ISOTONE EQUILIBRIUM IN GAMES OF INCOMPLETE INFORMATION

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An isotone pure strategy equilibrium exists in any game of incomplete information in which each player's action set is a finite sublattice of multidimensional Euclidean space, types are multidimensional and atomless, and each player's interim expected payoff function satisfies two "non-primitive conditions" whenever others adopt isotone pure strategies: (i) single-crossing in own action and type and (ii) quasisupermodularity in own action. Conditions (i,ii) are satisfied in supermodular and log-supermodular games given affiliated types, and in games with independent types in which each player's ex post payoff satisfies supermodularity in own action and non-decreasing differences in own action and type. This result is applied to provide the first proof of pure strategy equilibrium existence in the uniform price auction when bidders have multi-unit demand, non-private values, and independent types.

Keywords: Games of incomplete information, strategic complementarity, pure strategy equilibrium, isotone strategies, multi-unit auctions, uniform price auction

1. introduction

monotone methods have proven to be powerful in the study of games with strategic complementarity. For example, Milgrom and Roberts (1990) and Vives (1990) show that supermodular games possess several useful properties, including existence of pure strategy equilibrium, monotone comparative statics on equilibrium sets, and coincidence of the predictions of various solution concepts such as Nash equilibrium, correlated equilibrium, and rationalizability. Milgrom and Shannon (1994) generalize these results to games with strategic complementarity including, as Athey (1998) shows, log-supermodular games with affiliated types.

This paper adds to this literature by providing sufficient conditions for existence of *monotone* pure strategy equilibrium in games of incomplete information in which players have multidimensional actions and multidimensional types. A player's pure strategy is monotone (technically "isotone") when his action is non-decreasing along every dimension of his action space as his type increases along any dimension of his type space. The sufficient conditions for these existence results are satisfied in the two most widely studied sorts of games with strategic complementarity, supermodular games and log-supermodular games, given affiliated types. Isotonicity is important since it often provides testable empirical implications. For instance, in the Cournot with advertising example discussed in Section 2, lower production and advertising costs are each associated with (weakly) higher sales and advertising levels.

This paper departs from the usual strategic complements framework, however, and considers a broad class of games in which only some of the requirements of strategic complementarity are satisfied. For instance, Milgrom and Shannon (1994) require that each player's expected payoff function must satisfy single-crossing in own action and others' actions (informally, "complementarity across actions") and quasisupermodularity within own action (informally, "complementarity within own action"). This paper extends a new approach pioneered by Athey (2001) to develop monotone methods that apply to games of incomplete information which may fail to exhibit complementarity across actions but in which incremental expected payoffs to higher actions satisfy single-crossing in own type (informally, "monotone incremental returns in own type") when others adopt monotone strategies. Milgrom and Shannon (1994) do not require monotone incremental returns in own type to prove existence of a pure strategy equilibrium but, naturally, they can not guarantee existence of an *isotone* equilibrium.

In a setting with finitely many one dimensional actions and atomless one dimensional types, Athey (2001) shows that a non-decreasing pure strategy equilibrium exists when each player's interim expected payoff satisfies monotone incremental returns in own type given any non-decreasing strategies by others. This paper generalizes her result in a setting with multidimensional actions and multidimensional types, showing that an isotone pure strategy equilibrium exists when each player's interim expected payoff satisfies complementarity in own action and monotone incremental returns in own type given any isotone strategies by others. This result extends to games with a continuum action space whenever each player's ex post payoff is also continuous in his and others' actions, just as Athey (2001)'s results extend to this case.

The rest of the paper is organized as follows. Section 2 illustrates the main existence result by applying it to games with a continuum action space and differentiable payoffs. Section 3 lays out the basic model of incomplete information games with finite action spaces and atomless types while Section 4 states the main theorem and gives three sets of sufficient primitive conditions. Section 5 applies the main theorem to provide the first general pure strategy equilibrium existence result for the uniform price auction when bidders have non-private values and independent types. Section 6 explores "the heart of the contribution" while the Appendix provides proofs.

2. ILLUSTRATION GIVEN DIFFERENTIABLE PAYOFFS

Consider an incomplete information game in which n players each receive a signal $t_i = (t_i^1, ..., t_i^h) \in [0,1]^h$ and choose an action $a_i = (a_i^1, ..., a_i^k) \in [0,1]^k$. Define each player's interim expected payoff function π_i^{int} given others' pure strategies $a_{-i}(\cdot)$ as

$$\pi_i^{int}(a_i, t_i; a_{-i}(\cdot)) \equiv \int_{[0,1]^{h(n-1)}} \pi_i^{post}(a_i, a_{-i}(t_{-i}), t_i, t_{-i}) f(t_{-i}|t_i) dt_{-i}$$

where π_i^{post} is his ex post payoff and $f(\cdot|t_i)$ is the conditional p.d.f. of others' types given that player i's type is t_i. Suppose also that $\pi_i^{post}(\mathbf{a}, \mathbf{t}), f(t_{-i}|t_i)$ are smooth functions of $\mathbf{a}, \mathbf{t}$ and of t_i, respectively so that π_i^{int} is differentiable in a_i, t_i. (Bold notation is used to refer to vectors of all players' actions and types.) A specialized version of Corollary 1 of the main theorem stated on page 18 applies to this class of games.

COROLLARY: *Suppose that, for each bidder $i = 1, ..., n$ and all actions a_i, types t_i, and isotone strategy profiles $a_{-i}(\cdot)$ of others,*

$$(1) \qquad \frac{\partial^2 \pi_i^{int}}{\partial a_i^{j_1} \partial a_i^{j_2}}(\cdot, a_i^{-j_1,j_2}, t_i; a_{-i}(\cdot)) \geq 0 \qquad (1 \leq j_1 < j_2 \leq k)$$

$$(2) \qquad \frac{\partial^2 \pi_i^{int}}{\partial a_i^{j} \partial t_i^{m}}(\cdot, a_i^{-j}, t_i^{-m}; a_{-i}(\cdot)) \geq 0 \qquad (1 \leq j \leq k, 1 \leq m \leq h)$$

Then an isotone pure strategy equilibrium exists.[2]

For illustration purposes it is simplest to consider examples in which player types are independent, since then the cross-partial inequalities (1, 2) on expected payoffs are implied directly by the corresponding cross-partial inequalities on ex post payoffs.

EXAMPLE (Cournot with 2 advertising channels, n firms): Consider an undifferentiated product Cournot competition game in which n risk neutral firms each choose a quantity q_i and levels of two sorts of advertising e_i^1, e_i^2 to expand the size of the total market. In the pharmaceutical context, for example, drug companies advertise to patients through media advertising and to doctors through detailing (such as office visits from company reps). Firms also receive multidimensional independent private information t_i, where higher own type implies (weakly) lower own advertising and production costs. In particular, suppose that (i) $D(p; \mathbf{e}) = D(p) + \gamma(\mathbf{e})$ is total demand, (ii) $\phi_i(\mathbf{e}, \mathbf{t})$ is firm i's advertising cost function, and (iii) $c_i(q_i; \mathbf{t})$ is firm i's production cost function,[3] where $\mathbf{q}, \mathbf{e}, \mathbf{t}$ refer to vectors of all firms' quantities, advertising levels, and types. Firm i's ex post payoff is

$$\pi_i^{post}(\mathbf{q}, \mathbf{e}, \mathbf{t}) = \pi_i(\mathbf{q}, \mathbf{e}, \mathbf{t}) \equiv q_i p(\mathbf{q}, \mathbf{e}) - c_i(q_i; \mathbf{t}) - \phi_i(\mathbf{e}, \mathbf{t})$$

where $p(\mathbf{q}, \mathbf{e})$ is the market clearing price. If there were just one advertising channel, an isotone pure strategy equilibrium would always exist in this example since

$$\frac{\partial^2 \pi_i}{\partial q_i \partial e_i} \geq 0, \frac{\partial^2 \pi_i}{\partial q_i \partial t_i} \geq 0, \frac{\partial^2 \pi_i}{\partial e_i \partial t_i} \geq 0.$$

Given two advertising channels, similarly, an isotone equilibrium exists as long as $\frac{\partial^2 \pi_i}{\partial e_i^1 \partial e_i^2} \geq 0$. Note though that existence even of a unique isotone equilibrium does *not* provide the basis for monotone comparative statics. For example, suppose that a change in the tax code lowers all firms' production costs. In the new isotone equilibrium, some firms may produce and/or advertise less than they did in the original equilibrium.

3. MODEL: INCOMPLETE INFORMATION GAMES

This Section lays out the model of incomplete information games with atomless types and finite action spaces.

3.1. *Actions and Lattices*

DEFINITION ($\vee$, $\wedge$): Let $(L, \geq)$ be a partially ordered set and let $S \subset L$. The *least upper bound of* S, $\vee S$, is the unique element of L (if it exists) satisfying $\vee S \leq c \Leftrightarrow a \leq c$ for all $a \in S$ and all $c \in L$. The *greatest lower bound of* S, $\wedge S$, is the unique element of L satisfying $\wedge S \geq c \Leftrightarrow a \geq c$ for all $a \in S$ and all $c \in L$. When $S = \{a, b\}$, I use the standard notation $a \vee b$ and $a \wedge b$.

Definition (Lattice, Sublattice, Complete): A *lattice* $(L, \geq, \vee, \wedge)$ is a partially ordered set $(L, \geq)$ such that $a \vee b, a \wedge b \in L$ for all $a, b \in L$. $L_1 \subset L$ is a *sublattice* of L if and only if $a \vee b, a \wedge b \in L_1$ for all $a, b \in L_1$. L_1 is *complete* if and only if $\vee S, \wedge S \in L_1$ for every subset $S \subset L_1$. Every finite sublattice is complete (Birkhoff (1967)).

Assumption 1: *Each player $i = 1, ..., n$ has a common action set $L \subset \mathcal{R}^k$ that is a finite sublattice of k dimensional Euclidean space with respect to the product order on $\mathcal{R}^k$.*[4]

A typical action is $a_i \equiv \left(a_i^1, ..., a_i^k\right) \equiv \left(a_i^m, a_i^{-m}\right) \in L$ for $m = 1, ..., k$. A typical action profile is $\mathbf{a} \equiv (a_1, ..., a_n) \equiv (a_i, a_{-i}) \in \Pi_{i=1}^n L$. Similar subscript, superscript, and bold notation will be used consistently throughout the paper to refer to types and strategies as well as actions. For each $m = 1, ..., k$, define

$$L_m \equiv \left\{a_i^m \in \mathcal{R} : \left(a_i^m, a_i^{-m}\right) \in L \text{ for some } a_i^{-m} \in \mathcal{R}^{k-1}\right\}.$$

By definition, $L \subset \Pi_{m=1}^k L_m$, though I do not assume that $L = \Pi_{m=1}^k L_m$. Without loss, let $L_m = \{0, 1, ..., |L_m| - 1\}$.

3.2. *Types and Strategies*

Assumption 2: *Player i's type t_i is drawn from common support $T = [0,1]^h$. $f : \mathcal{R}^{nh} \to \mathcal{R}_+ +$, the joint density on type profiles (or states) $\mathbf{t} = (t_1, ..., t_n)$, is bounded above by $\overline{K}$ and bounded below by $\underline{K} > 0$.*[5] *The type space is endowed with the product order and the usual Euclidean topology and measure.*

Definition (Pure strategy, Isotone pure strategy): A *pure strategy* (PS) $a_i(\cdot) : T \to L$ is a measurable function mapping each type into an action $a_i(t_i)$. In an *isotone* pure strategy (IPS), $t_i' > t_i$ implies $a_i(t_i') \geq a_i(t_i)$.

$\mathcal{S}_i$ denotes the space of all of player i's PS, $\mathcal{S}_{-i} = \Pi_{j \neq i} \mathcal{S}_j$ the space of others' PS profiles, and $\mathcal{S} = \Pi_{i=1}^n \mathcal{S}_i$ the space of full PS profiles. Similarly, $\mathcal{I}_i$, $\mathcal{I}_{-i}$, and $\mathcal{I}$ are the spaces of own IPS, others' IPS profiles, and full IPS profiles.

3.3. *Payoffs*

Given a profile of actions $\mathbf{a}$ and types $\mathbf{t}$, player i's *ex post payoff* (or utility) is $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$.

Assumption 3: Π_i^{post} *is bounded and measurable.*

Interim expected payoff $\Pi_i^{int}(\cdot, \cdot; \cdot) : L \times T \times \mathcal{S}_{-i} \to \mathcal{R}$, similarly, depends on his own action, own type, and others' strategies:

$$\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot)) = E_{t_{-i}\big|t_i} \left[\Pi_i^{post}(a_i, a_{-i}(t_{-i}), \mathbf{t})|t_i\right].$$

For the most part, I restrict attention to settings in which others follow IPS, $a_{-i}(\cdot) \in \mathcal{I}_{-i}$.

Definition (Quasisupermodular in x (or $QSPM(x)$)): Let $(L, \geq, \vee, \wedge)$ be a lattice. $g : L \to \mathcal{R}$ is quasisupermodular in x if and only if

$$g(x') \geq (>) g(x' \wedge x) \Rightarrow g(x' \vee x) \geq (>) \; g(x)$$

for all $x', x \in L$. (Weak inequality implies weak inequality and strict inequality implies strict.)

Assumption 4: $\Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$ *satisfies* $QSPM(a_i)$ *for all* $t_i \in T$ *and all* $a_{-i}(\cdot) \in \mathcal{I}_{-i}$.

Definition (Single-crossing in (x,t) (or $SC(x,t)$) and in t (or $SC(t)$)): Let $(L, \geq, \vee, \wedge)$ be a lattice and $(T, \geq)$ a partially ordered set. $g : L \times T \to \mathcal{R}$ satisfies *single-crossing in* (x,t) if and only if

$$g(x', t) \geq (>) g(x, t) \Rightarrow g(x', t') \geq (>) \; g(x, t')$$

for all $x' > x \in L$ and all $t' > t \in T$. Similarly, $g : T \to \mathcal{R}$ satisfies *single-crossing in* t if and only if

$$g(t) \geq (>) 0 \Rightarrow g(t') \geq (>) \; 0$$

for all $t' > t$.

Assumption 5: Π_i^{int} *satisfying* $SC(a_i, t_i)$ *for all* $a_{-i}(\cdot) \in \mathcal{I}_{-i}$. *(This is equivalent to* $\Pi_i^{int}(a_i', t_i; a_{-i}(\cdot)) - \Pi_i^{int}(a_i, t_i; a_{-i}(\cdot))$*, the incremental expected payoff to* a_i' *versus* a_i*, satisfying* $SC(t_i)$ *for all* $a_i' > a_i$ *and* $a_{-i}(\cdot) \in \mathcal{I}_{-i}$*.)*

3.4. *Best Response and Equilibrium*

Let $BR_i\left(t_i, a_{-i}(\cdot)\right) \equiv \arg\max_{a \in L} \Pi_i^{int}\left(a, t_i; a_{-i}(\cdot)\right)$ denote player i's best response action set when others follow pure strategies $a_{-i}(\cdot)$. When it can not cause confusion, I simplify this notation to $BR_i(t_i)$.

Definition (Isotone pure strategy equilibrium): $\mathbf{a}^*(\cdot) \in \mathcal{S}$ is a *pure strategy equilibrium* (PSE) if and only if $a_i^*(t_i) \in BR_i\left(t_i, a_{-i}^*(\cdot)\right)$ for all i, t_i. Any PSE $\mathbf{a}^*(\cdot) \in \mathcal{I}$ is called an *isotone pure strategy equilibrium* (IPSE).

4. EXISTENCE OF ISOTONE EQUILIBRIUM

Theorem 1: *Under assumptions 1-5, an IPSE exists in games of incomplete information.*

The proof is in the Appendix. A straightforward extension in which actions sets $L = [0,1]^k$ and ex post payoffs are continuous in actions $\mathbf{a}$ is also provided in the Appendix.

4.1. *Sufficient primitive conditions*

I gather here three sets of primitive conditions that others' work proves are sufficient for interim expected payoff to satisfy quasisupermodularity in own action and single-crossing in own action and type (Assumptions 4,5). I refer the reader to this other work for the formal definitions of such standard terms as affiliated, supermodular, log-supermodular, and non-decreasing differences.

1. Types are affiliated and $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ is supermodular in $(\mathbf{a}, t_j)$ for all j. In this case, Athey (2002) proves that $\Pi_i^{int}\left(a_i, t_i; a_{-i}(\cdot)\right)$ is supermodular in $(\mathbf{a}, t_i)$ when $a_{-i}(\cdot) \in \mathcal{I}_{-i}$.

2. Types are affiliated and $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ is log-supermodular in $(\mathbf{a}, \mathbf{t})$. In this case, Athey (1998) proves that $\Pi_i^{int}\left(a_i, t_i; a_{-i}(\cdot)\right)$ is log-supermodular in $(\mathbf{a}, t_i)$ when $a_{-i}(\cdot) \in \mathcal{I}_{-i}$.

3. Types are independent and $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ is supermodular in a_i with non-decreasing differences in (a_i, t_i). Then expected payoff $\Pi_i^{int}\left(a_i, t_i; a_{-i}(\cdot)\right)$ is supermodular in a_i and has non-decreasing differences in (a_i, t_i) when $a_{-i}(\cdot) \in \mathcal{S}_{-i}$. (Others may follow any strategies.) See Topkis (1979).

In Milgrom and Roberts (1990) and Vives (1990), a supermodular game is one in which $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ is supermodular in $\mathbf{a}$, with no conditions placed on the distribution of types. Thus, the primitive conditions of cases 1,2 are only satisfied in a subclass of supermodular (and log-supermodular) games. This stands to reason, of course, since I prove that an *isotone* PSE exists whereas Milgrom and Roberts (1990) and Vives (1990) only prove existence of PSE. Case 3 makes the very strong requirement of independence but allows that players' payoffs may fail to exhibit complementarity across actions. If independence is replaced by affiliation in case 3, an isotone equilibrium may not exist. For example, McAdams (2002) provides a uniform price auction example with affiliated private values in which all case 3 requirements on ex post payoffs are satisfied but in which some bidders reduce their bids on *all* units as their values increase in *all* equilibria.

5. EXAMPLE: UNIFORM PRICE AUCTION

Proving existence of PSE in the uniform price auction with multi-unit demand is particularly challenging since payoffs fail to satisfy strategic complementarity (Milgrom and Shannon (1994) does not apply) and fail to satisfy diagonal quasiconcavity (Reny (1999)[6] does not apply). Indeed, the only general PSE existence theorems that I am aware of that apply to the uniform price auction require private values. Jackson and Swinkels (2001) prove existence of PSE with positive probability of trade in two sided[7] or one sided uniform price auctions given private values and a very general correlation structure. Bresky (2000) proves existence of IPSE given independent private values. Like Bresky (2000), my application of Theorem 1 requires independent types but proves IPSE existence in a setting that allows for a much more general structure of values.

MODEL: n bidders and S identical objects (or units) for sale.

Information and Payoffs: Bidder i receives value $V_i(\mathbf{q}, \mathbf{t})$ from the allocation $\mathbf{q} = (q_1, ..., q_n)$ in the state $\mathbf{t} = (t_1, ..., t_n)$, where t_i are i.i.d. with common support $[0, 1]^h$. V_i is piecewise continuous in $\mathbf{t}$ and $V_i(\mathbf{q}', \mathbf{t}) - V_i(\mathbf{q}, \mathbf{t})$ is non-decreasing in t_i whenever $q_i' > q_i$ and $q_j' \leq q_j$ for all $j \neq i$. (No other assumptions on values.) Bidders seek to maximize expected surplus, the difference between their value and payment.

Bids: A permissible bid is a vector $b_i = (b_i(1), ..., b_i(S))$ such that $b_i(q_1) \geq b_i(q_2)$ when $q_1 < q_2$ and $b_i(1), ..., b_i(S) \in \{p^{min}, p^{min} + 1, ..., p^{max}\}$.

Allocation: Let $b^k(\mathbf{b}(\cdot))$ (shorthand b^k) be the kth highest unit-bid across all bid schedules. Define $\underline{q}_i \equiv \max\{q : b_i(q) > b^S\}$ and $\overline{q}_i \equiv \max\{q : b_i(q) \geq b^S\}$. $\underline{q}_i$ ($\overline{q}_i$) is the least (greatest) quantity that bidder i can receive in any market clearing allocation. $\sum_{i=1}^n \underline{q}_i \leq S \leq \sum_{i=1}^n \overline{q}_i$ and quantity is rationed in the following manner:[8] Each bidder is assigned at least $\underline{q}_i$ and randomly ordered into a ranking ρ to ration the remaining quantity $r \equiv S - \sum_{i=1}^n \underline{q}_i$. If $r = 0$, stop. Else the first bidder in order, $i_1 = \rho(1)$, receives $q^*_{i_1} = \underline{q}_{i_1} + \min\{\overline{q}_{i_1} - \underline{q}_{i_1}, r\}$. Decrement r by $q^*_{i_1} - \underline{q}_{i_1}$ and repeat this process with bidder $i_2 = \rho(2)$ and so on until all quantity has been assigned.

Payment: A variety of uniform price payment rules have been considered in the literature. I study here the two most common: in the Sth (or $S + 1$st) price auctions, all bidders pay the lowest winning bid b^S (or highest losing bid b^{S+1}) on all units that they win, i.e. total payment $Z_i = q^*_i b^S$ (or $= q^*_i b^{S+1}$).

Several features of the model are worthy of note:

1. The formulation of bidder values includes as a special case the benchmark "interdependent values" in which bidder i's value for q_i units takes the form $V_i(q_i, \mathbf{t})$ and all incremental values $V_i(q'_i, \mathbf{t}) - V_i(q_i, \mathbf{t})$ (for $q'_i > q_i$) are typically assumed to be non-decreasing in $\mathbf{t}$ and strictly increasing in t_i.

2. Some sorts of externalities are permitted. Values take the form $V_i(\mathbf{q}, \mathbf{t})$, with the only monotonicity restriction being that incremental values $V_i(\mathbf{q}', \mathbf{t}) - V_i(\mathbf{q}, \mathbf{t})$ are non-decreasing in own type whenever $q'_i > q_i$ and $q'_j \leq q_j$ for all $j \neq i$. In other words, bidders may care about what other bidders win with the caveat that "own marginal values" are non-decreasing in own type. (Such a monotonicity assumption is present, for instance, in Jehiel, Moldavanu, and Stacchetti (1996).)

3. Bidders receive multidimensional private information and values need not be strictly increasing in own type. For example, part of a bidder's information may be relevant to own values given certain allocations but not given other allocations.

4. Values need not be monotone at all in others' private information.

5. Marginal values may be increasing in own quantity, allowing for increasing returns to scale in consumption. On the other hand, I continue to make the standard requirement that bids be non-increasing in quantity to guarantee existence of a market clearing allocation.[9]

THEOREM 2: *An IPSE exists in this model of the uniform price auction.*

Proof sketch: The set of all non-increasing bid schedules forms a lattice with respect to the product order. Thus, it suffices to check that Assumptions 4,5 are satisfied. In fact, I prove in the Appendix that two stronger conditions hold: (i) expected payoff $\Pi_i^{int}(\cdot, t_i; b_{-i}(\cdot;\cdot))$ is modular in own bid (see below) for all types t_i and all profiles of others' strategies (isotone or not) and (ii) $\Pi_i^{int}(\cdot,\cdot; b_{-i}(\cdot;\cdot))$ has non-decreasing differences in own bid and type for all profiles of others' strategies.[10]

Non-decreasing differences (NDD): The intuition behind NDD of expected payoffs is clear. First, note that ex post payment has zero differences in own bid and type since payment does not depend on type. Next, ex post values have NDD since submitting a higher bid (holding others' bids fixed) always leads one to win weakly greater quantity and others to win weakly less quantity. And by assumption ex post incremental value from such a change in the allocation is non-decreasing in own type. Finally, NDD is preserved under integration[11] so expected payoffs have NDD no matter what strategies others follow. (Independence is crucial in this step.)

DEFINITION (Modular (or Additively Separable)): Let $(L, \geq, \vee, \wedge)$ be a lattice. $g : L \to \mathcal{R}$ is modular (or additively separable) in x if and only if

$$g(x' \vee x) + g(x' \wedge x) = g(x') + g(x)$$

for all $x', x \in L$.

Modularity: Consider two bid schedules, $b_i^1(\cdot)$ and $b_i^2(\cdot)$. Their least upper bound $b_i^{1\vee 2}(\cdot)$ and greatest lower bound $b_i^{1\wedge 2}(\cdot)$ are their upper and lower envelopes, respectively. Given any state $\mathbf{t}$, profile of others' bids $b_{-i}(\cdot; t_{-i})$ and rationing ordering ρ, the auction mechanism maps each of these four bids into four allocations (shorthand $\mathbf{q}^1 \equiv \mathbf{q}^*(b_i^1(\cdot), b_{-i}(\cdot; t_{-i}), \rho)$ and so on) and into four uniform prices (shorthand $p^1 \equiv p^*(b_i^1(\cdot), b_{-i}(\cdot; t_{-i}))$ and so on). The key step of the modularity proof is to show that

$$\{(\mathbf{q}^1, p^1), (\mathbf{q}^2, p^2)\} = \{(\mathbf{q}^{1\vee 2}, p^{1\vee 2}), (\mathbf{q}^{1\wedge 2}, p^{1\wedge 2})\}.$$

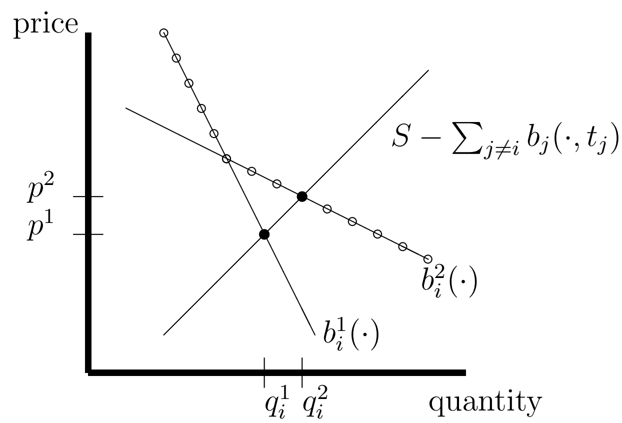

Figure 1: Modularity of ex post payoff given signals t_{-i}

Figure 1 illustrates why this is true in a special case in which (i) bids are made over a continuum of quantities and prices and (ii) bid schedules are continuous and strictly downward sloping. Simplification (ii) eliminates all subtleties that arise from the details of the pricing and/or rationing rules, but still the Figure conveys the basic idea. Two bid schedules for bidder i, $b_i^1(\cdot)$ and $b_i^2(\cdot)$, are labelled while their upper envelope $b_i^{1\vee 2}(\cdot)$ is traced with open circles. Price and bidder i's quantity are determined by where i's bid schedule crosses the residual supply schedule. In this case, price equals p^1 and bidder i gets quantity q_i^1 whether he submits $b_i^1(\cdot)$ or $b_i^{1\wedge 2}$. Furthermore, it is easy to observe that all other bidders receive the same quantity after either bid as well. Similarly, price and the allocation are identical whether bidder i submits $b_i^2(\cdot)$ or $b_i^{1\vee 2}(\cdot)$.

In other words, the *pair* of outcomes (allocation and price) of the auction after submitting bids $b_i^1(\cdot), b_i^2(\cdot)$ is identical to those after $b_i^{1\vee 2}(\cdot), b_i^{1\wedge 2}(\cdot)$. Consequently, bidder i's ex post surplus also "matches up" in this sense. Thus the expectation of any function of ex post surplus, taken with respect to any distribution over types, will itself be modular. In particular, as long as bidder utility takes the form $u_i(V_i, Z_i)$, then expected utility will be modular in own bid regardless of the type distribution.

6. HEART OF THE CONTRIBUTION

Theorem 1, the paper's main result, is essentially a corollary of the powerful Monotonicity Theorem of Milgrom and Shannon (1994) (hereafter MS). Indeed, in my view, the main contribution of this paper is to uncover the structure possessed by $\arg\max_x g(x,t)$ when x and t are multidimensional and g satisfies the conditions of the Monotonicity Theorem. This structure in turn happens to be exactly what is required to extend Athey (2001)'s ingenious approach to proving existence of monotone pure strategy equilibrium to a setting with multidimensional actions and multidimensional types. It seems worthwhile, then, to discuss these prior contributions and thereby indicate what I feel is at the heart of my contribution.

6.1. Monotonicity Theorem

First, I state a weakened version of the Monotonicity Theorem, for which two more definitions are needed.[12]

DEFINITION (Strong set order): Let $(L,\geq,\vee,\wedge)$ be a lattice. The *strong set order* $\geq_L$ is a partial ordering on $\mathcal{P}(L)$, the space of subsets of L. For $A, A' \subset L$, $A' \geq_L A$ if and only if $a' \in A', a \in A$ implies that $a' \vee a \in A', a' \wedge a \in A$.

DEFINITION (Increasing in the strong set order): Let $(L,\geq,\vee,\wedge)$ be a lattice and $(T,\geq)$ a partially ordered set. A correspondence $g : T \to \mathcal{P}(L)$ is *increasing in the strong set order* if and only if $g(t') \geq_L g(t)$ whenever $t' > t$.

THEOREM (Milgrom and Shannon (1994)): *Let $g : L \times T \to \mathcal{R}$, where $(L,\geq,\vee,\wedge)$ is a complete lattice and $(T,\geq)$ a partially ordered set. Then $\arg\max_{x\in L} g(x,t)$ is a complete sublattice for all t and increasing in the strong set order if g satisfies $QSPM(x)$ and $SC(x,t)$.*

Given the Monotonicity Theorem, it is not surprising that Assumptions 4 and 5 of my model (QSPM(own action) and SC(own action, own type) of expected payoffs whenever others follow isotone strategies) are associated with a result proving existence of IPSE. These conditions guarantee that each player always has an isotone pure best response strategy whenever others follow isotone pure strategies: For a given profile of others' isotone pure strategies $a_{-i}(\cdot)$, Assumptions 4,5 and the Monotonicity Theorem imply that player i's set of best response actions, $BR_i\left(t_i, a_{-i}(\cdot)\right)$, is a complete sublattice for all types t_i and that $BR_i(\cdot, a_{-i}(\cdot))$ is increasing in the strong set order. Since action sets are finite, also, these sets are non-empty. Consequently, an isotone selection exists from $BR_i(\cdot, a_{-i}(\cdot))$.

6.2. Athey's Vector Representation

Of course, existence of an isotone best response is far from guaranteeing isotone equilibrium. At the heart of the existence result is an extension of Athey (2001)'s remarkable proof that there is a sense in which each bidder's set of isotone pure best response strategies is convex. This convexity then is used to apply Glicksberg (1952)'s Fixed Point Theorem to a best response correspondence whose domain and range are restricted to the set of IPS profiles. To be more precise, given one dimensional finite action sets (say $\{0, 1, 2, ..., z\}$) and one dimensional atomless types (say drawn from $[0, 1]$), Athey observes that any non-decreasing strategy can be identified, up to the actions played by a zero measure set of types, with an z dimensional non-decreasing vector of types (perhaps with repetition) at which the player "increases" his action. For instance, when $z = 3$, the set of all strategies $a_i : [0, 1] \to \{0, 1, 2, 3\}$ such that $a_i(t_i) = 0$ for $t_i < 1/2$, $a_i(t_i) = 2$ for $t_i \in (1/2, 3/4)$, and $a_i(t_i) = 3$ for $t_i > 3/4$ gets mapped to the vector $(1/2, 1/2, 3/4)$. Say that two isotone strategies $a'_i(\cdot), a_i(\cdot)$ are equivalent if and only if $\Pr_{t_i}(a'_i(t_i) = a_i(t_i)) = 1$. It's easy to see that each such equivalence class of isotone strategies maps to a different vector, and that the range of this bijection is a compact, convex subset of $\mathcal{R}^z$. Furthermore, this mapping is a homeomorphism with respect to the usual Euclidean topology on $\mathcal{R}^z$ and the topology on strategies corresponding to the metric $|a'_i(\cdot) - a_i(\cdot)| = \Pr_{t_i}(a'_i(t_i) \neq a_i(t_i))$. An important property of this topology is that each bidder's expected payoff is continuous in others' strategies[13] whenever payoffs are bounded. (See Athey (2001).)

6.3. Convexity of Isotone Best Response Strategies

Why is the image of player i's set of isotone pure best response strategies under the Athey map a convex subset of $\mathcal{R}^z$? Take as known that his expected payoffs (given others' strategies $a_{-i}(\cdot)$) satisfy the requirements of the Monotonicity Theorem. Then the fact that his best response action set is increasing in the strong set order implies that $a_i \in BR(t_i)$ for all $t'_i < t_i < t''_i$ whenever $a_i \in BR(t'_i) \cap BR(t''_i)$. Now, convexity of the image of the isotone best responses is clear. For example, when $z = 2$, suppose that $w_1 = (1/2, 3/4)$ and $w_2 = (0, 1/4)$ both correspond to isotone best response strategies; a convex combination of these vectors is $w_1/2+w_2/2 = (1/4, 1/2)$. Revealed preference implies *directly* that all types in $[0, 1/2]$ find 0 to be a best response, all in $[0, 1/4) \cup (1/2, 3/4)$ find 1 a best response, and all in $(1/4, 1]$ find 2 a best response. To conclude that $w_1/2 + w_2/2$ corresponds to an isotone best response, however, we also need to know that types in $(1/4, 1/2)$ find 1 to be a best response. But the Monotonicity Theorem implies *indirectly* that all types in $[0, 3/4)$ find 1 to be a best response. It's easy to see that the same logic applies to *all* convex combinations of any two isotone pure best response strategies. (See Athey (2001).) Indeed, this approach applies as well to settings with multidimensional types and multidimensional actions. Surprisingly, the extension to multidimensional types is relatively straightforward while that to multidimensional actions is much more subtle and difficult.

Multidimensional types, one dimensional actions: When a player follows an IPS, his type space is divided into regions in which each action is played such that no type in the a_i'-region is less than any type in the a_i-region whenever $a_i' > a_i$. To represent player i's strategy as a vector, I partition player i's type space into many one dimensional subsets of the form $C(t_i^{-1}) = [0, 1] \times \{t_i^{-1}\}$. One may characterize the strategy over $C(t_i^{-1})$, up to what action is chosen by finitely many types, by a finite dimensional vector exactly as in Athey. And the vector of such vectors characterizes the strategy over the whole type space, again up to a zero measure set of types. After care is taken in properly defining the relevant topologies, I show in the Appendix that the induced bijection remains a homeomorphism between equivalence classes of strategies and a compact, convex subset of a convex linear topological space. A convex combination of elements in the image of the Athey map corresponds to taking the convex combination, "line by line" for each $C(t_i^{-1})$, of the boundaries between the type regions who play each action. For example, Figures 2 and 3 illustrate a convex combination of two strategies when $T = [0, 1]^2$ and $L = \{0, 1, 2\}$. The number 0,1,2 in each region of the type space is the action played by types in that region. Suppose that IPS corresponding to the two vectors illustrated in Figure 2 are both best responses. Revealed preference directly implies that all types play a best response action in any strategy corresponding to the convex combination vector illustrated in Figure 3 *except* for the upper left part of the type region playing action 1. On the other hand, note that for every type t_i in the interior of this "1-region" there is a pair of types $\{t_i', t_i''\}$ contained within the union of the 1-regions corresponding to the two original strategies such that $t_i' < t_i < t_i''$. Thus, again, the fact that $BR_i(t_i, a_{-i}(\cdot))$ is increasing in the strong set order implies that every such type t_i must find 1 to be a best response since both t_i', t_i'' do. Note that this verification of convexity, like Athey's, does not at all leverage the fact that $BR_i(t_i, a_{-i}(\cdot))$ is a lattice.

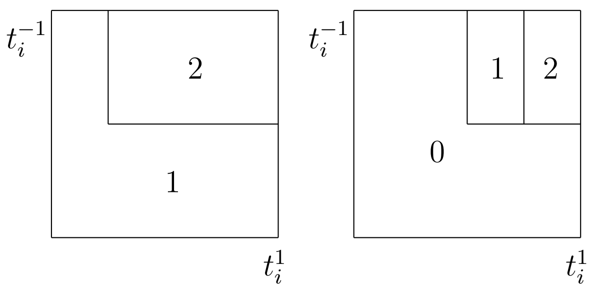

Figure 2: Two isotone strategies

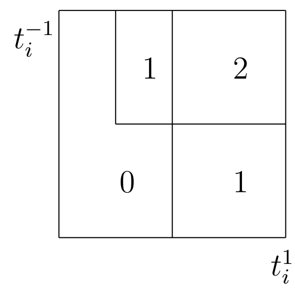

Figure 3: Convex combination

Multidimensional actions, one dimensional types: While the extension of Athey (2001)'s existence result to multidimensional types is relatively straightforward once viewed in the appropriate light, the generalization to multidimensional actions is remarkably subtle and complex. Defining a homeomorphism from the space of strategies to a convex, compact subset of a vector space is the relatively easy part. Consider the set of "projections" of an action onto each dimension of the action space. (Under projection $\phi^m : L \to L_m$, $\phi(a_i) \to a_i^m$ its mth coordinate.) Each strategy $a_i(\cdot)$ is characterized by its projections $\{a_i^m(\cdot) : m = 1, ..., k\}$, where $a_i^m(t_i) \equiv \phi^m(a_i(t_i))$ for all types. Furthermore, a pure strategy $a_i(\cdot)$ is isotone if and only if each $a_i^m(\cdot)$ is nondecreasing. We may therefore represent any isotone strategy as a vector of k vectors, each of which characterizes an isotone function mapping types into a one dimensional action space, as in the previous case.

The subtle and difficult part is proving that the image of the isotone pure best response strategies is convex. An example highlights some of the issues involved.

Example: $L = \{0, 1, 2\} \times \{0, 1\} \times \{0, 1, 2\}$, $T = [0, 1]$. Let $a_{-i}(\cdot)$ be a given profile of others' IPS. Consider two best response IPS of player i:

$$\begin{aligned} a_i'(t_i) &= (0,0,1) \text{ for all } t_i \in [0,1/2) \\ &= (1,1,2) \text{ for all } t_i \in [1/2,1] \\ a_i(t_i) &= (2,0,0) \text{ for all } t_i \in [0,1/2) \\ &= (2,1,0) \text{ for all } t_i \in [1/2,3/4) \\ &= (2,1,1) \text{ for all } t_i \in [3/4,1]. \end{aligned}$$

$a_i'(\cdot) \to ((1/2,1),1/2,(0,1/2)) = w_1$ and $a_i(\cdot) \to ((0,0),1/2,(3/4,1)) = w_2$ under the Athey map. $w_1/2 + w_2/2 = ((1/4,1/2),1/2,(3/8,3/4))$ is a convex combination of these vectors and maps back to an equivalence class of strategies having representative member

$$\begin{aligned} a_i(t_i; w_1/2 + w_2/2) &= (0,0,0) \text{ for all } t_i \in [0,1/4) \\ &= (1,0,0) \text{ for all } t_i \in [1/4,3/8) \\ &= (1,0,1) \text{ for all } t_i \in [3/8,1/2) \\ &= (2,1,1) \text{ for all } t_i \in [1/2,3/4) \\ &= (2,1,2) \text{ for all } t_i \in [3/4,1]. \end{aligned}$$

Several new actions are played in the strategy $a_i(t_i; w_1/2 + w_2/2)$ and no type t_i plays an action that he played under either original strategy. Thus, revealed preference tells us nothing about whether the new strategy is a best response. Even the fact that the set of types who find each action to be a best response is convex does not help us at all. Rather, to conclude that each type plays a best response action, one must *repeatedly* apply both the fact that the best response action set is a lattice and that it is increasing in the strong set order. For example, consider a type $t_i \in (3/8, 1/2)$. By the lattice property, type t_i finds $(2,0,1) = (0,0,1) \vee (2,0,0)$ to be a best response whereas types $t_i' \in (1/2, 3/4)$ (greater than t_i) find $(1,1,0) = (1,1,2) \wedge (2,1,0)$ to be a best response. Now we can use increasingness in the strong set order to conclude that $(2,0,1) \wedge (1,1,0) = (1,0,0) \in BR_i(t_i)$. Finally, again using the lattice property, $(1,0,0) \vee (0,0,1) = (1,0,1) \in BR_i(t_i)$ and we are done. The proof for the general case is very similar. For each type t_i, I prove *by induction* that the required action $a_i = (a_i^1, ..., a_i^k) \in BR_i(t_i)$: $BR_i(t_i)$ contains an element whose first coordinate equals a_i^1. Then, given that $BR_i(t_i)$ contains an element whose first j coordinates equal $(a_i^1, ..., a_i^j)$, $BR_i(t_i)$ contains an element whose first $j+1$ coordinates equal $(a_i^1, ..., a_i^{j+1})$.

7. CONCLUSION

This paper shows how two non-primitive conditions, quasisupermodularity in own action and single-crossing in own action and type of interim expected payoff whenever others follow isotone strategies, are sufficient for existence of an isotone pure strategy equilibrium in a very general setting with finitely many multidimensional actions and a continuum of multidimensional types. Furthermore, these conditions are satisfied in a variety of important classes of games such as supermodular and log-supermodular games with affiliated types as well as in some games in which strategic complementarity fails. For instance, as an application of the main theorem, I provide the first proof of equilibrium existence in pure strategies (indeed, isotone pure strategies) when bidders have non-private values and independent types in the uniform price auction.

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APPENDIX

ASSUMPTION 1': *Player i's action set is* $[0,1]^k$.

ASSUMPTION 6: $\Pi_i^{post}(\mathbf{a}, \mathbf{t})$ *is continuous in* $\mathbf{a}$ *for all* $\mathbf{t}$.

COROLLARY 1: *Under assumptions 1',2-6, an IPSE exists in games of incomplete information.*

Proof. The proof closely follows that of Theorem 2 in Athey (2001), and I refer to the reader to this proof for most details. The only potentially substantial difference is that each player's action is multidimensional, so one must argue that any sequence of IPS profiles $\mathbf{a}_j(\cdot)$ in a sequence of games having finer and finer action spaces has a subsequence that converges to an IPS profile $\mathbf{a}_*(\cdot)$ in the limiting game having a continuum action space.[14] But it is straightforward to apply Helly's Selection Theorem to the sequences $\mathbf{a}_j^m(\cdot)$ separately, each of which has a subsequence converging to $\mathbf{a}_*^m(\cdot)$. □

Proof of Theorem 1

Athey map: The Athey map A_i sends each IPS $a_i(\cdot) \in \mathcal{I}_i$ to a vector,

$$\begin{aligned} A_i :& \mathcal{I}_i \to \mathcal{A}_i \\ & a_i(\cdot) \mapsto \left(A_i(a_i(\cdot); m, j, t_i^{-1})\right)^{m=1,\ldots,k, j\in L_m, t_i^{-1}\in[0,1]^{h-1}} \end{aligned}$$

where

$$A_i(a_i(\cdot); m, j, t_i^{-1}) \equiv \sup\{t_i^1 \in [0,1] : a_i^m(t_i^1, t_i^{-1}) < j\}.$$

($a_i^m(\cdot) : [0,1]^h \to L_m$ was defined on page 15.) To avoid a notational mess, I will often refer to a typical element in the range of the Athey map, $\left(A_i(a_i(\cdot); m, j, t_i^{-1})\right)^{m=1,\ldots,k, j\in L_m, t_i^{-1}\in[0,1]^{h-1}}$, simply as $A_i(a_i(\cdot))$ or even more simply as A_i if there can be no confusion with the map itself.

Bijection between equivalence classes: Two strategies $a_i'(\cdot), a_i(\cdot)$ are *equivalent* if and only if $\Pr_{t_i}(a_i'(t_i) = a_i(t_i)) = 1$. Two vectors $A_i(a_i'(\cdot)), A_i(a_i(\cdot))$ in the range of the Athey map are *equivalent* if and only if

$$(3) \qquad E_{t_i^{-1}}\left[\max_{m, j\in L_m} \left|A_i(a_i'(\cdot); m, j, t_i^{-1}) - A_i(a_i(\cdot); m, j, t_i^{-1})\right|\right] = 0.$$

The Athey map induces a bijection between equivalence classes of IPS in $\mathcal{I}_i$ and equivalence classes of vectors in $\mathcal{A}_i$. (When I refer to "the Athey map" from here on I mean to refer to this induced bijection although for simplicity I will use notation as if the domain is $\mathcal{I}_i$ and the range $\mathcal{A}_i$). To see why, note first by the model's assumptions on the distribution of types there exist $0 < \underline{K} \leq \overline{K}$ so that $f_i(t_i^1|t_i^{-1}) \in [\underline{K}, \overline{K}]$ for all t_i^1, t_i^{-1}. Thus,

$$\begin{aligned} \left|A_i(a_i'(\cdot); m, j, t_i^{-1}) - A_i(a_i(\cdot); m, j, t_i^{-1})\right| &= \triangle \\ \Rightarrow \Pr_{t_i^1|t_i^{-1}} \left(a_i^{m'}(t_i^1, t_i^{-1}) \neq a_i^m(t_i^1, t_i^{-1})\right) &\in [\triangle\underline{K}, \triangle\overline{K}] \end{aligned}$$

for all t_i^{-1}. Consequently, (3) holds if and only if $\Pr_{t_i}(a_i'(t_i) = a_i(t_i)) = 1$.

Homeomorphism: Indeed, $\Pr_{t_i}(a_{i,n}(t_i) = a_{i,*}(t_i)) \to_{n\to\infty} 1$ if and only if

$$E_{t_i^{-1}}\left[\max_{m,j\in L_m}\left|A_i(a_{i,n}(\cdot); m, j, t_i^{-1}) - A_i(a_{i,*}(\cdot); m, j, t_i^{-1})\right|\right] \to_{n\to\infty} 0.$$

Thus, the Athey map is a homeomorphism with respect to the topologies on equivalence classes in $\mathcal{I}_i$ having metric $d(a_i'(\cdot), a_i(\cdot)) \equiv \Pr_{t_i}(a_i'(t_i) \neq a_i(t_i))$ and on equivalence classes in $\mathcal{A}_i \equiv A_i(\mathcal{I}_i)$ having metric

$$d(A_i', A_i) \equiv E_{t_i^{-1}}\left[\max_{m,j\in L_m}\left|A_i(a_{i,n}(\cdot); m, j, t_i^{-1}) - A_i(a_{i,*}(\cdot); m, j, t_i^{-1})\right|\right].$$

Closed Range: By homeomorphism, it suffices to show that any limit point of $\mathcal{I}_i$ is an element of $\mathcal{I}_i$. So suppose that $\{a_{i,n}(\cdot)\}$ is a sequence of IPS converging to $a_{i,*}(\cdot)$. Clearly, $a_{i,*}(\cdot)$ is isotone when restricted to the set of types at which it prescribes exactly the same action as $a_{i,n}(\cdot)$ (for all $n > N^*$ and any N^*). By convergence of $\{a_{i,n}(\cdot)\}$, then, $a_{i,*}(\cdot)$ must be isotone when restricted to some full measure set of types. And any such strategy can be modified on a zero measure set so that it becomes an IPS, i.e. the equivalence class containing $a_{i,*}(\cdot)$ includes an IPS.

Compact Range: By Tychonoff's Theorem,[15] closedness implies that the range is compact *with respect to the topology of pointwise convergence.* But the topology that I am using is coarser than this one, so the range must be compact with respect to my topology as well.[16]

Convex Range: Lemma 1 below characterizes the range of the Athey map. This range is convex since (4, 5) are preserved under convex combination. That is to say, if $A_i(a_i'(\cdot)), A_i(a_i(\cdot))$ each satisfy (4, 5), then so does $\alpha A_i(a_i'(\cdot)) + (1-\alpha)A_i(a_i(\cdot))$ for all $\alpha \in (0, 1)$.

Lemma 1: *$a_i(\cdot)$ is an isotone strategy if and only if*

(4) $A_i(a_i(\cdot); m, j', t_i^{-1}) \geq A_i(a_i(\cdot); m, j, t_i^{-1})$ *for all* $m, j' > j \in L_m, t_i^{-1}$

(5) $A_i(a_i(\cdot); m, j, t_i^{-1\prime}) \leq A_i(a_i(\cdot); m, j, t_i^{-1})$ *for all* $m, j \in L_m, t_i^{-1\prime} > t_i^{-1}$.

Proof. $a_i(\cdot)$ isotone if and only if $a_i^m(\cdot)$ non-decreasing for $m = 1, ..., k$.

$\Rightarrow$*:* Suppose $a_i^m(\cdot)$ non-decreasing. Then (4, 5) hold for all $j' > j \in L_m$ and all t_i^{-1} To prove (4), suppose otherwise that

$$A_i(a_i(\cdot); m, j', t_i^{-1}) < t_i^1 < A_i(a_i(\cdot); m, j, t_i^{-1}).$$

In this case, $a_i^m\left(t_i^1, t_i^{-1}\right) \geq j' > j > a_i^m\left(t_i^1, t_i^{-1}\right)$ by definition of the Athey map, a contradiction. To prove (5), suppose otherwise that

$$A_i(a_i(\cdot); m, j, t_i^{-1}) < t_i^1 < A_i(a_i(\cdot); m, j, t_i^{-1\prime}).$$

In this case, similarly, $a_i^m\left(t_i^1, t_i^{-1}\right) \geq a^m > a_i^m\left(t_i^1, t_i^{-1\prime}\right)$. But $a_i^m\left(t_i^1, t_i^{-1\prime}\right) \geq a_i^m\left(t_i^1, t_i^{-1}\right)$ since $a_i^m(\cdot)$ is non-decreasing, a contradiction.

$\Leftarrow$*:* Suppose that vector A_i satisfies (4, 5). Then $a_i(\cdot) \in \mathcal{I}_i$ exists so that $A_i = A_i\left(a_i(\cdot)\right)$. Consider the pure strategy $a_i(\cdot)$ defined as

$$a_i^m\left(\mathbf{t}\right) \equiv \max\left\{j \in L_{i,m} : A_i(m, j, t_i^{-1}) \leq t_i^1\right\}$$

for each $m = 1, ..., k$. It is easy to verify that $A_i = A_i(a_i(\cdot))$ when $a_i(\cdot)$ is so defined. Each such $a_i^m(\cdot)$ is non-decreasing: Let $\tilde{t}_i^1 \geq t_i^1$ and $\tilde{t}_i^{-1} \geq t_i^{-1}$ and suppose otherwise first that $a_i^m\left(\tilde{t}_i^1, t_i^{-1}\right) = j > a_i^m\left(\tilde{t}_i^1, \tilde{t}_i^{-1}\right)$. By construction of $a_i^m(\cdot)$, $a_i^m\left(\tilde{t}_i^1, \tilde{t}_i^{-1}\right) < j$ so that $A_i(m, j, \tilde{t}_i^{-1}) > \tilde{t}_i^1$ and $a_i^m\left(\tilde{t}_i^1, t_i^{-1}\right) = j$ so that $A_i(m, j, t_i^{-1}) \leq \tilde{t}_i^1$, contradicting (4). Similarly, $A_i(m, j, t_i^{-1}) \leq t_i^1$ and $A_i(m, j, \tilde{t}_i^{-1}) > \tilde{t}_i^1$ when $a_i^m\left(t_i^1, t_i^{-1}\right) = j > a_i^m\left(\tilde{t}_i^1, t_i^{-1}\right)$, contradicting (5). □

Closed Graph: Let $A_{-i} : \mathcal{I}_{-i} \rightarrow \mathcal{A}_{-i}$ and $\mathbf{A} : \mathcal{I} \rightarrow \mathcal{A}$ denote the composite Athey map taking IPS profiles into vector profiles. For each IPS profile $a_{-i}(\cdot)$, let $BR_i^{\geq}(a_{-i}(\cdot))$ denote the set of player i's best response IPS:

$$a_i(\cdot) \in BR_i^{\geq}(a_{-i}(\cdot)) \Leftrightarrow a_i(\cdot) \in \mathcal{I}_i \text{ and } a_i(t_i) \in BR_i(t_i, a_{-i}(\cdot)) \text{ for all } t_i.$$

Recall from Section 6.1 that $BR_i^{\geq}$ is non-empty-valued. Furthermore, note that player i's interim expected payoffs are continuous in $a_{-i}(\cdot)$ with respect to our topology. This guarantees that $BR_i^{\geq}$ has a closed graph.

Fixed Point: Define $\Lambda_i \equiv A_i \cdot BR_i^{\geq} \cdot A_{-i}^{-1} : \mathcal{A}_{-i} \to \mathcal{A}_i$ which maps the vector $A_{-i}(a_{-i}(\cdot))$ corresponding to a profile of others' IPS into the set of vectors corresponding to player i's IPS best responses. Finally, define $\Lambda : \mathcal{A} \to \mathcal{A}$ by

$$\Lambda\left(\mathbf{A}(\mathbf{a}(\cdot))\right) = \left(\Lambda_1(A_{-1}(a_{-1}(\cdot))), ..., \Lambda_n(A_{-n}(a_{-n}(\cdot)))\right).$$

The arguments presented so far imply that (i) $\mathcal{A}$ is a convex, compact subset of a convex topological linear space (indeed, of a vector space) and (ii) Λ is non-empty-valued (iii) with a closed graph. All that remains to be able to invoke Glicksberg Fixed Point Theorem is that Λ (or each Λ_i) is convex-valued. This is the most important technical result in the paper, so I label it as a theorem.

THEOREM 3: Λ_i *is convex-valued for all players* i.

IPSE exists: To complete the proof of Theorem 1 given Theorem 3, I need to show that the equivalence class of strategies corresponding to any fixed point of Λ contains an IPSE. Suppose that $\mathbf{A}(\mathbf{a}^*(\cdot))$ is a fixed point of Λ (for $\mathbf{a}^*(\cdot) \in \mathcal{I}$). This does *not* imply that $\mathbf{a}^*(\cdot)$ is an IPSE since some zero measure set of types may be playing a non-best response. But $\mathbf{a}^*(\cdot) \equiv \widehat{\mathbf{a}}(\cdot)$ for some profile $\widehat{\mathbf{a}}(\cdot)$ such that $\widehat{a}_i(\cdot) \in BR_i^{\geq}(a_{-i}^*(\cdot))$ for all i and $\widehat{\mathbf{a}}(\cdot)$ is an IPSE: Player i always plays a best response under $\widehat{a}_i(\cdot)$ and all others' best responses do not change as i modifies his action over a zero measure set of types.

Proof of Theorem 2

Proof. By the discussion in the text, it suffices to show that each bidder's ex post valuation has NDD in own bid and own type and both his ex post valuation and ex post payment are modular in own bid. In the following, I consider bidder 1 only and fix the profile of others' bids $b_{-1}(\cdot)$, the rationing ranking ρ, and the state $\mathbf{t}$. The analysis focuses on properties of the realized allocation and payment when bidder 1 submits one of two bids $b^1(\cdot)$ or $b^2(\cdot)$ or their join $b^{1\vee 2}(\cdot) \equiv b^1(\cdot) \vee b^2(\cdot)$ or meet $b^{1\wedge 2}(\cdot) \equiv b^1(\cdot) \wedge b^2(\cdot)$.

Shorthand notation: $q_j^1 \equiv q_j^*\left(b^1(\cdot), b_{-1}(\cdot); \rho\right)$ and so on for the other bids $b^2(\cdot)$, $b^{1\vee 2}(\cdot)$, and $b^{1\wedge 2}(\cdot)$. (Note that while bidder 1's bid varies, others' bids are held fixed.) Similarly, define $\overline{p}^1 \equiv b^S\left(b^1(\cdot), b_{-1}(\cdot)\right)$, $\underline{p}^1 \equiv b^{S+1}\left(b^1(\cdot), b_{-1}(\cdot)\right)$ and so on, where $b^S(\mathbf{b}(\cdot))$ and $b^{S+1}(\mathbf{b}(\cdot))$ are the Sth and $S+1$st highest unit bids given the profile of schedules $\mathbf{b}(\cdot)$. Lastly, define bidder j's "range of demand" at each price p by $\min D_j(p) = \max\{q : b_j(q) > p\}$ and $\max D_j(p) = \max\{q : b_j(q) \geq p\}$. For bidder 1, I will use shorthand $D_1^1(p)$ to refer to his range of demand given bid $b^1(\cdot)$ and so on for the other bids $b^2(\cdot)$, $b^{1\wedge 2}(\cdot)$, and $b^{1\vee 2}(\cdot)$.

Characterizing the allocation: Define bidder 1's *rationing function* to be

$$R_1^\rho(p) \equiv S - \sum_{\rho(j)=1}^{\rho(j)=\rho(1)-1} \max D_j(p) - \sum_{\rho(j)=\rho(1)+1}^{\rho(j)=n} \min D_j(p).$$

$R_1^\rho(p)$ is the amount that would be left for bidder 1 if all ahead of him in the rationing ranking ρ were given their maximum demand at price p and all behind him were given their minimal demand at that price. By design of the assumed rationing rule,[17]

$$\begin{aligned} q_1^* &= \min D_1(b^S) \text{ if } R_1^\rho(b^S) \leq \min D_1(b^S) \\ &= R_1^\rho(b^S) \text{ if } R_1^\rho(b^S) \in \left[\min D_1(b^S), \max D_1(b^S)\right] \\ &= \max D_1(b^S) \text{ if } R_1^\rho(b^S) \geq \max D_1(b^S). \end{aligned}$$

or, *equivalently*,

$$\begin{aligned} q_1^* &= \min D_1(b^{S+1}) \text{ if } R_1^\rho(b^{S+1}) \leq \min D_1(b^{S+1}) \\ &= R_1^\rho(b^{S+1}) \text{ if } R_1^\rho(b^{S+1}) \in \left[\min D_1(b^{S+1}), \max D_1(b^{S+1})\right] \\ &= \max D_1(b^{S+1}) \text{ if } R_1^\rho(b^{S+1}) \geq \max D_1(b^{S+1}). \end{aligned}$$

Both approaches, based on b^S and on b^{S+1}, lead to the same rationing outcome because rationing only occurs when $b^S = b^{S+1}$: $b^{S+1}(\mathbf{b}(\cdot)) \neq b^S(\mathbf{b}(\cdot))$ implies that there is a unique market clearing allocation, and both approaches lead to that allocation. More explicitly,

(6)
$$q_1^*(\mathbf{b}(\cdot);\rho) = \max\left\{\min D_1(b^S(\mathbf{b}(\cdot))), \min\left\{R_1^\rho(b^S(\mathbf{b}(\cdot))), \max D_1(b^S(\mathbf{b}(\cdot)))\right\}\right\}$$
(7)
$$= \max\left\{\min D_1(b^{S+1}(\mathbf{b}(\cdot))), \min\left\{R_1^\rho(b^{S+1}\mathbf{b}(\cdot)), \max D_1(b^{S+1}(\mathbf{b}(\cdot)))\right\}\right\}.$$

NDD in own bid and own type: By (6), observe that $q_1^*(b_1(\cdot), b_{-1}(\cdot); \rho)$ is non-decreasing and $q_j^*(\mathbf{b}(\cdot); \rho)$ $(j \neq i)$ non-increasing in own bid $b_1(\cdot)$ for any given $b_{-1}(\cdot), \rho$. Thus, bidder i's ex post valuation for the allocation, $V_1(\mathbf{q}^*(\mathbf{b}(\cdot); \rho), \mathbf{t})$, has non-decreasing differences in own bid and type.

Modularity in own bid: Let $b^1(\cdot), b^2(\cdot)$ be two permissible bids. By definition of the $\vee, \wedge$ operations, $\min\{\overline{p}^1, \overline{p}^2\} = \overline{p}^{1\wedge 2}$ and $\max\{\overline{p}^1, \overline{p}^2\} = \overline{p}^{1\vee 2}$. Similarly, $\min\{\underline{p}^1, \underline{p}^2\} = \underline{p}^{1\wedge 2}$ and $\max\{\underline{p}^1, \underline{p}^2\} = \underline{p}^{1\vee 2}$. Without loss, suppose that $\overline{p}^1 = \overline{p}^{1\wedge 2} \leq \overline{p}^2 = \overline{p}^{1\vee 2}$ and $q_1^1 \leq q_1^2$. (It is straightforward to see that $\overline{p}^1 < \overline{p}^2$ implies $q_1^1 \leq q_1^2$.) By the discussion in the text (page 10) note that, to prove modularity of ex post valuations and ex post payments in the Sth price auction, it suffices to show that $q_1^1 = q_1^{1\wedge 2}$ and $q_1^2 = q_1^{1\vee 2}$.

For this result, first, note that $q_1^1 = q_1^2$ implies $q_1^1 = q_1^2 = q_1^{1\wedge 2} = q_1^{1\vee 2}$: By (6), $q_1^1 = q_1^2$ if and only if either (A), (B), or (C) is satisfied.

$$
\begin{aligned}
&\text{(A)} \quad \max\{\min D^1(\overline{p}^1), \min D^2(\overline{p}^2)\} \leq R_1^\rho(b_{-1}(\cdot)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq \min\{\max D^1(\overline{p}^1), \max D^2(\overline{p}^2)\}, \\
&\text{(B)} \quad \max D^1(\overline{p}^1) = \max D^2(\overline{p}^2) < R_1^\rho(b_{-1}(\cdot)), \\
&\text{(C)} \quad \min D^1(\overline{p}^1) = \min D^2(\overline{p}^2)\} > R_1^\rho(b_{-1}(\cdot)).
\end{aligned}
$$

Further, if condition (A) holds for bids $b^1(\cdot), b^2(\cdot)$, then the analogous condition must hold for bids $b^{1\wedge 2}(\cdot), b^{1\vee 2}(\cdot)$, and similarly for conditions (B,C). So, without loss, suppose that $q_1^1 < q_1^2$.

Second, $\min D^1(\overline{p}^1) = \min D^{1\wedge 2}(\overline{p}^1)$: Else $\min D^1(\overline{p}^1) > \min D^{1\wedge 2}(\overline{p}^1)$, so that there exists some $q < \min D^1(\overline{p}^1) \leq q_1^1$ such that $b^2(q) < b^1(q)$. But this would imply $q_1^2 \leq q_1^1$, a contradiction.

Third, either both $\max D^1(\overline{p}^1), \max D^{1\wedge 2}(\overline{p}^1) \leq R_1^\rho(\overline{p}^1)$ or both $\geq R_1^\rho(\overline{p}^1)$: Else $\max D^1(\overline{p}^1) > R_1^\rho(\overline{p}^1) > \max D^{1\wedge 2}(\overline{p}^1)$, so that there exists $q < R_1^\rho(\overline{p}^1) \leq q_1^1$ such that $b^2(q) < \overline{p}^1$. But this would imply that $\max D^2(\overline{p}^1) \leq R_1^\rho(\overline{p}^1)$ implying (since $\overline{p}^2 \geq \overline{p}^1$) that $q_1^2 \leq q_1^1$, a contradiction.

All together, by (6), I have proven now that $q_1^1 = q_1^{1\wedge 2}$. The proof that $q_1^2 = q_1^{1\vee 2}$ is entirely analogous, the key steps being to show that $\max D^2(\overline{p}^2) = \max D^{1\vee 2}(\overline{p}^2)$ and that $\min D^2(\overline{p}^2), \min D^{1\vee 2}(\overline{p}^2) \leq R_1^\rho(\overline{p}^2)$ or both $\geq R_1^\rho(\overline{p}^2)$. This completes the proof for the Sth price auction. The proof for the $S+1$st price auction is entirely analogous, when $\overline{p}$ is replaced with $\underline{p}$. $\square$

Proof of Theorem 3

Preliminaries: The type space has partition $T = \{C(t_i^{-1})\}^{t_i^{-1}\in[0,1]^{h-1}}$, where $C(t_i^{-1}) \equiv [0,1] \times \{t_i^{-1}\}$. Let $a_i(\cdot;\alpha)$ be any isotone strategy in the equivalence class in the pre-image of $\alpha A_i(a_i'(\cdot)) + (1-\alpha)A_i(a_i(\cdot))$ with respect to the Athey map. All equivalent strategies specify the same action for all but the zero measure set of types $\mathcal{D}_i$ at which player i's action increases along some dimension in strategy $a_i(\cdot;\alpha)$:

$$\mathcal{D}_i \equiv \{t_i : t_i^1 = A_i(a_i(\cdot;\alpha); m, j, t_i^{-1}) \text{ for some } m, j \in L_m\}.$$

I need to prove only that, for all types $t_i \notin \mathcal{D}_i$, $a_i(t_i;\alpha)$ is a best response given that both $a_i'(\cdot), a_i(\cdot)$ are isotone best response strategies. What I will show is even stronger: $a_i(t_i;\alpha)$ is a best response given only that the actions played by types in $C(t_i^{-1})$ in strategies $a_i'(\cdot), a_i(\cdot)$ are all best response actions. *Without loss, then, I may focus entirely on the one dimensional set of types* $C(t_i^{-1})$ and, indeed, drop all reference to t_i^{-1}. Thus, subsequently, I will treat the notationally simpler case in which $T = [0,1]$: I will drop all superscripts, and any reference to the full set of player i's types refers instead to the one dimensional subset $C(t_i^{-1})$. For a given type $\hat{t}_i \notin \mathcal{D}_i$, define the shorthand $a_i(\hat{t}_i;\alpha) \equiv a(\alpha) = \left(a^1(\alpha), ..., a^k(\alpha)\right)$ and $a^{j_1,...,j_2}(\alpha) \equiv (a^{j_1}(\alpha), a^{j_1+1}(\alpha), ..., a^{j_2}(\alpha))$. (Subscripts denoting player identity are dropped when referring to actions for simplicity. This should not cause confusion since I only refer to player i throughout the entire proof.)

Part 1: In this part of the proof, I identify structure on bidder i's best response actions $BR_i(\cdot)$ that suffices for the convexity conclusion. (This structure is laid out here as Working Assumptions.) The second part then proves that this structure is present as long as the conditions of the Monotonicity Theorem are satisfied.

Working Assumption 1: *Player i's type* $\hat{t}_i$ *has best response actions* $\overline{a}, \underline{a}$ *such that* $\overline{a} \geq a(\alpha) \geq \underline{a}$.

Working Assumption 2: *For each dimension* $j = 1, ..., k$ *of the action space, there exist types* $\underline{t}_i^j, \overline{t}_i^j$ *such that (i)* $\overline{t}_i^j \geq \hat{t}_i \geq \underline{t}_i^j$, *(ii) type* $\overline{t}_i^j$ *has a best response action* $\grave{a}$ *such that* $\grave{a}^{1,...,j} \geq a^{1,...,j}(\alpha)$, $\grave{a}^{j+1} = a^{j+1}(\alpha)$, *and* $\grave{a}^{j+2,...,k} \leq a^{j+2,...,k}(\alpha)$, *and (iii) type* $\underline{t}_i^j$ *has a best response action* $\breve{a}$ *such that* $\breve{a}^{1,...,j} \leq a^{1,...,j}(\alpha)$, $\breve{a}^{j+1} = a^{j+1}(\alpha)$, *and* $\breve{a}^{j+2,...,k} \geq a^{j+2,...,k}(\alpha)$.

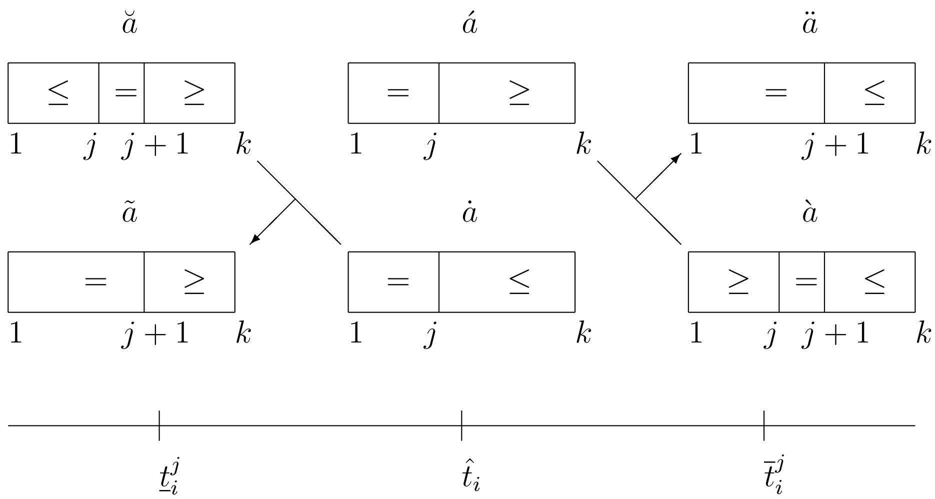

Figure 4: Illustration of induction step

Given these two working assumptions, an induction argument proves that $a(\alpha)$ is a best response action for player i's type $\hat{t}_i$, i.e. $a(\alpha) \in BR_i(\hat{t}_i)$.

> *Base step ($j = 0$):* $\overline{a}, \underline{a} \in BR_i\left(\hat{t}_i\right)$, where $\overline{a} \geq a_i(\alpha) \geq \underline{a}$.
>
> *Induction step:* Suppose $\dot{a}$, $\acute{a} \in BR_i\left(\hat{t}_i\right)$, where $\dot{a}^m = a^m(\alpha) = \acute{a}^m$ for $m = 1, .., j$ and $\dot{a}^m \leq a^m(\alpha) \leq \acute{a}^m$ for $m = j+1, .., k$. Then we may conclude $\ddot{a}, \tilde{a} \in BR_i\left(\hat{t}_i\right)$, where $\ddot{a}^m = a^m(\alpha) = \tilde{a}^m$ for $m = 1, ..., j+1$ and $\ddot{a}^m \leq a^m(\alpha) \leq \tilde{a}^m$ for $m = j+2, .., k$.

Base step is satisfied by Working Assumption 1. By Working Assumption 2 and the fact that $BR_i(\cdot)$ is increasing in the strong set order, $\ddot{a} \equiv \grave{a} \wedge \acute{a} \in BR_i(\hat{t}_i)$ and $\tilde{a} \equiv \breve{a} \vee \dot{a} \in BR_i(\hat{t}_i)$. (It is easily checked that $\ddot{a}^{1,...,j+1} = a^{1,...,j+1}(\alpha)$ and $\ddot{a}^{j+2,...,k} \leq a^{j+2,...,k}(\alpha)$ as well as that $\tilde{a}^{1,...,j+1} = a^{1,...,j+1}(\alpha)$ and $\tilde{a}^{j+2,...,k} \geq a^{j+2,...,k}(\alpha)$.) This notation heavy step is illustrated in Figure 4. The block from 1 to j is labelled $\leq$ in the $\breve{a}$ box to represent the fact that $\breve{a}^{1,...,j} \leq a^{1,...,j}(\alpha)$, and so on. The four actions $\tilde{a}, \acute{a}, \dot{a}, \ddot{a} \in BR_i(\hat{t}_i)$ whereas $\breve{a} \in BR_i(\underline{t}_i^j)$ and $\grave{a} \in BR_i(\overline{t}_i^j)$. This completes the induction step and hence the proof of Theorem 3 given the two Working Assumptions.

Part 2: Now I prove that Working Assumptions 1, 2 are satisfied given that $BR_i(\cdot)$ is non-empty- and lattice-valued and increasing in the strong set order. First, I develop some needed machinery that applies to any fixed dimension $m \in \{1,...,k\}$ of the action space. Let

$$BR_i^m(t_i) \equiv \left\{a^m \in L_m : \left(a^m, a^{-m}\right) \in BR_i(t_i) \text{ for some } a^{-m} \in L_{-m}\right\}.$$

First point: revealed preference. Given that $a_i(\cdot)$, $a_i'(\cdot)$ are best response strategies, revealed preference implies that $a^m(\alpha) \in BR_i^m(t_i)$ for all types t_i who play an action with $a^m(\alpha)$ as its mth coordinate in either strategy. This includes all types $t_i \in int\left(S^{a^m(\alpha)}(a_i(\cdot))\right) \cup int\left(S^{a^m(\alpha)}(a_i'(\cdot))\right)$ where

$$\begin{aligned} S^{a^m(\alpha)}(a_i(\cdot)) &\equiv [A_i(a_i(\cdot); m, a^m(\alpha)), A_i(a_i(\cdot); m, a^m(\alpha)+1)] \\ S^{a^m(\alpha)}(a_i'(\cdot)) &\equiv [A_i(a_i'(\cdot); m, a^m(\alpha)), A_i(a_i'(\cdot); m, a^m(\alpha)+1)]. \end{aligned}$$

$S^{a^m(\alpha)}(a_i(\cdot))$ is the closure of the order interval of types who play an action with mth coordinate $a^m(\alpha)$ in the strategy $a_i(\cdot)$. Similarly, $S^{a^m(\alpha)}(a_i'(\cdot))$ contains types who play an action with mth coordinate $a^m(\alpha)$ in the strategy $a_i'(\cdot)$. Define shorthand

$$\begin{aligned} H^m &\equiv \left[\hat{t}_i, 1\right] \cap \left(S^{a^m(\alpha)}(a_i'(\cdot)) \cup S^{a^m(\alpha)}(a_i(\cdot))\right) \\ L^m &\equiv \left[0, \hat{t}_i\right] \cap \left(S^{a^m(\alpha)}(a_i'(\cdot)) \cup S^{a^m(\alpha)}(a_i(\cdot))\right). \end{aligned}$$

H^m (L^m) is mnemonic for "types that are Higher (Lower) than $\hat{t}_i$ that play an action equal to $a^m(\alpha)$ on the mth dimension in either strategy $a_i'(\cdot)$ or $a_i(\cdot)$". (L^m should not be confused with the action lattice $L = \Pi_{m=1}^k L_m$.) Note that these sets are closed and that all types t_i in the interior of $H^m \cup L^m$ have a best response action whose mth coordinate equals $a^m(\alpha)$.

Second point: reduce to 1/2-1/2 *convex combinations.* The set of types $t_i \notin \mathcal{D}_i$ such that $a_i(t_i, \alpha) = a^m(\alpha)$ is the interior of the interval

$$S^{a^m(\alpha)}(\hat{a}_i(\cdot;\alpha)) \equiv \alpha S^{a^m(\alpha)}(a_i(\cdot)) + (1-\alpha) S^{a^m(\alpha)}(a_i'(\cdot))$$

where this is the usual convex combination of sets. In particular, for any such type t_i, the action $a_i(t_i;\alpha) = a_i(t_i;\tilde{\alpha})$ for all $\tilde{\alpha}$ in a neighborhood of α. Thus, I only need to prove that $a_i(t_i;\alpha) \in BR_i(t_i)$ for α belonging to a dense subset of $[0,1]$. By an induction argument, therefore, it suffices to prove that $a_i(t_i; 1/2) \in BR_i(t_i)$ (i.e. for $\alpha = 1/2$).

Third point: some type has best response action whose mth coordinate equals $a^m(1/2)$. Since $\hat{t}_i \notin \mathcal{D}_i$, one of the intervals $S^{a^m(1/2)}(a_i'(\cdot))$, $S^{a^m(1/2)}(a_i(\cdot))$ must have non-empty interior. Thus, there must be some type t_i so that either $a_i^m(t_i) = a^m(1/2)$ or $a_i'^m(t_i) = a^m(1/2)$, implying that $a^m(1/2) \in BR_i^m(t_i)$.

Fourth point: $\triangle_m$ properties. Since $\hat{t}_i \notin \mathcal{D}_i$,

$$\hat{t}_i \in int\left(S^{a^m(\alpha)}(a_i(\cdot;1/2))\right)$$

where $S^{a^m(1/2)}(a_i(\cdot;1/2))$ was defined in the first point. Thus,

$$W \equiv H^m \cap \left(2\hat{t}_i - L^m\right)$$

also has non-empty interior. Define

$$\triangle_m \equiv \max W - \hat{t}_i.$$

In words, $\triangle_m$ is the maximum length y such that $\hat{t}_i - y \in L^m$ and $\hat{t}_i + y \in H^m$. Key properties of $\triangle_m$ include:

1. *$\triangle_m > 0$:* Follows from the fact that W has non-empty interior and $\min W \geq \hat{t}_i$. (This fact will be used in Part 2 when I argue that types $\hat{t}_i - \triangle_m + \varepsilon$ and $\hat{t}_i + \triangle_m - \varepsilon$ have a best response action with mth coordinate equal to $a^m(1/2)$.)

2. *It can not be that both $\hat{t}_i - \triangle_m = \max L^m$ and $\hat{t}_i + \triangle_m = \min H^m$:* Otherwise, by definition of $\triangle_m$, one of the sets L^m, H^m must be a singleton and $\hat{t}_i \notin int\left(S^{a^m(\alpha)}(a_i(\cdot;1/2))\right)$, a contradiction. (For example, if $L^m = \{\widehat{t_i} - \triangle_m\}$ and $\hat{t}_i + \triangle_m = \min H^m$, then $\hat{t}_i = \min S^{a^m(\alpha)}(a_i(\cdot;1/2))$.)

3. $\max\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \leq a^m(1/2)$ *for all $t_i < \hat{t}_i - \triangle_m$:* This and the next facts follow immediately from the definition of $\triangle_m$.

4. $\min\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \leq a^m(1/2)$ *for all $t_i < \hat{t}_i + \triangle_m$.*

5. $\max\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \geq a^m(1/2)$ *for all $t_i > \hat{t}_i - \triangle_m$.*

6. $\min\left\{a_i^{m'}(t_i), a_i^m(t_i)\right\} \geq a^m(1/2)$ *for all $t_i > \hat{t}_i + \triangle_m$.*

By property 2, *either $\hat{t}_i - \triangle_m + \varepsilon \in L^m$ or $\hat{t}_i + \triangle_m - \varepsilon \in H^m$ for small enough ε.* This implies that *either*

$$\max\{a_i^m\left(\hat{t}_i - \triangle_m + \varepsilon\right), a_i^{'m}\left(\hat{t}_i - \triangle_m + \varepsilon\right)\} = a^m(1/2) \text{ or}$$
$$\min\{a_i^m\left(\hat{t}_i + \triangle_m - \varepsilon\right), a_i^{'m}\left(\hat{t}_i + \triangle_m - \varepsilon\right)\} = a^m(1/2).$$

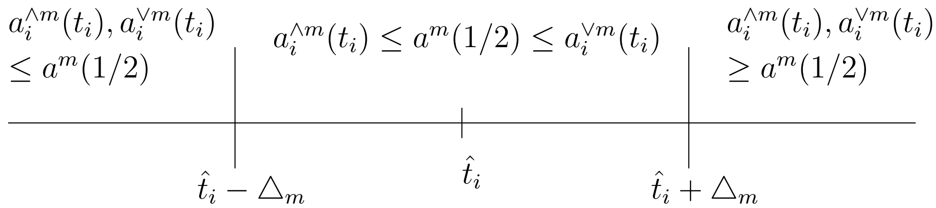

Figure 5: Illustration of properties of $a_i^{\vee}(\cdot), a_i^{\wedge}(\cdot)$, and $\triangle_m$

Fifth point: properties of meet and join strategies $a_i^{\wedge}(\cdot)$, $a_i^{\vee}(\cdot)$. Relabel the dimensions of player i actions so that $m_1 \geq m_2 \Leftrightarrow \triangle_{m_1} \geq \triangle_{m_2}$. Now, for each non-negative $z \notin \{\triangle_1, ..., \triangle_m\}$, note that there exists $m(z) \in \{1, ..., k\}$ such that

$$\triangle_m < z \text{ for all } m \leq m(z), \triangle_m > z \text{ for all } m > m(z).$$

For each such z, consider the actions

$$\begin{aligned} a_i^{\vee}\left(\hat{t}_i - z\right) &\equiv a_i'\left(\hat{t}_i - z\right) \vee a_i\left(\hat{t}_i - z\right) \\ a_i^{\wedge}\left(\hat{t}_i + z\right) &\equiv a_i'\left(\hat{t}_i + z\right) \wedge a_i\left(\hat{t}_i + z\right). \end{aligned}$$

Note that $a_i^{\vee}(\cdot)$ is defined only over the lower type range $[0, \hat{t}_i] \backslash \left\{\hat{t}_i - \triangle_1, ..., \hat{t}_i - \triangle_k\right\}$ whereas $a_i^{\wedge}(\cdot)$ is defined only over the higher type range $[\hat{t}_i, 1] \backslash \left\{\hat{t}_i + \triangle_1, ..., \hat{t}_i + \triangle_k\right\}$. $a_i^{\vee}\left(\hat{t}_i - z\right) \in BR_i\left(\hat{t}_i - z\right)$ and $a_i^{\wedge}\left(\hat{t}_i + z\right) \in BR_i\left(\hat{t}_i + z\right)$ since the set of best response actions is a lattice. Furthermore, by the fourth point,

$$\begin{aligned} a_i^{m\vee}\left(\hat{t}_i - \triangle_m - \varepsilon\right) \leq a^m(1/2), a_i^{m\vee}\left(\hat{t}_i - \triangle_m + \varepsilon\right) \geq a^m(1/2) \\ a_i^{m\wedge}\left(\hat{t}_i + \triangle_m - \varepsilon\right) \leq a^m(1/2), a_i^{m\wedge}\left(\hat{t}_i + \triangle_m + \varepsilon\right) \geq a^m(1/2) \end{aligned}$$

for all $\varepsilon > 0$. Since $m > m(z)$ implies that $\triangle_m > z$, $\hat{t}_i - \triangle_m < \hat{t}_i - z$ and $\hat{t}_i + \triangle_m > \hat{t}_i - z$ and therefore

$$\begin{aligned} a_i^{m\vee}\left(\hat{t}_i - z\right) &\geq a^m(1/2) \text{ for all } m > m(z) \\ a_i^{m\wedge}\left(\hat{t}_i + z\right) &\leq a^m(1/2) \text{ for all } m > m(z). \end{aligned}$$

Similarly, since $m \leq m(z)$ implies $\triangle_m < z$,

$$\begin{aligned} a_i^{m\vee}\left(\hat{t}_i - z\right) &\leq a^m(1/2) \text{ for all } m \leq m(z) \\ a_i^{m\wedge}\left(\hat{t}_i + z\right) &\geq a^m(1/2) \text{ for all } m \leq m(z). \end{aligned}$$

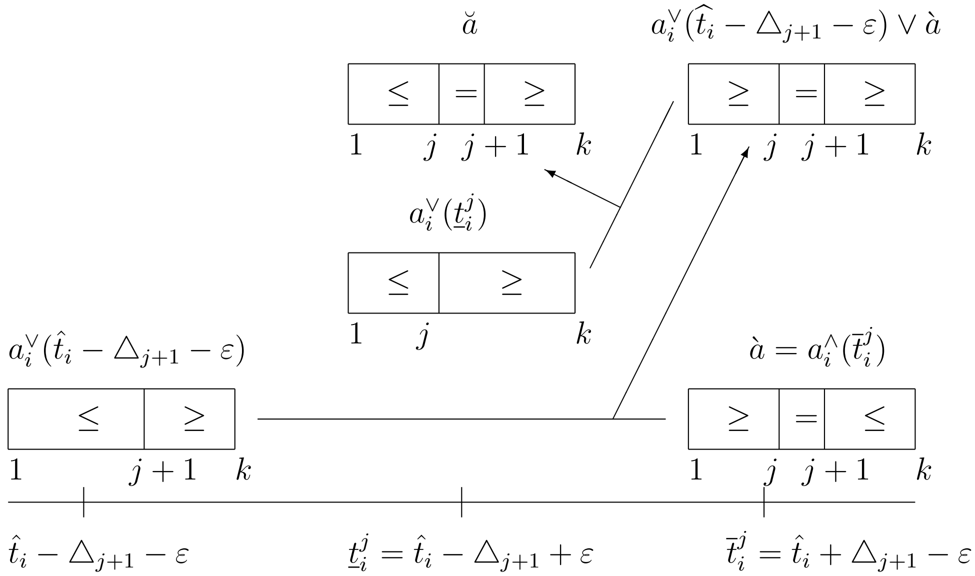

Figure 6: Illustration of Working Assumption 2 verification given (8)

Verify Working Assumption 1: By definition of $a_i^{\vee}(\cdot), a_i^{\wedge}(\cdot), a_i^{\vee}(\underline{z}) \geq a(1/2) \geq a_i^{\wedge}(\overline{z})$ for all $\underline{z} \in (\hat{t}_i - \triangle_1, \hat{t}_i)$ and all $\overline{z} \in (\hat{t}_i, \hat{t}_i + \triangle_1)$. Let $a \in BR_i(\hat{t}_i)$. By increasingness in the strong set order, $\overline{a}, \underline{a} \in BR_i(\hat{t}_i)$ where $\overline{a} \equiv a \vee a_i^{\vee}(\underline{z})$ and $\underline{a} \equiv a \wedge a_i^{\wedge}(\overline{z})$.

Verify Working Assumption 2: For each $j = 1, ..., k$, consider the four actions $a_i^{\vee}\left(\hat{t}_i - \triangle_{j+1} \pm \varepsilon\right)$ and $a_i^{\wedge}\left(\hat{t}_i + \triangle_{j+1} \pm \varepsilon\right)$. In the fourth point above, I proved that *either*

$$(8) \qquad a_i^{j+1\wedge}\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right) = a^{j+1}(1/2) \textit{ or}$$

$$(9) \qquad a_i^{j+1\vee}\left(\hat{t}_i - \triangle_{j+1} + \varepsilon\right) = a^{j+1}(1/2).$$

Suppose (8) holds. In this case, we may set $\grave{a} \equiv a_i^{\wedge j+1}\left(\hat{t}_i + \triangle_{j+1} - \varepsilon\right)$ and $\overline{t}_i^j \equiv \hat{t}_i + \triangle_{j+1} - \varepsilon$. So defined, $\grave{a} \in BR_i(\overline{t}_i^j)$ and $\grave{a}^{1,...,j} \geq a^{1,...,j}(\alpha)$, $\grave{a}^{j+1} = a^{j+1}(\alpha)$, and $\grave{a}^{j+2,...,k} \leq a^{j+2,...,k}(\alpha)$. Finally, set $\underline{t}_i^j \equiv \hat{t}_i - \triangle_{j+1} + \varepsilon$ and

$$\breve{a} \equiv \left(a_i^{\vee}\left(\hat{t}_i - \triangle_{j+1} - \varepsilon\right) \vee \grave{a}\right) \wedge a_i^{\vee}\left(\underline{t}_i^j\right).$$

So defined, $\breve{a} \in BR_i(\underline{t}_i^j)$ by repeated application of increasingness in the strong set order and, as can easily be checked, $\breve{a}^{1,...,j} \leq a^{1,...,j}(\alpha)$, $\breve{a}^{j+1} = a^{j+1}(\alpha)$, and $\breve{a}^{j+2,...,k} \geq a^{j+2,...,k}(\alpha)$. This step is illustrated in Figure 6. (Each diagram representing an action is placed above the type that finds that action to be a best response.) The argument when (9) holds is entirely symmetric. This completes the verification of Working Assumption 2 and hence the proof of Theorem 3. □

References

Athey, S. (1998): "Comparative Statics Under Uncertainty: Single Crossing Properties and Log-Supermodularity," MIT Working Paper Number 96-22R, Available at www.stanford.edu/˜athey.

——— (2001): "Single Crossing Properties and the Existence of Pure-Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69(4), 861–889.

——— (2002): "Monotone Comparative Statics under Uncertainty," *Quarterly Journal of Economics*, 117(1), 187–223.

Birkhoff, G. (1967): *Lattice Theory*. American Mathematical Society Colloquium Publications, 3rd edn.

Bresky, M. (2000): "Equilibria in Multi-Unit Auctions," manuscript.

Glicksberg, I. (1952): "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points," *Proceedings of the American Mathematical Society*, 38, 170–174.

Jackson, M., and J. Swinkels (2001): "Existence of Equilibrium in Single and Double Private Value Auctions," manuscript, Available at http://masada.hss.caltech.edu/jacksonm.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1996): "How (Not) to Sell Nuclear Weapons," *American Economic Review*, 86(4), 814–29.

McAdams, D. (2002): "Monotone Equilibrium in Multi-Unit Auctions," MIT Sloan Working Paper #4254-02, Available at http://www.mit.edu/˜mcadams.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 51, 981–994.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

Reed, M., and B. Simon (1980): *Functional Analysis*, vol. I. Academic Press, revised and enlarged edn.

Reny, P. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

Topkis, D. (1979): "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM Journal of Control and Optimization*, 17, 773–787.

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305–321.

Notes

[1]I am grateful to Susan Athey, Eddie Dekel, John McMillan, Barry Nalebuff, Robert Wilson, and two anonymous referees for helpful suggestions, and most especially to Paul Milgrom for introducing me to lattice theory. This research has been supported by the John Olin Foundation through a grant to the Stanford Institute for Economic Policy Research, as well as by the State Farm Companies Foundation. Email: mcadams@mit.edu. Post: MIT Sloan School of Management, E52-448, 50 Memorial Drive, Cambridge MA 02142

[2]The first condition implies that Assumption 4 is satisfied, the second Assumption 5.

[3] For simplicity, suppose further that all functions are smooth and that $\frac{\partial D}{\partial p} \leq 0$, $\frac{\partial D}{\partial \mathbf{e}} \geq 0$, $\frac{\partial c_i}{\partial t_i} \leq 0$, and $\frac{\partial \phi_i}{\partial t_i} \leq 0$. In particular, this implies that $\frac{\partial p}{\partial \mathbf{q}} \leq 0$ and $\frac{\partial p}{\partial \mathbf{e}} \geq 0$, hence that $\frac{\partial^2 \pi_i}{\partial q_i \partial t_i} \geq 0$ and $\frac{\partial^2 \pi_i}{\partial e_i \partial t_i} \geq 0$. Also, $\frac{\partial^2 \pi_i}{\partial e_i \partial q_i} \geq 0$ since advertising increases marginal revenue.

[4]Let $x' = (x'_1, ..., x'_k), x = (x_1, ..., x_k)$ be elements of $\mathcal{R}^k$. $x' \geq x$ in the *product order* if and only if $x'_m \geq x_m$ for all $m = 1, ..., k$. $x' > x$ if and only if $x' \geq x$ and $x' \neq x$. All results are easily generalizable to settings in which players have different action sets that may be of different dimensionality. Similarly, the assumption of a common type space is purely for simplicity.

[5]The assumption that density is bounded away from zero simplifies some arguments in the Appendix but does not appear to be essential.

[6]Reny proves existence of PSE in the as bid (or discriminatory) auction, but my understanding is that his paper does not prove PSE existence in other multi-unit auctions such as the uniform price auction in which payoffs are not diagonally quasiconcave. (See McAdams (2002).) Payoffs are trivially linear and hence diagonally quasiconcave when viewed as functions of distributional strategies, but then the result only implies existence of equilibrium in distributional strategies.

[7] In the two sided uniform price auction, when both buyers and sellers submit bids, the approach presented here could be used to prove that an IPSE exists given independent types and non-private values, but this could just be a no trade equilibrium.

[8]This rationing rule is a special case of the "randomized rationing rule" described in McAdams (2002) that applies for any supply correspondence.

[9]The same analysis goes through, however, if the permissible bid set is any sublattice of the set of non-increasing bid schedules. This allows for differing minimal bids on different quantities as well as making undifferentiated product Bertrand and Cournot competition special cases.

[10]Modularity implies quasisupermodularity and non-decreasing differences of expected payoff implies single-crossing of incremental expected payoff.

[11] If $f(\cdot,\cdot,\theta)$ has NDD for all $\theta \in \Theta$ and λ is a finite measure over Θ, then $\int_{\theta\in\Theta} f(\cdot,\cdot,\theta)d\lambda(\theta)$ has NDD.

[12] MS have a stronger "if and only if" formulation that also accounts for how the $\arg\max_x g(x,t)$ set varies with a constraint $S \subset L$. I do not leverage this aspect of their result, since each player's action set is fixed in my work.

[13]The metric on others' strategy profiles is $|a'_{-i}(\cdot) - a_{-i}(\cdot)| = \sum_{j\neq i} |a'_j(\cdot) - a_j(\cdot)|$

[14]For this limiting argument, the relevant topology on the space of isotone strategies is the topology of pointwise convergence inherited from the usual Euclidean topologies on the type space $[0,1]^k$ and the limiting action space $[0,1]^k$ (which contains all action spaces along the sequence).

[15]See for instance Reed and Simon (1980, p. 100).

[16] For each t_i^{-1} and $\varepsilon > 0$, $A_i(a_{i,n}(\cdot)) \to A_i(a_{i,*}(\cdot))$ in the topology of pointwise convergence implies that there exists $N(t_i^{-1},\varepsilon)$ so that

$$\max_{m,j\in L_m} \left|A_i(a_{i,n}(\cdot);m,j,t_i^{-1}) - A_i(a_{i,*}(\cdot);m,j,t_i^{-1})\right| < \varepsilon \text{ for all } n > N(t_i^{-1},\varepsilon).$$

Since $A_i(a_i(\cdot);m,j,t_i^{-1})$ is bounded (in $[0,1]$) for all $a_i(\cdot),m,j,t_i^{-1}$, this implies that there exists $N(\varepsilon)$ so that

$$E_{t_i^{-1}}\left[\max_{m,j\in L_m} \left|A_i(a_{i,n}(\cdot);m,j,t_i^{-1}) - A_i(a_{i,*}(\cdot);m,j,t_i^{-1})\right|\right] < \varepsilon \text{ for all } n > N(\varepsilon)$$

[17]$R_1^\rho(b^S) \leq \min D_1(b^S)$ ($R_1^\rho(b^S) \geq \max D_1(b^S)$) if and only if the rationing process described on page 9 ends before 1 is reached (after 1 is fully served)

in the rationing queue. Similarly, $R_1^\rho(b^S) \in \big(\min D_1(b^S), \max D_1(b^S)\big)$ if and only if 1 can only be partially served after those ahead of him have been fully served.