

# Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market

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## **Abstract**

We propose an estimation method to bound bidders' marginal valuations in discriminatory auctions using individual bid-level data, and apply the method to data from the Turkish Treasury auction market. Using estimated bounds on marginal values, we compute an upper bound on the inefficiency of realized allocations as well as bounds on how much additional revenue could have been realized in a counterfactual uniform price or Vickrey auction. We conclude that switching from a discriminatory auction to a uniform price or Vickrey auction would not significantly increase revenue. Moreover, such a switch would increase bidder expected surplus by at most 0.02%.

**Keywords:** Multi-unit auctions, divisible good or share auctions, Treasury auctions, structural econometrics, nonparametric identification and estimation

# 1 Introduction

What is the most effective way for a Treasury to sell government securities? Since governments sell about \$4 trillion dollars worth of securities every year (Bartolini and Cottarelli (1997)), many economists have tried to answer this question, interpreting “effectiveness” from both the revenue maximization and efficiency standpoints. At least since Friedman (1960), the unanimous suggestion of the profession has been to conduct the sale through an auction. Unfortunately there is much less of a consensus among economists as to what the optimal auction mechanism should be, with theory providing no clear answer.<sup>1</sup>

This paper begins to fill this gap in the literature by developing techniques to let the data decide on the relative ranking of different auction mechanisms. Cross-country studies of Treasury practices reveal that practitioners overwhelmingly prefer one mechanism over others: 39 out of 42 countries surveyed by Bartolini and Cottarelli (1997) use the *discriminatory auction* mechanism, also known as the “pay-as-bid” or “multiple-price” auction. In this auction format, bidders may submit multiple price-quantity pairs as their bids, which trace out *bid functions* on the price-quantity plane. Treasury officials aggregate individual bid functions and find where the *aggregate bid function* meets supply, as in Figure 1(a). The revenue of the auctioneer is the area under the aggregate bid function up to the market-clearing price. An alternative mechanism that has been the favorite of many economists, including Friedman, is the *uniform price auction*.<sup>2</sup> Here, winning bids are determined in the same manner as in the discriminatory auction, but bidders pay the market-clearing price for all the units they purchase. In this case, the auctioneer’s revenue is the rectangle defined by the total quantity being sold and the market-clearing price, as shown in Figure 1(b).

Building on the seminal work of Wilson (1979), we model bidder behavior in a discriminatory auction as an incomplete information game. As long as bids are strictly downward-sloping demand

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<sup>1</sup>See the survey by Nandi (1997). Ausubel and Cramton (1997) and Engelbrecht-Wiggans and Kahn (1998) show that the discriminatory and uniform price auctions cannot be generally ranked on either efficiency or revenue grounds, even given independent private values. In particular, the celebrated revenue equivalence results of Vickrey (1961) and the revenue ranking results of Milgrom and Weber (1982) do not apply to share auctions.

<sup>2</sup>An important advantage of the uniform price auction, cited by many economists, is that it invites wider participation since all winning bidders pay the same price and hence do not need to invest in predicting the market-clearing price. As Friedman (1991) argued, by contrast, “[the discriminatory auction] tends to limit the market to specialists” who may also be more likely to collude.

schedules, the model yields a first-order necessary condition in which the price bid for a given quantity equals the bidder’s marginal value less an oligopsonistic “mark-down” that depends on the inverse elasticity of the residual supply function each bidder expects to face. Inverting this mark-down rule identifies bidders’ unobserved marginal valuations, and allows an econometrician to conduct counterfactual calculations in which the revenue performance of a uniform price or Vickrey auction can be compared to the performance of a discriminatory auction. In practice, however, observed bids are not strictly downward-sloping and bidder values are not point-identified under the hypothesis of equilibrium play. Instead, we provide a methodology to compute upper and lower *bounds* on bidders’ valuations, and use these bounds for counterfactual analysis.

We apply this framework to data from the Turkish Treasury, covering 3-month T-bill auctions between 1991 and 1993, to explore how much additional revenue the Treasury might have generated if it had switched to either a uniform price or Vickrey auction. We conclude that switching to either of these auction formats would not have significantly increased revenue or (gross) bidder surplus. More precisely, we can not reject (ex ante) revenue equivalence between the discriminatory auction and a hypothetical auction that yields strictly higher revenue than the uniform price or Vickrey auction. By our point estimate, the switch from a discriminatory to a uniform price auction would lead to a gain of expected revenue that is *at most* 0.12% of the realized revenue (about \$14.4 million, considering that \$12 billion of debt was auctioned in this period). However, taking sampling variation into account, we can not reject the hypothesis that such a switch would lead to no difference in revenue, even under the best case scenario for the uniform price auction. We also find that the discriminatory auction itself was close to fully efficient, estimating that switching to an efficient mechanism would have increased bidder expected surplus by at most 0.02% over the 3-year period.

This paper differs from much of the past literature comparing Treasury auction mechanisms, in that we employ a structural model of strategic bidding. Past work has focused almost exclusively on “policy experiments” in which different auction formats have been used in different time periods or in the sale of securities of different maturities. These studies have compared the differential between the auction price and the resale or forward (“when-issued”) market price of the security across separate samples of discriminatory and uniform price auctions.<sup>3</sup> The assumption implicit in this line of research is that bidders’ true valuations for the security are better reflected in resale or forward markets than in

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<sup>3</sup>See Umlauf (1993), Simon (1994), Nyborg and Sundaresan (1996) and Malvey and Archibald (1998).

the auction. Hence, the claim is that smaller differentials between the auction price and the transaction prices in secondary markets reflects better surplus extraction by the auctioneer. The validity of this comparison relies heavily on the assumption that one can control for factors that may have changed between the end of the auction and the start of trade in the resale market, including the release of new information that might impact bidders' valuations of the security. The advantage of the structural approach taken here is that estimation of model primitives allows us to construct counterfactual simulations in which bidders' *ex ante* information sets are fixed. Furthermore, the policy experiment studies cited above have relied on aggregated price data from the auctions, whereas we utilize bidder level data. This allows us to assess the distributional aspects of a policy intervention that changes the mechanism.

The most closely related paper is Heller and Lengwiler (1998), who uses a similar first-order condition to estimate the amount of (counterfactual) bid-shading in a discriminatory auction.<sup>4</sup> However, their model is based on Nautz (1995), which abstracts from some important aspects of strategic bidding. In particular, their model predicts that all bidders will shade their bids by the same amount, regardless of their market power.<sup>5</sup> Also, Heller and Lengwiler (1998) model bidders' expectation about the market-clearing price distribution using the distribution of past market-clearing prices, and hence do not make use of the information contained in individual bids to investigate the strategic effect of each bidder on the equilibrium market-clearing price distribution.

Like this paper, Fevrier, Préget and Visser (2002) uses a structural approach to estimate a strategic model of bidding in the discriminatory auction. Their paper differs in that they estimate a parametric model with symmetric common values, while we estimate a non-parametric model assuming private values.

In work that follows and builds upon an earlier version of this paper (Hortaçsu (2002))<sup>6</sup>, Kang and Puller (2008) conducts a counterfactual revenue comparison of the discriminatory and uniform

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<sup>4</sup>The closely related idea of using first-order conditions to reconstruct unobserved marginal cost functions from observed supply decisions has been present in the industrial organization literature at least since Rosse (1970) and Bresnahan (1981).

<sup>5</sup>Other early work analyzed Treasury auctions within a strategic equilibrium framework but, due to the analytical and computational difficulties associated with divisible-good auctions, focused on models of single-unit auctions. See e.g. Nyborg, Rydqvist and Sundaresan (2002) and Gordy (1994).

<sup>6</sup>The earlier version of the paper developed identification results in the case analyzed by Wilson (1979) where bidders submit continuous demand curves. Here, we consider the more realistic situation where bidders submit step functions.

price auctions, using data from discriminatory Korean Treasury auctions. They find that the discriminatory auction yielded statistically greater expected revenue and a more efficient allocation of the securities at auction, although these effects were each fairly small. Also related is Kastl (2006), who uses a set of necessary conditions for optimal bidding to infer bounds on bidders' values in the Czech Treasury's uniform price auction. He concludes that the uniform price auction performs well, in that it "failed to extract at most 0.03% (in terms of the annual yield of T-bills) worth of expected surplus while implementing an allocation resulting in almost all of the efficient surplus".

This paper also adds to the literature on structural econometric modeling of auctions. In particular, we extend the nonparametric identification and estimation framework of Elyakime, Laffont, Loisel and Vuong (1994) and Guerre, Perrigne and Vuong (2000) to divisible good auctions, and offer an empirical algorithm with low computational demands. The statistical properties of our estimation algorithm are also discussed.

The outline of the paper is as follows. Section 2 presents a model of strategic bidding in a discriminatory auction. Section 3 discusses how this model can be used as an empirical device to estimate bidders' unobserved marginal valuations. Section 4 provides background information on the institutional setup of the Turkish Treasury auction market and reports descriptive statistics from the data, including evidence on inventory/liquidity concerns as a source of private information. In Section 5, we use bidding data from the Turkish Treasury bill auctions between October 1991 and October 1993 to estimate bidders' marginal valuations. Using these estimates, we conduct counterfactual comparisons of the discriminatory, uniform price and Vickrey auctions. Section 6 discusses several extensions, while Section 7 concludes.

## 2 Modeling and identification framework

Our theoretical model is based on Wilson (1979)'s share auction framework and, more specifically, Swinkels (2001)'s multi-unit auction model. There are  $T$  auctions. Each auction  $t = 1, \dots, T$  is a discriminatory auction of  $Q_t$  indivisible units with  $N$  potential bidders. As in Swinkels (2001), we allow for the possibility that supply is random, or that a random subset of the bidders may have zero values and choose not to participate. (By assumption, bidding is costless.)

**Assumptions on bidder values.** Bidders in each auction are symmetric and risk-neutral with independent private values (IPV). Each player desires up to  $y^{max}$  objects, with non-increasing marginal value schedule  $v_{it}(\cdot) = (v_{it}(1), \dots, v_{it}(y^{max})) \in \mathcal{V}$  having a well-defined density over  $\mathbf{R}^{y^{max}}$ . Let  $d_{it}(\cdot)$  denote the demand schedule corresponding to  $v_{it}(\cdot)$ , i.e.  $d_{it}(p) = \max\{y : v_{it}(y) \geq p\}$ .

**Discriminatory auction rules.** A bid is a non-increasing price schedule  $p_{it}(\cdot) = (p_{it}(1), \dots, p_{it}(y^{max}))$  or, equivalently, a non-increasing bid schedule  $y_{it}(\cdot)$  defined by  $y_{it}(p) \equiv \max\{0 \leq y \leq y^{max} : p_{it}(y) \leq p\}$ . (By convention, set  $p_{it}(0) = \infty$  and  $p_{it}(y) = 0$  for all  $y > y^{max}$ .) A range of consecutive quantities may be bid at the same price. If so, we refer to this range of quantities as a “step”.

Bidder  $i$  pays his unit-bid on every unit that he wins. At any price  $p$ , the (random) *residual supply* available to bidder  $i$  is  $Q_t - \sum_{j \neq i} y_{jt}(p, v_{jt}(\cdot))$ , where  $y_{jt}(\cdot, v_{jt}(\cdot))$  denotes bidder  $j$ 's strategy in auction  $t$ , as a function of his marginal value schedule. (We will also use inverse demand notation  $p_{it}(\cdot, v_{it}(\cdot))$  for strategies, as convenient.) In particular, if bidder  $i$  submits a bid  $p_{it}(\cdot)$  such that  $p_{it}(y) = p$ , bidder  $i$  wins at least quantity  $y$  as long as his residual supply at price  $p$  is greater than or equal to  $y$ , i.e. with probability<sup>7</sup>

$$G_{it}(y; p) = \Pr \left( Q_t - \sum_{j \neq i} y_{jt}(p, v_{jt}(\cdot)) \geq y \right). \quad (1)$$

**Assumptions on strategies.** In each auction  $t$ , bidders play symmetric pure strategies  $p_t(\cdot, v_{it}(\cdot))$  so that  $G_{it}(y; p) = G_t(y; p)$  for all  $i$ . Further, bidders' strategies constitute a Bayesian Nash equilibrium. Interim expected payoffs in the discriminatory auction take the form

$$\Pi_t(p(\cdot), v_{it}(\cdot)) = \sum_{y=1}^{y^{max}} G_t(y; p(y))(v_{it}(y) - p(y)). \quad (2)$$

The equilibrium assumption may be restated as  $\Pi_t(p_t(\cdot, v_{it}(\cdot)), v_{it}(\cdot)) \geq \Pi_t(p(\cdot), v_{it}(\cdot))$  for all bids  $p(\cdot)$  and all marginal value schedules  $v_{it}(\cdot) \in \mathcal{V}$ .

To simplify the exposition, we will henceforth drop notation specifying the bidder  $i$  and the auction  $t$ , except where such notation is needed for clarity.

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<sup>7</sup>Implicit in this formulation is a simplifying assumption that there are no “ties”, i.e. at most one bidder is rationed. In practice, given a finite set of prices so that such ties occur with positive probability, we modified each bidder's winning probability (1) to account for the effects of rationing.

**Assumptions on supply.** Supply is non-random, i.e. bidders know the quantity being supplied when they submit their bids. (In Section 5, we discuss an extension to settings with random supply.)

**Identifying bounds on bidder values.** To gain an intuition for our identification strategy, consider for the moment the special case in which a bidder submits a strictly decreasing bid schedule  $p(\cdot)$ , i.e.  $p(1) > p(2) > \dots > p(y^{max})$ , as a best response given some marginal values  $v(\cdot)$ . For each quantity  $y$ , this bidder could have increased or decreased its  $y$ -th unit-bid without changing its bid price for any other quantity. For instance, consider any prices  $p^h \in (p(y-1), p(y))$  and  $p^l \in (p(y), p(y+1))$ . Raising its  $y$ -th unit-bid from  $p(y)$  to  $p^h$  is a profitable deviation iff  $(v(y) - p^h)G(y; p^h) > (v(y) - p(y))G(y; p(\cdot))$ . In particular,  $p(\cdot)$  can only be a best response if the bidder's marginal value is small enough:

$$v(y) \leq p(y) + (p^h - p(y)) \frac{G(y; p^h)}{G(y; p^h) - G(y; p(y))}. \quad (3)$$

Similarly, lowering its  $y$ -th unit-bid from  $p(y)$  to  $p^l$  is profitable unless  $v(y)$  is large enough:

$$v(y) \geq p(y) + (p(y) - p^l) \frac{G(y; p^l)}{G(y; p(y)) - G(y; p^l)}. \quad (4)$$

Bounds of the form (3,4) are familiar in the literature on the econometrics of first-price auctions. Indeed, in the limit as  $p^h \searrow p(y)$  and  $p^l \nearrow p(y)$ , in the special case in which just one unit is sold, these bounds converge to the point-identifying “mark-down” relationship in Guerre, Perrigne, and Vuong (2000) (GPV):<sup>8</sup>

$$v(1) = p(1) + \frac{G(1; p(1))}{g(1; p(1))}. \quad (5)$$

Yet, in practice, observed bids are typically not strictly decreasing but rather *step-functions* specifying the same price over a range of quantities.<sup>9</sup> The presence of such steps poses an identification challenge. To see why, suppose for concreteness that two units are sold and that some bidder submits the same price for both units,  $p(1) = p(2) = p$ . Raising the first unit-bid from  $p$  to  $p^h$  is possible, so we may infer an upper bound on  $v(1)$  as in (3). On the other hand, lowering the first unit-bid to  $p^l$  is only possible if one also lowers the second unit-bid by at least as much. Lowering *both* unit-bids from  $p$  to

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<sup>8</sup>In (5),  $g(1; p(1)) = \frac{dG(1; p(1))}{dp}$  corresponds to the density of the maximal equilibrium bid submitted by any other bidder. In first-price auctions with independent private values, this density is known to exist and to be continuous.

<sup>9</sup>Step-functions are consistent with equilibrium bidding in the discriminatory auction. Indeed, Anwar (2007) shows that, when bidders have constant marginal values, bidders submit flat price schedules in the unique equilibrium.

$p^l = p$  is a profitable deviation iff  $\sum_{y=1,2}(v(1)-p^l)G(y;p^l) > \sum_{y=1,2}(v(y)-p)G(y;p)$ . Note that, since bidder values are by assumption non-increasing ( $v(1) \geq v(2)$ ), this deviation is profitable whenever  $v(1)$  is large enough that  $(v(1)-p^l) \sum_{y=1,2} G(y;p^l) > (v(1)-p) \sum_{y=1,2} G(y;p)$ . In particular,  $p(\cdot)$  can only be a best response if

$$v(1) \geq p + (p - p^l) \frac{\sum_{y=1,2} G(y;p^l)}{\sum_{y=1,2} G(y;p) - \sum_{y=1,2} G(y;p^l)}. \quad (6)$$

Unfortunately, as  $p^h \searrow p(y)$  and  $p^l \nearrow p(y)$ , the upper and lower bounds on  $v(1)$  derived in this way do *not* converge to one another.

In other words, because of the constraint that bids must be non-increasing schedules, bidder values in the discriminatory auction are not point-identified by the necessary conditions of optimal bidding given independent private values. Indeed, a bidder may find the same bid to be a best response given many different marginal value schedules. (For a simple example, see Example 2 in McAdams (2008).) Thus, the nature of the inference we are going to draw from our data regarding the distribution of the structural parameter  $v(\cdot)$  is *incomplete*, as in Haile and Tamer (2003).

Proposition 1 provides the upper and lower bounds on marginal values that we will use later in our empirical analysis.

**Proposition 1.** *Consider any bid  $p(\cdot)$  and any step  $[\underline{y}, \bar{y}]$  of that bid, i.e.  $p(\underline{y}-1) > p(\underline{y}) = p(\bar{y}) > p(\bar{y}+1)$ , an example of which is shown in Figure 2. Bid  $p(\cdot)$  can only be a best response if, for all  $y \in [\underline{y}, \bar{y}]$  and all  $0 < \Delta \leq \min\{p(\underline{y}-1) - p(\underline{y}), p(\bar{y}) - p(\bar{y}+1)\}$ , marginal value  $v(y)$  satisfies*

$$v(y) \leq \bar{v}(y; p(\cdot)) = p(y) + \Delta + \frac{\Delta \sum_{q=\underline{y}}^y G(q; p(y))}{\sum_{q=\underline{y}}^y [G(q; p(y) + \Delta) - G(q; p(y))]} \quad (7)$$

$$v(y) \geq \underline{v}(y; p(\cdot)) = p(y) + \frac{\Delta \sum_{q=y}^{\bar{y}} G(q; p(y) - \Delta)}{\sum_{q=y}^{\bar{y}} [G(q; p(y)) - G(q; p(y) - \Delta)]} \quad (8)$$

*Proof.* Suppose for the sake of contradiction that  $v(y) > \bar{v}(y; p(\cdot))$ . We will show that it is a profitable deviation for the bidder to raise his unit-bids on all quantities  $[\underline{y}, y]$  from  $p(y)$  to  $p(y) + \Delta$ , as illustrated by the deviation denoted by up-arrows in Figure 2. Such a deviation has no impact on the bidder's likelihood of winning or payment for any quantity outside  $[\underline{y}, y]$ , but both increases the likelihood of winning these units and increases the price paid for them upon winning. For each quantity  $q \in [\underline{y}, y]$ , bidding  $p(y)$  is enough to win that quantity with probability  $G(q; p(y))$ . In these cases, bidding  $p(y) + \Delta$  causes the bidder to pay  $\Delta$  more for that unit. On the other hand, if bidding  $p(y) + \Delta$  is enough

to win quantity  $q$  but bidding  $p(y)$  is not, which occurs with probability  $G(q; p(y) + \Delta) - G(q; p(y))$ , then bidding  $p(y) + \Delta$  allows the bidder to win for ex post surplus of  $v(q) - p(y) - \Delta$ . All together, the *change* in the bidder's expected surplus when bidding  $p(y) + \Delta$  on all quantities  $q \in [\underline{y}, y]$ , rather than  $p(y)$  on all such quantities, is equal to  $\sum_{q=\underline{y}}^y [G(q; p(y) + \Delta) - G(q; p(y))] (v(q) - p(y) - \Delta) - \Delta \sum_{q=\underline{y}}^y G(q; p(y))$ . Since marginal values are non-increasing,  $v(q) \geq v(y)$  for all  $q \in [\underline{y}, y]$ . Thus, this change in expected surplus is *at least*  $(v(y) - p(y) - \Delta) \sum_{q=\underline{y}}^y [G(q; p(y) + \Delta) - G(q; p(y))] - \Delta \sum_{q=\underline{y}}^y G(q; p(y))$ . Finally, the assumption that  $v(y) > \bar{v}(y; p(\cdot))$  and the definition of  $\bar{v}(y; p(\cdot))$  imply that  $(v(y) - p(y) - \Delta) \sum_{q=\underline{y}}^y [G(q; p(y) + \Delta) - G(q; p(y))] > \Delta \sum_{q=\underline{y}}^y G(q; p(y))$ . Thus, raising his unit-bids on all quantities  $[\underline{y}, y]$  from  $p(y)$  to  $p(y) + \Delta$  strictly increases the bidder's expected surplus, making this a profitable deviation.

This completes the proof of the upper bound (7). A parallel argument can be used to establish the lower bound (8). Namely, if  $v(y) < \underline{v}(y; p(\cdot))$ , one can show that a profitable deviation is to lower the price on all quantities  $[y, \bar{y}]$  from  $p(y)$  to  $p(y) - \Delta$ . (This deviation is denoted by down-arrows in Figure 2.) □

### 3 Estimation

In this section, we define estimators of the upper and lower bounds on bidder values derived in Section 2, and explore the asymptotics of these estimators as the number of auctions in the sample goes to infinity. In particular, for each of  $T$  auctions having  $N$  (potential) bidders, we assume that the econometrician observes all bids and a vector  $X_t$  of auction characteristics, or “covariates”, where realizations of the covariate vector  $X_t$  are iid across auctions. Further, we assume *no unobserved heterogeneity*, i.e. equilibrium bids are iid across auctions conditional on  $X_t$ . Given this assumption, notation specifying the auction  $t$  will be dropped whenever possible without creating confusion.

Inference about each bidder's marginal value schedule in each auction will take the form of upper and lower *bounds* as in (7,8). To estimate these bounds consistently, we need to estimate bidders' probability of winning a given quantity with a unit-bid equal to a given price, for a finite set of prices and quantities. For example, to estimate the lower bound (8) on a bidder's marginal value  $v(y)$  consistent with an observed bid schedule  $p(\cdot)$ , we need to estimate the winning probabilities  $G(q; p(y))$  and  $G(q; p(y) - \Delta)$  for all quantities  $q \in [y, \bar{y}]$ . To achieve this, we used a resampling

approach that provides consistent estimates of such winning probabilities, as the number of auctions  $T$  goes to infinity while holding fixed the number of bidders per auction.

### 3.1 A resampling approach

Since bidders are presumed to have private values, each bidder  $i$  cares about others' bidding strategies only insofar as they affect the distribution of bidder  $i$ 's residual supply. Our approach to estimate this distribution is based on resampling from the observed set of bids.<sup>10</sup> Consider for the moment the special case in which all  $T$  auctions in the sample have identical covariates. The following “resampling” procedure, illustrated in Figures 3-4, greatly simplifies the estimation problem. (Section 3.2 provides full details and generalizes this procedure to settings with auction-specific covariates.)

1. Fix bidder  $i$  and a bid  $p_i(\cdot)$  made by this bidder. (In Figures 3-4, this is bidder #2 and its bid in auction #1.)
2. Draw a random subsample of  $N - 1$  bid vectors with replacement from the sample of  $NT$  bids in the data set.
3. Construct bidder  $i$ 's realized residual supply were others to submit these bids, to determine the realized market-clearing price given  $i$ 's bid  $p_i(\cdot)$ , as well as whether bidder  $i$  would have won quantity  $y$  at price  $p$  for all  $(y, p)$ .

Repeating this process many times allows one to consistently estimate each of bidder  $i$ 's winning probabilities  $G(y; p)$ , simply as the fraction of all subsamples given which bidder  $i$  would have won a  $y$ -th unit at price  $p$ .

### 3.2 Estimation method

In this section, we formally define our estimators of the upper and lower bounds (7,8) on bidder  $i$ 's marginal value for quantity  $y$ , when observed submitting bid schedule  $p(\cdot)$ . We also discuss the asymptotic properties of these estimators, as the number of auctions  $T$  in the sample goes to infinity.

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<sup>10</sup>An alternative approach developed in the first-price auction literature (see e.g. Paarsch (1992) and Laffont, Ossard and Vuong (1995)) is to explicitly invert the equilibrium mapping from values to bids to form a likelihood function for the observed data. Unfortunately, such a “direct estimation strategy” is not yet feasible for the discriminatory auction since its equilibrium mapping has not been characterized.

**The estimator.** Let  $y(\cdot, v(\cdot), x, N)$  denote each bidder's bid (expressed as a demand function) given private marginal value schedule  $v(\cdot)$ , in an auction having covariates  $x$  and  $N$  bidders. Exploiting the symmetry assumption,<sup>11</sup> and assuming that supply is known, the winning probabilities in this auction can be expressed as

$$G(y; p|x, N) = \Pr \left( \sum_{j=1}^{N-1} y_j(p, v_j(\cdot), x) \leq Q - y|x \right) \quad (9)$$

$$= \underbrace{\int \dots \int}_{N-1} 1\{ \sum_{j=1}^{N-1} y_j \leq Q - y \} \prod_{j=1}^{N-1} dF(y_j; p|x, N) \quad (10)$$

$$= E[1\{ \sum_{j=1}^{N-1} y_j \leq Q - y \} | x, N] \quad (11)$$

where  $F(y; p|x) = \Pr(y(p, v(\cdot), x) \leq y|x)$  is the probability that a bidder will demand at most quantity  $y$  at price  $p$ , and  $dF(y; p|x) = \Pr(y(p, v(\cdot), x) = y|x)$ .

In the simplest case, where all  $T$  auctions in the data set have identical covariates and  $N$  bidders, we can estimate  $F(y; p)$  by the empirical cdf

$$\widehat{F}^T(y; p) = \frac{1}{NT} \sum_{t=1}^T \sum_{j=1}^N 1\{y_{jt}(p) < y\} \quad (12)$$

and plug the empirical cdf into equation (9) to obtain an estimate of the winning probability:

$$\widehat{G}^T(y; p) = \underbrace{\int \dots \int}_{N-1} 1\{ \sum_{j=1}^{N-1} y_j \leq Q - y \} \prod_{j=1}^{N-1} d\widehat{F}^T(y_j; p). \quad (13)$$

Note that with the empirical cdf  $\widehat{F}^T(y_j; p)$ , the above estimator is equivalent to:

$$\widehat{G}^T(y; p) = \frac{1}{(NT)^{N-1}} \sum_{j_1=1}^{NT} \dots \sum_{j_{N-1}=1}^{NT} 1\{y_{j_1}(p) + \dots + y_{j_{N-1}}(p) \leq Q - y\} \quad (14)$$

i.e. an average of all possible  $(N - 1)$ -fold sums of the  $NT$  bids,  $\{y_{jt}(p), j = 1 \dots N, t = 1 \dots T\}$ .

This is a *V-statistic* (Lehmann (1999), p. 387) with kernel function  $1\{ \sum_{j=1}^{N-1} y_j(p) \leq Q - y \}$ . Since this kernel function is bounded, as  $T \rightarrow \infty$ ,  $\widehat{G}^T(y; p)$  is pointwise (in  $y$ ) consistent and asymptotically normal (Lehmann (1999), Theorem 6.2.2, p.389). Asymptotic variance estimates are provided

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<sup>11</sup>Extending the method of this section to allow for asymmetric bidders and/or asymmetric strategies is straightforward. Simply estimate the bidder-specific winning probabilities  $G_{it}(y; p|x, N)$  in terms of bidder-specific  $F_{jt}(y; p|x, N)$  for all  $j \neq i$ .

by Lehmann (1999), though inference can also be conducted using the bootstrap (Bickel and Freedman (1981)).

In the more realistic case where auctions vary in their covariates  $X_t$  and number of bidders  $N_t$ , we can estimate the *conditional* cdf  $F(y; p|x, n)$  by

$$\widehat{F}^T(y; p|x, n) = \sum_{t=1}^T \frac{K\left(\frac{X_t - x}{h_T^x}, \frac{N_t - n}{h_T^n}\right)}{\sum_{t'=1}^T K\left(\frac{X_{t'} - x}{h_T^x}, \frac{N_{t'} - n}{h_T^n}\right)} \sum_{j=1}^{N_t} \frac{1\{y_{jt}(p) < y\}}{N_t} \quad (15)$$

where  $K(x, n)$  is a kernel function, and  $h_T^x$  and  $h_T^n$  are bandwidth parameters. The analog of (13) is then:

$$\widehat{G}^T(y; p|x, n) = \underbrace{\int \dots \int}_{n-1} 1\left\{\sum_{j=1}^{n-1} y_j \leq Q - y\right\} \prod_{j=1}^{n-1} d\widehat{F}^T(y_j; p|x, n). \quad (16)$$

This is equivalent to the *conditional V-statistic*,

$$\widehat{G}^T(y; p|x, n) = \frac{\sum_{j_1=1}^{\bar{N}(T)} \dots \sum_{j_{n-1}=1}^{\bar{N}(T)} 1\{y_{j_1}(p) + \dots + y_{j_{n-1}}(p) \leq Q - y\} \prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)}{\sum_{j_1=1}^{\bar{N}(T)} \dots \sum_{j_{n-1}=1}^{\bar{N}(T)} \prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)} \quad (17)$$

as in Example 4 of Falk and Reiss (1992), where  $\bar{N}(T) = \sum_{t=1}^T N_t$  and  $(X_{j_k}, N_{j_k})$  are the covariate/number of bidders corresponding to the bid function  $y_{j_k}(p)$ .

In the appendix, we show that (17) has the same probability limit and limiting distribution as the closely related *conditional U-statistic*:

$$\check{G}^T(y; p|x, n) = \frac{\sum_{(j_1, \dots, j_{n-1})} 1\{y_{j_1}(p) + \dots + y_{j_{n-1}}(p) \leq Q - y\} \prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)}{\sum_{(j_1, \dots, j_{n-1})} \prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)} \quad (18)$$

where the sum is indexed over all  $(n - 1)$ -tuples  $(j_1, \dots, j_{n-1})$  with *distinct* elements taken from  $1, \dots, \bar{N}(T)$ . The argument is based on observing that (17) and (18) can be expressed as the ratio of two unconditional V- and U-statistics, respectively. The asymptotic equivalence of unconditional U- and V- statistics are well-known (see, e.g. Lehmann (1999)).

The representation of the estimator as a conditional U-statistic is convenient, as Stute (1991) obtains the asymptotic normality (Theorem 1) and consistency (Theorem 2) of (18) around  $G(y; p|x, n)$  (equation 11). The main conditions are that: (i) the density of  $y_j(p)$  is continuous and strictly positive, (ii)  $(h_T^n, h_T^x) \rightarrow 0$ ,  $(Th_T^n, Th_T^x) \rightarrow \infty$ , and (iii)  $(T^{1/5}h_T^n, T^{1/5}h_T^x) \rightarrow 0$  for the asymptotic bias of the kernel estimator to vanish.<sup>12</sup> It is important to note that even with moderate sized  $T$ , computing (18)

<sup>12</sup>Stute (1991) also imposes a number of mild restrictions on the form of the kernel function  $K(\cdot)$  for consistency – see conditions (ii) and (iii) of his Theorem 2.

or (17) is infeasible due to the large number of combinations/permutations. We therefore approximate (17) by resampling with replacement from the set of all observed bids across all auctions, weighting each bid  $y_{jt}(p)$  by  $\frac{1}{N_t} \frac{K(\frac{x-X_t}{h_T^x}, \frac{n-N_t}{h_T^n})}{\sum_{t'=1}^T K(\frac{x-X_{t'}}{h_T^x}, \frac{n-N_{t'}}{h_T^n})}$  (i.e. bids were more likely to be drawn from auctions with covariates that were closer to those in auction  $t$ .)

For inference on (18) and (17), which are ratios of unconditional U- and V-statistics, we once again use the bootstrap. However,  $\widehat{G}^T(y; p, x)$  is not the main object of estimation and inference; what we are mainly interested in are the upper and lower bounds (7,8) on bidders' marginal values, which we construct as follows:

$$\underline{\widehat{v}}^T(y, p(\cdot)|x) = p(y) + \frac{\Delta \sum_{q=\underline{y}}^{\overline{y}} \widehat{G}^T(q; p(y) - \Delta|x)}{\sum_{q=\underline{y}}^{\overline{y}} [\widehat{G}^T(q; p(y)|x) - \widehat{G}^T(q; p(y) - \Delta|x)]} \quad (19)$$

$$\widehat{\underline{v}}^T(y, p(\cdot)|x) = p(y) + \Delta + \frac{\Delta \sum_{q=\underline{y}}^y \widehat{G}^T(q; p(y)|x)}{\sum_{q=\underline{y}}^y [\widehat{G}^T(q; p(y) + \Delta|x) - \widehat{G}^T(q; p(y)|x)]} \quad (20)$$

For a fixed  $\Delta$ , consistency of the bounds (as  $T \rightarrow \infty$ ) follows from the consistency of  $\widehat{G}^T(q; p(y)|x)$ . Since  $\overline{v}(y; p(\cdot))$  and  $\underline{v}(y; p(\cdot))$  in (7) and (8) are smooth functions of  $\{G(q; p(y)) : q = [\underline{y}, \overline{y}], p \in \{p - \Delta, p, p + \Delta\}\}$ , and there are a finite number of these of components, we invoke the “delta method for bootstrap” (van der Vaart (2000), Theorem 23.5, page 331), and compute bootstrap confidence intervals for  $\underline{\widehat{v}}^T(y, p(\cdot))$  and  $\widehat{\underline{v}}^T(y, p(\cdot))$ .

## 4 The Turkish Treasury auction market and the data

Our data consists of all bidding information (including coded bidder identity) in Turkish Treasury auctions of 13-week Treasury bills between October 1991 and October 1993, as well as the quantity of T-bills awarded to each bidder in each of these auctions.

The Turkish Treasury has auctioned government debt since May 1985. Its short-term securities are pure discount bills and bonds and do not bear coupons. Auctions are held on Wednesdays, rotating through maturities. The 13-week bills studied here comprise about three-eighths of the bond issues of the period.

Table 1 displays various summary statistics for the auctions in the data set. Macroeconomic data at different frequencies was obtained from the Central Bank of Turkey. The “total number of

bidders” is the number of unique bidders who appear in the data set. Revenues are converted to US dollars using daily exchange rate data. Cover ratio is the ratio of the number of T-bills sold in the auction to the number of T-bills demanded by bidders. The auction yield is the quantity weighted average yield of the T-bills that were sold in the auction. To get an indication of the dispersion of opinion among bidders regarding the auction interest rate, we calculate the quantity-weighted variance of the price bids. As one can see from Table 1, the ex post *real* interest rates realized in the auction are high (about 23%).<sup>13</sup> The average variance in bid prices translates to about 2% in annual yield. This bid spread is very high compared to U.S. standards, where the spread is several basis points (one-hundredth of a percent).

**Auction format.** The Turkish Treasury uses the discriminatory auction format. Each bidder is asked to specify a set of prices and quantities demanded at these prices. Prices are quoted as the amount a bidder is willing to pay for an imaginary T-bill with face value of 100 Turkish Liras (TL). Prices can be specified up to three significant digits, although the smallest bid increment in the data was 0.02. Quantities are specified in terms of the face value of T-bills the bidder wants to buy. The minimum quantity a bidder can request to buy is 50 million TL (about \$6000). There is no limit to the number of price-quantity pairs submitted by a bidder, or limits to the maximum quantity that a bidder can demand. In fact, the average number of price-quantity pairs submitted in the sample is 6.9, with one bidder submitting over 60 price-quantity pairs per auction. Bids are submitted by noon on Wednesdays. Auction results are announced at 5:00 PM and winning bids are settled one week after the auction.<sup>14</sup>

**Bidder values.** A survey of 51 banks and other financial institutions conducted by Alkan (1991) in late 1989 revealed that 42% of total auction purchases were made to meet the *liquid asset reserve requirements* monitored by the Central Bank, while 37% were for resale in the secondary market,

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<sup>13</sup>Berument and Malatyali (1999) argue that the high interest rates in this era are due to a significant inflation/devaluation risk premium, which was partly justified by a devaluation of the Turkish lira in May 1994.

<sup>14</sup>Unlike the U.S., there is no when-issued (forward) trading in the Turkish T-bill market. Non-competitive bids (bids that do not specify a price, and are automatically filled at the prevailing market-clearing price) are not allowed in the Turkish Treasury auctions.

10% were to fill customer orders, and the remaining 10% were to fulfill collateral requirements,<sup>15</sup> for investment funds administered by the bank, and for buy-and-hold purposes.

In the period studied, at least 30% of bank portfolios had to be held as government bonds and bills, with an average maturity of 210 days. Failure to comply with this requirement resulted in monetary fines and could lead to a suspension of bank operations. Bidders' idiosyncratic inventory/liquidity needs therefore constitute an important source of private information in these auctions. Such private information generates a *private value* if – conditional on publicly available information on the state of the economy and especially the money markets/banking sector – learning about competing bidders' inventory/liquidity needs does not affect a bidder's assessment of its own inventory/liquidity needs going into an auction.<sup>16</sup>

While resale incentives are also important for bidders' demand of Treasury securities generally, such incentives might be less pronounced in 13-week T-bill auctions due to their short maturity.<sup>17</sup> The main venue for trading in government securities is the Istanbul Stock Exchange Bonds and Bills Market (ISEBBM). Using daily transaction data from the ISEBBM, we calculated that, on average, 4% of the total volume of auctioned Treasury bills are traded on the ISE in a 3-day post-auction window. If, as this seems to suggest, there is not a lot of liquidity in the resale market for T-bills compared to the primary auction market, common values might be a poor model of bidder values for these securities. We should note, however, that trading volumes on the ISEBBM showed a strong growth trend during the period. Further, ISEBBM was not the only resale market available to the bidders, who could also participate in over-the-counter deals. Unfortunately, we do not have access to data on these transactions to assess their importance.

Although we believe that a private-values model of bidder values is a reasonable approximation in the context of 13-week Turkish T-bill auctions, we would like to let the data decide on the appropriate specification. Unfortunately, existing approaches to testing the private-values assumption do not apply

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<sup>15</sup>The primary source of short-term funding for Turkish banks is the interbank money market, where banks engage in short-term borrowing (“repo”) or lending (“reverse repo”), using Treasury securities as collateral.

<sup>16</sup>In a private-values environment, bidders may still find information about their competitors' reserve requirements and liquidity states valuable, as noted by Alkan (1991). Although such information does not affect a bidder's demand for the security being auctioned, it gives the bidder a strategic advantage in the auction, as observed by Hortaçsu and Sareen (2005) and Hortaçsu and Kastl (2008).

<sup>17</sup>Haile (2001) provides related evidence in the context of timber auctions, that the role of resale decreases when time horizons are shorter.

in our setting. In a recent paper, Hortaçsu and Kastl (2008) provides a formal method to test for the null hypothesis of private values in the Bank of Canada’s three-month treasury-bill auctions, and fail to reject private values in that application. However, Hortaçsu and Kastl’s testing strategy relies heavily on a specific institutional feature of the Canadian market that is not present in the Turkish market. In single-unit auctions, various tests of common-value and private-value specifications have been developed that exploit the “winner’s curse” effect. These tests examine how the distribution of bids changes in response to exogenous changes in the number of bidders in the auction. See Paarsch (1992), Hendricks, Pinkse and Porter (2003), Haile, Hong and Shum (2003), Athey and Haile (2002) for discussions and various implementations. However, an extension of this testing strategy to divisible good/multi-unit auction models has not been developed.

Alkan (1991) reports as well that bidders find the following bits of information as being useful in their bidding decision (most important listed first): Treasury’s borrowing requirements and repayment schedule, liquidity in money markets, the bidder’s own liquid asset reserve requirement and the reserve positions of other bidders, conversations with other bidders, and results of previous auctions. The fact that information about *supply* was most important to bidders could be consistent with either private or common values. In particular, banks who enter the auction primarily to meet their liquid asset reserve requirements need this information as a critical input when determining what price to bid for their minimum required quantity.

**Supply.** In the period we study, October 1991 to October 1993, the Treasury followed two different procedures when determining the quantity of bills to supply. Beginning in February 1993, in auctions #18-#27, the Treasury pre-announced the quantity in its 3- and 6-month Treasury bill auctions.<sup>18</sup> Before then, in auctions #1-#17, the total quantity of bills to be sold was not announced until after bids were submitted. Nonetheless, bank managers that we have interviewed claimed they could estimate the supply of Treasury bills quite accurately by tracking debt service requirements of the Treasury, and through contacts in the Treasury. As a reasonable approximation, we will assume that bidders know the quantity to be supplied before the bidding in all auctions. However, since uncertainty about supply could have been important in auctions #1-#17, we will also discuss an extension in which we

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<sup>18</sup>The Treasury’s decision to pre-commit to quantity was widely viewed as an attempt to commit to restricting the supply of short term securities, and to increase the average maturity of outstanding government debt. (February 3, 1993 issue of *Cumhuriyet*.)

relax this assumption for those auctions.

**Bidders.** Bidding is open to the general public. However, banks are the main players in the market, capturing 93% of the bills sold in the sample. Brokerages buy 6%, and other bidders (institutional investors, insurance firms, pension funds) share the remainder. The top 5 bidders capture 30% about evenly, while the top 15 capture about 70% of the quantity sold, with an average market share difference of no more than 3% across the top 15. We assume symmetric bidders in the base-case of our empirical analysis, but we also discuss an extension in which we allow for bidder asymmetries.

**Equilibrium bidding.** Conversations among bidders up to minutes before the auction is a shared characteristic of Treasury auction markets around the world. Although one might be inclined to suspect collusion in the presence of such pre-auction communication, market participants and Treasury officials we have interviewed have expressed that they did not think collusion was an issue in this market. Therefore, in our analysis we will assume bidding is competitive. Further, since the government securities market is the largest organized financial market in Turkey, bidders expend significant resources to strategize. Personal interviews with managers of two mid-scale private banks have revealed that these banks have developed proprietary analytic and decision support software to aid their bidding decisions.

## 5 Empirical Results

Our empirical approach estimates bounds on each bidder’s value in every auction in which it submitted a bid. For illustration purposes, we focus first on *bidder #2 in auction #1*. This is the first auction in our data, held in October 1991. 67 bidders submitted bids for a total of 1959.5 billion Turkish liras worth of 3-month Treasury bills – about 400 million U.S. dollars. 58% of the submitted bids were successful. The market-clearing price was 84.388 TL for a 100 TL face value 3-month T-bill. Bidder #2 submitted 14 price-quantity pairs totalling to a demand of 12% of the issue.

1000 iterations of the resampling procedure of Section 3 yielded the market-clearing price distribution that is plotted in Figure 4. We see that all bids lie within the support of the resampled market-clearing price distribution. In the algorithm, we utilize the supply of Treasury bills and the number of bids in each auction as covariates to construct the kernel-based resampling weights in

equation (15).<sup>19</sup>

Looking at the figure, we also see that the high bid at 84.825 TL looks like an outlier among the next 13 bids, and that the probability of the market-clearing price being above this bid is quite small. An explanation for this high bid is that the bidder has a very high value for the initial “step” of quantity. Such a high valuation makes sense when we consider that banks in Turkey have to satisfy a liquid asset reserve requirement which is monitored very closely by the Central Bank. If a bank cannot win enough T-bills in the auction, then it has to buy securities in the resale market or in the following week’s auction to close its reserve shortfall. But since the resale market had much less volume than the primary market in this period, banks could be willing to pay a significant premium to satisfy their reserve requirement through the auction.

To calculate bounds on bidder #2’s marginal valuation  $v(y)$  for every quantity  $y$ , we used equations (7) and (8). Specifically, we assume  $\Delta = 0.02$ , which corresponds to the smallest bid price increment seen in the data. We then used the resampling algorithm in Section 3 to calculate  $G(q, \cdot)$  for the intermediate quantities  $q$  in the “step” bid at the same price as  $y$ .<sup>20</sup> Standard errors were computed using 200 bootstrap resamples of the bid data.

Figure 5 displays the estimation results for bidder #2, using 1000 realizations of the “resampled” market-clearing price. The horizontal axis is the quantity (as percent of total supply) that bidder #2 requested, with the prices on the vertical axis. Once again, we have the “staircase” representation of the bid vector. We plot the point estimates of the upper and lower bounds on bidder 2’s marginal valuation for various quantities, as well as the 5-95% confidence band around these point estimates.

Note that while the upper and lower bounds on the marginal value estimates and their associated confidence bounds are relatively tight for bids on quantities corresponding to 4% of the supply and up, the marginal values “rationalizing” the initial step are higher, and somewhat less precisely estimated. The lower precision should not be surprising given Figure 4, which suggests that the marginal value estimates for especially the initial steps can rely heavily on being able to pin down the upper tail of

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<sup>19</sup>We checked the robustness of our results with different covariate vectors, but did not find important differences in our revenue comparisons. Note that our ability to construct robustness checks is limited by the fact that we have only 27 auctions in the data set – thus a high-dimensional covariate set is infeasible to incorporate in a nonparametric fashion.

<sup>20</sup>Since the quantity grid allowed by the auction rules was very fine, it was not feasible to compute  $G(q, \cdot)$  for every intermediate quantity. Instead, we bisected each horizontal step and calculated  $G(q, \cdot)$  at the endpoints of the step as well as the midpoint of the step.

the market clearing price distribution.

**“Ex post” revenue comparisons.** Repeating the above analysis for every bidder in auction #1 yields upper and lower bounds on the marginal valuations that rationalize each bid in that auction. These bounds on marginal values in turn permit a counterfactual comparison of the revenue that would have been generated in the uniform price or Vickrey auction versus in the discriminatory auction given the realized marginal values in auction #1, which we refer to as “ex post” revenue in that auction.

We find it useful to consider a hypothetical uniform price auction with truthful bidding (“UP-ATB”). Since bidders bid less than true values in equilibrium in the uniform price auction,<sup>21</sup> the auctioneer’s revenue in the UPATB provides an upper bound on the counterfactual revenue in the uniform price auction. Similarly, revenue in the UPATB provides an upper bound on Vickrey auction revenue.<sup>22</sup> Since truthful bids are increasing in bidder values, an upper bound on counterfactual revenue in both the uniform price and Vickrey auctions is that generated in the UPATB in which bidders’ marginal values are equal to the upper bound in (7).

The top panel of Figure 6 illustrates the result of this procedure for auction #1. In this figure, we overlay the aggregated bid schedule and the aggregated upper and lower bounds of the marginal valuations, with supply normalized to 1. Revenue in the UPATB is never greater than the area of the rectangle formed by the intersection of the marginal valuation upper bound schedule (delineated with open circles) with supply, and never less than the area of the rectangle formed by the intersection of the marginal valuation lower bound schedule (delineated with inverted triangles) with supply.

Since the estimated upper bound marginal valuations are quite close to the bids, it appears from the top panel of Figure 6 that discriminatory auction revenue is higher than the counterfactual UPATB revenue. However, our estimate of the UPATB revenue is subject to sampling variation, for which we use the bootstrap. Indeed, for auction #1, the realized discriminatory auction revenue is

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<sup>21</sup>Kastl (2006) considers an alternative model in which bidders incur an additional cost when submitting a bid having more steps. Given such bid preparation costs, which lead bidders to “bundle” their optimal bids over ranges of quantities, some bidders may, in equilibrium, bid more than their marginal value on some units. By contrast, bidders in our model may submit bids having any number of steps (at no cost), in which case a straightforward dominance argument shows that equilibrium bids must be less than or equal to value.

<sup>22</sup>In a private values setting, it is well-known that the Vickrey auction achieves truthful revelation of marginal values. However, given the same truthful bids, payments are higher in the uniform price auction than in the Vickrey auction since bidders typically pay less than the market-clearing price in the Vickrey auction.

0.26% higher than the 95th percentile of (the bootstrap distribution of) UPATB revenue when values are equal to our estimated upper bounds on values. (See the first row of Table 2.) Thus, using our methodology, we may conclude that switching from the discriminatory auction to a (hypothetical) UPATB would have *lowered* ex post revenue in this auction. Since ex post revenue is weakly higher in the UPATB than in either the uniform-price or Vickrey auctions, we conclude that switching from the discriminatory to either of these formats would have lowered ex post revenue.

Not all auctions in our data set yield the same result. As an illustration, in the bottom panel of Figure 6 we plot the aggregate bid and valuation schedules for auction #12 in the data set. This is an auction where bidders appear to “shade” their bids a lot more in the discriminatory auction, thus the revenue comparison is not at all evident from the plot. In fact, calculated at the point estimate of the upper bound marginal valuations, the counterfactual UPATB revenue is 0.08% higher than the realized discriminatory auction revenue (row 12 of Table 2). However, when we account for sampling variability, we find that we can not reject revenue equivalence between the UPATB and discriminatory auctions.

Table 2 presents the results of similar calculations for all 27 auctions.

In five auctions (#1, #2, #8, #9 and #19), we conclude that ex post revenue in the discriminatory auction is greater than it would have been in a counterfactual UPATB. In these auctions, the 95th percentile of the bootstrap distribution of counterfactual UPATB revenue gain is negative, even when bidders values are set equal to our estimated upper bounds; see the third column of Table 2. Since UPATB revenue is non-decreasing in bidders’ values, we conclude that ex post revenue in the discriminatory auction is greater than it would have been in a counterfactual UPATB, given *any* marginal values in the band between our estimated upper and lower bounds. Finally, since the UPATB raises greater ex post revenue than either the uniform-price auction (“UPA”) or the Vickrey auction, we conclude *a fortiori* that the discriminatory auction yielded greater ex post revenue than either of these formats would have in the realized environments of auctions #1, #2, #8, #9 and #19.

By contrast, there is no auction in our sample for which we can conclude that ex post revenue in the discriminatory auction is less than it would have been in a counterfactual UPATB. In particular, there is no auction in our sample for which we can conclude that switching to a UPA or a Vickrey auction would have increased ex post revenue. That said, in seven auctions (#15,#20,#21,#23,#24,#25 and #27), the UPATB would have yielded significantly greater ex post revenue than the discrimina-

tory auction *if bidders' marginal values were equal to our estimated upper bounds*; see the first column of Table 2. However, bidders' marginal values lie within the band *between* our estimated upper and lower bounds, and there is not a single auction in our sample for which the UPATB would have yielded significantly greater ex post revenue than the discriminatory auction if values were set equal to our estimated lower bounds; see the fourth column of Table 2.

**“Ex ante” revenue comparisons.** Another approach to counterfactual revenue analysis is based on the idea that the observed bids and our estimated bounds on marginal values are in fact random variables. To compare the *expected* revenues under both mechanisms, we use the empirical distribution of bids and estimated marginal valuation bounds. For the discriminatory price auction, we draw 500 iid resamples from the original set of bid vectors to generate 500 bid “data sets,” and calculate the auctioneer’s revenue with each such data set. We also calculate the counterfactual revenue in the uniform price auction with truthful bidding (UPATB). We then average the percentage difference between the actual revenue and counterfactual revenue across the 500 data sets to arrive at our point estimate of the “ex ante” revenue difference.<sup>23</sup>

Our ex ante revenue counterfactual findings are detailed in Table 3. If bidders were to have values equal to our point estimates of the upper bounds on those values, UPATB ex ante revenue would have on average been .12% *higher* than in the discriminatory auction. However, if we look at the 5th percentile of the estimates, we cannot reject ex-ante revenue equivalence between the discriminatory auction and the UPATB when bidders bid the upper bound to their marginal valuations. The same result obtains for the lower bounds on marginal valuations as well.

**Discussion of revenue results.** Would the auctioneer’s expected revenue have been higher had a uniform price auction been used to sell Turkish Treasury T-bills during our sample period, rather than a discriminatory auction? If so, a (hypothetical) uniform price auction with truthful bidding (UPATB) must generate greater expected revenue than the discriminatory auction. According to our “ex ante” revenue analysis, one cannot reject the hypothesis that UPATB and the discriminatory

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<sup>23</sup>To compute confidence intervals on the ex ante revenue difference, we also repeat this ex ante revenue calculation for each bootstrap replication of the marginal valuation upper and lower bound estimates. This generates the bootstrap distribution of the ex ante revenue difference.

auction generated the same expected revenue.<sup>24</sup> Does this failure to reject expected revenue equivalence merely reflect an inherent laxness of our derived bounds on marginal values, or imprecision of our estimates of those bounds? If so, one would expect our “ex post” analysis to be similarly inconclusive. However, we do sometimes reject ex post revenue equivalence. Namely, in auctions #1,#2, #8, #9 and #19, we found that ex post revenue was greater in the discriminatory auction than it would have been in the UPATB (with 95% confidence for each separate auction). All together, we conclude that a switch to a uniform price auction would not have increased expected revenue.

**Efficiency comparisons.** Maximizing the efficiency of the allocation may be an important objective of the Turkish Treasury. We used our bounds on bidders’ marginal valuations to compute an upper bound on the discriminatory auction’s efficiency. Note that an inefficiency occurs if some bidder’s marginal value for a unit that he won is less than another bidder’s marginal values for a unit that she did not win. However, since our empirical approach only provides a band in which marginal values must lie, we cannot hope to compute exact efficiency losses in the discriminatory auction. Even so, there *must* be an inefficiency whenever the upper bound on the marginal value of a quantity that was won is less than the lower bound on the marginal value of a quantity that was not won. Conversely, there *might* be an inefficiency whenever the lower bound on the marginal value of a quantity that was won is less than the upper bound on the marginal value of a quantity that was not won. Exploiting this idea, an *upper bound* on the inefficiency of the discriminatory auction can be computed by comparing the lower bounds on marginal values for quantities that were won with the upper bounds on marginal values for quantities that were not won. Namely, suppose that we assign to each bidder his minimal marginal value on quantities that he won, and his maximal marginal value on quantities that he did not win. The upper bound that we estimate on the inefficiency of the discriminatory auction is just the inefficiency that results when bidders have these marginal values at the top and bottom of the estimated band.

This upper bound on the inefficiency of the discriminatory auction can be computed in “ex post” or “ex ante” terms, much like our revenue counterfactual bounds. In auction #1, our point-

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<sup>24</sup>Since the UPATB revenue-dominates the uniform price auction, revenue equivalence of the UPATB and discriminatory auctions does not imply revenue equivalence of the uniform price and discriminatory auctions. However, unfortunately, computing equilibrium strategies in the uniform price auction is extremely challenging and we have not been able to conduct counterfactual analysis directly with the uniform price auction.

estimate of the upper bound on the ex post inefficiency of the discriminatory auction is .000089% of total surplus (reported as .00% in Table 2). Across all 27 auctions in the sample, the average point-estimate of the upper bound on ex post inefficiency was .02% of total surplus. Our point-estimate of the upper bound on *ex ante* inefficiency was also approximately .02%. At a 95%-confidence level, we conclude that the expected efficiency gains from switching to an auction with truthful bidding (whether UPATB or Vickrey) would have been at most .035% of the total surplus. We thus conclude that the discriminatory auction led to very little loss in allocational efficiency.

## 6 Extensions

**Supply uncertainty.** The results reported above do not take into account any supply uncertainty that bidders may have faced when submitting their bids. However, in auctions #1-#17, the Treasury did not preannounce the total quantity of T-bills for sale. Suppose that, from the bidders’ perspective, the total supply of T-bills in auction  $t$  is a random variable,  $Q_t$ , whose distribution is common knowledge among the bidders.<sup>25</sup> The estimation method now has to incorporate this additional random variable. In an extension of our analysis, we modeled supply in the first 17 auctions as having a time series structure. Namely, we fit an AR(1) process for  $QSOLD$ , the quantity of T-bills sold in auction.<sup>26</sup> We used this estimated AR(1) specification of the quantity sold to generate random draws for the total supply, and used these draws in the resampling procedure. Results were qualitatively quite similar to those obtained when we did not account for supply uncertainty, and we do not report detailed results to save space.

**Asymmetric bidders.** The data suggests that there can be potentially important asymmetries among bidders. Fortunately, the methodology developed here can easily accommodate certain kinds of asymmetries. In particular, suppose that bidders belong to one of finitely many “symmetry classes” (all members of each class have values drawn from the same distribution) and that the econometrician knows to which class each bidder belongs. Then, when resampling from a distribution of bids, we need

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<sup>25</sup>In practice, the discretion of the Treasury is limited by the requirements of its repayment calendar. Bidders may thus have information about the targeted supply of debt in each auction, but there is residual uncertainty about the exact amount of supply.

<sup>26</sup>Alternative specifications for the supply process, including those with additional variables to proxy for the borrowing requirement of the Treasury, did not yield more informative results.

simply make sure to draw for each bidder a bid made by another bidder in his symmetry class. In a second extension of our analysis, we divided the bidders into three classes: those who participated in more than 21 auctions, those who participated in 11 to 20 auctions, and those who participated in less than 11 auctions. (This categorization matched quite closely with an alternative categorization based on market shares.) Results were qualitatively similar to those without asymmetric bidders, and are not reported here to save space.

**Random participation.** By including the number of bids as an auction covariate, our empirical approach implicitly assumes that the set of participating bidders is known to all bidders when they submit their bids. Assuming that the econometrician can control for all sources of auction heterogeneity that might drive participation, the distribution of the number of bidders (or, more generally, of the set of bidders) can be estimated from the empirical distribution across auctions, and can be incorporated into the resampling algorithm with the modification that, as a first step, a set of bidders is drawn before resampling their bids. We leave the implementation of this extension to future research.

**Affiliated private values.** Our empirical approach can be adapted to accommodate (asymmetric) affiliated private values. For example, suppose that each bidder's marginal values take the form  $v_i(q, t_i)$  where  $v_i(\cdot, t_i)$  is strictly increasing in  $t_i$  and  $t_1, \dots, t_N$  are affiliated with a symmetric joint distribution. As in first-price auctions with affiliated private values (Li, Perrigne and Vuong (2002)), for each bid  $p(\cdot)$  in the support of the empirical distribution of bids, one may estimate the distribution of others' bids conditional on  $p_i(\cdot) = p(\cdot)$  by the empirical distribution of others' bids conditional on this bid. This conditional empirical distribution may also be viewed as an estimate of bidder  $i$ 's belief about the distribution of others' bids when he bids  $p(\cdot)$ . One may then proceed to simulate the winning probability for each bid using draws from this conditional empirical distribution of competing bids, and compute estimates of the upper and lower bounds on bidder  $i$ 's marginal values that could have rationalized the bid  $p(\cdot)$ .

**Unobserved auction heterogeneity and multiple equilibria.** What if the data is subject to unobserved heterogeneity across auctions? For example, the discriminatory auction may have multiple equilibria, and there is no guarantee that bidders will play the same equilibrium across otherwise

identical auctions.<sup>27</sup> If bidders are in fact playing different equilibria at different auctions in the sample, then the empirical distribution of bids across auctions will lead to invalid estimates of bidder values. More broadly, one may suspect that the auctions in the sample differ in ways commonly known to the bidders but unknown to the econometrician. If so, the empirical distribution of bids across a set of auctions having similar observable characteristics will lead to invalid estimates of bidder values in those auctions.

For simplicity and concreteness, we shall focus here on the possibility of multiple equilibria as the source of unobserved heterogeneity. Suppose that the econometrician observes a sample of  $T$  identical auction environments having (i)  $N$  bidders with iid private values and (ii) potentially random supply  $Q$  that is iid across auctions, but in which bidders may play different symmetric equilibria across auctions. To avoid trivial equilibrium outcomes in the limit as the number of bidders goes to infinity, consider a sequence of auctions in which *per-bidder supply* is distributed, up to an integer constraint, exactly as in the observed auction having  $N$  bidders and total supply  $Q$ , i.e. total supply when there are  $M$  bidders is  $Q^M \sim \lfloor \frac{M}{N}Q \rfloor$ .

Swinkels (2001) characterizes equilibrium bidding strategies in the discriminatory auction in the  $M \rightarrow \infty$  limit, which we shall denote as  $p^*(., v_i(.))$ :

$$p^*(y, v_i(.)) = E[p^c | p^c \leq v_i(y)] \tag{21}$$

where  $p^c$  is the “competitive market-clearing price in the limit” that would result if all  $M \rightarrow \infty$  bidders submitted bids equal to their true demand, defined implicitly by  $E[d_i(p^c)] = \lim_{M \rightarrow \infty} \frac{Q^M}{M} = \frac{Q}{N}$ . Whether bidder values can be identified from observed limiting equilibrium bids as in (21) depends crucially on whether supply is non-random as in our baseline model, or random as in the extension discussed earlier.

Since supply is iid across auctions, the distribution of supply can be consistently estimated using the empirical distribution of supply across the sample of  $T$  auctions, as  $T \rightarrow \infty$ . Since bidders have iid marginal values,  $\lim_{M \rightarrow \infty} \frac{\sum_i d_i(p)}{M} = E[d_i(p)]$  by the Law of Large Numbers. That is, the average quantity that bidders truly demand at any price is degenerate in the  $M \rightarrow \infty$  limit. If per-bidder supply  $\lim_{M \rightarrow \infty} \frac{Q^M}{M} = \frac{Q}{N}$  is also non-random, the limiting market-clearing price  $p^c$  that results when all bidders submit truthful bids will be non-random as well. By (21), this in turn implies that

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<sup>27</sup>Equilibrium uniqueness in the discriminatory auction remains an open question, even given symmetric IPV bidders.

$p^*(y, v_i(y)) = p^c$  for all bidders  $i$  and quantities  $y$  such that  $v_i(y) > p^c$ . In other words, all winning bidders pay exactly the same price in the limit! Consequently, inference about bidders' values from their bids is extremely limited in the  $M \rightarrow \infty$  limit when supply is non-random. All that one can conclude is that bidders must have had marginal values of at least  $p^c$  on quantities that they won and marginal values of at most  $p^c$  on quantities that they did not win. In particular, one cannot infer bidder values using a mark-down formula as in (5).

On the other hand, if per-bidder supply is random, quantities with different marginal values will be bid at different prices in the limit. In this case, the realized market-clearing price in the discriminatory auction will *not* equal  $p^c$  in the limit, as every bidder will be shading its bid. However, since no bidder has any effect on the market-clearing price in this limit, the mapping from unit-bids to marginal values takes the same, simple form as in GPV's mark-down formula (5) for the first-price auction. In particular, bid schedule  $p(\cdot)$  can only be a best response for bidder  $i$  given marginal value  $v_i(y) = p(y) + \frac{G^*(p(y))}{g^*(p(y))}$  for each quantity  $y$ , where  $G^*(p) = \Pr\left(E[y^*(p, v_i(\cdot))] < \frac{Q}{N}\right)$  is the probability that per-bidder supply will be sufficient to meet *expected per-bidder limiting equilibrium* quantity demanded at price  $p$ , and  $y^*(\cdot, v(\cdot))$  is the  $M \rightarrow \infty$  limit of bidders' equilibrium bids, as derived from Swinkels' characterization (21).

This "winning probability" can also be written as  $G^*(p) = 1 - F_{Q/N}(E[y^*(p, v_i(\cdot))])$ , where  $F_{Q/N}(\cdot)$  is the cdf of per-bidder supply. As above,  $F_{Q/N}(\cdot)$  can be estimated consistently using data on (iid) supply realizations across auctions. However, to estimate  $E[y^*(p, v_i(\cdot))]$  consistently, data from a single auction with  $M \rightarrow \infty$  suffices. Thus, one may consistently estimate bidder values in each individual auction regardless of whether different equilibrium strategies are played in different auctions.

## 7 Conclusion

This paper extends Guerre, Perrigne, and Vuong (2000)'s identification approach for the first-price auction to settings in which many objects are sold and bidders have multi-unit demand. Unlike in the first-price auction, the distribution of bidder values is not point identified from the distribution of bids in the discriminatory auction. Rather, bidder values are only partially identified to lie in a band between upper and lower bounds. Nonetheless, these bounds on bidder values can be used to

perform meaningful analysis of the discriminatory auction's performance, as they bound inefficiency losses in the discriminatory auction and bound counterfactual revenue in the uniform price or Vickrey auctions.

We applied these methods to data from the Turkish Treasury auction market. Using our estimated bounds on marginal values, we found that the discriminatory auctions used by the Treasury were very close to being fully efficient, and that switching to a uniform price or Vickrey auction would not significantly increase revenue.

## 8 Appendix

### 8.1 Asymptotic equivalence of $\check{G}^T(y; p|x, n)$ and $\widehat{G}^T(y; p|x, n)$ in Section 3.2

Following Stute (1991) page 813, we can write the conditional U-statistic  $\check{G}^T(y; p|x, n)$  in equation (18) as

$$\check{G}^T(y; p|x, n) = \frac{U(h, x, n)}{U(1, x, n)} \quad (22)$$

with

$$U(h, x, n) = \frac{(\bar{N}(T) - (n - 1))!}{\bar{N}(T)!} \sum_{(j_1, \dots, j_{n-1})} h(y_{j_1}, \dots, y_{j_{n-1}}) \frac{\prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)}{\prod_{k=1}^{n-1} EK\left(\frac{x - X_1}{h_T^x}, \frac{n - N_1}{h_T^n}\right)} \quad (23)$$

where  $h(y_{j_1}, \dots, y_{j_{n-1}}) = 1\{y_{j_1}(p) + \dots + y_{j_{n-1}}(p) \leq Q - y\}$  and the sum is once again over  $(n - 1)$  tuples with distinct elements. Stute (1991) points out that both  $U(h, x, n)$  and  $U(1, x, n)$  are unconditional U-statistics.

We can similarly write the conditional V-statistic  $\widehat{G}^T(y; p|x, n)$  in equation (17) as the ratio of unconditional V-statistics:

$$\widehat{G}^T(y; p|x, n) = \frac{V(h, x, n)}{V(1, x, n)} \quad (24)$$

with

$$V(h, x, n) = \frac{1}{\bar{N}(T)^{n-1}} \sum_{j_1} \dots \sum_{j_{n-1}} h(y_{j_1}, \dots, y_{j_{n-1}}) \frac{\prod_{k=1}^{n-1} K\left(\frac{x - X_{j_k}}{h_T^x}, \frac{n - N_{j_k}}{h_T^n}\right)}{\prod_{k=1}^{n-1} EK\left(\frac{x - X_1}{h_T^x}, \frac{n - N_1}{h_T^n}\right)} \quad (25)$$

We then utilize the asymptotic equivalence of unconditional U- and V- statistics (Lehmann (1999), Theorem 6.2.2, p.389), i.e. that  $V(h, x, n)$  and  $U(h, x, n)$  have the same asymptotic distribution and that  $V(1, x, n)$  and  $U(1, x, n)$  have the same probability limit.

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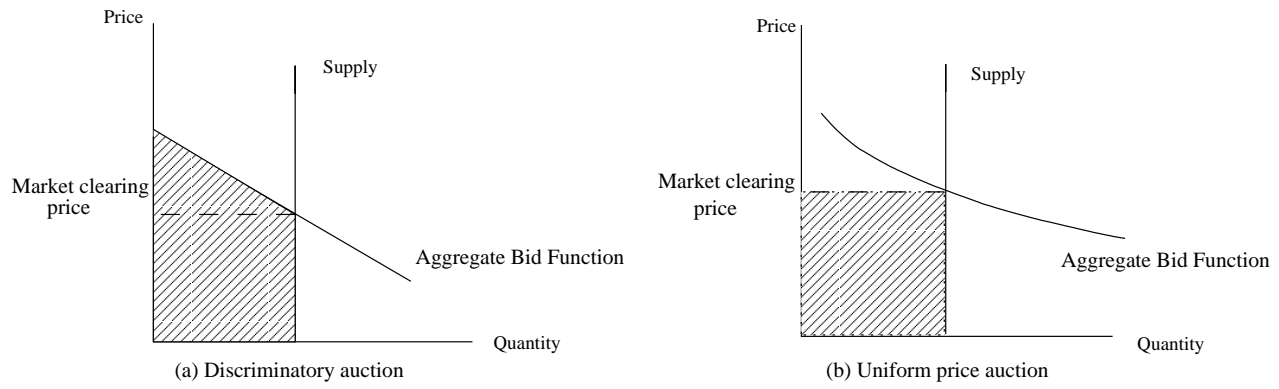


Figure 1: Discriminatory and uniform price auctions

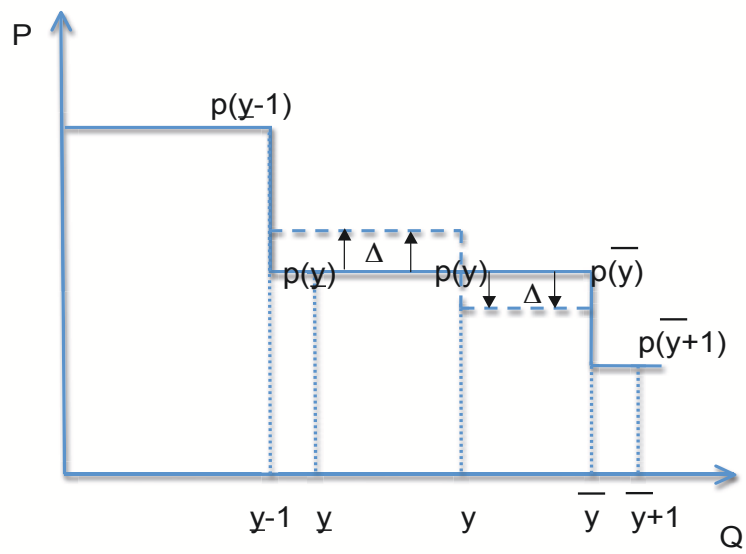


Figure 2: Illustration of a bid step and feasible  $\Delta$  deviations considered in Proposition 1

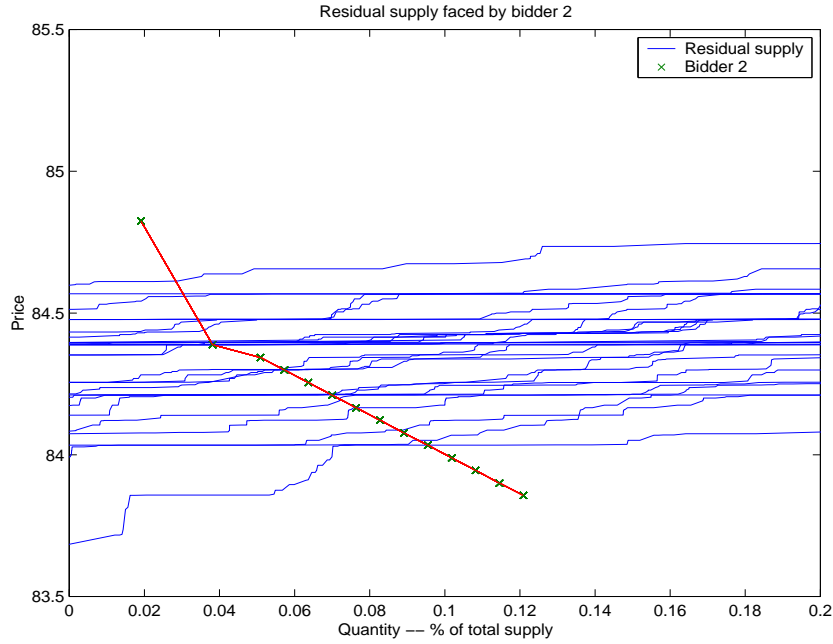


Figure 3: Illustration of the “resampling approach” (Section 3.1): bidder #2’s bid in auction #1, and realizations of residual supply generated by resampling

**Table 1: Summary statistics for 3-month T-bill auctions**

Variable	Average	Std.dev.	Min	Max	Total
Inflation <sup>a</sup>	67.8%	5.6%	58.0%	78.7%	
No. of bidders	69	21.02	32	110	134 <sup>b</sup>
Revenue (million \$) <sup>c</sup>	456.3	443.1	31.4	1560.9	12369.1
Cover ratio <sup>d</sup>	28.9%	24.9%	0.2%	84.3%	
Auction yield <sup>e</sup>	90.8%	6.9%	79.0%	101.4%	
Variance of bids <sup>f</sup>	0.035	0.031	0.005	0.134	

<sup>a</sup>Monthly CPI from the Central Bank of Turkey

<sup>b</sup>Total number of unique bidders who participated in the auctions.

<sup>c</sup>Converted using daily exchange rate data

<sup>d</sup>Quantity sold/Quantity demanded

<sup>e</sup>Quantity weighted average yield

<sup>f</sup>Calculated using the formula:  $\frac{\sum_{i=1}^N (p_i - p_{avg})^2 q_i}{\sum_{i=1}^N q_i}$

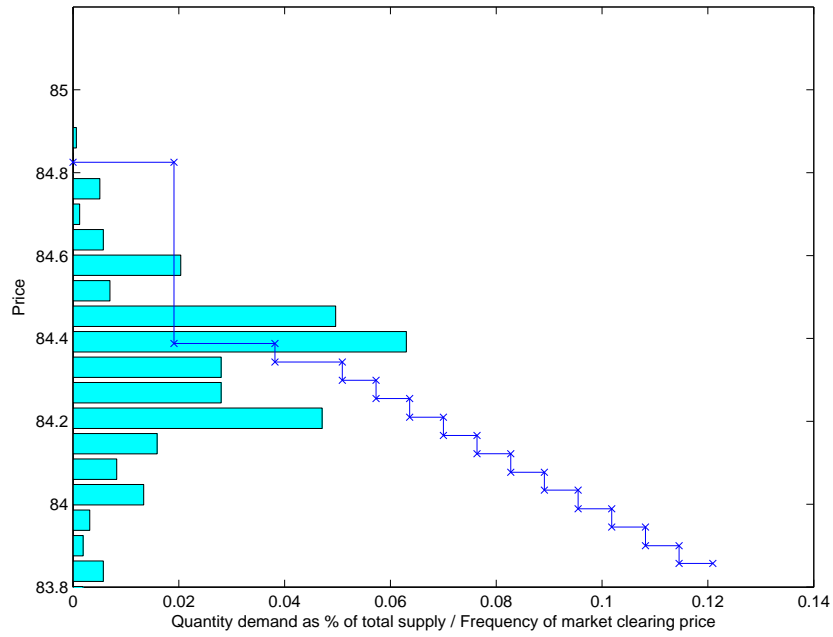


Figure 4: Bidder #2 in auction #1 and market-clearing price distribution generated by resampling procedure

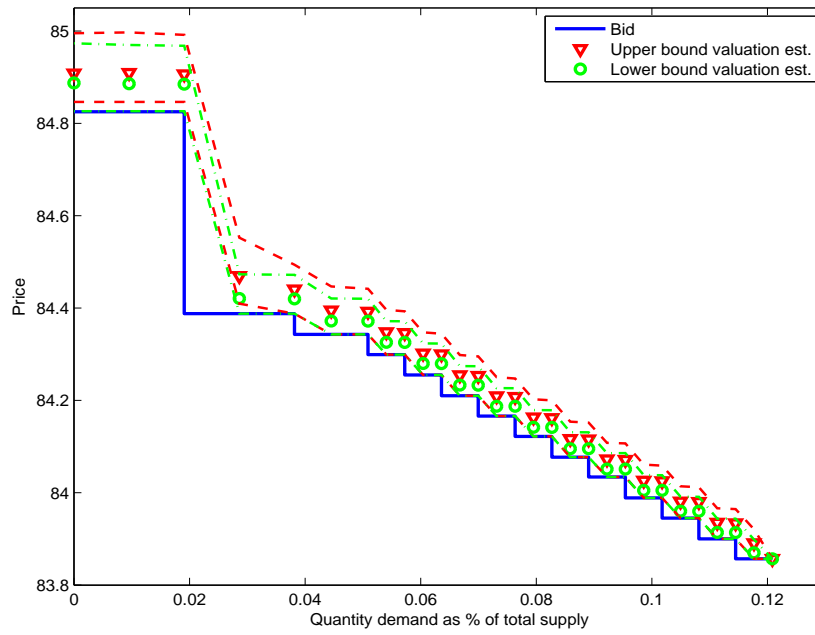


Figure 5: Estimation results for bidder #2 in auction #1

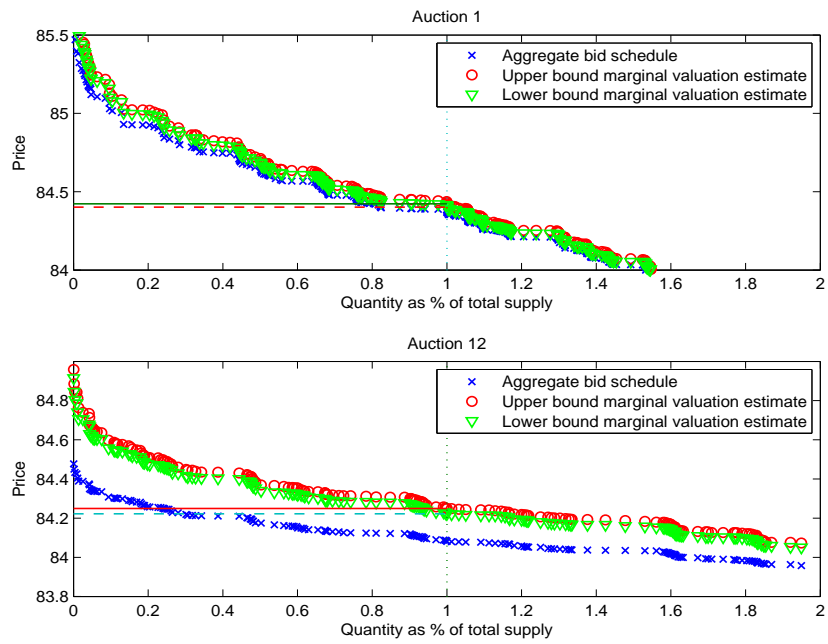


Figure 6: Counterfactual revenue comparison for auctions #1 and #12

**Table 2: Counterfactual Ex Post Revenue and Efficiency Comparisons**

Auction	UPATB revenue gain (upper bnd)			UPATB revenue gain (lower bnd)			UPATB efficiency gain		
	5th pc.	Pt. est.	95th pc.	5th pc.	Pt. est.	95th pc.	5th pc.	Pt. est.	95th pc.
1	-0.35%	-0.31%	-0.26%	-0.35%	-0.33%	-0.28%	0.00%	0.00%	0.00%
2	-0.08%	-0.08%	-0.05%	-0.08%	-0.08%	-0.08%	0.00%	0.00%	0.00%
3	-0.31%	0.23%	0.53%	-0.34%	0.17%	0.42%	0.00%	0.04%	0.05%
4	-0.15%	-0.02%	0.09%	-0.18%	-0.07%	0.00%	0.00%	0.00%	0.01%
5	-0.06%	-0.03%	0.03%	-0.09%	-0.06%	-0.01%	0.00%	0.00%	0.01%
6	-0.02%	0.02%	0.06%	-0.05%	0.00%	0.03%	0.00%	0.00%	0.00%
7	-0.13%	-0.03%	0.03%	-0.15%	-0.06%	0.00%	0.00%	0.00%	0.01%
8	-0.10%	-0.08%	-0.03%	-0.12%	-0.10%	-0.06%	0.00%	0.00%	0.00%
9	-0.22%	-0.19%	-0.14%	-0.24%	-0.21%	-0.16%	0.00%	0.00%	0.00%
10	-0.16%	-0.03%	0.11%	-0.19%	-0.06%	0.08%	0.00%	0.01%	0.01%
11	-0.13%	0.04%	0.21%	-0.13%	0.01%	0.16%	0.00%	0.00%	0.00%
12	-0.07%	0.08%	0.25%	-0.10%	0.05%	0.21%	0.00%	0.00%	0.00%
13	-0.06%	0.09%	0.13%	-0.10%	0.05%	0.10%	0.00%	0.00%	0.00%
14	-0.02%	0.03%	0.13%	-0.05%	0.00%	0.08%	0.00%	0.00%	0.00%
15	0.01%	0.11%	0.19%	-0.04%	0.06%	0.15%	0.00%	0.00%	0.00%
16	-0.01%	0.05%	0.18%	-0.04%	0.01%	0.15%	0.00%	0.00%	0.01%
17	-0.03%	-0.02%	0.07%	-0.06%	-0.06%	0.03%	0.00%	0.00%	0.00%
18	-0.04%	0.42%	0.48%	-0.08%	0.24%	0.36%	0.00%	0.04%	0.09%
19	-0.25%	-0.17%	-0.01%	-0.30%	-0.26%	-0.20%	0.00%	0.00%	0.10%
20	0.06%	1.34%	1.12%	-0.03%	0.41%	0.35%	0.02%	0.22%	0.26%
21	0.04%	0.31%	0.62%	-0.06%	0.12%	0.34%	0.00%	0.00%	0.01%
22	-0.06%	-0.03%	0.02%	-0.09%	-0.07%	-0.03%	0.01%	0.01%	0.03%
23	0.03%	0.11%	0.13%	-0.01%	0.03%	0.04%	0.00%	0.00%	0.00%
24	0.02%	0.11%	0.16%	-0.02%	0.05%	0.10%	0.00%	0.00%	0.01%
25	0.01%	0.23%	0.24%	-0.04%	0.12%	0.10%	0.00%	0.00%	0.01%
26	-0.01%	0.03%	0.29%	-0.04%	0.00%	0.17%	0.00%	0.00%	0.01%
27	0.00%	0.47%	0.57%	-0.14%	0.01%	0.03%	0.03%	0.10%	0.12%
Average	-0.08%	0.10%	0.19%	-0.12%	0.00%	0.08%	0.00%	0.02%	0.03%

**Table 3: Counterfactual Ex Ante Revenue and Efficiency Comparisons**

UPATB revenue gain (upper bnd)			UPATB revenue gain (lower bnd)			UPATB efficiency gain		
5th pc.	Pt. est.	95th pc.	5th pc.	Pt. est.	95th pc.	5th pc.	Pt. est.	95th pc.
-0.07%	0.12%	0.23%	-0.11%	0.02%	0.11%	0.002%	0.020%	0.035%