Identification and Testable Restrictions in Private Value Multi-Unit Auctions

David McAdams, MIT*

June 8, 2005

Abstract

This paper studies discriminatory and uniform price auctions, the two most common "multi-unit auctions" for selling multiple identical objects. In such auctions, the distribution of bidder values is only partially identified from the distribution of bids. Given (asymmetric unobserved) correlated private values, sufficient conditions are provided for a given bid distribution to be rationalized by equilibrium behavior. Given independent private values, *all* value distributions that rationalize the data are identified. Given non-increasing marginal values, the best response hypothesis can be tested and lower bounds obtained on the extent to which each bidder fails to play a best response.

*MIT Sloan School of Management E52-448, 50 Memorial Drive, Cambridge, MA 02142. E-mail: mcadams@mit.edu. I thank Susan Athey, Ali Hortacsu, Harry Paarsch, Isabelle Perrigne, and Quang Vuong for helpful conversations and seminar audiences at Berkeley, Chicago, Harvard/MIT, Northwestern Kellogg, and Penn State. This material is based upon work supported by the National Science Foundation under Grant No. 0241468.

1 Introduction

This paper studies auctions of multiple identical objects (so-called "multi-unit auctions" or "split-award auctions") which have been used to model games as varied as oligopolistic competition (Klemperer and Meyer (1989)) and procurement (Anton and Yao (1989)). Real-world multi-unit auctions also provide abundant data. In such an auction, a bid is a demand *schedule* and in many applications each bidder's entire schedule is observed.

Multi-unit auctions differ in practice along several dimensions. They may be one-sided (bidders are all buyers or all sellers) or two-sided (each bidder may buy and/or sell units). They may have different payment rules, with the two most common being the uniform price and discriminatory rules. In a discriminatory auction, each bidder pays the price that he bid on each unit that he wins. In a uniform price auction, all bidders pay the same price for every unit. Finally, bidders may have private or common values as well as independent or correlated types. This paper focuses on one-sided multi-unit auctions when bidders have asymmetric (possibly correlated) private values.[1] All results extend trivially to two-sided auctions, however, once demand schedules are interpreted more broadly as net demand schedules.

Most Treasuries around the world use multi-unit auctions to issue bonds. For instance, Hortacsu (2002a, 2002b) studies the Turkish Treasury's one-sided discriminatory auction to sell new bonds and models bidders as having independent private values. (In some Treasury auctions, it is more natural to model bidders as having common values. Whether private values or common values is more appropriate depends on whether bonds are purchased for

[1]As in single-object auctions, the presence of common values presents additional, fundamental difficulties for identification. (See the discussion in Athey and Haile (2002).) While important, the case of common values is beyond the scope of this paper. Similarly, I shall restrict attention to risk-neutral bidders.

resale in a secondary market or to meet private reserve requirements.) One-sided uniform price auctions have been studied extensively in the context of electricity procurement[2] and two-sided uniform price auctions have been used to model emissions permit trading,[3] among other applications. In these settings, bidders are usually either assumed to have independent private values or no private information (which is an extreme form of correlated private values).

This paper has three main results.

(A) *Consistency with equilibrium.* When bidders have correlated private values, sufficient conditions are provided for an observed distribution of bids to be consistent with equilibrium.

(B) *Robust policy recommendations.* When bidders have independent private values, necessary *and* sufficient conditions are provided for an observed distribution of bids to be consistent with equilibrium.[4] Furthermore, *all* value distributions that could generate the data in a (mixed strategy) equilibrium are identified, along with the corresponding equilibrium strategies. This characterization allows one to determine whether any given policy conclusion is "robust" in the sense that it holds for all value distributions that might rationalize the data.

The standard approach is not robust in this sense. Multi-unit auction identification techniques in the existing literature have extended the well understood single-object auc-

[2] In Australia (Wolak (2003)), England and Wales (Wolfram (1998, 1999), Wolak and Patrick (2001)), California (Borenstein, Bushnell, and Wolak (2002), Joskow and Kahn (2001), and Harvey and Hogan (2001)), and Texas (Hortacsu and Puller (2004)), among others.

[3] For sulfur dioxide (Cason and Plott (1996), Ellerman, et al (2000)), for nitrous oxide (Joskow and Kahn (2001)), and potentially for carbon dioxide (Ellerman, Jacoby, and Decaux (1998)).

[4] Results (B,C) apply whenever an "observable beliefs property" is satisfied. This includes the case of independent private values but also some settings with correlation. See the discussion in Section 2.

tion approach.[5] Each bidder's unobserved marginal value schedule (or "values") is inferred from his observed demand schedule (or "bid") through a system of first-order equations. For instance, see Hortacsu (2002b) and Wolak (2003). Yet these first-order equations only provide necessary conditions for a given distribution of bids to be rationalized by equilibrium behavior. Furthermore, multi-unit auction models are typically not uniquely identified. The first-order equation approach identifies one distribution of bidder values that could generate the observed bids in equilibrium, but other distributions could as well. By contrast, this paper characterizes the full set of such value distributions.

Section 5 provides an example showing how and why this sort of robustness matters. In that example, an auctioneer who currently sells two objects separately must decide whether to sell them as a package instead. The standard identification approach leads one to predict a 15.5% increase in revenue from selling them as a package. There are many different value distributions that can rationalize the data other than the one identified by the standard approach, however, and the revenue increase could be more or less than 15.5%. (Numerical computations show that the revenue increase can be anywhere from 15.0% to 19.0%.) This ambiguity is not due to having a finite sample but to the more fundamental limitation that multi-unit auction models are only partially identified.

(C) *Testing the best response hypothesis.* Given auction data, perhaps the most basic question is whether bids could be best responses. Most existing tests of the equilibrium and/or best response hypotheses require the model to be over-identified.[6] This paper introduces a new sort of test of the best response hypothesis that applies to multi-unit auction

[5] See e.g. Guerre, Perrigne, and Vuong (2000) and Athey and Haile (2002) for non-parametric identification and e.g. Donald and Paarsch (1996) for parametric identification in the first-price auction. Athey and Haile (2005) Theorem 5.2 provides necessary and sufficient conditions for bids to be rationalized.

[6] See Athey and Haile (2005) for an excellent overview and discussion.

models despite the fact that such models are only partially identified.

This test leverages a standard assumption in many applications, that each bidder's marginal value is non-increasing in quantity. This assumption has bite in multi-unit auctions because it allows us to transform information about one unit's marginal value into restrictions on other units' marginal values. For instance, suppose that an observed bid can only be a best response for bidder i if his marginal value for the q-th unit is less than $\overline{v}_q$. This implies that his marginal value on for every unit $q' > q$ must be less than $\overline{v}_q$ as well, and such restrictions have testable implications. Indeed, in the example of Section 5, the non-increasing marginal values assumption leads us to conclude that bidder 2 *never* plays a best response!

If some bidder sometimes fails to play a best response, furthermore, this paper shows how to compute a lower bound on his foregone profit. Thus, on a bidder-by-bidder basis, one can both test whether best response is played and, if it is not, measure the extent to which best response must fail to describe real-world behavior. Previously, this has only been possible in experimental or other settings in which bidder values are known. The novel feature of this paper's approach is that one can detect and measure best response deviations without observing bidder values.

The rest of the paper is organized as follows. Section 2 describes the model of discriminatory and uniform price auctions. (Theory and techniques for both auction formats will be developed in parallel.) Section 3 presents the results on identification while Section 4 presents the results on testable restrictions. Section 5 illustrates the paper's main ideas and results in the context of an example. Some remarks in Section 6 conclude the paper.

2 Model

Rules. Each bidder $i = 1, ..., n$ submits a permissible bid schedule $\mathbf{b}_i$ which, given others' bid schedules $\mathbf{b}_{-i}$ and the reserve price schedule $\mathbf{r}$, determines his quantity $q_i(\mathbf{b}_i; \mathbf{b}_{-i}, \mathbf{r})$ and total payment $z_i(\mathbf{b}_i; \mathbf{b}_{-i}, \mathbf{r})$.[7]

Permissible bids. A permissible bid schedule $\mathbf{b}_i$ (or simply "bid") is a vector of unit-bids $b_{i,1}, ..., b_{i,\overline{S}} \in \{NO\} \cup [0, \infty)$ satisfying two requirements. (i) $b_{i,q} \geq b_{i,q'}$ when $q < q'$ and (ii) $q^1 < ... < q^{K^*+1}$ do not exist such that $b_{i,q^1} > ... > b_{i,q^{K^*+1}}$. The set of all permissible bids will be denoted $\mathfrak{B}$. *Notes:* (a) $b_{i,q} = NO$ guarantees that bidder i will always win less than q units. Thus, participation is voluntary: bidding $\mathbf{b}_i = (NO, ..., NO)$ is equivalent to not participating. (b) A permissible bid can be interpreted as a demand schedule that is a non-increasing step function with at most K^* steps, where K^* is pre-specified. (c) Extending the analysis to allow for a discrete set of prices is trivial but complicates notation.

Supply. There is an exogenous, possibly random reserve price schedule (or simply "supply") $\mathbf{r} = (r_1, ..., r_{\overline{S}})$, where $r_q \in [0, \infty]$ specifies the auctioneer's reserve price for the q-th unit and $\overline{S} < \infty$ is the most that can be produced at any price. By assumption, $r_q \leq r_{q'}$ for all $q < q'$. Each bidder knows the distribution but not the realization of $\mathbf{r}$ when making his bid. *Notes:* (a) Reserve price $r_q = \infty$ guarantees that less than q units will be sold. (b) Perfectly inelastic supply S is a special case in which $r_q = 0$ for all $q \leq S$ and $r_q = \infty$ for all $q > S$. (c) The "supply function equilibrium" literature typically assumes that supply is drawn from a one-dimensional family. (See e.g. Klemperer and Meyer (1989) and Hortacsu and Puller (2004).) Here supply may have any distribution. (d) Two-sided auctions can be analyzed using this paper's approach when we modify the model slightly. Suppose that

[7] Throughout the paper, vectors are bolded and scalars are not bolded.

a bid for bidder i is a vector $\mathbf{b}_i = (b_{i,-\overline{S}}, ..., b_{i,0}, b_{i,1}, ..., b_{i,\overline{S}})$ and that total supply is zero. For $q \geq 1$, unit-bid $b_{i,q}$ is an offer to buy a q-th unit while, for $q \geq 0$, unit-bid $b_{i,-q}$ is an offer sell a $(q+1)$-st unit. (e) The alternative possibility that bidders observe supply before submitting their bids can also be treated by slightly modifying the analysis.[8]

Allocation rule. The uniform price and discriminatory auctions have the same allocation rule: the highest $\overline{S}$ unit-bids win. Some definitions are useful.

Definition 1 (Auctioneer's bid $\mathbf{b}_0$). Define the "auctioneer's bid" $\mathbf{b}_0 = (b_{0,1}, ..., b_{0,\overline{S}})$ by the relation $b_{0,q} = r_{\overline{S}+1-q}$.

There is no difference between having reserve price schedule $\mathbf{r}$ and having fixed supply $\overline{S}$ with auctioneer bid $\mathbf{b}_0$. (Since $\mathbf{r}$ is random, of course, so is $\mathbf{b}_0$.) Thus, without loss I shall focus on the case of fixed supply $\overline{S}$, use notation $\mathbf{b}_{-i} \equiv (\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, ..., \mathbf{b}_n)$, and drop notation $\mathbf{r}$.

Definition 2 (Residual supply). Let $s_{i,q}(\mathbf{b}_{-i})$ denote the $(\overline{S} - q + 1)$-st highest unit-bid submitted by players other than bidder i. (By convention, let $s_{i,q}(\mathbf{b}_{-i}) = 0$ when this unit-bid is *NO*.) The vector $\mathbf{s}_i(\mathbf{b}_{-i}) \equiv \left(s_{i,1}(\mathbf{b}_{-i}), ..., s_{i,\overline{S}}(\mathbf{b}_{-i})\right)$ is bidder i's *"residual supply"*.

For brevity, I will adopt the notation $\mathbf{s}_i$ for bidder i's residual supply whenever this is unlikely to cause confusion.

Bidder i's quantity $q_i(\mathbf{b}_i; \mathbf{b}_{-i})$ is determined as follows. If $b_{i,q} > s_{i,q}$, then $q_i(\mathbf{b}_i; \mathbf{b}_{-i}) \geq q$. If $b_{i,q} < s_{i,q}$ or $b_{i,q} = NO$, then $q_i(\mathbf{b}_i; \mathbf{b}_{-i}) < q$. If $b_{i,q} = s_{i,q}$ and $b_{i,q} \neq NO$, finally, a tie-breaking rule determines whether bidder i wins the q-th unit. (See the discussion of ties below.)

[8] Unless best responses are ex post optimal, different value distributions will be inferred from the same observed bid distribution depending other whether supply is known or unknown to the bidders when bids are submitted.

Payment rule. The discriminatory and uniform price auctions differ in their payment rules. (The bulk of this paper's analysis applies to the discriminatory and uniform price auctions simultaneously. Superscripts "D" or "U" are used only when the analysis applies specifically to the discriminatory or uniform price auction, respectively.) In the discriminatory auction, bidder i pays the sum of his unit-bids on what he wins,

$$z_i^D(\mathbf{b}_i;\mathbf{b}_{-i}) \equiv \sum_{x=1}^{q_i(\mathbf{b}_i;\mathbf{b}_{-i})} b_{i,x}$$

In the uniform price auction,[9] bidder i pays

$$z_i^U(\mathbf{b}_i;\mathbf{s}_i) \equiv q_i(\mathbf{b}_i;\mathbf{b}_{-i}) \max\left\{b_{i,q_i(\mathbf{b}_i;\mathbf{b}_{-i})+1}, s_{i,q_i(\mathbf{b}_i;\mathbf{b}_{-i})}\right\} \tag{1}$$

To make sense of (1), note that the highest losing unit-bid when bidder i wins exactly q units is either $b_{i,q+1}$ or $s_{i,q}$.

Observables. This paper is not about estimation. In particular, I shall *assume* that the joint distribution of $(\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_n)$ is observed.

For simplicity in most of the analysis, furthermore, I will assume that there are no atoms or kinks in the distribution of each bidder's residual supply, i.e. that $\Pr(b_{i,q} > s_{i,q})$ is continuously differentiable in $b_{i,q}$ for every bidder i and quantity q. Among other things, this assumption implies that ties occur with zero probability so that the details of the tie-breaking rule are not relevant to best responses. In particular, bidder i's residual supply is a sufficient statistic for his quantity and payment, i.e. $q_i(\mathbf{b}_i;\mathbf{b}'_{-i}) = q_i(\mathbf{b}_i;\mathbf{b}_{-i})$ and $z_i(\mathbf{b}_i;\mathbf{b}'_{-i}) = z_i(\mathbf{b}_i;\mathbf{b}_{-i})$ whenever $\mathbf{s}_i(\mathbf{b}'_{-i}) = \mathbf{s}_i(\mathbf{b}_{-i})$. As is well known and briefly reviewed in Section 4.1, the presence of atoms and/or of kinks creates additional testable restrictions. Assuming

[9] This paper studies the uniform $(\overline{S}+1)$-st price auction in which all bidders pay the highest losing unit-bid. Closely related techniques apply to other uniform price auctions such as the $\overline{S}$-th price auction (price equals lowest winning unit-bid) and $(\overline{S}+1/2)$-th price auction (price is the average of $\overline{S}$-th and $(\overline{S}+1)$-st).

away these complicating factors allows us to focus on the most novel aspects of the present analysis.

Unobservables. Each bidder i has private value schedule (or simply "values") $\mathbf{v}_i = (v_{i,1}, ..., v_{i,\overline{S}})$ where $v_{i,q}$ is bidder i's marginal value for the q-th unit. If bidder i wins q_i units and makes total payment z_i, his ex post payoff is $\sum_{x=1}^{q_i} v_{i,x} - z_i$. In addition to learning his values, each bidder i may receive a payoff-irrelevant (multi-dimensional) signal θ_i. Bidder i's type $\mathbf{t}_i = (\mathbf{v}_i, \theta_i)$. Assumptions about the joint distribution of $(\mathbf{b}_0, \mathbf{t}_1, ..., \mathbf{t}_n)$ will be discussed below. For now, note that each bidder i's marginal values $(v_{i,1}, ..., v_{i,\overline{S}})$ may be correlated.

Each bidder adopts a mixed strategy that is also unobserved.

Definition 3 (Mixed strategy). $\mathbf{b}_i(\cdot)$ denotes bidder i's *mixed strategy*, with $\mathbf{b}_i(\mathbf{t}_i)$ being his (random) bid given type $\mathbf{t}_i$. $\mathbf{b}_i \in \mathbf{b}_i(\mathbf{t}_i)$ means that $\mathbf{b}_i$ is in the support of $\mathbf{b}_i(\mathbf{t}_i)$.

Assumptions about unobservables.

Bids are best responses. For identification purposes, we shall assume that each observed bid is a best response, i.e. bidders play a mixed strategy equilibrium. A few definitions are needed.

Definition 4 ("Belief"). When others adopt mixed strategies $\mathbf{b}_{-i}(\cdot)$, bidder i's *"belief"* $B_i(\mathbf{t}_i, \mathbf{b}_{-i}(\cdot))$ is the distribution of bidder i's residual supply that is induced by these strategies and the joint distribution of $(\mathbf{b}_0, \mathbf{t}_{-i})$ conditional on bidder i having type $\mathbf{t}_i$.

When it is unlikely to cause confusion, I will use shorter notation B_i for beliefs.

Bidder i's ex post payoff given bid $\mathbf{b}_i$, values $\mathbf{v}_i$, and residual supply $\mathbf{s}_i$ is $\Pi_i^{expost}(\mathbf{b}_i, \mathbf{v}_i; \mathbf{s}_i) \equiv \sum_{x=1}^{q_i(\mathbf{b}_i;\mathbf{s}_i)} v_{i,x} - z_i(\mathbf{b}_i, \mathbf{s}_i)$. Given beliefs B_i, his interim expected payoff similarly only depends on his bid and values: $\Pi_i(\mathbf{b}_i, \mathbf{v}_i; B_i) \equiv E_{B_i}\left[\Pi_i^{expost}(\mathbf{b}_i, \mathbf{v}_i; \mathbf{s}_i)\right]$

Definition 5 (Best response). A bid $\mathbf{b}_i$ is a *best response* for bidder i given values $\mathbf{v}_i$ and beliefs B_i when $\mathbf{b}_i \in BR_i(\mathbf{v}_i; B_i) \equiv \arg\max_{\mathbf{b}_i \in \mathfrak{B}} \Pi_i(\mathbf{b}_i, \mathbf{v}_i; B_i)$.

Definition 6 (Mixed strategy equilibrium). A profile of mixed strategies $(\mathbf{b}_1(\cdot), ..., \mathbf{b}_n(\cdot))$ is a *mixed strategy equilibrium* when, for all bidders $i = 1, ..., n$ and types $\mathbf{t}_i = (\mathbf{v}_i, \theta_i)$, $\mathbf{b}_i \in \mathbf{b}_i(\mathbf{t}_i)$ implies that $\mathbf{b}_i \in BR_i(\mathbf{v}_i; B_i(\mathbf{t}_i, \mathbf{b}_{-i}(\cdot)))$.

Distributional assumptions. This paper provides two sets of results. First, *sufficient conditions* are provided for the observed joint distribution of $(\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_n)$ to be consistent with mixed strategy equilibrium given correlated private values.

1. *Correlated private values.* $(\mathbf{b}_0, \mathbf{t}_1, ..., \mathbf{t}_n)$ may have any joint distribution.

The paper's most powerful results, however, apply in more specialized environments having the following "observable beliefs property".

Definition 7 (Observable beliefs property). The observable beliefs property is satisfied when $B_i(\mathbf{t}'_i; \mathbf{b}_{-i}(\cdot)) = B_i(\mathbf{t}_i; \mathbf{b}_{-i}(\cdot))$ for all $\mathbf{t}'_i, \mathbf{t}_i$ such that $\mathbf{b}_i \in \mathbf{b}_i(\mathbf{t}'_i) \cap \mathbf{b}_i(\mathbf{t}_i)$.

When the observable beliefs property is satisfied, all types who sometimes bid $\mathbf{b}_i$ face the same distribution of residual supply. Thus, the observed distribution of others' bids $\mathbf{b}_{-i}$ *conditional* on $\mathbf{b}_i$ allows us to infer bidder i's belief when he bids $\mathbf{b}_i$. Before proceeding, consider two specific environments that satisfy the observable beliefs property.

2. *Independent private values.* $\mathbf{t}_i = \mathbf{v}_i$ and $(\mathbf{b}_0, \mathbf{v}_1, ..., \mathbf{v}_n)$ are independently distributed.

3. *Sparse support.* If $(\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_n)$ and $(\mathbf{b}'_0, \mathbf{b}'_1, ..., \mathbf{b}'_n)$ are both in the support of the joint distribution of bids, then either $\mathbf{b}'_i = \mathbf{b}_i$ for all $i \neq 0$ or $\mathbf{b}'_i \neq \mathbf{b}_i$ for all $i \neq 0$. (The auctioneer's bid $\mathbf{b}_0$ may have any distribution conditional on $(\mathbf{b}_1, ..., \mathbf{b}_n)$.)

Remark 1. *The observable beliefs property is satisfied when bidders have independent private values or when the observed joint distribution of bids has sparse support.*

Proof. In the Appendix. □

Sparse support is a property of the observed distribution of bids, *not* an assumption about the unobserved distribution of values. To see how sparse support may arise, consider electricity procurement auctions. Bidding in these auctions is usually modelled as a "supply function equilibrium" à la Klemperer and Meyer (1989), see e.g. Borenstein, Bushnell, and Wolak (2002). Since these are procurement auctions, I will use cost notation $c_{i,q}$ in place of values. In particular, generators are assumed to know each others' costs when submitting their bids, i.e. generator i has costs $\mathbf{c}_i(\theta) = (c_{i,1}(\theta), ..., c_{i,\overline{S}}(\theta))$ where all generators observe θ. For instance, the parameter θ might include each generator's production technology (including whether it will be shut down for repairs) and the relevant prices of natural gas and other inputs, as well as possibly some information about the uncertain demand for electricity. Sparse support will be satisfied as long as each bidder does not submit the *exact same* supply function given any two different θ.

2.1 Preliminaries

Before proceeding to the analysis, it is useful to collect some basic facts about payoffs in discriminatory and uniform price auctions. Most importantly, each bidder's interim expected payoff given his values is additively separable across unit-bids as well as across marginal values. (Additive separability holds regardless of the joint distribution of bids, i.e. independence is not needed.) This result is relatively well understood for discriminatory auctions but appears to be new for the uniform price auction.

Proposition 1 (Discriminatory auction payoffs). *In the discriminatory auction, bidder i's interim expected profit* $\Pi_i^D(\mathbf{b}_i, \mathbf{v}_i; B_i)$ *for a given (fixed) belief* B_i *takes the form*

$$\Pi_i^D(\mathbf{b}_i, \mathbf{v}_i; B_i) = \sum_{q=1}^{\overline{S}} \Pi_{i,q}^D(b_{i,q}, v_{i,q}; B_i)$$

where, for all $1 \leq q \leq \overline{S}$*,*

$$\Pi_{i,q}^D(b_{i,q}, v_{i,q}; B_i) \equiv (v_{i,q} - b_{i,q}) \Pr{}_{B_i} (b_{i,q} > s_{i,q}) \tag{2}$$

Proof. The proof is omitted. For instance, see the analysis in Hortacsu (2002a). □

Proposition 2 (Uniform price auction payoffs). *In the uniform price auction, bidder i's interim expected profit* $\Pi_i^U(\mathbf{b}_i, \mathbf{v}_i; B_i)$ *for a given (fixed) belief* B_i *takes the form*

$$\Pi_i^U(\mathbf{b}_i, \mathbf{v}_i; B_i) = \sum_{q=1}^{\overline{S}} \Pi_{i,q}^U(b_{i,q}, v_{i,q}; B_i)$$

where, for all $1 \leq q \leq \overline{S}$*,*

$$\begin{aligned} \Pi_{i,q}^U(b_{i,q}, v_{i,q}; B_i) = {} & (v_{i,q} - E[s_{i,q} | b_{i,q} > s_{i,q}]) \Pr{}_{B_i} (b_{i,q} > s_{i,q}) \\ & - (q-1) E[s_{i,q} - s_{i,q-1} | b_{i,q} > s_{i,q}] \Pr{}_{B_i} (b_{i,q} > s_{i,q}) \\ & - (q-1) E[b_{i,q} - s_{i,q-1} | s_{i,q} \geq b_{i,q} > s_{i,q-1}] \Pr{}_{B_i} (s_{i,q} \geq b_{i,q} > s_{i,q-1}) \end{aligned} \tag{3}$$

Proof. In the Appendix. □

3 Identification

This section investigates whether and how a given distribution of observed bids can be rationalized by equilibrium behavior. Sections 3.1-3.2 begin by considering the problem of rationalizing a particular bid $\mathbf{b}_i$ given that bidder i has belief B_i about the distribution of residual supply. In particular, these sections characterize $V_i^*(\mathbf{b}_i, B_i) \equiv \{\mathbf{v}_i : \mathbf{b}_i \in BR_i(\mathbf{v}_i; B_i)\}$, the set of values such that bidder i finds $\mathbf{b}_i$ to be a best response. Section 3.3 then presents the identification results, both when the observable beliefs property is satisfied and when it is not.

3.1 Local deviations

Before one can identify which values $\mathbf{v}_i$ find a given bid $\mathbf{b}_i$ to be a best response, one must characterize all permissible local deviations from $\mathbf{b}_i$. Clearly, all such deviations must be unprofitable for any $\mathbf{v}_i \in V_i^*(\mathbf{b}_i, B_i)$.

Step-bids. It is useful to decompose every bid into "step-bids".

Definition 8 (Step, step-bids). A *step* in bid $\mathbf{b}_i$ is a maximal interval $[q, q']$ of quantities having the property that $b_{i,q} = ... = b_{i,q'} > NO$, in which case $\mathbf{b}_{i,[q,q']} \equiv (b_{i,q}, ..., b_{i,q'})$ is called a step-bid. (Figure 1 shows a bid having two steps, $[1, 3]$ at step-bid \$4 and $[4, 5]$ at step-bid \$2.) $K(\mathbf{b}_i)$ denotes the number of steps in bid $\mathbf{b}_i$.

By additive separability (Propositions 1,2), the problem of whether bidder i has any profitable local deviation separates across steps. That is to say, it suffices to consider local deviations in which bidder i only raises and/or lowers unit-bids on quantities in a single step. Furthermore, whether any such deviation is profitable depends only on his marginal values for units in that step.

Case I: $K(\mathbf{b}_i) < K^*$. First suppose that bidder i does not use the maximum permitted number of steps in the observed bid $\mathbf{b}_i$.

Feasible local deviations from step-bid $\mathbf{b}_{i,Q}$. For a specific step Q and step-bid $\mathbf{b}_{i,Q}$, consider all possible "directions" in which deviations can feasibly be made.

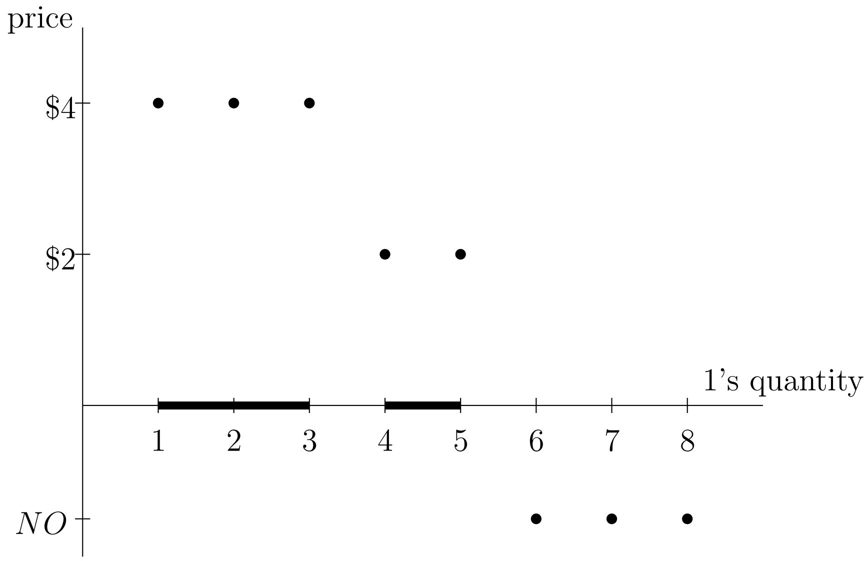

Figure 1: Bid $\mathbf{b}_i = (\$4, \$4, \$4, \$2, \$2, NO, NO, NO)$ has two steps.

Definition 9 (Deviation direction). Let $\widehat{\mathbf{b}}_Q$ be a deviation from the step-bid $\mathbf{b}_{i,Q}$. The *direction* of this deviation is $\mathbf{d}_Q(\widehat{\mathbf{b}}_Q, \mathbf{b}_{i,Q}) = \frac{(\widehat{\mathbf{b}}_Q - \mathbf{b}_{i,Q})}{\max_{q \in Q} |\widehat{b}_q - b_q)|}$. A generic deviation direction from step-bid $\mathbf{b}_{i,Q}$ will be referred to as $\mathbf{d}_Q$.

(The normalization in Definition 9 scales all coordinates of the vector $\mathbf{d}_Q(\widehat{\mathbf{b}}_Q, \mathbf{b}_{i,Q})$ to lie in $[-1, 1]$ with at least one element in $\{1, -1\}$.) For example, the deviation in which the $(\min Q)$-th unit-bid is raised by 2ε, the $(\min Q + 1)$-st unit-bid is raised by ε, and all other unit-bids stay the same has direction $(1, .5, 0, ..., 0)$. Not all possible deviation directions are feasible. Since bids must be non-increasing and $\mathbf{b}_{i,Q}$ is flat, any feasible deviation direction $\mathbf{d}_Q = (d_{\min Q}, ..., d_{\max Q})$ must satisfy $d_{\min Q} \geq ... \geq d_{\max Q}$.

Necessary and sufficient deviation directions to consider. Every feasible deviation direction can be constructed as a convex combination of directions in the following family:

$$d_x^+(q) = 1 \text{ for all } x \leq q \text{ and } d_x^+(q) = 0 \text{ for all } x > q$$

$$d_x^-(q) = 0 \text{ for all } x < q \text{ and } d_x^-(q) = -1 \text{ for all } x \geq q$$

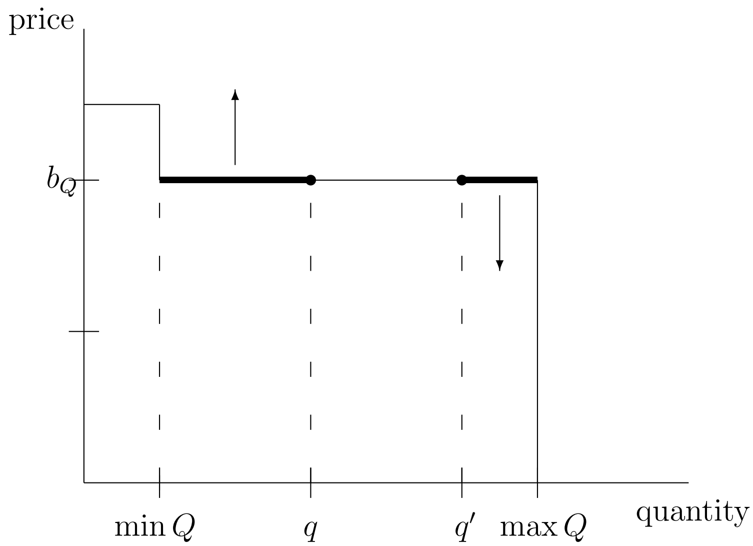

Figure 2: Deviation directions $\mathbf{d}_Q^+(q)$ and $\mathbf{d}_Q^-(q')$ are illustrated.

So, if deviating in any direction $\widehat{\mathbf{d}}$ is profitable locally, then deviating in one of these directions must also be profitable locally. For each deviation direction $\mathbf{d}_Q^+(q)$, we have inequality restriction

$$\sum_{x\in[\min Q,q]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, v_{i,x}; B_i)}{\mathrm{d}b_{i,x}} \leq 0 \tag{4}$$

and for each deviation direction $\mathbf{d}_Q^-(q)$, the inequality restriction

$$\sum_{x\in[q,\max Q]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, v_{i,x}; B_i)}{\mathrm{d}b_{i,x}} \geq 0 \tag{5}$$

Discriminatory auction. Substituting (2) into (4,5) yields

$$0 \geq \sum_{x\in[\min Q,q]} \left((v_{i,x} - b_{i,x}) \frac{\mathrm{d}\Pr_{B_i}(b_{i,x} > s_{i,x})}{\mathrm{d}b_{i,x}} \right) - \sum_{x\in[\min Q,q]} \Pr_{B_i}(b_{i,x} > s_{i,x}) \tag{6}$$

$$0 \leq \sum_{x\in[q,\max Q]} \left((v_{i,x} - b_{i,x}) \frac{\mathrm{d}\Pr_{B_i}(b_{i,x} > s_{i,x})}{\mathrm{d}b_{i,x}} \right) - \sum_{x\in[q,\max Q]} \Pr_{B_i}(b_{i,x} > s_{i,x}) \tag{7}$$

Uniform price auction. Similarly, substituting (3) into (4,5) and cancelling terms yields

$$0 \geq \sum_{x \in [\min Q, q]} \left((v_{i,x} - b_{i,x}) \frac{\mathrm{d} \Pr_{B_i} (b_{i,x} > s_{i,x})}{\mathrm{d} b_{i,x}} \right) \tag{8}$$

$$- \sum_{q \in [\min Q, q]} (q-1) \Pr_{B_i} (s_{i,q} > b_{i,q} \geq s_{i,q+1})$$

$$0 \leq \sum_{x \in [q, \max Q]} \left((v_{i,x} - b_{i,x}) \frac{\mathrm{d} \Pr_{B_i} (b_{i,x} > s_{i,x})}{\mathrm{d} b_{i,x}} \right) \tag{9}$$

$$- \sum_{q \in [q, \max Q]} (q-1) \Pr_{B_i} (s_{i,q} > b_{i,q} \geq s_{i,q+1})$$

Equality constraint on step-values $\mathbf{v}_{i,Q}$ that rationalize step-bid $\mathbf{b}_{i,Q}$. In addition to the inequality constraints described above, there is one equality constraint per step-bid. Bidder i must be indifferent to raising (or lowering) all of the unit-bids in step-bid $\mathbf{b}_{i,Q}$:

$$\sum_{q \in Q} \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}, v_{i,q}; B_i)}{\mathrm{d} b_{i,q}} = 0 \tag{10}$$

Discriminatory auction. Substituting (2) into (10), we get

$$0 = \sum_{q \in M} \left((v_{i,q} - b_{i,q}) \frac{\mathrm{d} \Pr_{B_i} (b_{i,q} > s_{i,q})}{\mathrm{d} b_{i,q}} \right) - \sum_{q \in Q} \Pr_{B_i} (b_{i,q} > s_{i,q}) \tag{11}$$

Uniform price auction. Substituting (3) into (10) now, we get

$$0 = \sum_{q \in M} \left((v_{i,q} - b_{i,q}) \frac{\mathrm{d} \Pr_{B_i} (b_{i,q} > s_{i,q})}{\mathrm{d} b_{i,q}} \right) \tag{12}$$

$$- \sum_{q \in Q} (q-1) \Pr_{B_i} (s_{i,q} > b_{i,q} \geq s_{i,q+1})$$

Theorem 1 (Necessary first-order conditions, discriminatory). *Suppose that $K(\mathbf{b}_i) < K^*$ in the discriminatory auction. For each step Q in the bid $\mathbf{b}_i$, let $\hat{V}^D_{i,Q}(\mathbf{b}_{i,Q}, B_i)$ be the set of step-values that satisfy (6,7,11), and let $\hat{V}^D_i(\mathbf{b}_i, B_i) \equiv \times_{steps\ Q} \hat{V}^D_{i,Q}(\mathbf{b}_{i,Q}, B_i)$. $\mathbf{v}_i \in \hat{V}^D_i(\mathbf{b}_i, B_i)$ is a necessary condition for type $\mathbf{v}_i$ to find bid $\mathbf{b}_i$ to be a best response given beliefs B_i.*

Theorem 2 (Necessary first-order conditions, uniform price). *Suppose that* $K(\mathbf{b}_i) < K^*$ *in the uniform price auction. For each step* Q *in the bid* $\mathbf{b}_i$*, let* $\hat{V}^U_{i,Q}(\mathbf{b}_{i,Q}, B_i)$ *be the set of step-values that satisfy (8,9,12), and let* $\hat{V}^U_i(\mathbf{b}_i, B_i) \equiv \times_{steps\ Q} \hat{V}^U_{i,Q}(\mathbf{b}_{i,Q}, B_i)$*.* $\mathbf{v}_i \in \hat{V}^U_i(\mathbf{b}_i, B_i)$ *is a necessary condition for type* $\mathbf{v}_i$ *to find bid* $\mathbf{b}_i$ *to be a best response given beliefs* B_i*.*

Case II: $K(\mathbf{b}_i) = K^*$**.** What if the constraint on the number of steps is binding? The set of feasible local deviations shrinks dramatically to those in which all unit-bids in any given step are raised or lowered by the same amount. Consequently, Theorems 1, 2 must be modified as follows: "For each step Q in the bid $\mathbf{b}_i$, let $\hat{V}_{i,Q}(\mathbf{b}_{i,Q}, B_i)$ be the set of step-values that satisfy (11) or (12)" in the discriminatory and uniform price auction, respectively.

On unique identification. In both Case I and Case II, the set $\hat{V}_i(\mathbf{b}_i, B_i)$ is at least $(\overline{S} - K(\mathbf{b}_i))$-dimensional. (If bidder i is indifferent to some local deviations, then this set may have even higher dimension.) Thus, multi-unit auction models can only be uniquely identified if the rules permit strictly downward-sloping bids ($K^* = \overline{S}$) and all observed bids are strictly downward-sloping.

3.2 Global deviations

Section 3.1 identified a set of values $\hat{V}_i(\mathbf{b}_i, B_i)$ given which bidder i *might* find bid $\mathbf{b}_i$ to be a best response. This section provides a method to check whether bidder i has a profitable global deviation given values $\mathbf{v}_i \in \hat{V}_i(\mathbf{b}_i, B_i)$. To address this issue, it suffices to compute bidder i's supremum expected payoff given values $\mathbf{v}_i$, evaluated over all permissible bids. Then we can simply compare bidder i's expected profit from $\mathbf{b}_i$ versus this supremum payoff. If they are the same, then $\mathbf{b}_i$ is a best response. Otherwise bidder i has a profitable global deviation.

Computing bidder i's supremum payoff is difficult since he faces a multi-dimensional constrained optimization problem. Bids are constrained in two ways: (i) *non-decreasing bid constraint*, $b_{i,q'} \leq b_{i,q}$ for all $q < q'$, and (ii) *at most K^* steps constraint*, there does not exist $q^1 < ... < q^{K^*+1}$ such that $b_{i,q^1} > ... > b_{i,q^{K^*+1}}$. The contribution here is to simplify this computational problem. In particular, my approach requires only that one solve a sequence of *one-dimensional* problems. In the text, I consider the special case $K^* = \overline{S}$ in which strictly downward-sloping bids are permitted. An extension to the more general case with $K^* \leq \overline{S}$ is included in the Appendix.

Special case: $K^* = \overline{S}$. The logic of the algorithm is by induction. For every $1 \leq q \leq \overline{S}$ and every $b \geq 0$, we will find

$$\pi_i(\mathbf{v}_i; q, b) \equiv \sup_{\mathbf{b}_i \in \mathcal{B},\ b_{i,q} \geq b,\ b_{i,q+1}=0} \Pi_i\left(\mathbf{b}_i, \mathbf{v}_i; B_i\right) \tag{13}$$

That is to say, suppose that bidder i is constrained to bid zero on units $q+1, ..., \overline{S}$ and to bid at least b on unit q, in addition to the other constraints imposed by the rules. Our goal is to calculate $\pi_i(\mathbf{v}_i; \overline{S}, 0)$ since this is type $\mathbf{v}_i$'s supremum payoff over all permissible bids.

Recall from Section 2.1 that expected payoffs are additively separable, $\Pi_i(\mathbf{b}_i, \mathbf{v}_i; B_i) = \sum_{x=1}^{\overline{S}} \Pi_{i,x}(b_{i,x}, v_{i,x}; B_i)$. Thus, for all $b \geq 0$ and all $1 \leq q < \overline{S}$,

$$\pi_i(\mathbf{v}_i; q+1, b) = \sup_{b_{i,q+1} \geq b} \left(\pi_i(\mathbf{v}_i; q, b_{i,q+1}) + \Pi_{i,q+1}(b_{i,q+1}, v_{i,q+1}; B_i)\right) \tag{14}$$

Thus, to compute $\pi_i(\mathbf{v}_i; q+1, b)$ we need only solve the one-dimensional optimization problem (14) in addition to having previously computed $\pi_i(\mathbf{v}_i; q, b')$ for all $b' \geq b$.

3.3 Mixed strategy equilibria that rationalize the bids

With observable beliefs. When the observable beliefs property is satisfied, characterizing the set of all mixed strategy equilibria that rationalize the observed distribution of bids – along with the corresponding distributions of bidder types – is straightforward. A definition will be needed.

Definition 10 (Inverse mixed strategy). $\phi_i(\cdot)$ denotes bidder i's *inverse mixed strategy.* $\mathbf{t}_i \in \phi_i(\mathbf{b}_i)$ whenever $\mathbf{b}_i \in \mathbf{b}_i(\mathbf{t}_i)$, with $\phi_i(\mathbf{b}_i)$ interpreted as a random draw from all types that bid $\mathbf{b}_i$.

Consider any observed bid $\mathbf{b}_i$. (i) The observable beliefs property gives us a belief about residual supply (call it $B_i(\mathbf{b}_i)$) that bidder i must have had when bidding $\mathbf{b}_i$. Without loss of generality, let the payoff-irrelevant part θ_i of bidder i's type $\mathbf{t}_i = (\mathbf{v}_i, \theta_i)$ just be his belief about the distribution of residual supply, i.e. $\theta_i = B_i(\mathbf{b}_i)$. (ii) Earlier analysis gives us a set $V_i^*(\mathbf{b}_i, B_i(\mathbf{b}_i))$ of values that bidder i might have had when bidding $\mathbf{b}_i$. The only restriction on inverse mixed strategies imposed by equilibrium is that $\mathbf{t}_i \in \phi_i(\mathbf{b}_i)$ implies $\mathbf{v}_i \in V_i^*(\mathbf{b}_i, B_i(\mathbf{b}_i))$.

Given the observed joint distribution of bids, any profile of inverse mixed strategies mapping bids into types induces an inferred joint distribution of types as well as an inferred profile of mixed strategies mapping types into bids. By construction, these inferred mixed strategies constitute a mixed strategy equilibrium that rationalizes the observed distribution of bids as long as each inverse mixed strategy has the property that $\mathbf{t}_i \in \phi_i(\mathbf{b}_i)$ implies $\mathbf{v}_i \in V_i^*(\mathbf{b}_i, B_i(\mathbf{b}_i))$. Furthermore, any such equilibrium can be constructed in this way.

Without observable beliefs. By definition, when the observable beliefs property fails there exists two different types $\mathbf{t}_i'$ and $\mathbf{t}_i$ that make the same bid $\mathbf{b}_i$ and that face different distributions of residual supply. For example, an important class of models in which the observable beliefs property could fail are those with unobserved heterogeneity. Suppose that all bidders observe a signal, $\theta_i = \theta$ for all i, which is informative about the joint distribution of bidder values. Given different realizations of θ, bidder i may make the same bid given different values. When the distribution of θ is unobserved, there is no way to identify the distribution of bidder values from the distribution of bids, either conditional or unconditional on θ.

To make progress in this case, I shall restrict attention to a subclass of mixed strategy equilibria in which every bidder plays a "one-to-one strategy".

Definition 11 (One-to-one strategy). A mixed strategy $\mathbf{b}_i(\cdot)$ is *one-to-one* when $\mathbf{b}_i \in \mathbf{b}_i(\mathbf{t}_i)$ for some $\mathbf{t}_i$ implies that $\mathbf{b}_i \notin \mathbf{b}_i(\mathbf{t}'_i)$ for all $\mathbf{t}'_i \neq \mathbf{t}_i$.

Since only one bidder type $\mathbf{t}_i$ ever makes any particular bid $\mathbf{b}_i$, the distribution of others' bids conditional on $\mathbf{b}_i$ is the same as the conditional distribution that bidder i faced given type $\mathbf{t}_i$. In other words, the observable beliefs property is satisfied for bidder i when bidder i adopts a one-to-one strategy. As long as all bidders adopt such strategies, we may therefore apply the previous analysis to characterize all mixed strategy equilibria *in one-to-one strategies* that rationalize the bids. Of course, if no such equilibrium exists, some equilibrium not in one-to-one strategies could still exist that rationalizes the bids. Given correlated private values, then, this paper provides sufficient but not necessary conditions for observed bids to be consistent with equilibrium behavior.

4 Testable restrictions

There are several ways to test the hypothesis that bidder i plays a best response strategy given beliefs B_i. First, the identification procedure of Section 3 may fail (Section 4.1). Second, we may find it reasonable to impose extra assumptions about values. Even if identification does not fail, all identified value distributions may violate these extra assumptions. In particular, this paper explores the implications of assuming that each bidder has non-increasing marginal values (Section 4.2).

4.1 Identification failure

Failure due to atoms or upward kinks. Some bids may not be a best response for bidder i regardless of his values. Two simple first-price auction examples illustrate this point. (A first-price auction is a discriminatory auction with one unit.) The details are straightforward to check and so omitted.

Example 1 (Atom $\Rightarrow$ profitable global deviation). Two bidders in first-price auction, independent private values. Bidder 1 is observed sometimes bidding .45. The distribution of bidder 2's bid b_2 puts a 50%-probability atom at .5 with the other 50% uniformly spread over $[0, 1]$. For bidder 1 not to have a profitable local deviation from .45, it must be that $v_1 = .9$. In this case, however, bidder 1's expected profit is only $22.5\% * (.9 - .45) \approx .10$ which is less than the payoff from bidding .51.

Example 2 (Kink $\Rightarrow$ profitable local deviation). Two bidders in first-price auction, independent private values. Bidder 1 is observed sometimes bidding 2. The distribution of bidder 2's bid b_2 has 50% mass spread uniformly over $[0, 2]$ and 50% spread uniformly over $[2, 3]$, i.e. there is an "upward kink" in the c.d.f. of b_2 at 2. For bidder 1 not to have a profitable local *downward* deviation from 2, it must be that $v_i \geq 4$. But in this case he has a profitable local *upward* deviation.

Definition 12 (Monotone pure strategy). A mixed strategy is a monotone pure strategy when (i) no types mix and (ii) $\mathbf{b}_i(\mathbf{v}_i') \geq \mathbf{b}_i(\mathbf{v}_i)$ in the product order whenever $\mathbf{v}_i' > \mathbf{v}_i$ in the product order.[10]

Failure of monotonicity. Given the assumption of independent private values, bidder i's best response bids in the uniform price or discriminatory auction are monotone with respect

[10] $\mathbf{v}_i' > \mathbf{v}_i$ in the product order iff $v_{i,q}' \geq v_{i,q}$ for all q and $v_{i,q}' > v_{i,q}$ for some q, and likewise for bid vectors.

to his values (McAdams (2002)). (More precisely, the *set* of best response bids is increasing in Veinott's strong set order with respect to $\mathbf{v}_i$.) Consequently, Guerre, Perrigne, and Vuong (2000)'s observation regarding monotone strategy restrictions still applies, although now in a multi-dimensional form. For any two bids $\mathbf{b}_i', \mathbf{b}_i$, let $\mathbf{b}_i' \wedge \mathbf{b}_i$ and $\mathbf{b}_i' \vee \mathbf{b}_i$ denote, respectively, their greatest lower bound and least upper bound in the product order. Recall that $\hat{V}_i(\mathbf{b}_i, B_i)$ is the set of bidder i's values given which the first-order inequalities are satisfied for bid $\mathbf{b}_i$ to be a best response.

Theorem 3. *For any bids* $\mathbf{b}_i' \not> \mathbf{b}_i$ *and belief* B_i*, suppose that* $\mathbf{v}_i' \geq \mathbf{v}_i$ *for all* $\mathbf{v}_i' \in \hat{V}_i(\mathbf{b}_i', B_i)$ *and* $\mathbf{v}_i \in \hat{V}_i(\mathbf{b}_i, B_i)$*. Then one of the following must be true:*

(a) bidder i *has a profitable global deviation from* $\mathbf{b}_i$ *given all values in* $\hat{V}_i(\mathbf{b}_i, B_i)$

(b) bidder i *has a profitable global deviation from* $\mathbf{b}_i'$ *given all values in* $\hat{V}_i(\mathbf{b}_i', B_i)$

(c) bidder i *is indifferent between bids* $\mathbf{b}_i$ *and* $\mathbf{b}_i' \wedge \mathbf{b}_i$ *given values* $\mathbf{v}_i$ *and between bids* $\mathbf{b}_i'$ *and* $\mathbf{b}_i' \vee \mathbf{b}_i$ *given values* $\mathbf{v}_i'$

Proof. In the Appendix. □

Theorem 3 provides a shortcut for determining that bidder i must have a global deviation, without having to implement the more conclusive algorithm of Section 3.2.

4.2 Testing best response when bidder beliefs are known

For the rest of Section 4, I shall add the *assumption* that each bidder has non-increasing marginal values, i.e. $v_{i,q} \geq v_{i,q'}$ for all $q' > q$. Non-increasing marginal values imposes testable restrictions since all values that rationalize a given bid may be increasing over some quantity range. (See Section 5 for an example with two units.) I shall focus on the case in which values exist that rationalize $\mathbf{b}_i$ given belief B_i, i.e. $V^*(\mathbf{b}_i, B_i) \neq \emptyset$. The question addressed here is whether any of these values satisfy the non-increasing marginal values assumption.

Let $Q(q)$ denote the step containing quantity q and let $b_{i,Q}$ be the level of all unit-bids on units in step Q. Bidder i must not prefer to make either of the following specific sorts of deviations, for any pair of quantities $q \leq q'$. (A) "Upward deviation at q": raise unit-bids on units $[\min Q(q), q]$ by same amount. (B) "Downward deviation at q'": Lower unit-bids on $[q', \max Q(q')]$ by same amount. Define

$$\overline{v}_q(B_i) \equiv \max \left\{ v : \sum_{x \in [\min Q(q), q]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, v; B_i)}{\mathrm{d}b_{i,x}} \leq 0 \right\} \tag{15}$$

$$\underline{v}_{q'}(B_i) \equiv \min \left\{ v : \sum_{x \in [q', \max Q(q')]} \frac{\mathrm{d}\Pi_{i,x}(b_{i,x}, v; B_i)}{\mathrm{d}b_{i,x}} \geq 0 \right\} \tag{16}$$

Now suppose that $\overline{v}_q(B_i) < \underline{v}_{q'}(B_i)$ for some $q \leq q'$. In order for the upward deviation at q to be unprofitable, there must be *some* quantity $\underline{q} \in [\min Q(q), q]$ such that $v_{i,\underline{q}} \leq \overline{v}_q(B_i)$. Similarly, for the downward deviation at q' to be unprofitable, there must be *some* quantity $\overline{q} \in [q', \max Q(q')]$ such that $v_{i,\overline{q}} \geq \underline{v}_{q'}(B_i)$. Overall, then, $v_{i,\overline{q}} > v_{i,\underline{q}}$. Since $q' \geq q$, however, $\overline{q} > \underline{q}$, violating the assumption of non-increasing marginal values. See Figures 3, 4.

Theorem 4 summarizes what we have just proven.

Theorem 4 (Testable restrictions). *The assumption of non-increasing marginal values is violated whenever there exists a pair of quantities $q \leq q'$ such that $\overline{v}_q(B_i) < \underline{v}_{q'}(B_i)$, where these terms are defined in (15,16).*

Discriminatory auction: In the discriminatory auction, Theorem 4 leads to the inequality

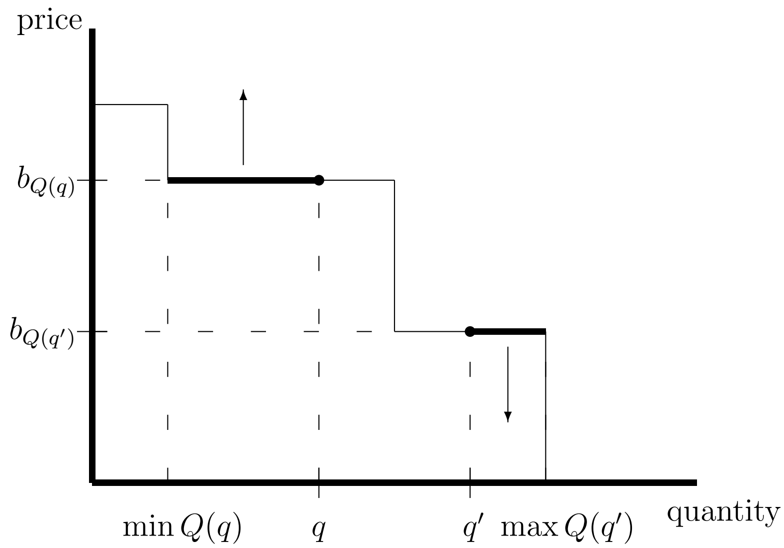

Figure 3: "Upward deviation at q" and "downward deviation at q'" for $q \leq q'$.

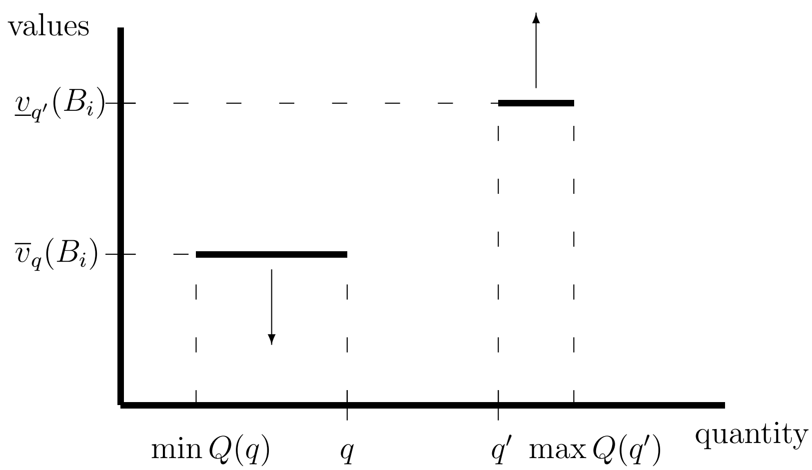

Figure 4: Best response imposes constraints on marginal values that can be mutually inconsistent with non-increasing marginal values.

restriction $\overline{v}_q^D(B_i) \geq \underline{v}_{q'}^D(B_i)$ for all $q' > q$ where

$$\overline{v}_q^D(B_i) = b_{Q(q)} + \frac{\sum_{x \in [\min Q(q), q]} \Pr_{B_i}\left(b_{Q(q)} > s_{i,x}\right)}{\sum_{x \in [\min Q(q), q]} \frac{\mathrm{d} \Pr_{B_i}\left(b_{Q(q)} > s_{i,x}\right)}{\mathrm{d} b_{i,x}}}$$

$$\underline{v}_{q'}^D(B_i) = b_{Q(q')} + \frac{\sum_{x \in [q', \max Q(q')]} \Pr_{B_i}\left(b_{Q(q')} > s_{i,x}\right)}{\sum_{x \in [q', \max Q(q')]} \frac{\mathrm{d} \Pr_{B_i}\left(b_{Q(q')} > s_{i,x}\right)}{\mathrm{d} b_{i,x}}}$$

Uniform price auction: Similarly, in the uniform price auction, Theorem 4 leads to the inequality restriction $\overline{v}_q^U(B_i) \geq \underline{v}_{q'}^U(B_i)$ for all $q' > q$ where

$$\overline{v}_q^U(B_i) = b_{Q(q)} + \frac{\sum_{x \in [\min Q(q), q]} (x-1) \Pr_{B_i}\left(s_{i,x} > b_{Q(q)} > s_{i,x-1}\right)}{\sum_{x \in [\min Q(q), q]} \frac{\mathrm{d} \Pr_{B_i}\left(b_{Q(q)} > s_{i,x}\right)}{\mathrm{d} b_{i,x}}}$$

$$\underline{v}_{q'}^U(B_i) = b_{Q(q')} + \frac{\sum_{x \in [q', \max Q(q')]} (x-1) \Pr_{B_i}\left(s_{i,x} > b_{Q(q')} > s_{i,x-1}\right)}{\sum_{x \in [q', \max Q(q')]} \frac{\mathrm{d} \Pr_{B_i}\left(b_{Q(q')} > s_{i,x}\right)}{\mathrm{d} b_{i,x}}}$$

4.3 Testing best response when bidder beliefs are not known

Theorem 4 can be interpreted as providing restrictions on observed bids for any given belief (the interpretation so far) *or* as providing restrictions on unobserved beliefs for any given bid. In particular, suppose that we have α-confidence that bidder i's belief $B_i(\mathbf{b}_i)$ about residual supply when he bids $\mathbf{b}_i$ belongs to a set of distributions Y. Theorem 4 allows us to reject the hypothesis that bidder i plays a best response with α-confidence as long as, for all $B_i \in Y$, $\overline{v}_q(B_i) < \underline{v}_{q'}(B_i)$ for some $q \leq q'$. In other words, the analysis here provides $\overline{S}(\overline{S}+1)/2$ distinct inequality restrictions on beliefs that must simultaneously hold for any given bid to be a best response.

4.4 Bounding the magnitude of "errors"

When Theorem 4 allows us to reject the best response hypothesis, we can go further to put a *lower bound* on bidder i's foregone expected profit when he bids $\mathbf{b}_i$. (For this section, I shall return to the case in which bidder i's belief B_i about residual supply is known. Of course, based on this analysis, we could easily formulate analogous results for the case when one only has α-confidence that B_i belongs to some set of distributions.)

More precisely, under the maintained assumption of non-increasing marginal values, we may conclude that bidder i must lose at least $d_i^*(\mathbf{b}_i; B_i)$ relative to a best response:

$$d_i^*(\mathbf{b}_i; B_i) \equiv \min_{\mathbf{v}_i : v_{i,1} \geq \ldots \geq v_{i,\overline{S}}} \left(\Pi_i(\mathbf{b}_i^*(\mathbf{v}_i; B_i), \mathbf{v}_i; B_i) - \Pi_i(\mathbf{b}_i, \mathbf{v}_i; B_i)\right) \text{ where} \tag{17}$$

$$\mathbf{b}_i^*(\mathbf{v}_i; B_i) \equiv \arg\max_{\mathbf{b}_i \in \mathcal{B}_i} \Pi_i(\mathbf{b}_i, \mathbf{v}_i; B_i) \tag{18}$$

$\mathbf{b}_i^*(\mathbf{v}_i; B_i)$ a best response bid for bidder i given values $\mathbf{v}_i$ and belief B_i. Thus, $d_i^*(\mathbf{b}_i; B_i)$ is a lower bound on bidder i's expected loss (in absolute terms) from bidding $\mathbf{b}_i$ rather than a best response. Let $\mathbf{v}_i^*(\mathbf{b}_i; B_i)$ be some values given which bidding $\mathbf{b}_i$ leads to this minimal possible loss. By definition, then, $d_i^*(\mathbf{b}_i; B_i)$ is the amount of "money left on the table" by type $\mathbf{v}_i^*(\mathbf{b}_i; B_i)$ when bidding $\mathbf{b}_i$ given belief B_i. (In terms of the notation of Section 3, $d_i^*(\mathbf{b}_i; B_i) = 0$ iff $\mathbf{v}_i^*(\mathbf{b}_i; B_i) \in V_i^*(\mathbf{b}_i; B_i)$, in which case we can not reject the best response hypothesis.) Given other values, bidding $\mathbf{b}_i$ involves even more foregone expected profit.

Solving (17) is not a trivial numerical exercise. To evaluate $\Pi_i(\mathbf{b}_i^*(\mathbf{v}_i; B_i), \mathbf{v}_i; B_i)$ for any given type $\mathbf{v}_i$, we must compute that type's payoff from a best response, i.e. run the algorithm described in Section 3.2. Furthermore, there is a different (17) problem for every observed bid. Fortunately, there are some simplifying features to the problem as well.

First-order conditions for computing $\mathbf{v}_i^*(\mathbf{b}_i; B_i)$. Additive separability simplifies (17) to

$$d_i^*(\mathbf{b}_i; B_i) \equiv \min_{\mathbf{v}_i : v_{i,1} \geq \ldots \geq v_{i,\overline{S}}} \sum_{q=1}^{\overline{S}} \left(\Pi_{i,q}(b_{i,q}(\mathbf{v}_i), v_{i,q}; B_i) - \Pi_{i,q}(b_{i,q}, v_{i,q}; B_i)\right) \tag{19}$$

Fix any $1 \leq \hat{q} \leq \overline{S}$ and suppose that $v_{i,\hat{q}} < v_{i,\hat{q}-1}$ so that it is feasible to increase bidder i's marginal value $v_{i,\hat{q}}$ without running afoul of non-increasing values. This will decrease the objective in (19) unless

$$\frac{\partial\Pi_{i,\hat{q}}(b_{i,\hat{q}},v_{i,\hat{q}};B_i)}{\partial v_{i,\hat{q}}} \leq \sum_q \frac{\mathrm{d}\Pi_{i,q}(b_{i,q}(\mathbf{v}_i),v_{i,q};B_i)}{\mathrm{d}v_{i,\hat{q}}} = \frac{\partial\Pi_{i,\hat{q}}(b_{i,\hat{q}}(\mathbf{v}_i),v_{i,\hat{q}};B_i)}{\partial v_{i,\hat{q}}} \tag{20}$$

where (20) follows from the Envelope Theorem. Similarly, if $v_{i,\hat{q}} > v_{i,\hat{q}+1}$ then we can decrease $v_{i,\hat{q}}$ without creating increasing values. In this case, a necessary condition for any solution to (17) is

$$\frac{\partial\Pi_{i,\hat{q}}(b_{i,\hat{q}},v_{i,\hat{q}};B_i)}{\partial v_{i,\hat{q}}} \geq \frac{\partial\Pi_{i,\hat{q}}(b_{i,\hat{q}}(\mathbf{v}_i),v_{i,\hat{q}};B_i)}{\partial v_{i,\hat{q}}}$$

More generally, suppose that Q is a step in bidder i's marginal value schedule (i.e. $v_{i,\min Q-1} > v_{i,\min Q} = ... = v_{i,\max Q} > v_{i,\max Q+1}$) and that $\hat{q} \in [\min Q, \max Q]$. It is feasible to simultaneously increase $(v_{i,\min Q},...,v_{i,\hat{q}})$ or to simultaneously decrease $(v_{i,\hat{q}},...,v_{i,\max Q})$.

Overall, we get the following $2\overline{S}$ necessary first-order inequality restrictions. For $\hat{q} = 1,...,\overline{S}$,

$$\sum_{q=\hat{q}}^{\max Q(\hat{q})} \frac{\partial\Pi_{i,q}(b_{i,q},v_{i,q};B_i)}{\partial v_{i,q}} \geq \sum_{q=\hat{q}}^{\max Q(\hat{q})} \frac{\partial\Pi_{i,q}(b_{i,q}(\mathbf{v}_i),v_{i,q};B_i)}{\partial v_{i,q}} \tag{21}$$

$$\sum_{q=\min Q(\hat{q})}^{\hat{q}} \frac{\partial\Pi_{i,q}(b_{i,q},v_{i,q};B_i)}{\partial v_{i,q}} \leq \sum_{q=\min Q(\hat{q})}^{\hat{q}} \frac{\partial\Pi_{i,q}(b_{i,q}(\mathbf{v}_i),v_{i,q};B_i)}{\partial v_{i,q}} \tag{22}$$

The $2\overline{S}$ inequalities (21,22) here are very similar to the $2\overline{S}$ inequalities (4,5) used for identification purposes, and with good reason. The identification problem can be viewed as a special case of the present problem in which it turns out that $d_i^*(\mathbf{b}_i;B_i) = 0$.

5 An Example with Two Asymmetric Bidders

This example illustrates the paper's main techniques and results.

Setup. Consider a uniform third-price auction of two units with two bidders having independent private values. A bid by bidder i is a two-dimensional vector of unit-bids, $\mathbf{b}_i = (b_{i,1}, b_{i,2})$, satisfying the restriction (i.e. rule) that $b_{i,1} \geq b_{i,2}$. The highest two unit-bids win and winners pay the third-highest unit-bid for each unit. Note that bidder i wins a first unit when $b_{i,1} > b_{-i,2}$ and wins the second unit when $b_{i,2} > b_{-i,1}$. (Equivalently, residual supply takes the simple form $s_{i,q} = b_{-i,-q}$.)

Observed bids. Bidder 1 submits all bids in the triangle $\{\mathbf{b}_1 : 1 \geq b_{1,1} \geq b_{1,2} \geq 0\}$, each equally likely. Bidder 2 submits all "flat" bids along the diagonal of this triangle, $\{\mathbf{b}_2 : 1 \geq b_{2,1} = b_{2,2} \geq 0\}$, with $b_2 \sim U[0,1]$. $\mathbf{b}_1$ and $\mathbf{b}_2$ are independent.

Analysis of bidder 1. Since $\mathbf{b}_2$ is always flat, bidder 1's second unit-bid $b_{1,2}$ never sets the price. Hence, bidder 1's weakly dominant strategy is to bid his true value on both units. Thus, any bid $\mathbf{b}_1$ can be rationalized as a best response given values $\mathbf{v}_1 = \mathbf{b}_1$. The rest of the discussion focuses on bidder 2.

Payoffs are additively separable across quantities. By Proposition 2, bidder 2's expected payoffs take the form $\Pi_2(\mathbf{b}_2, \mathbf{v}_2) = \Pi_{2,1}(b_{2,1}, v_{2,1}) + \Pi_{2,2}(b_{2,2}, v_{2,2})$, where

$$\Pi_{2,1}(b_{2,1}, v_{2,1}) = (v_{2,1} - E[b_{1,2}|b_{2,1} > b_{1,2}]) \Pr(b_{2,1} > b_{1,2}) \tag{23}$$

$$\begin{aligned} \Pi_{2,2}(b_{2,2}, v_{2,2}) = {} & (v_{2,2} - E[b_{1,1}|b_{2,2} > b_{1,1}]) \Pr(b_{2,2} > b_{1,1}) \\ & - E[b_{1,1} - b_{1,2}|b_{2,2} > b_{1,1}] \Pr{}_{B_i}(b_{2,2} > b_{1,1}) \\ & - E[b_{2,2} - b_{1,2}|b_{1,1} \geq b_{2,2} > b_{1,2}] \Pr(b_{1,1} \geq b_{2,2} > b_{1,2}) \end{aligned} \tag{24}$$

Equations (23,24) arise from the following thought experiment. Suppose that bidder 2 sets his unit-bids one at a time, starting from zero. When he sets only his first unit-bid, bidder 2 wins a first unit when $b_{2,1} > b_{1,2}$ and pays $b_{1,2}$ on that unit, and otherwise wins nothing. Next, when he raises his second unit-bid, two things may happen. (i) He wins a second unit when $b_{2,2} > b_{1,1}$ and pays $b_{1,1}$ on that unit and/or (ii) He raises the price paid on the first unit from $b_{1,2}$ to $b_{1,1}$ when $b_{2,2} > b_{1,1}$ or from $b_{1,2}$ to $b_{2,2}$ when $b_{2,2} \in (b_{1,2}, b_{1,1})$.

Example-specific facts. Collecting facts for later, observe that for all $x \in [0,1]$, $\Pr(x > b_{1,2}) = x(2-x)$, $\Pr(x > b_{1,1}) = x^2$, and $\Pr(b_{1,1} > x > b_{1,2}) = 2x(1-x)$ so $\frac{\mathrm{d}\Pr(x>b_{1,2})}{\mathrm{d}x} = 2(1-x)$, $\frac{\mathrm{d}\Pr(x>b_{1,1})}{\mathrm{d}x} = 2x$, and $\frac{\mathrm{d}\Pr(b_{1,1}>x>b_{1,2})}{\mathrm{d}x} = 2(1-2x)$. Furthermore, for $x \in [0,1]$ routine calculations show $E[b_{1,1}|x > b_{1,1}] = 2x/3$, $E[b_{1,2}|x > b_{1,2}] = \frac{x^2(1-2x/3)}{x(2-x)}$, $E[b_{1,2}|b_{1,1} > x > b_{1,2}] = x/2$, and $E[b_{1,1} - b_{1,2}|x > b_{1,1}] = x/3$. When $x > 1$, finally, these probabilities and expectations are the same as if $x = 1$.

Thus, unit-payoffs and their derivatives take the form

$$\Pi_{2,1}(b_{2,1}, v_{2,1}) = \left(v_{2,1} - \frac{b_{2,1}^2(1-2b_{2,1}/3)}{b_{2,1}(2-b_{2,1})}\right) b_{2,1}(2-b_{2,1})$$

$$= b_{2,1}(2-b_{2,1})v_{2,1} - b_{2,1}^2(1-2b_{2,1}/3) \tag{25}$$

$$\Pi_{2,2}(b_{2,2}, v_{2,2}) = (v_{2,2} - 2b_{2,2}/3)b_{2,2}^2 - (b_{2,2}/3)b_{2,2}^2 - (b_{2,2} - b_{2,2}/2)2b_{2,2}(1-b_{2,2})$$

$$= b_{2,2}^2(v_{2,2} - 1) \tag{26}$$

$$\frac{\mathrm{d}\Pi_{2,1}(b_{2,1}, v_{2,1})}{\mathrm{d}b_{2,1}} = (v_{2,1} - b_{2,1})2(1-b_{2,1}) \tag{27}$$

$$\frac{\mathrm{d}\Pi_{2,2}(b_{2,2}, v_{2,2})}{\mathrm{d}b_{2,2}} = 2b_{2,2}(v_{2,2} - 1) \tag{28}$$

Local indifference condition. Let $\mathbf{b}_2$ be an observed bid that we would like to rationalize as a best response. A necessary condition of best response is that bidder 2 must not prefer to slightly (a) raise his first unit-bid, (b) lower his second unit-bid, or (c) raise or lower both

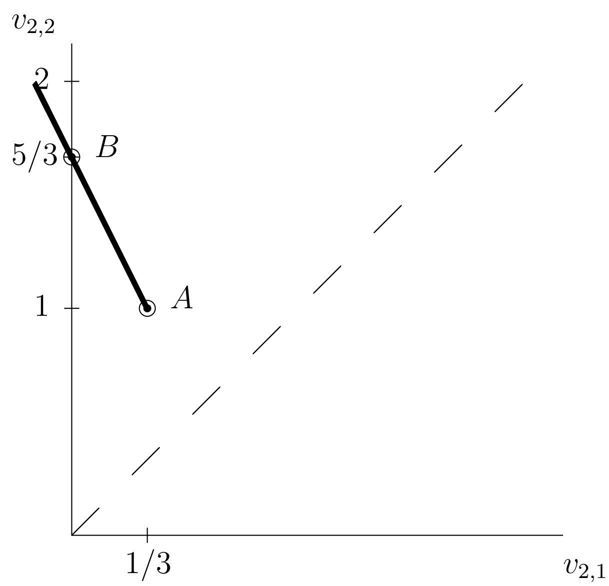

Figure 5: Values $\mathbf{v}_2$ locally indifferent to deviating from bid $\mathbf{b}_2 = (1/3, 1/3)$.

unit-bids:

$$\frac{\mathrm{d}\Pi_{2,1}(b_{2,1}, v_{2,1})}{\mathrm{d}b_{2,1}} \leq 0 \tag{a}$$

$$\frac{\mathrm{d}\Pi_{2,2}(b_{2,2}, v_{2,2})}{\mathrm{d}b_{2,2}} \geq 0 \tag{b}$$

$$\frac{\mathrm{d}\Pi_{2,1}(b_{2,1}, v_{2,1})}{\mathrm{d}b_{2,1}} + \frac{\mathrm{d}\Pi_{2,2}(b_{2,2}, v_{2,2})}{\mathrm{d}b_{2,2}} = 0 \tag{c}$$

Furthermore, if $b_{2,1} > b_{2,2}$, then (a,b) can be strengthened to equalities:

$$\frac{\mathrm{d}\Pi_{2,1}(b_{2,1}, v_{2,1})}{\mathrm{d}b_{2,1}} = 0 \text{ when } b_{2,1} > b_{2,2} \tag{a'}$$

$$\frac{\mathrm{d}\Pi_{2,2}(b_{2,2}, v_{2,2})}{\mathrm{d}b_{2,2}} = 0 \text{ when } b_{2,1} > b_{2,2} \tag{b'}$$

Case I: Downward-sloping bid. If $b_{2,1} > b_{2,2}$, then the first-order equalities (27,28) have a unique solution, $v_{2,1} = b_{2,1}$ and $v_{2,2} = 1$, corresponding to point A in Figure 5.

Case II: Flat bid. When $b_{2,1} = b_{2,2} \equiv b_2$, the equalities (a',b') still identify particular values given which bidder 2 does not have a profitable local deviation. There are many other values, however, that satisfy the relevant system of *inequalities* (a,b,c):

$$\left\{ \mathbf{v}_2 : v_{2,1} \leq b_2,\ v_{2,2} = 1 + (b_2 - v_{2,1}) \frac{1 - b_2}{b_2} \right\} \tag{29}$$

Figure 5 illustrates the case $b_2 = 1/3$, given which the set of values (29) is the ray $\overrightarrow{AB}$. (There is no a priori restriction that marginal values must be positive.)

Second-order condition / global deviations. Next consider the relevant second-order conditions, noting that $\frac{\mathrm{d}^2 \Pi_{2,1}(b_{2,1}, v_{2,1})}{\mathrm{d}b_{2,1}^2} = -2(1 - 2b_{2,1} + v_{2,1})$ and $\frac{\mathrm{d}^2 \Pi_{2,2}(b_{2,2}, v_{2,2})}{\mathrm{d}b_{2,2}^2} = -2(1 - v_{2,2})$. For the bid $\mathbf{b}_2 = (1/3, 1/3)$, consider first the values $(1/3, 1)$ represented by point A in Figure 5. (a,b) hold with equality and the relevant second-order conditions are satisfied: $\frac{\mathrm{d}^2 \Pi_{2,1}(1/3, 1/3)}{\mathrm{d}b_{2,1}^2} = -4/3$ and $\frac{\mathrm{d}^2 \Pi_{2,2}(1/3, 1)}{\mathrm{d}b_{2,2}^2} = 0$. Consider next the values $(0, 5/3)$ represented by point B in Figure 5. Since (a,b) hold with inequality, the only relevant second-order condition is $\frac{\mathrm{d}^2 \Pi_{2,1}(1/3,0) + \Pi_{2,2}(1/3,5/3)}{\mathrm{d}b^2} \leq 0$. This condition fails since $\frac{\mathrm{d}^2 \Pi_{2,1}(1/3,0)}{\mathrm{d}b^2} + \frac{\mathrm{d}^2 \Pi_{2,1}(1/3,0)}{\mathrm{d}b^2} = -2/3 + 4/3 > 0$. Indeed, along the ray $\overrightarrow{AB}$, the second-order condition is only satisfied for those values for which $v_{2,1} > 1/9$ and $v_{2,2} < 13/9$, i.e. on the line segment $\overline{AB'}$ in Figure 6. Finally, all of these remaining types do not have any profitable global deviation (details omitted to save space).

Partial identification. Bidder 2 finds bid $\mathbf{b}_2 = (1/3, 1/3)$ to be a best response whenever his values lie on the line segment $\overline{AB'}$. More generally, for any $b_2 \in (0, 1)$, bidder 2 finds bid $\mathbf{b}_2 = (b_2, b_2)$ to be a best response whenever his values lie on the line segment between $(b_2, 1)$ to $(b_2^2, 1 + (1 - b_2)^2)$. This allows us to characterize all possible value distributions for $\mathbf{v}_2$ that might rationalize the observed distribution of bids.

Robust policy recommendations. While bidder 2's values are not uniquely identified, we can conclude that $\Pr(v_{2,1} < v_{2,2}) = 1$, i.e. that bidder 2 has increasing returns to consumption. A natural policy alternative to consider, then, is whether the auctioneer

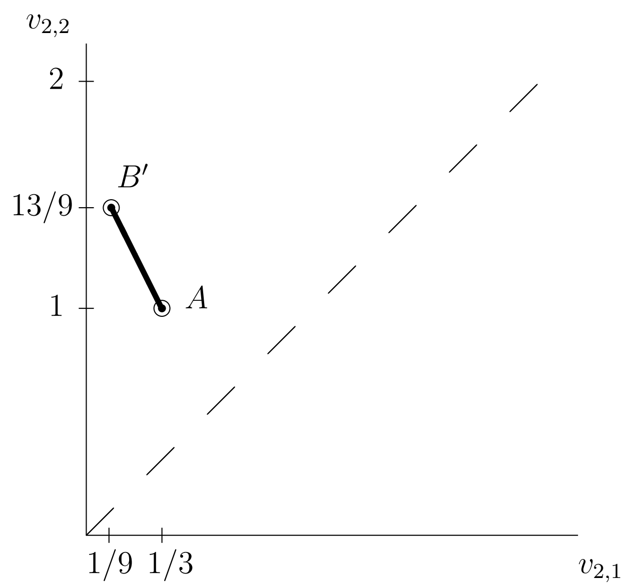

Figure 6: Values $\mathbf{v}_2$ with no profitable local deviation from bid $\mathbf{b}_2 = (1/3, 1/3)$.

should bundle the units together. In particular, will the auctioneer raise more revenue if she packages the two units together and sells them using a second-price auction?

To perform this counterfactual experiment, we need to know the distribution of each bidder's value for the bundle, i.e. $v_{1,1} + v_{1,2}$ and $v_{2,1} + v_{2,2}$. Recall that bidder 1 is observed making all bids in the triangle $\{\mathbf{b}_1 : 1 \geq b_{1,1} \geq b_{1,2} \geq 0\}$, each equally likely, and that we have inferred that bidder 1 has values $(b_{1,1}, b_{1,2})$ when he bids $(b_{1,1}, b_{1,2})$. This nails down the distribution of $v_{1,1} + v_{1,2}$. Since bidder 2's values are only partially identified, however, there are a few ways to proceed. Recall that bidder 2 is only observed making flat bids $\{\mathbf{b}_2 : b_{2,1} = b_{2,2} \equiv b_2 \in [0, 1]\}$, each equally likely.

The standard approach identifies each observed bid (b_2, b_2) with values $(b_2, 1)$. Under this approach, $v_{2,1} + v_{2,2} \sim U[1, 2]$ and numerical computations show that expected revenue will be about .96 in a second-price auction but only about .83 in the given third-price auction. (See the Appendix for computation details.) *Tentative* policy conclusion: bundle the units. Yet this policy conclusion might be sensitive to the implicit choice of value distribution.

More generally, we have seen how $(v_{2,1}, v_{2,2})$ rationalizes the bid (b_2, b_2) as long as it lies anywhere on the interval $\overline{AB'}$ where $A = (b_2, 1)$ and $B' = (b_2^2, 1+(1-b_2)^2)$. This puts upper and lower bounds on $v_{2,1} + v_{2,2}$ corresponding to each bid (b_2, b_2):

$$\min\{b_2 + 1, b_2^2 + 1 + (1 - b_2)^2\} \leq v_{2,1} + v_{2,2} \leq \max\{b_2 + 1, b_2^2 + 1 + (1 - b_2)^2\}$$

Thus, we may identify "the most pessimistic" distribution and "the most optimistic" distribution and compute upper and lower bounds for expected revenue in the counterfactual. In particular, expected revenue in the second-price auction will be at least about .955 and at most about .99. Since this is always greater than .83, the recommendation to bundle is robust. Indeed, if anything, the standard approach may somewhat understate the potential advantage of bundling.

Remark 2. *Suppose that the observed distribution of bids is generated in an equilibrium. Then the auctioneer can unambiguously increase revenue by bundling the two units together and selling them in a second-price auction, assuming that bidders continue to play an equilibrium.*

Proof. Spreadsheets that implement these computations are posted at www.mit.edu/˜mcadams.

□

Testing the best response hypothesis. What if bidder 2's marginal values are known to be non-increasing? This contradicts our previous finding that $\Pr(v_{2,1} < v_{2,2}) = 1$. Unless we are willing to relax this or some other assumption, we must conclude that bidder 2 fails to play a best response with probability one!

Bounding bidder losses relative to best response ("errors"). Even though bidder 2's value is only partially identified, it is possible to put a *lower bound* on how much lower his expected profit must be with each bid than in a best response. Let $\tilde{\mathbf{b}}_2$ be any observed bid. Then

$$\min_{\mathbf{v}_2 : v_{2,1} \geq v_{2,2}} \left(\Pi_2(\mathbf{b}_2^*(\mathbf{v}_2), \mathbf{v}_2) - \Pi_2(\tilde{\mathbf{b}}_2, \mathbf{v}_2) \right) \tag{30}$$

is the minimal difference between the payoff from a best response and from $\tilde{\mathbf{b}}_2$, evaluated over all non-increasing values, where $\mathbf{b}_2^*(\mathbf{v}_2)$ denotes a best response given values $\mathbf{v}_2$. Once we have solved (30), we will also implictly have found values given which this minimal "error" is realized.

First, we need to compute best responses. This is easy in this example. Whenever $v_{2,1} \geq v_{2,2}$ and $v_{2,2} \leq 1$, a best response bid is $\mathbf{b}_2^*(\mathbf{v}_2) = (v_{2,1}, 0)$. On the other hand, $v_{2,1} \geq v_{2,2} \geq 1$ implies that bid $(1, 1)$ is a best response. Thus, (30) becomes

$$\min \Bigg\{ \min_{\mathbf{v}_2 : \min\{1, v_{2,1}\} \geq v_{2,2}} \left(\Pi_{2,1}(v_{2,1}, v_{2,1}) + \Pi_{2,2}(0, v_{2,2}) - \Pi_{2,1}(\tilde{b}_{2,1}, v_{2,1}) - \Pi_{2,2}(\tilde{b}_{2,2}, v_{2,2}) \right), \\ \min_{\mathbf{v}_2 : v_{2,1} \geq v_{2,2} > 1} \left(\Pi_{2,1}(1, v_{2,1}) + \Pi_{2,2}(1, v_{2,2}) - \Pi_{2,1}(\tilde{b}_{2,1}, v_{2,1}) - \Pi_{2,2}(\tilde{b}_{2,2}, v_{2,2}) \right) \Bigg\}$$

Fortunately, this problem can be simplified in a few ways. First, $\Pi_{2,2}(0, v_{2,2}) = 0$. Second, without loss we can focus on the case in which $1 \geq v_{2,1} \geq v_{2,2}$. (If $v_{2,1} > 1$ and/or $v_{2,2} > 1$, then the objective (30) can be reduced by replacing $v_{2,1}$ with $\min\{1, v_{2,1}\}$ and $v_{2,2}$ with $\min\{1, v_{2,2}\}$.[11] Third, without loss we can restrict attention to the boundary at which $v_{2,1} = v_{2,2} \equiv v_2 \in [0, 1]$. (For any given $v_{2,1}$, $\Pi_{2,2}(\tilde{b}_{2,2}, v_{2,2})$ is maximized when the constraint $v_{2,2} = v_{2,1}$ is binding.) Applying (25,26) and collecting terms, (30) becomes

$$\min_{v_2 \in [0,1]} \left(-v_2^3/3 + v_2^2 - v_2 \left(2\tilde{b}_{2,1} - \tilde{b}_{2,1}^2 + \tilde{b}_{2,2}^2 \right) + \left(\tilde{b}_{2,1}^2 - 2\tilde{b}_{2,1}^3/3 + \tilde{b}_{2,2}^2 \right) \right) \tag{31}$$

The minimand of (31) $v_2^*(\tilde{\mathbf{b}}) = 1 - \sqrt{1 - (2\tilde{b}_{2,1} - \tilde{b}_{2,1}^2 + \tilde{b}_{2,2}^2)}$ when $2\tilde{b}_{2,1} - \tilde{b}_{2,1}^2 + \tilde{b}_{2,2}^2 \leq 1$; otherwise $v_2^*(\tilde{\mathbf{b}}_2) = 1$. By construction, bidder 2's inferred error associated with bid $\tilde{\mathbf{b}}_2$ is smallest given marginal values $(v_2^*(\tilde{\mathbf{b}}_2), v_2^*(\tilde{\mathbf{b}}_2))$.

[11]This step uses the fact that each unit-payoff function $\Pi_{2,q}$ has non-decreasing differences in $(b_{2,q}, v_{2,q})$.

Magnitude of inferred "error". How large are these inferred errors? Recall that bidder 2 makes bids of form (b_2, b_2) where $b_2 \sim U[0,1]$. On an *absolute* basis, the magnitude of (31) ranges from zero (when $b_2 \in \{0,1\}$) to approximately .090 (when $b_2 \approx .45$). On a *percentage* basis relative to the best response bid $\mathbf{b}_2^*(v_2^*(b_2,b_2), v_2^*(b_2,b_2))$, the inferred loss ranges from 0% (when $b_2 = 1$) up to approximately 100% (when $b_2 \approx 0$). For instance, consider $b_2 = .33$. Bidder 2's absolute loss relative to best response is minimized given values $(v_2^*(1/3,1/3), v_2^*(1/3,1/3))$ where $v_2^*(1/3,1/3) = 1 - 1/\sqrt{3} \approx .423$. This type's best response is to bid close to $(.423, 0)$ which gives expected payoff about .153. Given bid $(1/3,1/3)$, however, his expected payoff is only about .084, for an absolute loss of at least .069 and percentage loss of 45%.

Remark 3 gathers some facts gotten by numerical computation.

Remark 3. *Suppose that bidder 2 has non-increasing marginal values. Then none of bidder 2's observed bids can be a best response, except for bids* $(0,0)$ *and* $(1,1)$ *which are made with zero probability. Indeed, we can put lower bounds on how much bidder 2 must lose relative to playing best response.*[12]

- *Bidder 2's minimal inferred loss relative to best response is highest for the bid* $(.45,.45)$*, in which case bidder 2 is inferred to lose at least* .090 *for percentage loss 25.0% by failing to play a best response.*

- *Averaged over the event in which he bids* (b_2,b_2) *for* $b_2 \in [.3,.6]$*, bidder 2 is inferred to lose at least* .074 *for percentage loss of 18.2% by failing to play a best response.*

[12]For these computations, each observed bid is matched with a type who finds that bid to have the smallest *absolute* payoff difference from a best response. We then compute that type's expected payoff from the observed bid and from a best response. A similar exercise that seeks to minimize percentage losses (rather than absolute losses) adds no new insight.

- *Averaged over all observed bids, bidder 2 is inferred to lose at least* .033 *for percentage loss of 8.3% by failing to play a best response.*

Proof. Spreadsheets that implement these computations are posted at www.mit.edu/˜mcadams.

□

6 Concluding remarks

When faced with data generated in a game, a natural first question is whether that data could be consistent with equilibrium behavior. This paper makes progress on two fronts in answering this question for discriminatory and uniform price multi-unit auctions.

First, this paper provides a more complete answer than the existing literature to the question of whether observed bids can be rationalized by equilibrium behavior. Given correlated private values, I provide sufficient conditions for a given distribution of bids to be rationalized as well as techniques to construct corresponding equilibrium strategies. In settings satisfying an "observable beliefs property" (including the important special case of independent private values), furthermore, a full characterization is provided of all distributions of bidder values that could rationalize the bids.

Second, a novel test of the best response hypothesis is introduced that does not require bidder values to be observed, if one is willing to impose the extra assumption that bidders have non-increasing marginal values. The strength of this test is that we do not need to make any assumptions about others' strategies to test whether bidder i plays a best response. Furthermore, when we are led to conclude that bidder i fails to play a best response, methods are provided to put a lower bound dollar figure on how much bidder i must be losing by failing to play a best response. In the prior literature, measuring the magnitude of such losses required that bidder values be observed. My approach puts a *lower bound* on such losses without observing anything about values.

The techniques developed here allow one to detect whether some bidders are not playing a best response in the sense of short-run profit-maximization. Much work remains to be done, however, to distinguish between competing alternative hypotheses regarding the nature of "inferred bidder errors". For instance, it could be that bidders literally make mistakes from time to time, that they have a non-monetary preference for submitting "simple" bids, or that they stop thinking at some point about what exactly to bid. Or, some bidder(s) might be "first-movers" in the sense of committing to a bidding strategy that is not a best response. Or, in a richer setting allowing for repeated interactions, bidders could be playing a dynamic best response.

Appendix

Proof of Remark 1

Independent private values: Since $(\mathbf{b}_0, \mathbf{v}_1, ..., \mathbf{v}_n)$ are independent, bidder i faces the same residual supply distribution regardless of his values. *Sparse bids:* Whenever bidder i bids $\mathbf{b}_i$, all others' bids are determined (say $\mathbf{b}_j(\mathbf{b}_i)$ for all $j \neq i, 0$) by sparse support. Consequently, residual supply can be inferred from the distribution of the auctioneer's bid $\mathbf{b}_0$ conditional on $(\mathbf{b}_1(\mathbf{b}_i), ..., \mathbf{b}_{i-1}(\mathbf{b}_i), \mathbf{b}_i, \mathbf{b}_{i+1}(\mathbf{b}_i), ..., \mathbf{b}_n(\mathbf{b}_i))$.

Proof of Proposition 2

By induction, it suffices to prove that for all $1 \leq q \leq \overline{S}$,

$$\Pi^U_{i,q}(b_{i,q}, v_{i,q}; B_i) = \Pi^U_i((b_{i,1}, ..., b_{i,q}, 0, ..., 0), \mathbf{v}_i; B_i) - \Pi^U_i((b_{i,1}, ..., b_{i,q-1}, 0, ..., 0), \mathbf{v}_i; B_i)$$

What is the incremental effect of bidding $(b_{i,1},...,b_{i,q},0,...,0)$ rather than $(b_{i,1},...,b_{i,q-1},0,...,0)$?

Case I: When $b_{i,q} > s_{i,q}$. In this case, bidder i wins $q-1$ units and pays $s_{i,q-1}$ when he bids $(b_{i,1},...,b_{i,q-1},0,...,0)$ ("the old bid") and wins q units and pays $s_{i,q}$ when he bids $(b_{i,1},...,b_{i,q},0,...,0)$ ("the new bid"). The total impact on expected profit in this case is

$$(v_{i,q} - E[s_{i,q}|b_{i,q} > s_{i,q}] - (q-1)E[s_{i,q} - s_{i,q-1}|b_{i,q} > s_{i,q}]) \Pr{}_{B_i}(b_{i,q} > s_{i,q}) \quad (32)$$

Case II: When $s_{i,q} \geq b_{i,q} > s_{i,q-1}$. Here bidder i wins $q-1$ units when he makes the old bid or the new bid, but the price increases from $s_{i,q-1}$ to $b_{i,q}$ under the new bid. The total impact on expected profit in this case is

$$-(q-1)E[b_{i,q} - s_{i,q-1}|s_{i,q} \geq b_{i,q} > s_{i,q-1}] \Pr{}_{B_i}(s_{i,q} \geq b_{i,q} > s_{i,q-1}) \quad (33)$$

Case III: When $s_{i,q-1} \geq b_{i,q}$. Here bidder i wins $q-1$ units and pays the same price when he makes the old bid or the new bid. Thus there is no incremental impact on expected profit.

Summing the terms (32,33) yields (3). This completes the proof.

Extension of Algorithm in Section 3.2

General case: $K^* \leq \overline{S}$. The algorithm is inductive. For every $1 \leq q \leq \overline{S}$, every $b \geq 0$, and every $K \leq K^*$, we will find both

$$\pi_i(\mathbf{v}_i; q, b, K, \geq) \equiv \sup_{\mathbf{b}_i \in \mathcal{B},\ b_{i,q} \geq b,\ K(b_{i,1},...,b_{i,q},0,...,0) \leq K} \Pi_i\left((b_{i,1},...,b_{i,q},0,...,0), \mathbf{v}_i; B_i\right) \quad (34)$$

$$\pi_i(\mathbf{v}_i; q, b, K, =) \equiv \sup_{\mathbf{b}_i \in \mathcal{B},\ b_{i,q} = b,\ K(b_{i,1},...,b_{i,q},0,...,0) \leq K} \Pi_i\left((b_{i,1},...,b_{i,q},0,...,0), \mathbf{v}_i; B_i\right) \quad (35)$$

where $K(\mathbf{b}_i)$ is the number of steps in the bid $\mathbf{b}_i$. In words, $\pi_i(\mathbf{v}_i; q, b, \geq)$ is the most expected payoff that bidder i can get given values $\mathbf{v}_i$ when he is constrained to submit a permissible bid with at most K steps in which $b_{i,q} \geq b$ and $b_{i,q+1} = ... = b_{i,\overline{S}} = 0$. Similarly, $\pi_i(\mathbf{v}_i; q, b, =)$ is the most that he can get if his bid must satisfy the further constraint that $b_{i,q} = b$. Our goal is to calculate $\pi_i(\mathbf{v}_i; \overline{S}, 0, K^*, \geq)$.

First, by definition, $\pi_i(\mathbf{v}_i; q, b, K, \geq) = \sup_{b' \geq b} \pi_i(\mathbf{v}_i; q, b, K, =)$. Thus, it suffices to compute $\pi_i(\mathbf{v}_i; q, b, K, =)$ for all $1 \leq q \leq \overline{S}$, all $b \geq 0$, and all $K \leq K^*$. To complete the algorithm by induction, note that

$$\pi_i(\mathbf{v}_i; q+1, b, K, =) = \Pi_{i,q+1}(b, v_{i,q+1}; B_i) + \max\left\{\pi_i(\mathbf{v}_i; q, b, K, =), \pi_i(\mathbf{v}_i; q, b, K-1, \geq)\right\} \tag{36}$$

In words, when bidder i bids b on unit $q+1$, he has two sorts of choices regarding what to bid on unit q. First, he can bid b on unit q, in which case his bid on units $[1, q]$ can have up to K steps. His expected payoff on those units can then be as high as $\pi_i(\mathbf{v}_i; q, b, K, =)$. Second, he can bid more than b on unit q, in which case his bid on units $[1, q]$ can have up to $K-1$ steps. His expected payoff on those units can then be as high as $\pi_i(\mathbf{v}_i; q, b, K-1, \geq)$. This completes the algorithm.

Proof of Theorem 3

Suppose that (a,b) fail, i.e. $\mathbf{v}'_i \in V_i(\mathbf{b}'_i, B_i)$ finds $\mathbf{b}'_i$ to be a best response and $\mathbf{v}_i \in V_i(\mathbf{b}_i, B_i)$ finds $\mathbf{b}_i$ to be a best response. By assumption, $\mathbf{v}'_i \geq \mathbf{v}_i$ but $\mathbf{b}'_i \not\geq \mathbf{b}_i$. For the set of best response bids to be increasing in the strong set order, it must be the case that these bids' greatest lower bound $\mathbf{b}'_i \wedge \mathbf{b}_i$ is a best response given values $\mathbf{v}_i$ and their least upper bound $\mathbf{b}'_i \vee \mathbf{b}_i$ is a best response given values $\mathbf{v}'_i$. That is, (c) is satisfied.

References

Anton, J., and D. Yao (1989): "Split-Awards, Procurement, and Innovation," *RAND Journal of Economics*, 20(4), 538–552.

Athey, S., and P. Haile (2002): "Identification of Standard Auction Models," *Econometrica*, 70, 2107–2140.

Athey, S., and P. Haile (2005): "Nonparametric Approaches to Auctions," Handbook of Econometrics, Volume 6 (forthcoming).

Borenstein, S., J. Bushnell, and F. Wolak (2002): "Measuring Market Inefficiencies in California's Restructured Wholesale Electricity Market," *American Economic Review*, 92(5), 1376–1405.

Cason, T., and C. Plott (1996): "EPA's New Emissions Trading Mechanism: A Laboratory Evaluation," *Journal of Environmental Economics and Management*, 30, 133–160.

Donald, S., and H. Paarsch (1996): "Identification, Estimation, and Testing in Parametric Empirical Models of Auctions within the Independent Private Values Paradigm," *Econometric Theory*, 12, 517–567.

Ellerman, D., H. Jacoby, and A. Decaux (1998): "The Effects on Developing Countries of the Kyoto Protocol and Carbon Dioxide Emissions Trading," NBER Working Paper No. 2019.

Ellerman, D., P. Joskow, R. Schmalensee, J.-P. Montero, and E. Bailey (2000): *Markets for clean air: the U.S. acid rain program.* Cambridge University Press, Cambridge, UK.

Guerre, E., I. Perrigne, and Q. Vuong (2000): "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68(3), 525–574.

Harvey, S., and W. Hogan (2001): "On the Exercise of Market Power Through Strategic Withholding in California," manuscript, available at http://ksghome.harvard.edu/~whogan.cbg.ksg.

Hortacsu, A. (2002a): "Bidding Behavior in Divisible Good Auctions: Theory and Evidence from the Turkish Treasury Auction Market," Working Paper (University of Chicago).

——— (2002b): "Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market," Working Paper (University of Chicago).

Hortacsu, A., and S. Puller (2004): "Testing Strategic Models of Bidding in Deregulated Electricity Markets: A Case Study of ERCOT," Working Paper (University of Chicago).

Joskow, P., and E. Kahn (2001): "A Quantitative Analysis of Pricing Behavior in California's Wholesale Electricity Market During Summer 2000," *Energy Journal*, 23(4), 1–35.

Klemperer, P., and M. Meyer (1989): "Supply Function Equilibria in Oligopoly under Uncertainty," *Econometrica*, 57(6), 1243–1277.

McAdams, D. (2002): "Monotone Equilibrium in Multi-Unit Auctions," MIT Sloan Working Paper #4254-02, available at http://www.mit.edu/~mcadams.

Wolak, F. (2003): "Identification and Estimation of Cost Functions Using Observed Bid Data: An Application to Competitive Electricity Markets," in *Advances in Economics and*

Econometrics: Theory and Applications, Eighth World Congress, ed. by M. Dewatripont, L. Hansen, and S. Turnovsky, vol. 2, pp. 115–149. Cambridge University Press, Cambridge.

Wolak, F., and R. Patrick (2001): "The Impact of Market Rules and Market Structure on the Price Discrimination Process in the England and Wales Electricity Market," NBER Working Paper 8248.

Wolfram, C. (1998): "Strategic Bidding in a Multi-Unit Auction: An Empirical Analysis of Bids to Supply Electricity in England and Wales," *RAND Journal of Economics*, 29(4), 703–25.

——— (1999): "Measuring Duopoly Power in the British Electricity Spot Market," *American Economic Review*, 89(4), 805–26.