for all* i*, where* $\max\{\underline{b}, r\} < \hat{b} \leq \bar{b}$*. Then there exists* $\gamma > 0$ *such that* $b_i(\phi_i(b)) = b$ *for all* i *and all* $b \in (\hat{b} - \gamma, \min\{\hat{b} + \gamma, \bar{b}\})$*.*

Claim 3 is the most technically challenging result in the paper.

Claim 4 *Suppose* $b_i(\hat{t}+) = \hat{b}$ *for all* i*, where* $\max\{\underline{b}, r\} < \hat{b} \leq \bar{b}$*. Then there exists* $\gamma > 0$ *so that, for all* i*,* $\phi_i(b)$ *is continuously differentiable at all* $b \in (\hat{b} - \gamma, \min\{\hat{b} + \gamma, \bar{b}\})$*.*

Local uniqueness near where symmetric

Claim 5 *Suppose* $b_i(\hat{t}+) = \hat{b}$ *for all* i*, where* $\max\{\underline{b}, r\} < \hat{b} \leq \bar{b}$*. Then there exists a strictly increasing, continuously differentiable function* $\phi(\cdot)$ *and* $\gamma > 0$ *such that* $\phi_i(b) = \phi(b)$ *for all* i *and all* $b \in (\hat{b} - \gamma, \min\{\hat{b} + \gamma, \bar{b}\})$*.*

Proof of Claim 5. *Preliminaries.*

Definition 1 **(**$a_{ij}(b), c_i(b)$**)** *For all* $i, j \neq i$*,* b*, and* $\boldsymbol{\phi} \in [0,1]^n$*, define*

$$
\begin{aligned}
a_{ij}(b, \boldsymbol{\phi}) &\equiv \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-ij}} u(\phi_i; \phi_j, \mathbf{t}_{-ij}; b) f(\phi_j, \mathbf{t}_{-ij} | \phi_i) \mathrm{d}\mathbf{t}_{-ij} \\
a_{i,i}(b, \boldsymbol{\phi}) &\equiv 0 \\
c_i(b, \boldsymbol{\phi}) &\equiv - \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-i}} \frac{\partial u(\phi_i; \mathbf{t}_{-i}; b)}{\partial b} f(\mathbf{t}_{-i} | \phi_i) \mathrm{d}\mathbf{t}_{-i}
\end{aligned}
$$

Similarly, define $a_{ij}(b) \equiv a_{ij}(b, \boldsymbol{\phi}(b))$ *and* $c_i(b) \equiv c_i(b, \boldsymbol{\phi}(b))$*.*

Given **(A3,4)**, it is easy to check that $a_{ij}(b, \boldsymbol{\phi})$ and $c_i(b, \boldsymbol{\phi})$ are continuously differentiable (in all variables).

By assumption, $\hat{b}$ has the property that $\phi_i(\hat{b}) = \hat{t}$ for all i. This symmetry implies that $a_{ij}(\hat{b}) \equiv \hat{a}$ and $c_i(\hat{b}) \equiv \hat{c}$ for all i, j. Furthermore, $\hat{b} > \underline{b}$ implies $\hat{a}, \hat{c} > 0$. To see this, let $\hat{\mathbf{t}}_I$ denote a vector of types for bidders I all equal to $\hat{t}$. Then

$$\begin{aligned} a(\hat{b}) &= \int_{\mathbf{0}}^{\hat{\mathbf{t}}_{-ij}} u(\hat{t}; \hat{t}, \mathbf{t}_{-ij}; b) f(\hat{t}, \mathbf{t}_{-ij} | t_i = \hat{t}) \mathrm{d}\mathbf{t}_{-ij} \\ &\geq \int_{\mathbf{0}}^{\hat{\mathbf{t}}_{-i}} u(\hat{t}; \mathbf{t}_{-i}; b) f(\mathbf{t}_{-i} | t_i = \hat{t}) \mathrm{d}\mathbf{t}_{-i} > 0 \end{aligned}$$

The weak inequality follows from Lemma 1(a). (Set $X \equiv \{\mathbf{t} : t_i = \hat{t}, t_k \leq \hat{t} \ \forall k \neq i\}$ and $X \backslash A \equiv \{\mathbf{t} : t_i = \hat{t}, t_j = \hat{t}, t_k \leq \hat{t} \ \forall k \neq i, j\}$.) The strict inequality follows from Lemma 5, since $b_i(\hat{t}-) = \hat{b} > \underline{b}$ implies that type $\hat{t}$ must get positive expected utility. ($b_i(\hat{t}-) = \hat{b} > \underline{b}$ implies that there exists t_i such that $b_i(t_i) > \underline{b}$ and $\hat{t} > t_i$.) $\hat{c} > 0$ by **(A5)** since $\frac{\partial u}{\partial b} < 0$.

Definition 2 **($A(b), C(b)$)** *For all i, b, and $\boldsymbol{\phi} \in [0,1]^n$, define matrix $A(b, \boldsymbol{\phi}) \equiv (a_{ij}(b, \boldsymbol{\phi}) : 1 \leq i, j \leq n)$ and row vector $C(b, \boldsymbol{\phi}) \equiv (c_i(b, \boldsymbol{\phi}) : 1 \leq i \leq n)$. Similarly, define $A(b) \equiv A(b, \boldsymbol{\phi}(b))$ and $C(b) \equiv C(b, \boldsymbol{\phi}(b))$.*

Using shorthand $\hat{a}$ defined above, the matrix $A(\hat{b}) = A(\hat{b}, \hat{t}, ..., \hat{t})$ takes the special symmetric form

$$A(\hat{b}) = \begin{pmatrix} 0 & \hat{a} & ... & \hat{a} & \hat{a} \\ \hat{a} & 0 & ... & \hat{a} & \hat{a} \\ ... & ... & ... & ... & ... \\ \hat{a} & \hat{a} & ... & 0 & \hat{a} \\ \hat{a} & \hat{a} & ... & \hat{a} & 0 \end{pmatrix}$$

Since $\hat{a} > 0$, this matrix is invertible (straightforward proof omitted). Thus, its determinant is non-zero (Anton [1], Theorem 2.3.4). Since each $a_{ij}(b, \boldsymbol{\phi})$ is continuously differentiable, so is the determinant of $A(b, \boldsymbol{\phi})$. In particular, the inverse $A^{-1}(b, \boldsymbol{\phi})$ exists for all $(b, \boldsymbol{\phi})$ in a neighborhood of $(\hat{b}, \hat{t}, ..., \hat{t})$, and the entries $a_{ij}^{-1}(b, \boldsymbol{\phi})$ of this inverse matrix are continuously differentiable as well. Finally, by Lemma 3, $\hat{b} > \underline{b}$ implies that $\phi_i(\cdot)$ is continuous in a neighborhood of $\hat{b}$. Thus, the inverse $A^{-1}(b) = A^{-1}(b, \boldsymbol{\phi}(b))$ exists for all b in a neighborhood of $\hat{b}$.

First-order conditions on bidding. Consider any bid-level $b \in (\hat{b} - \gamma, \hat{b} + \gamma)$. By Claim 4, derivatives $\phi_i'(b)$ exist for all i. Thus,

$$\frac{\partial \Pi_i(t_i, b)}{\partial b} = \sum_{j \neq i} \left\{ \phi_j{}'(b) \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-ij}(b)} u(t_i; \phi_j(b), \mathbf{t}_{-ij}; b) f(\phi_j(b), \mathbf{t}_{-ij} | t_i) \mathrm{d}\mathbf{t}_{-ij} \right\} + \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-i}(b)} \frac{\partial u(t_i; \mathbf{t}_{-i}; b)}{\partial b} f(\mathbf{t}_{-i} | t_i) \mathrm{d}\mathbf{t}_{-i}$$

Since type $\phi_i(b)$ finds bid b to be a best response, the following system of n equations must be satisfied by $\boldsymbol{\phi}(b) = (\phi_1(b), ..., \phi_n(b))$: for all i,

$$0 = \sum_{j \neq i} \left\{ \phi_j{}'(b) \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-ij}(b)} u(\phi_i(b); \phi_j(b), \mathbf{t}_{-ij}; b) f(\phi_j(b), \mathbf{t}_{-ij} | \phi_i(b)) \mathrm{d}\mathbf{t}_{-ij} \right\} + \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-i}(b)} \frac{\partial u(\phi_i(b); \mathbf{t}_{-i}; b)}{\partial b} f(\mathbf{t}_{-i} | \phi_i(b)) \mathrm{d}\mathbf{t}_{-i} \quad (3)$$

As long as $A(b)$ is invertible, we may express system (3) as:

$$(\phi_1'(b), ..., \phi_n'(b)) \equiv (g_1(b, \boldsymbol{\phi}(b)), ..., g_n(b, \boldsymbol{\phi}(b))) = C(b) * A^{-1}(b)$$

Note that $g_i(b, \boldsymbol{\phi}) = C(b, \boldsymbol{\phi}) * A^{-1}(b, \boldsymbol{\phi})$ is continuously differentiable in a neighborhood of $(\hat{b}, \hat{\mathbf{t}})$ for all i. This is more than enough to imply the Lipschitz condition needed to apply the Fundamental Theorem of Differential Equations

(FTODE).[8] All bidders' inverse bid functions $\phi_i(b)$ are uniquely determined and continuously differentiable over a neighborhood $(\hat{b}-\gamma, \min\{\hat{b}+\gamma, \bar{b}\})$ for some $\gamma > 0$. Uniqueness implies symmetry of this local solution since any asymmetric solution would lead to another asymmetric solution after permuting the identities of the bidders. So, $\phi_i(b) = \phi(b)$ for all $b \in (\hat{b}-\gamma, \min\{\hat{b}+\gamma, \bar{b}\})$ where $\phi(\cdot)$ is strictly increasing and continuously differentiable.

All MPSE are symmetric.

Recall our shorthand for the lowest winning bid $\underline{b} \equiv \max_i b_i(0+)$. For each bidder i, define $\underline{t}_i \equiv \phi_i(\max\{\underline{b}, r\}+)$.

Claim 6 *$\underline{t} \in [0,1]$ and $b(\cdot)$ exist such that (i) $\underline{t}_i = \underline{t}$ for all i and (ii) $b_i(t) = b(t)$ for all i and all $t \in [0, \underline{t}) \cup (\underline{t}, 1)$, where (iii) $b(\cdot)$ is strictly increasing and continuously differentiable over $(\underline{t}, 1)$ and (iv) $b(t) = OUT$ for all $t \in (0, \underline{t})$.*

Proof of Claim 6. If $\bar{b} = OUT$, (i)-(iv) are immediate: set $\underline{t} = 1$ and $b(t) = OUT$ for all $t < 1$. Otherwise, $b_i(1-) = \bar{b} > \max\{\underline{b}, r\}$ for all i by Claim 1. By Claim 5, furthermore, there exists strictly increasing, continuously differentiable $\phi(\cdot)$ and $\gamma^1 > 0$ such that $\phi_i(b) = \phi(b)$ for all $b \in (\bar{b}-\gamma^1, \bar{b})$. Define $b^* \equiv \min\{\tilde{b} \in [\max\{\underline{b}, r\}, \bar{b}) : \phi_i(b) = \phi(b)$ for all i and all $b \in (\tilde{b}, \bar{b})$, where $\phi(\cdot)$ is strictly increasing and continuously differentiable$\}$. (So far, we have shown

[8] The Lipschitz condition in our case requires that $M < \infty$ exists such that, for all i, $\frac{|g_i(b,\boldsymbol{\phi})-g_i(b,\tilde{\boldsymbol{\phi}})|}{\max_{1\le i\le n}|\phi_i-\tilde{\phi}_i|} \le M$ for all b in a neighborhood of $\hat{b}$ and all $\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}$ in a neighborhood of $\hat{\mathbf{t}}$. (See Theorem 2′ from Kolmogorov and Fomin [8], p. 72.) Continuous differentiability of each g_i implies that $\frac{|g_i(b,\boldsymbol{\phi})-g_i(b,\tilde{\boldsymbol{\phi}})|}{\max_{1\le i\le n}|\phi_i-\tilde{\phi}_i|} < 2\sum_{1\le j\le n}\frac{\partial g_i}{\partial \phi_j}(\hat{b}, \hat{\mathbf{t}})$ for all i when these neighborhoods are small enough. Thus, we may set $M = \max_i\left(2\sum_{1\le j\le n}\frac{\partial g_i}{\partial \phi_j}(\hat{b}, \hat{\mathbf{t}})\right) < \infty$.

that $b^* \leq \overline{b} - \gamma^1$.)

I claim that $b^* = \max\{\underline{b}, r\}$. Suppose otherwise that $b^* > \max\{\underline{b}, r\}$. By construction, $b_i(t) = b(t)$ for all $t > \phi(b^*)$, where $b(\cdot)$ is strictly increasing and continuously differentiable over this range of types. In particular, $b_i(\phi(b^*)+) = b^*$ for all i. Claim 5 then implies that we can (uniquely) extend $\phi(\cdot)$ to the wider range of bid-levels $(b^* - \gamma^2, \overline{b})$ for some $\gamma^2 > 0$. But then $b^* \leq b^* - \gamma^2$, a contradiction. We conclude that

$$\phi_i(b) = \phi(b) \text{ for all } b > \max\{\underline{b}, r\}$$

where $\phi(\cdot)$ is strictly increasing and continuously differentiable over $(\max\{\underline{b}, r\}, \overline{b})$. Equivalently, $b_i(t) = b(t)$ for all $t \in (\underline{t}, 1)$, where $\underline{t} = \phi(\max\{\underline{b}, r\})$ and $b(\cdot)$ is strictly increasing and continuously differentiable over $(\underline{t}, 1)$.

To complete the proof, we need to show that $b_i(t_i) = OUT$ for all $t_i < \underline{t}$. There are two cases to consider.

First, suppose that $\max\{\underline{b}, r\} = \underline{b}$. By definition, there exists j^* such that $\underline{b} = b_{j^*}(0+)$. This implies $\phi_{j^*}(\underline{b}) = 0$ so that $\underline{t} = 0$. $b_i(t_i) = OUT$ for all $t_i < \underline{t}$ is vacuous in this case.

Second, suppose that $\underline{b} = OUT$ so that $\max\{\underline{b}, r\} = r$. In this case, $b_i(t_i) \in \{OUT, r\}$ for all i and all $t_i < \underline{t}$. By Lemma 2, at most one bidder (say bidder 1) can have an atom at r, so $b_i(t_i) = OUT$ for all $i \neq 1$ and all $t_i < \underline{t}$. Thus, $\sup_b \Pi_i(t_i, b) = 0$ for all $i \neq 1$ and almost all $t_i < \underline{t}$. By symmetry of bidder strategies above r, $\sup_b \Pi_i(t, b) \geq \lim_{\delta \to 0} \Pi_i(t, r + \delta) = \lim_{\delta \to 0} \Pi_1(t, r + \delta)$ for all i and all t. Since no bidder $i \neq 1$ has an atom at r, $\lim_{\delta \to 0} \Pi_1(t_1, r + \delta) = \Pi_1(t_1, r)$ for all t_1. Lastly, by **(A3-4)**, $\Pi_1(t_1, r)$ is continuous in t_1. All together, we conclude that $\Pi_1(\underline{t}, r) \leq 0$. Since $P_1(\underline{t}, r) > 0$, $\Pi_1(\underline{t}, r) \leq 0$

implies $U_1(\underline{t}, r) \equiv E\left[u(\underline{t}; \mathbf{t}_{-1}; r) | t_1 = \underline{t}, \mathbf{t}_{-1} \leq \mathbf{1}\right] \leq 0$. Since $U_1(t_1, r)$ is strictly increasing in t_1 (Lemma 1(b)), however,

$$\Pi_1(t_1, r) = P_1(t_1, r) U_1(t_1, r) < 0 \text{ for all } t_1 < \underline{t}. \tag{4}$$

All types $t_1 < \underline{t}$ strictly prefer not to participate rather than bid r, a contradiction.

Unique 'minimal winning type' and 'minimal winning bid'.

Define $h(t, b) \equiv E[u(t_1; \mathbf{t}_{-1}; b) | t_1 = t, t_1 \leq t \text{ for all } i \neq 1]$. $h(t, b)$ is strictly increasing in t and strictly decreasing in b. (The proof of this is very similar to that of Lemma 1(b) and omitted to save space.)

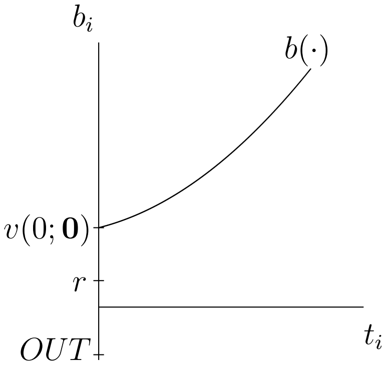
Claim 7 *Every MPSE has the same 'minimal winning type' $\underline{t}$ and the same 'minimal winning bid' $\underline{b}$, where these depend on the environment. Case I: If $h(0, r) \geq 0$, then $\underline{t} = 0$ and $\underline{b} \geq r$ solves $h(0, \underline{b}) = 0$. Case II: If $h(0, r) < 0$ and $h(1, r) > 0$, then $\underline{t}$ solves $h(\underline{t}, r) = 0$ and $\underline{b} = OUT$. Case III: If $h(1, r) \leq 0$, then $\underline{t} = 1$ and $\underline{b} = OUT$.*

Figures 1, 2 illustrate Cases I,II given risk-neutral bidders, i.e. $u(t_i; \mathbf{t}_{-i}; b) = v(t_i; \mathbf{t}_{-i}) - b$. In this setting, $h(0, r) = v(0; \mathbf{0}) - r$.

Proof of Claim 7. Each bidder gets zero payoff given type $\underline{t}$, since he gets zero payoff given any type less than $\underline{t}$ and payoffs are continuous in t_i (Lemma 4(a)). Thus,

$$\Pr(\mathbf{t}_{-i} < \underline{\mathbf{t}} | t_i = \underline{t}) E[u_i(\underline{t}; \mathbf{t}_{-i}; b(\underline{t}+)) | t_i = \underline{t}, \mathbf{t}_{-i} < \underline{\mathbf{t}}] = 0$$

and either $\underline{t} = 0$ or $E[u_i(\underline{t}; \mathbf{t}_{-i}; b(\underline{t}+)) | t_i = \underline{t}, \mathbf{t}_{-i} < \underline{\mathbf{t}}] = 0$. When $\underline{t} = 0$, further, $u(0; \mathbf{0}; b(0+)) = 0$ else each bidder i would prefer to deviate given types $t_i \approx 0$.



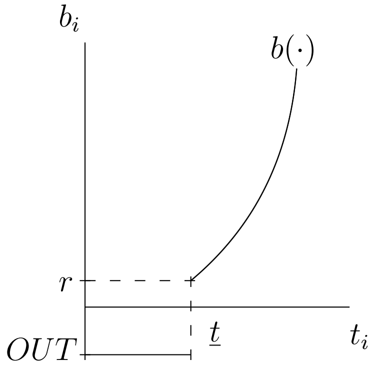


Fig. 1. If $v(0;\mathbf{0}) > r$, then $\underline{t} = 0$ and $b(0+) = v(0;\mathbf{0})$.

Fig. 2. If $v(0;\mathbf{0}) \leq r$, then $b(\underline{t}+) = r$ and $E\left[v(t_i;\mathbf{t}_{-i})|t_i = \underline{t}, \mathbf{t}_{-i} < \underline{\mathbf{t}}\right] = r$.

Thus, in either case, equilibrium requires that $h(\underline{t}, b(\underline{t}+)) = 0$.

Furthermore, from the proof of Claim 6, $b(\underline{t}+) = \max\{\underline{b}, r\}$ and $\underline{b} > r$ is only possible when $\underline{t} = 0$. Thus, equilibrium requires $h(\underline{t}, \max\{\underline{b}, r\}) = 0$ and either $\underline{t} = 0$ or $\underline{b} = OUT$.

Case I: $h(0, r) \geq 0$. $\underline{t} = 0$ in this case: otherwise, $\max\{\underline{b}, r\} = r$ so that $h(\underline{t}, \max\{\underline{b}, r\}) \geq h(\underline{t}, r) > h(0, r) \geq 0$. $\underline{b} \geq r$ is then uniquely determined by $h(0, \underline{b}) = 0$.

Case II: $h(0, r) < 0$ *and* $h(1, r) > 0$. Now $\underline{t} \in (0, 1)$. Otherwise, $h(\underline{t}, \max\{\underline{b}, r\}) \leq h(0, r) < 0$. Thus, $\underline{b} = OUT$ and $\underline{t}$ is uniquely determined by $h(\underline{t}, r) = 0$.

Case III: $h(1, r) \leq 0$. Here $\underline{t} = 1$ and $\underline{b} \leq r$. Otherwise, $h(\underline{t}, \max\{\underline{b}, r\}) < h(1, r) \leq 0$. As discussed earlier, $\underline{b} \leq r$ implies $\underline{b} = OUT$. Thus, $\underline{t} = 1$ and $\underline{b} = OUT$. This completes the proof.

Uniqueness of MPSE.

Consider Case I in which $\underline{t} = 0$, i.e. the reserve price is not binding. (The proof for Cases II, III proceeds in a similar way and is omitted.)

So far, we have shown that any MPSE must be symmetric: $b_i(t) = b(t)$ for all i and all $t \in (0,1)$, where $b(\cdot)$ is strictly increasing and continuously differentiable. More precisely, given only that $\max_i b_i(1-) = \bar{b}$, we showed by construction that there is at most one such bidding function that is consistent with MPSE. Thus, if there are multiple MPSE $\mathbf{b}^1(\cdot) = (b^1(\cdot), ..., b^1(\cdot))$ and $\mathbf{b}^2(\cdot) = (b^2(\cdot), ..., b^2(\cdot))$, then it must be that $b^1(1-) \neq b^2(1-)$.

Further, all MPSE must be strictly ordered in the sense that $b^1(1-) > b^2(1-)$ implies $b^1(t) > b^2(t)$ for all $t \in (\underline{t}, 1)$. Suppose to the contrary that $b^1(1-) > b^2(1-)$ but $\hat{b} = b^1(\hat{t}) = b^2(\hat{t})$ for some $\hat{t} \in (\underline{t}, 1)$. By Claim 4, $\gamma > 0$ exists so that $b^1(t) = b^2(t)$ for all $t \in (\hat{t} - \gamma, \hat{t} + \gamma)$. Indeed, by this logic $b^1(t) = b^2(t)$ for all $t \in (\hat{t} - \gamma, 1)$ so that $b^1(1-) = b^2(1-)$,[9] a contradiction. On the other hand, $b^1(0+) = b^2(0+) = \underline{b}$ where $\underline{b}$ is defined implicitly by $u(0; \mathbf{0}; \underline{b}) = 0$.

Finally, suppose that $(b^1(\cdot), ..., b^1(\cdot))$ and $(b^2(\cdot), ..., b^2(\cdot))$ are two MPSE where $b^1(t) > b^2(t)$ for all $t \in (0,1)$ and $b^1(0+) = b^2(0+) = \underline{b}$. Fix any type $\tilde{t} \in (0,1)$ and let $b^1 \equiv b^1(\tilde{t})$ and $b^2 \equiv b^2(\tilde{t})$. By symmetry, each bidder's first-order condition (3) in each equilibrium can be re-arranged as:

$$\phi^{1\prime}(b^1) = \frac{-\int_{\mathbf{0}}^{\tilde{\mathbf{t}}_{-1}} \frac{\partial u(\tilde{t}; \mathbf{t}_{-1}; b^1)}{\partial b} f(\mathbf{t}_{-1} | t_1 = \tilde{t}) \mathrm{d}\mathbf{t}_{-1}}{(n-1) \int_{\mathbf{0}}^{\tilde{\mathbf{t}}_{-12}} u(\tilde{t}; \tilde{t}, \mathbf{t}_{-12}; b^1) f(t_2 = \tilde{t}, \mathbf{t}_{-12} | t_1 = \tilde{t}) \mathrm{d}\mathbf{t}_{-12}} \tag{5}$$

$$\phi^{2\prime}(b^2) = \frac{-\int_{\mathbf{0}}^{\tilde{\mathbf{t}}_{-1}} \frac{\partial u(\tilde{t}; \mathbf{t}_{-1}; b^2)}{\partial b} f(\mathbf{t}_{-1} | t_1 = \tilde{t}) \mathrm{d}\mathbf{t}_{-1}}{(n-1) \int_{\mathbf{0}}^{\tilde{\mathbf{t}}_{-12}} u(\tilde{t}; \tilde{t}, \mathbf{t}_{-12}; b^2) f(t_2 = \tilde{t}, \mathbf{t}_{-12} | t_1 = \tilde{t}) \mathrm{d}\mathbf{t}_{-12}} \tag{6}$$

where bolded notation $\tilde{\mathbf{t}}_I = (t_i = \tilde{t} : i \in I)$.

By **(A5)**, u is strictly decreasing in b. Since $b^1 > b^2$, the denominator of (5) is strictly less than the denominator of (6). (One can show that the denom-

[9] Define $t^* \equiv \min\{\tilde{t} > \hat{t} : b^1(t) = b^2(t) \text{ for all } t \in (\hat{t}, \tilde{t})\}$. One shows $t^* = 1$ by repeating, with minor modifications, the argument in the proof of Claim 6 that $b^* = \max\{\underline{b}, r\}$.

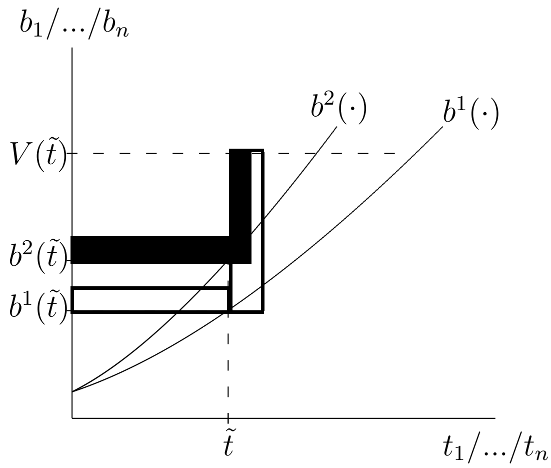


Fig. 3. Graphical intuition why equilibria can not be ordered.

inators in (5,6) are positive, in the same way that we showed $\hat{a} > 0$ in the proof of Claim 5.) By **(A6)**, $\partial u/\partial b$ is non-increasing in b and negative, so the numerator of (5) is weakly greater than the numerator of (6) (and both are positive). Thus, $\phi^{1\prime}(b^1) > \phi^{2\prime}(b^2)$ so that $b^{2\prime}(\tilde{t}) > b^{1\prime}(\tilde{t})$ for all $\tilde{t} > \underline{t}$. But this implies $b^1(\tilde{t}) = b(\underline{t}) + \int_{\underline{t}}^{\tilde{t}} b^{1\prime}(t)\mathrm{d}t < b(\underline{t}) + \int_{\underline{t}}^{\tilde{t}} b^{2\prime}(t)\mathrm{d}t = b^2(\tilde{t})$, a contradiction.

Graphical intuition why strictly ordered MPSE can not exist. Consider bidders' first-order conditions in both equilibria given type $\tilde{t} > 0$. Figure 3 summarizes bidder i's trade-off associated with bidding slightly higher than $b^1(\tilde{t})$ in equilibrium 1 and/or slightly higher than $b^2(\tilde{t})$ in equilibrium 2. The extra expected payment from bidding higher is the 'area' of a horizontal rectangle; the extra expected surplus from the marginal winning event is the 'area' of a vertical rectangle. (Rectangles corresponding to equilibrium 2 are filled. $\max_{j \neq i} t_j = \tilde{t}$ in the marginal winning event, so bidder i's conditional expected value is $V(\tilde{t}) \equiv E[v(\tilde{t}; \mathbf{t}_{-1}) | t_2 = \tilde{t}, \mathbf{t}_{-12} \leq \tilde{\mathbf{t}}_{-12}]$.) For the same small increase in the bids, the two horizontal rectangles have the same area. Since $b^2(\tilde{t}) > b^1(\tilde{t})$, the vertical rectangle for equilibrium 1 has more height. Since bidder 1 must be indifferent to raising its bid in each equilibrium, the vertical and horizontal areas must be the same in each equilibrium, implying that the vertical rectangle for equilibrium 1 has less width. That is to say, $b^{1\prime}(\tilde{t}) > b^{2\prime}(\tilde{t})$ for all $\tilde{t} > 0$. But this contradicts the presumption that $b^1(\tilde{t}) < b^2(\tilde{t})$ since $b^1(0) = b^2(0)$ and these bid functions are continuous.

4 Extension: All-pay auctions

The first-price auction has the property that bidders get the same utility from losing as from not participating. This property is not essential to my analysis. What is essential is that each bidder's payoff does not depend on others' bids (except insofar as others' bids determine the winner). More precisely, suppose that bidders have different utilities from winning and losing the object, $u_i^W(t_i; \mathbf{t}_{-i}; b_i; \mathbf{b}_{-i})$ and $u_i^L(t_i; \mathbf{t}_{-i}; b_i; \mathbf{b}_{-i})$. The analysis depends on: (i) u_i^W, u_i^L each do not depend on $\mathbf{b}_{-i}$, (ii) u_i^W is strictly decreasing in b_i while u_i^L is non-increasing in b_i, and (iii) $u_i^W(t_i; \mathbf{t}_{-i}; b_i) - u_i^L(t_i; \mathbf{t}_{-i}; b_i)$ is strictly increasing in t_i and non-decreasing in t_{-i}.

All-pay auctions: In the all-pay auction, losers pay their own bid and one can check that (i,ii,iii) are satisfied. Thus, the proof of Theorem 1 implies that the all-pay auction has a unique monotone pure strategy equilibrium.[10] Yet whether the all-pay auction has mixed strategy or non-monotone pure strategy equilibria is unknown, so the question of uniqueness remains partially unresolved.

[10] To apply to the all-pay auction, the equations for bidders' first-order conditions (see e.g. (3,20,27)) must be modified slightly to reflect the fact that each bidder always pays its bid. This does not change the argument in any substantive way.

Appendix

Proof of Lemma 1

(a) follows immediately from MW Theorem 23, since u_i is non-decreasing in $\mathbf{t}$ for all i. For (b), fix $\tilde{t} \in [0,1]$ and $\tilde{b} \geq r$. Define an intermediate function $g_i(t_i;\mathbf{t}_{-i}) \equiv u_i(\tilde{t}_i;\mathbf{t}_{-i};\tilde{b})$ that is non-decreasing in $\mathbf{t}$ and constant in t_i. Consider any $t_i' \geq \tilde{t}_i$ and $b' \leq \tilde{b}$ such that $(t_i', b') \neq (\tilde{t}_i, \tilde{b})$.

$$\begin{aligned} U_i(t_i', b') &= E\left[u_i(t_i';\mathbf{t}_{-i};b')|t_i = t_i', \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(b')\right] \\ &> E\left[u_i(\tilde{t}_i;\mathbf{t}_{-i};\tilde{b})|t_i = t_i', \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(b')\right] = E\left[g_i(t_i';\mathbf{t}_{-i})|t_i = t_i', \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(b')\right] \\ &\geq E\left[g_i(t_i';\mathbf{t}_{-i})|t_i = t_i', \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(\tilde{b})\right] \geq E\left[g_i(\tilde{t}_i;\mathbf{t}_{-i})|t_i = \tilde{t}_i, \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(\tilde{b})\right] \\ &= E\left[u_i(\tilde{t}_i;\mathbf{t}_{-i};\tilde{b})|t_i = \tilde{t}_i, \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(\tilde{b})\right] = U_i(\tilde{t}_i, \tilde{b}) \end{aligned}$$

The strict inequality holds since u_i is strictly increasing in t_i and strictly decreasing in t_i.[11] The weak inequalities hold by successive applications of MW Theorem 23.[12]

Proof of Lemma 2

Preliminaries: By definition, $b_j(t_j) < b$ when $t_j \in [0, \phi_j(b)]$ and $b_j(t_j) = b$ when $t_j \in [\phi_j(b), \phi_j(b+)]$ (up to zero measure boundaries), where $\phi_j(b+) \equiv$

[11] $E\left[u_i(\tilde{t}_i;\mathbf{t}_{-i};\tilde{b})|t_i = t_i', \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(b')\right] \equiv \frac{\int_{\mathbf{0}}^{\boldsymbol{\phi}_{-i}(b')} u_i(\tilde{t}_i;\mathbf{t}_{-i};\tilde{b}) f(\mathbf{t}_{-i}|t_i=t_i') d\mathbf{t}_{-i}}{\int_{\mathbf{0}}^{\boldsymbol{\phi}_{-i}(b')} f(\mathbf{t}_{-i}|t_i=t_i') d\mathbf{t}_{-i}}$ and likewise for similar conditional expectations.

[12] g_i is non-decreasing in $\mathbf{t}$. For the first inequality, consider sublattice $X \equiv \{t_i'\} \times [\mathbf{0}, \boldsymbol{\phi}_{-i}(b')]$ and decreasing subset $A \equiv \{t_i'\} \times [\mathbf{0}, \boldsymbol{\phi}_{-i}(\tilde{b})]$. For the second inequality, consider sublattice $X \equiv \{t_i', \tilde{t}_i\} \times [\mathbf{0}, \boldsymbol{\phi}_{-i}(\tilde{b})]$ and decreasing subset $A \equiv \{\tilde{t}_i\} \times [\mathbf{0}, \boldsymbol{\phi}_{-i}(\tilde{b})]$.

$\lim_{\delta\to 0}\phi_j(b{+}\delta)$. The assumption $\Pr\left(b \geq \max_j b_j(t_j)\right) > 0$ implies that $\phi_j(b+) > 0$ for all j.

Consider any $b \geq r$. The result is immediate if every bidder bids b with zero probability. Suppose without loss that bidder 1 bids b with positive probability, so that $\phi_1(b+) > \phi_1(b)$. All types in $(\phi_1(b), \phi_1(b+))$ bid b and almost all of them find b to be a best response. Let $\tilde{t}_1 \in (\phi_1(b), \phi_1(b+))$ be some type for which b is a best response. Interim expected payoff for this type is

$$\Pi_1(\tilde{t}_1, b) \equiv \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b+)} \frac{1}{1 + \#\left\{j \neq 1 : b_j(t_j) = b\right\}} u_1(\tilde{t}_1; \mathbf{t}_{-1}; b) f(\mathbf{t}_{-1}|\tilde{t}_1) d\mathbf{t}_{-1} \tag{7}$$

Define the following shorthand:

$$\begin{aligned} G(k) &\equiv \left\{\mathbf{t}_{-1} \leq \boldsymbol{\phi}_{-1}(b+) : \#\left\{j \neq 1 : b_j(t_j) = b\right\} = k - 1\right\}. \\ A(k) &\equiv \Pr\left(G(k)\middle|\tilde{t}_1\right) E\left[u_1(\tilde{t}_1; \mathbf{t}_{-1}; b)\middle|\tilde{t}_1, \mathbf{t}_{-1} \in G(k)\right] \end{aligned}$$

$G(k)$ is the event in which bidder 1 wins the object with probability $1/k$ because he ties with $k-1$ others at b. Let $k^* - 1$ be the number of other bidders who bid b with positive probability. Thus, $A(k) = 0$ for all $k > k^*$. In terms of this shorthand, note that

$$\Pi_1(\tilde{t}_1, b) = \sum_{k=1}^{k^*} \frac{A(k)}{k}, \quad \Pi_1(\tilde{t}_1, OUT) = 0, \quad \lim_{\delta\to 0} \Pi_1(\tilde{t}_1, b + \delta) = \sum_{k=1}^{k^*} A(k) \tag{8}$$

Step 1: Restrictions imposed by best response. Since b is a best response, type $\tilde{t}_1$ does not prefer to submit the null bid OUT nor to bid slightly more than b. Thus,

$$\sum_{k=1}^{k^*} \frac{1}{k} A(k) \geq 0, \ \sum_{k=2}^{k^*} \frac{k-1}{k} A(k) \leq 0 \tag{9}$$

Step 2: Either $k^* = 1$ or $\sum_{k=1}^{k^*} A(k) = 0$. In words, either no other bidder bids b with positive probability or bidder 1 gets approximately zero expected utility given type $\tilde{t}_1$ when he bids slightly more than b. When $k^* = 1$, the first inequality (9) is satisfied when $\sum_{k=1}^{k^*} A(k) \geq 0$ while the second inequality is vacuously satisfied.

Suppose instead that $k^* > 1$. Note that $\left[\mathbf{0}, \boldsymbol{\phi}_{-1}(b+)\right]$ is a lattice and that, for all $m \in \{1, ..., k^*\}$, $\cup_{k=1}^{m} G(k)$ is a decreasing subset of $\left[\mathbf{0}, \boldsymbol{\phi}_{-1}(b+)\right]$.[13] Thus, Lemma 1(a) implies that $\sum_{k=1}^{k^*} A(k) \geq 0$ since

$$\sum_{k=1}^{k^*} A(k) < 0 \Rightarrow \sum_{k=1}^{m} A(k) < 0 \text{ for all } m = 1, ..., n$$
$$\Rightarrow \sum_{k=1}^{k^*} \frac{1}{k} A(k) = \frac{1}{k^*} \sum_{k=1}^{k^*} A(k) + \sum_{m=1}^{n-1} \left(\left(\frac{1}{m} - \frac{1}{m+1} \right) \sum_{k=1}^{m} A(k) \right) < 0$$

contradicting (9). Similarly, $\sum_{k=1}^{k^*} A(k) \leq 0$ since

$$\sum_{k=1}^{k^*} A(k) > 0 \Rightarrow \sum_{k=m}^{k^*} A(k) > 0 \text{ for all } m = 1, ..., k^*$$
$$\Rightarrow \sum_{k=2}^{k^*} \frac{k-1}{k} A(k) = \frac{1}{2} \sum_{k=2}^{k^*} A(k) + \sum_{m=3}^{k^*} \left(\left(\frac{m-1}{m} - \frac{m-2}{m-1} \right) \sum_{k=m}^{k^*} A(k) \right) > 0$$

contradicting (9). Finally, $E\left[u_1(t; \mathbf{t}_{-1}; b) \middle| t_1 = t\right] = \sum_{k=1}^{k^*} A(k)$ is strictly increasing in t by Lemma 1(b). Thus, at most one bidder 1-type finds b to be a best response. But almost all types in $(\phi_1(b), \phi_1(b+))$ find b to be a best response, a contradiction.

Proof of Lemma 3

By assumption, $\Pr(b > \max_i b_i(t_i)) > 0$, so $\phi_i(b) > 0$ for all i. Furthermore,

[13] Each set in this union has the form, up to a zero measure set, of

$$G(k) = \bigcup_{J \subset \{2,...,n\}: \#(J) = k-1} \left(\prod_{j \in J} [\phi_j(b), \phi_j(b+)] \prod_{j \in \{2,...,n\} \setminus J} [0, \phi_j(b)] \right)$$

So, if $\mathbf{t}_{-1} \in G(k)$ and $\mathbf{t}'_{-1} < \mathbf{t}_{-1}$, then $\mathbf{t}'_{-1} \in G(k')$ for some $k' \leq k$.

given no ties (Lemma 2) there is at most one bidder (say bidder 1) with an atom at b.

Step 1: Someone else bids just below b. There can not be a gap in the distribution of $\max_{j\neq 1} b_j(t_j)$ below b. If there were, bid b would be strictly dominated for bidder 1, contradicting the presumption that almost all types $t_1 \in (\phi_1(b), \phi_1(b+))$ find b to be a best response. Thus, there exists a bidder $j^* \neq 1$ with a convergent sequence of types $\{t_{j^*,k}\}^{k=1,2,\dots} \nearrow t_j^*$ such that $b - b_{j^*}(t_{j^*,k}) \equiv \delta_k \searrow 0$. Without loss, we can select this sequence so that type $t_{j^*,k}$ finds $b_{j^*}(t_{j^*,k})$ to be a best response for all k.

For sufficiently large K, $P_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K})) > 0$. Thus, $\Pi_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K})) > 0$ by Lemma 5. (The proof of Lemma 5 does not depend on Lemma 3.) Indeed, profits of slightly higher types are strictly bounded above zero: for all $k > K$,

$$\begin{aligned}
\Pi_{j^*}(t_{j^*,k}, b_{j^*}(t_{j^*,k})) &\geq \Pi_{j^*}(t_{j^*,k}, b_{j^*}(t_{j^*,K})) = U_{j^*}(t_{j^*,k}, b_{j^*}(t_{j^*,K}))P_{j^*}(t_{j^*,k}, b_{j^*}(t_{j^*,K})) \\
&> U_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K}))P_{j^*}(t_{j^*,k}, b_{j^*}(t_{j^*,K})) \\
&\geq \frac{f_{low}}{f_{high}} U_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K}))P_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K})) = \frac{f_{low}}{f_{high}} \Pi_{j^*}(t_{j^*,K}, b_{j^*}(t_{j^*,K}))
\end{aligned}$$

The first weak inequality is by revealed preference, the strict inequality is by Lemma 1(b), and the last inequality follows from assumption (A2).

Step 2: No one else bids just below b. Each type $t_{j^*,k}$ must at least weakly prefer to bid $b_{j^*}(t_{j^*,K}) = b - \delta_k$ rather than $b + \delta_k$. Since bidder 1 has an atom at b and all other bidders bid strictly less than b with positive probability, raising his bid by $2\delta_k$ allows bidder j^* to increase his probability of winning by an amount that does not disappear as $\delta_k \to 0$.

For these deviations to be unprofitable, then, bidder j^* must not strictly prefer to win the object at price b conditional on tying (in the limit):

$$\begin{aligned}
0 &\geq \lim_{k\to\infty} E\left[u\left(t_{j^*};\mathbf{t}_{-j^*};b\right)\Big|t_{j^*}=t_{j^*,k},\mathbf{t}_{-j^*}:\max_{i\neq j^*} b_i(t_i)\in(b-\delta_k,b+\delta_k)\right] \\
&\geq \lim_{k\to\infty} E\left[u\left(t_{j^*};\mathbf{t}_{-j^*};b\right)\Big|t_{j^*}=t_{j^*,k},\mathbf{t}_{-j^*}\in\max_{i\neq j^*} b_i(t_i)\leq b-\delta_k\right] \\
&= \lim_{k\to\infty} U_{j^*}(t_{j^*,k},b_{j^*}(t_{j^*,k}))
\end{aligned}$$

The second inequality follows from Lemma 1(a). (Set $X\equiv\times_{i=1}^{n}[0,\phi_i(b+\delta_k)]$ and $A\equiv\times_{i=1}^{n}[0,\phi_i(b-\delta_k)]$.) In this case, however, $\lim_{k\to\infty}\Pi_{j^*}(t_{j^*,k},b_{j^*}(t_{j^*,k}))\leq 0$, a contradiction.

Proof of Lemma 4

Proof of (a). By (A3-4), $\Pi_i(t_i,b)=\int_{\mathbf{t}_{-i}\leq\boldsymbol{\phi}_{-i}(b)} u_i(t_i;\mathbf{t}_{-i};b)f(\mathbf{t}_{-i}|t_i)\mathrm{d}\mathbf{t}_{-i}$ is absolutely continuous in t_i for each fixed b and

$$\begin{aligned}
\frac{\partial\Pi_i(t_i,b)}{\partial t_i} &= \frac{\partial\left(\int_{\mathbf{t}_{-i}\leq\boldsymbol{\phi}_{-i}(b)} u_i(t_i;\mathbf{t}_{-i};b)f(\mathbf{t}_{-i}|t_i)\mathrm{d}\mathbf{t}_{-i}\right)}{\partial t_i} \\
&= \int_{\mathbf{t}_{-i}\leq\boldsymbol{\phi}_{-i}(b)}\left(\frac{\partial u_i(t_i;\mathbf{t}_{-i};b)}{\partial t_i}f(\mathbf{t}_{-i}|t_i)+u_i(t_i;\mathbf{t}_{-i};b)\frac{\partial f(\mathbf{t}_{-i}|t_i)}{\partial t_i}\right)\mathrm{d}\mathbf{t}_{-i}
\end{aligned}$$

exists for each fixed b. Indeed, by (A3-4), $u_i(t_i;\mathbf{t}_{-i};b)$ and $f(\mathbf{t}_{-i}|t_i)$ are continuously differentiable, so $\frac{\partial u_i(t_i;\mathbf{t}_{-i};b)}{\partial t_i}f(\mathbf{t}_{-i}|t_i)+u_i(t_i;\mathbf{t}_{-i};b)\frac{\partial f(\mathbf{t}_{-i}|t_i)}{\partial t_i}$ is continuous and hence uniformly bounded above and below for all $\mathbf{t}\in[0,1]^n$ and all $b\leq\bar{b}$. Thus, there exists an integrable function $x:[0,1]\to\mathbf{R}_+$ such that $\left|\frac{\partial\Pi_i(t_i,b)}{\partial t_i}\right|\leq x(t_i)$ for all $b\leq\bar{b}$.

By Theorem 2 of Milgrom and Segal [14], we may therefore conclude that $\sup_{b\leq\bar{b}}\Pi_i(t_i,b)$ is absolutely continuous in t_i. Since all bids strictly greater than $\bar{b}$ are strictly dominated, finally, $\sup_b\Pi_i(t_i,b)=\sup_{b\leq\bar{b}}\Pi_i(t_i,b)$. Thus, $\sup_b\Pi_i(t_i,b)$ is (absolutely) continuous in t_i.

Proof of (b). Consider any type t_i such that $b_i(t_i-)>\max\{\underline{b},r\}$ and any increasing sequence $\{t^k\} \nearrow t_i$ such that $b_i(t^k)$ is a best response for type t^k for all k. $\Pi_i(t,b)$ is continuous in t by assumptions **(A3-4)**, $\sup_b \Pi_i(t,b)$ is continuous in t by Lemma 4(a), and $\Pi_i(t,b)$ is continuous in b at bid-level $b_i(t_i-)$ by Lemma 3. (Since $b_i(t_i-) > \max\{\underline{b}, r\}$, Lemma 3 implies that there are no atoms at $b_i(t_i-)$.) Thus, $\sup_b \Pi_i(t_i,b) = \lim_{k\to\infty} \sup_b \Pi_i(t^k,b) = \lim_{k\to\infty} \Pi_i(t^k, b_i(t^k)) = \Pi_i(t_i, b_i(t_i-))$ and bid $b_i(t_i-)$ is a best response.

The proof that $b_i(t_i+)$ is a best response when $b_i(t_i+) > \max\{\underline{b}, r\}$ is very similar and omitted to save space.

Proof of Lemma 5

Consider any type t_i such that $P_i(t_i, b_i(t_i)) > 0$, any $t_i' > t_i$, and any $\tilde{t}_i \in (t_i, t_i')$ having a best response. $f(\mathbf{t}) \in [f_{low}, f_{high}]$ for all $\mathbf{t}$ by **(A2)**, implying $f(\mathbf{t}_{-i}|t_i) \in [f_{low}, f_{high}]$ for all $\mathbf{t}$. In particular, $P_i(t_i', b_i(t_i)), P_i(\tilde{t}_i, b_i(t_i)) \geq \frac{f_{low}}{f_{high}} P_i(t_i, b_i(t_i)) > 0$. Further, $b_i(\tilde{t}_i) \geq b_i(t_i)$ implies $P_i(t_i', b_i(\tilde{t}_i)), P_i(\tilde{t}_i, b_i(\tilde{t}_i)) > 0$. By revealed preference, $\Pi_i(\tilde{t}_i, b_i(\tilde{t}_i)) \geq \Pi_i(\tilde{t}_i, OUT) = 0$ so $P_i(\tilde{t}_i, b_i(\tilde{t}_i)) > 0$ implies $U_i(\tilde{t}_i, b_i(\tilde{t}_i)) \geq 0$. By Lemma 1(b), $U_i(t,b) \equiv E\left[u(t;\mathbf{t}_{-i};b)|t_i = t, \mathbf{t}_{-i} \leq \boldsymbol{\phi}_{-i}(b))\right]$ is strictly increasing in t for all b. Thus $U_i(t_i', b_i(\tilde{t}_i)) > 0$. All together,

$$\sup_b \Pi_i(t_i', b) \geq \Pi_i(t_i', b_i(\tilde{t}_i)) = U_i(t_i', b_i(\tilde{t}_i)) P_i(t_i', b_i(\tilde{t}_i)) > 0$$

Proof of Lemma 6

If $\bar{b} = OUT$ we are done, so suppose that $\bar{b} \geq r$ for the rest of the proof. Without loss, suppose that $\bar{b}_1 = \bar{b}$.

Step I: $\bar{b} \neq \underline{b}$. Suppose otherwise. Since $\bar{b} \geq b_1(1-) \geq b_1(0+) = \underline{b} = \bar{b}$, we conclude that $b_1(t_1) = \underline{b}$ for all $t_1 \in (0,1)$. By Lemma 2, no other bidder can have an atom at this bid-level. Thus, $b_i(t_i) < \underline{b}$ and $\sup_b \Pi_i(t_i, b) = \Pi_i(t_i, b_i(t_i)) = 0$ for all $i \neq 1$ and almost all $t_i < 1$, and bidder 1 wins the object with probability one when bidding $\underline{b}$. Repeating the argument in the text leading to (4), replacing type $\underline{t}$ with type 1 and bid r with bid $\underline{b} = \overline{b}$, we conclude that $\Pi_1(t_1, \underline{b}) < 0$ for all $t_1 < 1$. Thus, all types $t_1 < 1$ strictly prefer not to participate rather than bid $\underline{b}$, a contradiction.

Step II: $\overline{b} \neq r$. Suppose otherwise. Since $b_1(1-) = r$, bidder 1 must have an atom at bid-level r while (by Lemma 2) $b_i(t_i) = OUT$ for all $i \neq 1$ and all $t_i < 1$. This leads to a contradiction, as in Step I.

Step III: $\sup_b \Pi_i(1, b) \geq \Pi_1(1, \overline{b}_1) = \sup_b \Pi_1(1, b) > 0$ *for all* i. By Steps I-II, $\overline{b}_1 > \max\{\underline{b}, r\}$. Thus, by Lemma 3 there are no atoms at bid-level $\overline{b}_1$ and any bidder can win with probability one by bidding $\overline{b}_1$. Thus, $\sup_b \Pi_i(1, b) \geq \Pi_i(1, \overline{b}_1) = \Pi_1(1, \overline{b}_1) > 0$. The equality $\Pi_i(1, \overline{b}_1) = \Pi_1(1, \overline{b}_1)$ holds by symmetry while the inequality $\Pi_1(1, \overline{b}_1) > 0$ follows from Lemma 5. (Since $\overline{b}_1 > \max\{\underline{b}, r\}$, $P_1(1-\varepsilon, b_1(1-\varepsilon)) > 0$ for all small enough $\varepsilon > 0$.) Finally, consider any sequence $\{t^k\} \nearrow 1$ such that, for all k, every player plays a best response given type t^k. $\Pi_1(1, \overline{b}_1) = \lim_{k\to\infty} \Pi_1(t^k, b_1(t^k)) = \lim_{k\to\infty} \sup_b \Pi_1(t^k, b) = \sup_b \Pi_1(1, b)$. The first equality holds since there are no atoms at $\overline{b}_1$, the second since each type t^k plays a best response, and the third by Lemma 4.

Step IV: $\overline{b}_i > \max\{\underline{b}, r\}$ for all i. By Step I-II, $\overline{b} > \max\{\underline{b}, r\}$. Without loss, suppose that $\overline{b}_1 = \overline{b}$ and $\overline{b}_2 \leq \max\{\underline{b}, r\}$. By definition, bidder 2 never wins the object and gets zero payoff when he bids less than $\max\{\underline{b}, r\}$. By Lemma 4, $\sup_b \Pi_2(1, b) = \lim_{k\to\infty} \sup_b \Pi_2(t^k, b) = \lim_{k\to\infty} \Pi_2(t^k, b_2(t^k))$. If $\overline{b}_2 < \max\{\underline{b}, r\}$, then all types $t_2 < 1$ bid less than $\max\{\underline{b}, r\}$. Consequently,

$\Pi_2(t^k, b_2(t^k)) = 0$ for all k and $\sup_b \Pi_2(1, b) = 0$, a contradiction of Step III.

Finally, suppose that $\bar{b}_2 = \max\{\underline{b}, r\}$. Again, $\Pi_2(t^k, b_2(t^k)) = 0$ for all k and a contradiction is reached *unless* $b_2(t^k) = \bar{b}_2$ for all large enough k, i.e. bidder 2 has an atom at $\max\{\underline{b}, r\}$. In this case, Lemma 2 implies that no other bidder has an atom at $\bar{b}_2$. But then $\sup_b \Pi_1(1, b) \geq \lim_{\delta \to 0} \Pi_1(1, \bar{b}_2 + \delta) > \Pi_2(1, \bar{b}_2)$, since

$$\begin{aligned}
\lim_{\delta \to 0} \Pi_1(1, \bar{b}_2 - \delta) &= \Pr\left(W_1(\bar{b}_2) | t_1 = 1\right) E\left[u_1(t_1; \mathbf{t}_{-1}; \bar{b}_2) | t_1 = 1, W_1(\bar{b}_2)\right] \\
&\geq \Pr\left(W_1(\bar{b}_2) | t_1 = 1\right) E\left[u_2(t_2; \mathbf{t}_{-2}; \bar{b}_2) | t_2 = 1, W_2(\bar{b}_2)\right] \qquad (10) \\
&> \Pr\left(W_2(\bar{b}_2) | t_2 = 1\right) E\left[u_2(t_2; \mathbf{t}_{-2}; \bar{b}_2) | t_2 = 1, W_2(\bar{b}_2)\right] = \Pi_2(1, \bar{b}_2) \qquad (11)
\end{aligned}$$

For the first equality, note that $W_1(\bar{b}_2)$ is by definition the event in which all bidders $i \neq 1$ bid strictly less than $\bar{b}_2$; inequalities (10,11) are then identical to (1,2) in the text. This again contradicts Step III.

Step V: $\Pi_i(1, \bar{b}_i) = \sup_b \Pi_i(1, b)$ *for all* i. Since $\bar{b}_i > \max\{\underline{b}, r\}$, we may simply repeat for all bidders $i \neq 1$ the argument used in Step III to show that $\Pi_1(1, \bar{b}_1) = \sup_b \Pi_1(1, b)$. This completes the proof.

Proof of Claim 2

Without loss, assume that $b_1(\hat{t}-) \geq ... \geq b_n(\hat{t}-)$.

Suppose that $b_1(\hat{t}-) < \hat{b}$. Since $\hat{b} > r$, there must be a gap in the distribution of bids below $\hat{b}$, so that $\hat{b}$ is strictly dominated for all bidders. Since $\hat{b} > \underline{b}$, however, Lemma 4 implies that all bidders must find $\hat{b} = b_i(\hat{t}+)$ to be a best response given type $\hat{t}$. This is a contradiction, so $b_1(\hat{t}-) = \hat{b}$.

Suppose that $b_n(\hat{t}-) < \hat{b}$. Several steps establish a contradiction.

First, $\Pi_1(\hat{t}, b_1(\hat{t})) = \Pi_1(\hat{t}, \hat{b}) = \lim_{\varepsilon\to 0} \Pi_n(\hat{t}-\varepsilon, b_n(\hat{t}-\varepsilon))$. As mentioned above, bidder 1 finds $\hat{b}$ to be a best response given type $\hat{t}$, so $\Pi_1(\hat{t}, b_1(\hat{t}) = \Pi_1(\hat{t}, \hat{b})$. By Lemma 4, $\lim_{\varepsilon\to 0} \Pi_n(\hat{t} - \varepsilon, b_n(\hat{t} - \varepsilon)) = \lim_{\varepsilon\to 0} \Pi_n(\hat{t} + \varepsilon, b_n(\hat{t} + \varepsilon))$. There are no atoms at $\hat{b}$ by Lemma 3, so $\lim_{\varepsilon\to\infty} b_n(\hat{t}+\varepsilon) = \hat{b}$ implies $\lim_{\varepsilon\to 0} \Pi_n(\hat{t} + \varepsilon, b_n(\hat{t} + \varepsilon)) = \Pi_n(\hat{t}, \hat{b})$. Since $\phi_i(\hat{b}) = \hat{t}$ for all i, finally, symmetry implies $\Pi_n(\hat{t}, \hat{b}) = \Pi_1(\hat{t}, \hat{b})$.

Second, $\Pi_n(\hat{t}-\varepsilon, b_n(\hat{t}-\varepsilon)) > 0$ for all small enough $\varepsilon > 0$. Since $b_1(\hat{t}-) = \hat{b} > \underline{b}$, $b_1(\hat{t} - \varepsilon) > \underline{b}$ for small enough $\varepsilon > 0$. Hence bidder 1 wins the object with positive probability given type $\hat{t} - \varepsilon$. By Lemma 5, then, $\Pi_1(\hat{t}, \hat{b}) > 0$. The desired result follows now from the first point above.

Third, (i) $\phi_n(b_n(\hat{t}-)) = \hat{t}$, (ii) $\phi_1(b_n(\hat{t}-)) < \hat{t}$, and (iii) $\phi_i(b_n(\hat{t}-)) > 0$ for all $i \neq n$. (i) holds since $b_n(t_n) \geq \hat{b} > b_n(\hat{t}-)$ for all $t_n > \hat{t}$ while $b_n(t_n) \leq b_n(\hat{t}-)$ for all $t_n < \hat{t}$. (ii) is immediate from $b_n(\hat{t}-) < b_1(\hat{t}-)$. (Bidder 1 has types less than $\hat{t}$ that bid more than $b_n(\hat{t}-)$.) To prove (iii), suppose that $\phi_{i^*}(b_n(\hat{t}-)) = 0$ for some i^*, i.e. $b_{i^*}(t_{i^*}) \geq b_n(\hat{t}-)$ for all $t_{i^*} > 0$. Bidder n wins the object with zero probability with any bid $b < b_n(\hat{t}-)$. As we have seen, however, $\Pi_n(\hat{t} - \varepsilon, b_n(\hat{t} - \varepsilon)) > 0$ for some $\varepsilon > 0$. Since $b_n(\hat{t} - \varepsilon) \leq b_n(\hat{t}-)$, we conclude that $b_n(t_n) = b_n(\hat{t}-)$ for all $t_n \in (\hat{t}-\varepsilon, \hat{t})$, i.e. bidder n has an atom at bid-level $b_n(\hat{t}-)$. By Lemma 2, no other bidder can have an atom at $b_n(\hat{t}-)$, including bidder i^*. Thus, $b_{i^*}(t_{i^*}) > b_n(\hat{t}-)$ for all $t_{i^*} > 0$. Hence, all bidder-n types in $(\hat{t} - \varepsilon, \hat{t})$ win the object with probability zero and get zero profit. This is a contradiction.

Fourth, no bidder $i \neq n$ has an atom at bid-level $b_n(\hat{t}-)$. There are two cases to consider. ($\phi_i(b_n(\hat{t}-) > 0$ for all i implies $b_n(\hat{t}-) \geq \underline{b}$.) (A) If $b_n(\hat{t}-) > \underline{b}$, no bidder has an atom at $b_n(\hat{t}-)$ by Lemma 3. (B) If $b_n(\hat{t}-) = \underline{b}$, it must be that

$b_n(\hat{t}-\varepsilon) = \underline{b}$ for all small enough ε, i.e. bidder n has an atom at $\underline{b}$. But then no *other* bidder can have an atom at $\underline{b}$ by Lemma 2.

Fifth, $\Pi_1(\hat{t}, b_1(\hat{t})) > \lim_{\varepsilon\to 0} \Pi_n(\hat{t}-\varepsilon, b_n(\hat{t}-\varepsilon))$, contradicting the first point. By bidder 1's revealed preference, $\Pi_1(\hat{t}, b_1(\hat{t})) \geq \Pi_1(\hat{t}, b)$ for all b. Thus, $\Pi_1(\hat{t}, b_1(\hat{t})) \geq \lim_{\varepsilon\to 0} \Pi_1(\hat{t}, b_n(\hat{t}-) + \varepsilon)$.

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \Pi_1(\hat{t}, b_n(\hat{t}-) + \varepsilon) \\
&= \Pr\left(\mathbf{t}_{-1} \leq \phi_{-1}(b_n(\hat{t}-)+) | t_1 = \hat{t}\right) E\left[u_1(\hat{t}; \mathbf{t}_{-1}; b_n(\hat{t}-)) | t_1 = \hat{t}, \mathbf{t}_{-1} \leq \phi_{-1}(b_n(\hat{t}-)+))\right] \\
&\geq \Pr\left(\mathbf{t}_{-1} \leq \phi_{-1}(b_n(\hat{t}-)) | t_1 = \hat{t}\right) E\left[u_1(\hat{t}; \mathbf{t}_{-1}; b_n(\hat{t}-)) | t_1 = \hat{t}, \mathbf{t}_{-1} \leq \phi_{-1}(b_n(\hat{t}-)))\right] \\
&> \Pr\left(\mathbf{t}_{-n} \leq \phi_{-n}(b_n(\hat{t}-)) | t_n = \hat{t}\right) E\left[u_1(\hat{t}; \mathbf{t}_{-1}; b_n(\hat{t}-)) | t_1 = \hat{t}, \mathbf{t}_{-1} \leq \phi_{-1}(b_n(\hat{t}-)))\right] \\
&\geq \Pr\left(\mathbf{t}_{-n} \leq \phi_{-n}(b_n(\hat{t}-)) | t_n = \hat{t}\right) E\left[u_n(\hat{t}; \mathbf{t}_{-n}; b_n(\hat{t}-)) | t_n = \hat{t}, \mathbf{t}_{-n} \leq \phi_{-n}(b_n(\hat{t}-)))\right] \\
&= \lim_{\varepsilon\to 0} \Pi_n(\hat{t}-\varepsilon, b_n(\hat{t}-\varepsilon))
\end{aligned}
$$

The first equality holds by definition: $\phi_i(b) \equiv \sup\{t_i : b_i(t_i) \geq b\}$, so $\phi_i(b+) = \phi_i(b)$. The two inequalities are essentially identical to (1,2); see their discussion in the text. ($b_n(\hat{t}-) < b_1(\hat{t}-)$ implies that $\phi_n(b_n(\hat{t}-)) = \hat{t} > \phi_1(b_n(\hat{t}-))$.) The last equality is valid since no bidder $i \neq n$ has an atom at $b_n(\hat{t}-)$. This completes the proof.

Proof of Claim 3

Part I: preliminaries. $b_i(\hat{t}+) = \hat{b}$ for all i implies $b_i(\hat{t}-) = \hat{b}$ and hence $b_i(\hat{t}) = \hat{b}$ for all i (Claim 2). Since $\hat{b} > \underline{b}$, there are no atoms at $\hat{b}$ by Lemma 3, so $b_i(\phi_i(\hat{b})) = \hat{b}$ for all i. To complete the proof, it suffices to show that (A) $\gamma > 0$ exists such that $b_i(\hat{t}-) = \hat{b}$ implies $b_i(\phi_i(b)) = b$ for all $b \in (\hat{b}-\gamma, \hat{b})$ and (B) when $\hat{b} \neq \overline{b}$, $\gamma > 0$ exists such that $b_i(\hat{t}+) = \hat{b}$ for all i implies $b_i(\phi_i(b)) = b$ for all $b \in (\hat{b}, \hat{b}+\gamma)$. (If $\hat{b} = \overline{b}$, only (A) is relevant.) The rest of the proof establishes (A). The proof of (B) is symmetrical and omitted to save space.

First, I show that it suffices to establish (A') $\gamma > 0$ exists such that $b_i(t_i-) \in (\hat{b}-\gamma, \hat{b})$ for some t_i implies $b_i(t_i-) = b_i(t_i+)$. When (A') holds with respect to $\gamma > 0$, I claim that (A) holds with respect to $\gamma^* \equiv \hat{b} - \max_i \check{b}_i$, where $\check{b}_i \equiv b_i(\phi_i(\hat{b}-\gamma)+)$ for all i. Before proceeding, observe that $\hat{b}-\gamma \leq \check{b}_i < \hat{b}$ for all i, so that $(\check{b}_i, \hat{b}) \subset (\hat{b}-\gamma, \hat{b})$.[14]

Suppose for the sake of contradiction that $b_i(\phi_i(\tilde{b})) \neq \tilde{b}$ for some i and some $\tilde{b} \in (\hat{b}-\gamma^*, \hat{b})$. This is only possible if $b_i(\phi_i(\tilde{b})-) < b_i(\phi_i(\tilde{b})+)$. To reach a contradiction, it suffices to show that $b_i(\phi_i(\tilde{b})-) \in (\check{b}_i, \hat{b})$: if so, (A') implies that $b_i(\phi_i(\tilde{b})-) = b_i(\phi_i(\tilde{b})+)$.

By definition of γ^*, $\tilde{b} \in (\hat{b}-\gamma^*, \hat{b})$ implies $\tilde{b} \in (\max_i \check{b}_i, \hat{b}) \subset (\check{b}_i, \hat{b})$. Since $\check{b}_i = b_i(\phi_i(\hat{b}-\gamma)+)$, $\tilde{b} > \check{b}_i$ implies $\phi_i(\tilde{b}) > \phi_i(\hat{b}-\gamma)$. In particular, $b_i(\phi_i(\tilde{b})-) \geq \check{b}_i = b_i(\phi_i(\hat{b}-\gamma)+)$. By the argument of footnote 14, further, $b_i(\phi_i(\tilde{b})-) \neq \check{b}_i$ since there can not be an atom at bid-level $\check{b}_i$. Similarly, $\hat{b} = b_i(\hat{t}-)$ and $\tilde{b} < \hat{b}$ implies that $\phi_i(\tilde{b}) < \hat{t}$. Thus, $b_i(\phi_i(\tilde{b})-) \leq b_i(\hat{t}-) = \hat{b}$. Again by the argument of footnote 14, $b_i(\phi_i(\tilde{b})-) \neq \hat{b}$ since there can not be an atom at bid-level $\hat{b}$. All together, we conclude that $b_i(\phi_i(\tilde{b})-) \in (\check{b}_i, \hat{b})$. This yields the desired contradiction.

Before continuing, I collect important notation in a series of 'definitions'.

Definition 3 (Bidder i_γ and bid-level b_γ) *For any given $\gamma > 0$, let i_γ*

[14] $\check{b}_i \geq \hat{b}-\gamma$ by definition of $\check{b}_i$. Since $\hat{b} = b_i(\hat{t}-)$, $\phi_i(b') < \hat{t}$ for all $b' < \hat{b}$. In particular, $\phi_i(\hat{b}-\gamma) < \hat{t}$ and $\check{b}_i = b_i(\phi_i(\hat{b}-\gamma)+) \leq b_i(\hat{t}-) = \hat{b}$. Thus, $\check{b}_i = \hat{b}$ is only possible if $b_i(t_i) = \hat{b}$ for all $t_i \in (\phi_i(\hat{b}-\gamma), \hat{t})$. Since $\hat{b} > \underline{b}$, however, Lemma 3 rules out such atoms. We conclude that $\hat{b}-\gamma \leq \check{b}_i < \hat{b}$ for all i.

denote any bidder for whom $b_{i_\gamma}(t_{i_\gamma}-) = b_\gamma \in (\hat{b}-\gamma, \hat{b})$ *for some* t_{i_γ} *but* $b_{i_\gamma}(t_{i_\gamma}+) > b_\gamma$.

To complete the proof, we need to show that such a bidder i_γ *does not exist* for small enough $\gamma > 0$.

Recall from Definition 1 the shorthand notation $a_{ij}(b)$ and $c_i(b)$. Recall from the text that $a_{ij}(b)$ and $c_i(b)$ are each continuous when $b > \bar{b}$. Thus, $\varepsilon(\gamma)$ is well-defined for all $\gamma > 0$ small enough that $\hat{b} - \gamma > \underline{b}$:

Definition 4 ($\varepsilon(\gamma)$) *Define* $\varepsilon(\gamma) > 0$ *so that* $\lim_{\gamma\to 0}\varepsilon(\gamma) = 0$ *and*

$$\left|\frac{a_{ij}(b)}{(n-1)c_i(b)} - \frac{a(\hat{b})}{(n-1)c(\hat{b})}\right|, \left|a_{ij}(b) - a(\hat{b})\right|, \left|c_i(b) - c(\hat{b})\right| < \varepsilon(\gamma) \tag{12}$$

for all i, j *and all* $b \in (\hat{b}-\gamma, \hat{b})$.

Definition 5 (Bidder i^* and sequences $\{b_{i^*}^k\}$,$\{t_{i^*}^k\}$) *Let* $\{b_{i^*}^k\}$ *be a decreasing sequence such that (i)* $\lim_{k\to\infty} b_{i^*}^k = b_\gamma$ *and there exists bidder* i^* *and a decreasing sequence of types* $\{t_{i^*}^k\}$ *such that (ii)* i^* *finds* $b_{i^*}^k$ *to be a best response given type* $t_{i^*}^k$ *for all* k *and (iii)* $\lim_{k\to\infty} \frac{\phi_{i^*}(b_{i^*}^k) - \phi_{i^*}(b_\gamma)}{b_{i^*}^k - b_\gamma} > \frac{c(b_\gamma)}{(n-1)a(b_\gamma)} - \varepsilon(\gamma)$.

Part II: constructing $\{b_{i^*}^k\}$*,* $\{t_{i^*}^k\}$*, and* i^**.* We will show that such a bidder i^* exists. This is the hardest part of the proof and essential for reaching a contradiction in Part III. Say that bidder i is *'active above* b*'* if $b_i(t_i+) = b$ for some type t_i. Observe that $b_\gamma > \underline{b}$ implies that at least two bidders must be active above b_γ. (This is a standard argument, so some steps are sketched without full details.) First, if no one is active above b_γ, then there is a gap in the distribution of $\max_i b_i(t_i)$ from b_γ up to $\min_i b_i(\phi_i(b_\gamma)+)$. This leads a contradiction, since bid $\min_i b_i(\phi_i(b_\gamma)+)$ is strictly dominated for all bidders, but some bidder must find it to be a best response (Lemma 4). Second, suppose that only bidder j is active above b_γ. Now there is a gap in the distribution of $\max_{i\neq j} b_j(t_j)$ from b_γ up to $\min_{i\neq j} b_i(\phi_i(b_\gamma)+)$. This leads to a contradiction, since all bids in between b_γ and $\min_{i\neq j} b_i(\phi_i(b_\gamma)+)$ are strictly dominated for bidder j.

Without loss, suppose that bidder 1 is one of the bidders who is active above b_γ. For any decreasing sequence of types $\{t_1^k\} \searrow \phi_1(b_\gamma)$, define a sequence of bids $\{b_1^k\}$ by $b_1^k = b_1(t_1^k)$ for all k. $\{b_1^k\} \searrow b_\gamma$ since bidder 1 is active above b_γ. Without loss, we may assume that $\lim_{k\to\infty} \frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k - b_\gamma}$ exists for all i (possibly infinite). (Otherwise, select a subsequence such that this limit exists for bidder 1, a further subsequence so that the limit exists for bidder 2, and so on.)

Revealed preference of bidder 1. By construction, bidder 1 finds b_1^k to be a best response for type t_1^k for all k. Consequently, $\Pi_1(t_1^k, b_1^k) - \Pi_1(t_1^k, b_1^{k'}) \geq 0$ and $\Pi_1(t_1^{k'}, b_1^k) - \Pi_1(t_1^{k'}, b_1^{k'}) \leq 0$ for all k and all $k' > k$. (Note: $b_1^k > b_1^{k'}$ and $\phi_i(b_1^k) \geq \phi_i(b_1^{k'})$ for all i.)

$\Pi_1(t_1^k, b_1^k) - \Pi_1(t_1^k, b_1^{k'}) \geq 0$ implies

$$
\begin{aligned}
0 &\leq \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^k)} u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) f(\mathbf{t}_{-1}|t_1^k) \mathrm{d}\mathbf{t}_{-1} - \int_0^{\boldsymbol{\phi}_{-1}(b_1^{k'})} u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'}) f(\mathbf{t}_{-1}|t_1^k) \mathrm{d}\mathbf{t}_{-1} \\
&= \int_{\mathbf{t}_{-1} \in [\mathbf{0}, \boldsymbol{\phi}_{-1}(b_1^k)] \setminus [\mathbf{0}, \boldsymbol{\phi}_{-1}(b_1^{k'})]} u_1(t_1^k; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i}|t_1^k) \mathrm{d}\mathbf{t}_{-1} \\
&\qquad + \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} \left(u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'}) \right) f(\mathbf{t}_{-1}|t_1^k) \mathrm{d}\mathbf{t}_{-1}
\end{aligned}
$$

Now $[\mathbf{0}, \boldsymbol{\phi}_{-1}(b_1^k)] \setminus [\mathbf{0}, \boldsymbol{\phi}_{-1}(b_1^{k'})] \equiv \cup_{J \subsetneq \{2,\dots,n\}} Z_J$, where the sets $Z_J \equiv \left(\times_{i\in J}[0, \phi_i(b_1^k)]\right) \times \left(\times_{i\notin J}[\phi_i(b_1^{k'}), \phi_i(b_1^k)]\right)$ have measure zero intersection. For each $J \subsetneq \{2, ..., n\}$, define shorthand $W_J \equiv \int_{\mathbf{t}_{-1}\in Z_J} u_1(t_1^k; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i}|t_1^k) \mathrm{d}\mathbf{t}_{-1}$. By Lemma 1(a),

$$
W_J \geq \int_0^{\boldsymbol{\phi}_{-1}(b_1^{k'})} u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'}) f(\mathbf{t}_{-1}|t_1^k) \mathrm{d}\mathbf{t}_{-1} = \Pi_1(t_1^k, b_1^{k'})
$$

(Observe that $[\mathbf{0},\boldsymbol{\phi}_{-1}(b_1^{k'})] \cup Z_J$ is a lattice with decreasing subset $[\mathbf{0},\boldsymbol{\phi}_{-1}(b_1^{k'})]$.) Furthermore, $\Pi_1(t_1^k, b_1^{k'}) > 0$. (Recall that $b_\gamma > \underline{b}$ so $P_1(t_1^{k'}, b_1^{k'}) > 0$. By the proof of Lemma 5, we may then conclude that $P_1(t_1^k, b_1^{k'}) > 0$ and $U_1(t_1^k, b_1^{k'}) > 0$.) We conclude that $W_J > 0$ for all $J \subsetneq \{2,...,n\}$. Thus, in particular, $\sum_{J \subsetneq \{2,\dots,n\}} W_J \leq \sum_{i \neq 1} \sum_{J \subset \{2,\dots,n\}: i \notin J} W_J$.

Next, observe that $[\phi_i(b_1^{k'}), \phi_i(b_1^k)] \times [\mathbf{0}, \boldsymbol{\phi}_{-1i}(b_1^k)] = \cup_{J \subset \{2,\dots,n\}: i \notin J} Z_J$. Thus

$$\int_{\phi_i(b_1^{k'})}^{\phi_i(b_1^k)} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_1^k)} u_1(t_1^k; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i} | t_1^k) \mathrm{d}\mathbf{t}_{-1i} \mathrm{d}t_i = \sum_{J \subset \{2,\dots,n\}: i \notin J} W_J$$

for all i. All together, we conclude that

$$0 \leq \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^k)} u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) f(\mathbf{t}_{-1} | t_1^k) \mathrm{d}\mathbf{t}_{-1} - \int_{0}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'}) f(\mathbf{t}_{-1} | t_1^k) \mathrm{d}\mathbf{t}_{-1} \tag{13}$$

$$\leq \sum_{i \neq 1} \int_{\phi_i(b_1^{k'})}^{\phi_i(b_1^k)} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_1^k)} u_1(t_1^k; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i} | t_1^k) \mathrm{d}\mathbf{t}_{-1i} \mathrm{d}t_i \tag{14}$$

$$+ \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} \left(u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'}) \right) f(\mathbf{t}_{-1} | t_1^k) \mathrm{d}\mathbf{t}_{-1}$$

$$\leq \sum_{i \neq 1} \left(\phi_i(b_1^k) - \phi_i(b_1^{k'}) \right) Y_{1,i}^{k,k'} + \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} (u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'})) f(\mathbf{t}_{-1} | t_1^k) \mathrm{d}\mathbf{t}_{-1} \tag{15}$$

for all $k' > k$, where

$$Y_{1,i}^{k,k'} \equiv \max_{t_i \in [\phi_i(b_1^{k'}), \phi_i(b_1^k)]} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_1^k)} u_1(t_1^k; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i} | t_1^k) \mathrm{d}\mathbf{t}_{-1i}$$

Since these inequalities hold for all k, they must also hold when we divide both sides by $b_1^k - b_1^{k'} > 0$. In particular, for any sequence $\{(k_l, k_l')\}$ such that

$k_l \to \infty$ and $k_l' > k_l$ for all l (shorthand '$k, k' \to \infty$'), we have

$$\lim_{k,k'\to\infty} \sum_{i\neq 1} \frac{\phi_i(b_1^k) - \phi_i(b_1^{k'})}{b_1^k - b_1^{k'}} Y^{k,k'} + \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} \frac{u_1(t_1^k; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^k; \mathbf{t}_{-1}; b_1^{k'})}{b_1^k - b_1^{k'}} f(\mathbf{t}_{-1}|t_1^k) \mathrm{d}\mathbf{t}_{-1}$$

$$= \sum_{i\neq 1} \lim_{k\to\infty} \frac{\phi_i(b_1^k) - \phi_i(b_\gamma)}{b_1^k - b_\gamma} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_\gamma)} u_1(\phi_1(b_\gamma); \phi_i(b_\gamma), \mathbf{t}_{-1i}; b_\gamma) f(\phi_i(b_\gamma), \mathbf{t}_{-1i}|\phi_1(b_\gamma)) \mathrm{d}\mathbf{t}_{-1i}$$

$$+ \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_\gamma)} \frac{\partial u_1(\phi_1(b_\gamma); \mathbf{t}_{-1}; b_\gamma)}{\partial b} f(\mathbf{t}_{-1}|\phi_1(b_\gamma)) \mathrm{d}\mathbf{t}_{-1}$$

$$\equiv \sum_{i\neq 1} \lim_{k\to\infty} \frac{\phi_i(b_1^k) - \phi_i(b_\gamma)}{b_1^k - b_\gamma} a_{1i}(b_\gamma) - c_1(b_\gamma) \geq 0 \tag{16}$$

The first equality relies on the fact that each bidder's inverse bid function is continuous (Lemma 3) and that $\lim_{k\to\infty} b_1^k = b_\gamma$ and $\lim_{k\to\infty} t_1^k = \phi_1(b_\gamma)$, as well as our assumption that u is continuously differentiable and f continuous. The second equality is by definition.

Similarly, $\Pi_1(t_1^{k'}, b_1^k) - \Pi_1(t_1^{k'}, b_1^{k'}) \leq 0$ implies

$$0 \geq \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^k)} u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^k) f(\mathbf{t}_{-1}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1} - \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^{k'}) f(\mathbf{t}_{-1}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1} \tag{17}$$

$$\geq \sum_{i\neq 1} \int_{\phi_i(b_1^{k'})}^{\phi_i(b_1^k)} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_1^{k'})} u_1(t_1^{k'}; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1i} \mathrm{d}t_i \tag{18}$$

$$+ \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} \left(u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^{k'}) \right) f(\mathbf{t}_{-1}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1}$$

$$\geq \sum_{i\neq 1} \left(\phi_i(b_1^k) - \phi_i(b_1^{k'}) \right) \min_{t_i \in [\phi_i(b_1^{k'}), \phi_i(b_1^k)]} \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_1^k)} u_1(t_1^{k'}; t_i, \mathbf{t}_{-1i}; b_1^k) f(t_i, \mathbf{t}_{-1i}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1i}$$

$$+ \int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_1^{k'})} \left(u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^k) - u_1(t_1^{k'}; \mathbf{t}_{-1}; b_1^{k'}) \right) f(\mathbf{t}_{-1}|t_1^{k'}) \mathrm{d}\mathbf{t}_{-1}$$

The sets $[\phi_i(b_1^{k'}), \phi_i(b_1^k)] \times [\mathbf{0}, \boldsymbol{\phi}_{-1i}(b_1^{k'})]$ are disjoint and their union is a strict subset of $[\mathbf{0}, \boldsymbol{\phi}_{-1i}(b_1^k)] \backslash [\mathbf{0}, \boldsymbol{\phi}_{-1i}(b_1^{k'})]$. Thus, (18) follows from (17) since we are now 'under-counting' types.[15] We conclude that

$$\sum_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}\int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1i}(b_\gamma)}u_1(\phi_1(b_\gamma);\phi_i(b_\gamma),\mathbf{t}_{-1i};b_\gamma)f(\phi_1(b_\gamma),\phi_i(b_\gamma),\mathbf{t}_{-1i})\mathrm{d}\mathbf{t}_{-1i}$$

$$+\int_{\mathbf{0}}^{\boldsymbol{\phi}_{-1}(b_\gamma)}\frac{\partial u_1(\phi_1(b_\gamma);\mathbf{t}_{-1};b_\gamma)}{\partial b}f(\phi_1(b_\gamma),\mathbf{t}_{-1})\mathrm{d}\mathbf{t}_{-1}\leq 0$$

$$\equiv\sum_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}a_{1i}(b_\gamma)-c_1(b_\gamma)\leq 0 \tag{19}$$

Combined with (16), we conclude

$$0=\sum_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}a_{1i}(b_\gamma)-c_1(b_\gamma) \tag{20}$$

In particular, since $b_\gamma\in(\hat{b}-\gamma,\hat{b})$ by presumption,

$$\max_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}\geq\frac{c_1(b_\gamma,\boldsymbol{\phi}(b_\gamma))}{(n-1)\max_{i\neq 1}a_{1i}(b_\gamma)}>\frac{c(\hat{b})}{(n-1)a(\hat{b})}-\varepsilon(\gamma)$$

The weak inequality follows from (20); the strict inequality is by definition of $\varepsilon(\gamma)$.

We are now ready to define bidder i^* and the sequences $\{t_{i^*}^k\}$,$\{b_{i^*}^k\}$:

$$i^*\equiv\arg\max_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}$$

Next, define $t_{i^*}^k\equiv\phi_{i^*}(b_1^k)$ and $b_{i^*}^k=b_{i^*}(t_{i^*}^k-)$ for all k. By Lemma 4, bidder i^* finds $b_{i^*}^k$ to be a best response given type $t_{i^*}^k$ for all k. Furthermore, note that $\phi_{i^*}(b_{i^*}^k)=\phi_{i^*}(b_1^k)$ while $b_{i^*}^k\leq b_1^k$ for all k.[16] Consequently, by our choice of bidder i^*

$$\lim_{k\to\infty}\frac{\phi_{i^*}(b_{i^*}^k)-\phi_{i^*}(b_\gamma)}{b_{i^*}^k-b_\gamma}\geq\max_{i\neq 1}\lim_{k\to\infty}\frac{\phi_i(b_1^k)-\phi_i(b_\gamma)}{b_1^k-b_\gamma}>\frac{c(\hat{b})}{(n-1)a(\hat{b})}-\varepsilon(\gamma) \tag{21}$$

[15] The formal derivation of inequality (18) is very similar to that of (14); (14) holds since we 'overcount types'.

[16] Since $b_\gamma>\underline{b}$, Lemma 3 implies that $\phi_{i^*}(b_{i^*}(t-))=t$ whenever $b_{i^*}(t)>b_\gamma$. In particular, $\phi_{i^*}(b_{i^*}^k)=\phi_{i^*}(b_{i^*}(\phi_{i^*}(b_1^k))-)=\phi_{i^*}(b_1^k)$.

Revealed preference of bidder i^*. By construction, bidder i^* finds $b_{i^*}^k$ to be a best response for type $t_{i^*}^k$ for all k. Repeating the steps used to derive equation (20), we conclude

$$0 = \sum_{i \neq i^*} \lim_{k \to \infty} \frac{\phi_i(b_{i^*}^k) - \phi_i(b_\gamma)}{b_{i^*}^k - b_\gamma} a_{i^* i}(b_\gamma) - c_{i^*}(b_\gamma) \tag{22}$$

Part III: Profitable deviation for bidder i_γ.

Revealed preference of bidder i_γ. By construction, $b_\gamma = b_{i_\gamma}(t_{i_\gamma}-)$ so bidder i_γ finds b_γ to be a best response given type t_{i_γ} (Lemma 4). In particular, $\Pi_{i_\gamma}(t_{i_\gamma}, b^k) - \Pi_{i_\gamma}(t_{i_\gamma}, b_\gamma) \leq 0$ for all k. Repeating the steps used to derive inequality (19), we conclude

$$0 \geq \sum_{i \neq i_\gamma} \lim_{k \to \infty} \frac{\phi_i(b_{i^*}^k) - \phi_i(b_\gamma)}{b_{i^*}^k - b_\gamma} a_{i_\gamma i}(b_\gamma) - c_{i_\gamma}(b_\gamma) \tag{23}$$

Bidder i_γ strictly prefers $b_{i^*}^k$ over b_γ given type t_{i_γ}, for all large enough k. By presumption, bidder i_γ is not active above b_γ, which implies $\phi_{i_\gamma}(b) = \phi_{i_\gamma}(b_\gamma)$ for all b in a neighborhood above b_γ. Thus, $\lim_{k \to \infty} \frac{\phi_{i_\gamma}(b_{i^*}^k) - \phi_{i_\gamma}(b_\gamma)}{b_{i^*}^k - b_\gamma} = 0$ and (22) becomes

$$0 = \sum_{i \neq i^*, i_\gamma} \lim_{k \to \infty} \frac{\phi_i(b_{i^*}^k) - \phi_i(b_\gamma)}{b_{i^*}^k - b_\gamma} a_{i^* i}(b_\gamma) - c_{i^*}(b_\gamma) \tag{24}$$

Subtracting (24) from (23),

$$\begin{aligned} 0 \geq \lim_{k \to \infty} \frac{\phi_{i^*}(b_{i^*}^k) - \phi_{i^*}(b_\gamma)}{b_{i^*}^k - b_\gamma} a_{i_\gamma i}(b_\gamma) + \sum_{i \neq i^*, i_\gamma} \lim_{k \to \infty} \frac{\phi_i(b_{i^*}^k) - \phi_i(b_\gamma)}{b_{i^*}^k - b_\gamma} \Big(a_{i_\gamma i}(b_\gamma) - a_{i^* i}(b_\gamma) \Big) \\ - \Big(c_{i_\gamma}(b_\gamma) - c_{i^*}(b_\gamma) \Big) \quad (25) \\ > \left(\frac{c(\hat{b})}{(n-1)a(\hat{b})} - \varepsilon(\gamma) \right) a_{i_\gamma i}(b_\gamma) - \varepsilon(\gamma) \sum_{i \neq i^*, i_\gamma} \lim_{k \to \infty} \frac{\phi_i(b_{i^*}^k) - \phi_i(b_\gamma)}{b_{i^*}^k - b_\gamma} \quad (26) \end{aligned}$$

(26) follows from (25) by definition of $\varepsilon(\gamma)$ and i^*, see (12,21). By (24), the

sum

$$\sum_{i \neq i^*, i_\gamma} \lim_{k\to\infty} \frac{\phi_i(b^k_{i^*}) - \phi_i(b_\gamma)}{b^k_{i^*} - b_\gamma} \leq \frac{c_{i^*}(b_\gamma)}{\min_{i\neq i^*, i_\gamma} a_{i^*i}(b_\gamma)} < \frac{c(\hat{b})}{a(\hat{b})} + \delta(\gamma)$$

where $\lim_{\gamma\to 0} \delta(\gamma) = 0$. Since $\hat{a}, \hat{c} > 0$, the right-hand-side of (26) is positive for all small enough $\gamma > 0$. This is a contradiction and completes the proof.

Proof of Claim 4

Let $\gamma > 0$ be that identified by Claim 3.

Right-derivatives $\phi_i'(b+)$ well-defined. Consider any bid-level $b_\gamma \in (\hat{b}-\gamma, \max\{\hat{b}+\gamma, \bar{b}\})$ and any decreasing sequence $\{b^k\} \searrow b_\gamma$ such that $\lim_{k\to\infty} \frac{\phi_j(b^k)-\phi_j(b_\gamma)}{b^k - b_\gamma}$ exists (possibly infinite) for all i. (This involves no loss of generality; see the proof of Claim 3.) By Claim 3, bidder i bids b^k and hence finds b^k to be a best response given type $t_i^k \equiv \phi_i(b^k)$ for all i and all k. (Without loss, we may assume that $b^k \in (b_\gamma, \max\{\hat{b}+\gamma, \bar{b}\})$ so that Claim 3 applies to bid-level b^k.)

We can now repeat that part of the proof of Claim 3 labeled 'Revealed preference for bidder 1' *for each bidder i*. As in equation (20), we conclude that

$$0 = \sum_{j\neq i} \lim_{k\to\infty} \frac{\phi_j(b^k) - \phi_j(b_\gamma)}{b^k - b_\gamma} a_{ij}(b_\gamma) - c_i(b_\gamma) \text{ for all } i = 1, ..., n \tag{27}$$

As long as $A(b_\gamma)$ is invertible, there is a unique solution

$$\left(\lim_{k\to\infty} \frac{\phi_1(b^k)-\phi_1(b_\gamma)}{b^k-b_\gamma} \; ... \; \lim_{k\to\infty} \frac{\phi_n(b^k)-\phi_n(b_\gamma)}{b^k-b_\gamma} \right) = C(b_\gamma) * A^{-1}(b_\gamma)$$

(Recall the meaning of $C(b_\gamma)$, $A^{-1}(b_\gamma)$ from definition 2 in the text.) We conclude $\inf_{k\to\infty} \frac{\phi_j(b^k)-\phi_j(b_\gamma)}{b^k-b_\gamma} = \sup_{k\to\infty} \frac{\phi_j(b^k)-\phi_j(b_\gamma)}{b^k-b_\gamma}$ *and does not depend on the chosen sequence* $\{b^k\}$. Hence, the right-derivatives

$$\phi_i'(b_\gamma+) \equiv \lim_{\varepsilon\to 0} \frac{\phi_i(b_\gamma+\varepsilon) - \phi_i(b_\gamma)}{\varepsilon}$$

exist for all i and all $b_\gamma \in (\hat{b} - \gamma, \max\{\hat{b} + \gamma, \bar{b}\})$.

Derivatives $\phi_i'(b_\gamma)$ well-defined and continuous. Similarly, to prove that left-derivatives $\phi_i'(b_\gamma -)$ are well-defined, consider any *increasing* sequence $\{b^k\} \nearrow b_\gamma$ such that $\lim_{k\to\infty} \frac{\phi_j(b^k)-\phi_j(b_\gamma)}{b^k - b_\gamma}$ exists (possibly infinite) for all i. Repeating the same steps, the same solution is the unique solution

$$\left(\lim_{k\to\infty} \frac{\phi_1(b_\gamma)-\phi_1(b^k)}{b_\gamma - b^k} \; ... \; \lim_{k\to\infty} \frac{\phi_n(b_\gamma)-\phi_n(b^k)}{b_\gamma - b^k} \right) = C(b_\gamma) * A^{-1}(b_\gamma)$$

Thus, the left-derivative and derivative exists:

$$\phi_i'(b_\gamma -) \equiv \lim_{\varepsilon \to 0} \frac{\phi_i(b_\gamma) - \phi_i(b_\gamma - \varepsilon)}{\varepsilon} = \phi_i'(b_\gamma +)$$

Finally, the derivative is continuous at b_γ since $C(b_\gamma) * A^{-1}(b_\gamma)$ is continuous. Thus, $\phi_i(\cdot)$ is continuously differentiable over $(\hat{b} - \gamma, \max\{\hat{b} + \gamma, \bar{b}\})$ for all i.

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