Monotone Equilibrium in Multi-Unit Auctions

David McAdams*
MIT

January 2006

Abstract

In two-sided multi-unit auctions having a variety of payment rules, including uniform-price and discriminatory auctions, a monotone pure strategy equilibrium (MPSE) exists when bidders are risk-neutral with independent multi-dimensional types and interdependent values. In fact, all mixed strategy equilibria are ex post allocation- and interim expected payment-equivalent to MPSE. Thus, for standard expected surplus / revenue analysis, there is no loss restricting attention to monotone strategies.

JEL classification numbers: D44, D62, D82. Keywords: multi-unit auctions, uniform-price auction, discriminatory auction, monotone equilibrium, modularity, single-crossing condition.

1 INTRODUCTION

The U.S. Treasury's bond issue auctions and the NYSE's opening batch auctions are just two among many examples of real-world auctions of multiple

*David McAdams, MIT Sloan School of Management E52-448, 50 Memorial Drive, Cambridge, MA 02142. E-mail: mcadams@mit.edu. I thank John McMillan, Robert Wilson, seminar participants at Carnegie Mellon GSIA, MIT Sloan, NYU Stern, Northwestern, Penn State, WUSL Olin, U Michigan, U North Carolina, and Yale, and especially Susan Athey and Paul Milgrom for their helpful suggestions on the November 2000 version of this paper. An editor and two referees helped to substantially improve the paper. Financial support by the National Science Foundation grant #SES-0241468, the Olin Foundation, and the State Farm Companies Foundation is gratefully acknowledged.

identical objects (so-called 'multi-unit auctions'). A bid in such an auction is a demand and/or supply *schedule*, specifying a price (or 'unit-bid') for each unit, and a bidder receives or keeps a given unit if his bid on that unit is among the highest unit-bids. There are a wide variety of payment rules for such auctions, but two sorts are the most common in practice: discriminatory and uniform-price auctions. Until the late 1990s, the U.S. Treasury used a discriminatory auction to issue bonds: bidders' paid their bids for each unit that they won. Recently the Treasury changed its auction to a uniform $(S+1)$-st price format: all bidders pay a price equal to the highest losing bid (S is the number of units). Every morning at the beginning of trading, NYSE market-makers run a two-sided auction (bids are submitted overnight) and all trades at that time execute at the same price. The Paris and Amsterdam exchanges also open trading with batch auctions which may be thought of, at least approximately, as uniform $(S + 1/2)$-th price auctions: bidders pay (if buying) or receive (if selling) the average market-clearing price.[1]

Despite their practical importance, the theory of multi-unit auctions is quite incomplete. Worse still, single-object auction theory provides an unreliable guide in settings with multi-unit demand. For example, critics of the Treasury's plan to switch to the uniform-price auction noted that, even though this auction appears similar to a second-price auction (which is an $(S + 1)$-st price auction with $S = 1$), bidders will not submit truthful demand schedules in equilibrium. Indeed, Back and Zender (1993) expanded an example in Wilson (1979) showing that the uniform $(S + 1)$-st price auction can have equilibria with arbitrarily low revenue! Since intuitions drawn from single-object auction theory can be deceptive, we need to develop a free-standing multi-unit auction theory.

This paper establishes basic structural properties of equilibria in multi-unit auctions when bidders have multi-unit demand, independent multidimensional types, and valuations that are increasing in own type and nondecreasing in others' types ('interdependent values'). Every mixed strategy equilibrium is shown to be equivalent to a monotone pure strategy equilibrium ('MPSE'), in the sense of ex post allocation- and interim expected payment-equivalence. (Two equilibria are *ex post allocation-equivalent* if their induced allocations are the same with probability one and *interim expected*

[1]To the best of my knowledge, NYSE does not have a formally defined price-setting rule. Since "specialists must maintain a *fair*, competitive, orderly and efficient market" (NYSE website, italics added), however, the $(S+1/2)$-price auction is a natural modelling candidate since it treats buyers and sellers symmetrically.

payment-equivalent if each bidder makes the same interim expected payment with probability one. See Section 2 for formal definitions.)

While my results are cast in a setting with multiple identical units, this characterization result is new even for the well-studied case of single-object auctions. For example, in the second-price auction with interdependent values, this paper is the first to show that every mixed strategy equilibrium is ex post allocation- and interim expected payment-equivalent to a MPSE. One reason why this is important is that symmetric single-object auctions are efficient iff bidders adopt symmetric monotone strategies. Thus, to prove efficiency of all equilibria in such an auction, one only needs to prove that all MPSE are symmetric.

Monotonicity is important in multi-unit auctions for several reasons. First, by this paper's characterization result, all mixed strategy equilibria lead to the same bidder surplus and the same auctioneer revenue as if all bidders adopted monotone pure strategies. In other words, there is no loss in restricting attention to MPSE for the purpose of standard surplus and revenue analysis. Second, monotonicity is a natural property that can serve as an intuition check as we learn how multi-unit auctions differ from single-object auctions. This paper explores the domain in which monotonicity is preserved, but there are other settings in which it does not continue to hold. For example, when bidders have private values, monotonicity can fail given risk-aversion and/or affiliation. In first-price or second-price auctions, on the other hand, risk-aversion and affiliation do not lead monotonicity to fail.[2] (See McAdams (2002a) for a variety of examples with non-monotone strategies and discussion.) Finally, monotonicity puts testable restrictions on observed bids since higher types must make higher bids (see McAdams (2005), Theorem 3).

This paper complements a growing literature studying existence and monotonicity of equilibrium in multi-unit auctions. There are several important dimensions to models of such auctions so, perhaps not surprisingly, the models in this literature are typically incomparable in their generality. Each paper makes a contribution by relaxing certain assumptions but at the cost of requiring a relatively strong assumption elsewhere. In the brief (and non-exhaustive) review to follow, I italicize the key aspects in each paper that are

[2]For first-price auctions with risk-aversion and affiliated types (including the case of affiliated private values), see Reny and Zamir (2004) and McAdams (2003b). Reny and Zamir (2004) prove existence of MPSE while McAdams (2003b) shows that every equilibrium is ex post allocation- and ex post payment-equivalent to some MPSE.

less general than in the others, summarizing at the end how this paper relates. Reny (1999) shows existence of MPSE in the one-sided discriminatory auction given multi-dimensional *independent private values* ('IPV'). Using an entirely different approach, Jackson and Swinkels (2005) (hereafter 'JS') establish existence of equilibrium in *distributional strategies* in a wide variety of auctions when bidders have multi-dimensional correlated *private values*. Also given *private values* in *large* auctions, Cripps and Swinkels (2005) ('CS') show that all equilibria of the uniform-price auction are asymptotically efficient. In yet another vein, Fudenberg, Mobius, and Sziedl (2003) ('FMS') and Reny and Perry (2003) ('RP') establish existence and asymptotic efficiency of MPSE in *large* uniform-price two-sided auctions when bidders have *single-unit demand*. FMS also require that bidders have *private values*, while RP allow for interdependent values given affiliated signals drawn from a *symmetric distribution*. This paper takes another sort of approach, exploiting the lattice structure of bids in multi-unit auctions, to establish existence and that all equilibria are equivalent to MPSE in a variety of multi-unit auctions when bidders receive multi-dimensional *independent signals* and there may be just a few bidders having multi-unit demand and interdependent values. The working-paper version also proves existence of MPSE given certain sorts of allocative externalities à la Jehiel, Moldavanu, and Stacchetti (1996). In short, this paper relies on independence while JS, CS, and FMS rely on private values and CS, FMS, and RP on there being sufficiently many bidders.

Another paper with closely related results is Bresky (2000) who shows that all equilibria are MPSE given *independent private values* in *discriminatory and all-pay auctions* (but not in uniform-price auctions). For these sorts of multi-unit auctions, monotonicity comes fairly immediately for the same reason that it does so in the first-price auction: given IPV, every best response is monotone regardless of what strategies others adopt. In the second-price auction as in uniform-price auctions, however, bidders can have many non-monotone best responses even in the IPV case and it is not obvious that *all* equilibria must be equivalent to MPSE. The possibility of interdependent values creates even more challenges not addressed by Bresky. Lastly, the existence result here applies and extends techniques developed in McAdams (2003a). McAdams proves MPSE existence in the uniform S-th and $(S+1)$-st price auctions given independent types, but my argument there does not apply to uniform $(S+\alpha)$-th price auctions when $\alpha \in (0,1)$ nor to other sorts of auctions. Furthermore, that paper does not have any characterization result.

The remainder of the paper is organized as follows: Section 2 lays out the model. Section 3 presents and proves the main results and discusses key assumptions. Some remarks conclude the paper.

2 MODEL

For clarity and brevity, the analysis here will be restricted to the two-sided uniform $(S+\alpha)$-th price auction.[3] See the working-paper version for an extended analysis that applies to other sorts of auctions.

Bidder $i = 1, ..., n$ gets *value* $V_i(q_i, \mathbf{t})$ from q_i units in the state $\mathbf{t} = (t_1, ..., t_n)$, where *types* $t_i = (t_{i,1}, ..., t_{i,h})$ are multi-dimensional with support $[0,1]^h$ and joint density function $f(\mathbf{t})$.[4] For technical convenience I assume that, for some $\overline{f} > 1$, $f(\mathbf{t}) \in (1/\overline{f}, \overline{f})$ for all $\mathbf{t}$. Types are independent across bidders, i.e. $f(\mathbf{t}) = \Pi_{i=1}^n f_i(t_i)$, though $\{t_{i,1}, ..., t_{i,h}\}$ need not be independent for each bidder i. V_i is bounded and piecewise continuous in $\mathbf{t}$ and $V_i(q_i', \mathbf{t}) - V_i(q_i, \mathbf{t})$ is increasing in t_i and non-decreasing in $\mathbf{t}_{-i}$ whenever $q_i' > q_i$. (Values may be negative and/or decreasing in quantity and marginal values may be non-monotone in q_i.) Bidders are risk-neutral, i.e. seek to maximize expected surplus, the difference between their value and payment.

Each bidder i is endowed with $e_i \in \mathcal{Z}_+$ units where $\sum_i e_i = S$ the total number of units. Each bidder may potentially receive more than his endowment ('buy units') or less than his endowment ('sell units') in a final allocation and thus, for the purposes of payment, may win a negative number of units. Unit-bids on $q \leq e_i$ are bids to *sell*; those on $q > e_i$ are bids to *buy*.

There are S indivisible units. A *permissible bid* is a vector $b_i = (b_{i,1}, ..., b_{i,S})$ of prices (or 'unit-bids'), one for each unit. Unit-bid $b_{i,q}$ can be interpreted as an offer to buy when $q > e_i$ and as an offer to sell when $q \leq e_i$. Permissible bids must satisfy a few requirements. (i) Each unit-bid $b_{i,q} \in \{-\infty\} \cup \mathfrak{p}_i \cup \{\infty\}$ where $\mathfrak{p}_i \subset \mathbf{R}$ is finite. ($-\infty$ is the lowest possible unit-bid while ∞ is the highest.) To avoid complications relating to ties, I will follow Reny and Zamir (2004) and assume that $\mathfrak{p}_i \cap \mathfrak{p}_j = \emptyset$ for all bidders $i \neq j$[5] (see proof of

[3] Translating the model to incorporate one-sided auctions is straightforward: the auctioneer has endowment S while other bidders have zero endowments and the auctioneer's 'bid' consists of reserve prices for each unit.

[4] Vectors are bolded throughout the paper. The exceptions to this rule are bidder types t_i and bids b_i which are multi-dimensional but unbolded to distinguish them from profiles of bids and types.

[5] In most real-world applications, each bidder may submit bids from the same set of

Theorem 2.1 on pages 1121-1122). (ii) $b_{i,q} > -\infty$ for all $q \leq e_i$ and $b_{i,q} < \infty$ for all $q > e_i$. Trading is voluntary. In particular, a bidder who bids $-\infty$ on a unit $q > e_i$ (or ∞ on a unit $q \leq e_i$) guarantees that it will not buy (or sell) that unit. (iii) $b_{i,q_1} \geq b_{i,q_2}$ for all $q_1 < q_2$, i.e. bids are non-increasing in quantity. Let $\mathcal{B}_i$ denote the space of all permissible bids for bidder i. For purposes of the proof, all that is important is that each $\mathcal{B}_i$ is a lattice with respect to the product order.[6] Let $b_I^k(\mathbf{b}_I)$ be the k-th highest unit-bid submitted across all bidders in $I \subset \{1, ..., n\}$, with shorthand $b_{\{1,...,n\}}^k(\mathbf{b}) \equiv b^k$ for the k-th highest unit-bid overall.

Bidder i receives quantity $q_i(\mathbf{b}) \equiv \max\{q : b_{i,q} \geq b^S\}$.

In the uniform $(S+\alpha)$-th price auction, each bidder pays $p^\alpha(\mathbf{b})$ on each unit that it wins, where $p^\alpha(\mathbf{b}) \equiv \alpha b^{S+1} + (1-\alpha)b^S$ (Given the restriction (ii) on bids, $b^S = \infty$ and/or $b^{S+1} = -\infty$ implies that there are no trades and hence no payments. So without loss we may suppose that $b^S, b^{S+1} \in [\min_i \min \mathfrak{p}_i, \max_i \max \mathfrak{p}_i]$.) Bidder i's total payment is

$$Z_i(\mathbf{b}) \equiv p^\alpha(\mathbf{b})\,(q_i(\mathbf{b}) - e_i)$$

A *pure strategy* $b_i : [0,1]^h \rightarrow \mathcal{B}_i$ for bidder i maps each type into a permissible bid. Let $\mathcal{S}_i$ be the set of bidder i's pure strategies and $\mathcal{S} \equiv \Pi_{j=1}^n \mathcal{S}_j$ the pure strategy profiles.

Given bids $\mathbf{b} = (b_1, ..., b_n)$ in state $\mathbf{t}$, bidder i's *ex post payoff* takes the form $V_i\,(q_i(\mathbf{b}), \mathbf{t}) - Z_i(\mathbf{b})$. Given others' pure strategies $\mathbf{b}_{-i}(\cdot)$, bidder i's *interim expected payoff* depends only on his own bid and type:

$$\Pi_i(b_i, t_i; \mathbf{b}_{-i}(\cdot)) = E_{\mathbf{t}_{-i}}\left[V_i\,(q_i(b_i, \mathbf{b}_{-i}(\mathbf{t}_{-i})), \mathbf{t}) - Z_i(b_i, \mathbf{b}_{-i}(\mathbf{t}_{-i}))\right]$$

Let $BR_i(t_i; \mathbf{b}_{-i}(\cdot)) \equiv \arg\max_{b_i \in \mathcal{B}_i} \Pi_i(b_i, t_i; \mathbf{b}_{-i}(\cdot))$ be bidder i's set of best response bids to the pure strategy profile $\mathbf{b}_{-i}(\cdot)$ given type t_i. Let $BR_i(\mathbf{b}_{-i}(\cdot))$ map others' pure strategy profiles into sets of bidder i's best response pure strategies. A profile $\mathbf{b}^*(\cdot) \in \mathcal{S}$ is a *pure strategy equilibrium* iff

$$b_i^*(\cdot) \in BR_i\left(\mathbf{b}_{-i}^*(\cdot)\right) \text{ for all } i.$$

prices. Fortunately, the feature of the model that $\mathfrak{p}_i \cap \mathfrak{p}_j = \emptyset$ does not appear to be crucial to the results, at least not if the bidders share a very fine grid of prices. See the discussion of Reny and Perry (2003) in Section 3.3 and of Kazumori (2002) in Section 4.

[6] Let x_1, x_2 be any two elements in K-dimensional Euclidean space. Their least-upper-bound (or 'join') is their coordinate-wise maximum, $x_1 \vee x_2 = (\max\{x_{1,k}, x_{2,k}\} : k = 1, ..., K)$, while their greatest-lower-bound (or 'meet') is their coordinate-wise minimum.

A *mixed strategy* $\lambda_i : \mathcal{B}_i \times [0,1]^h \to [0,1]$ specifies a mixture over bids for each type $t_i \in [0,1]^h$, i.e. $\lambda_i(b_i; t_i)$ is the probability that type t_i bids b_i. A *mixed strategy equilibrium* is a profile of mixed strategies such that, for all t_i and each element b_i in the support of $\lambda_i(\cdot; t_i)$, b_i is a best response to strategies $\lambda_{-i}(\cdot; \cdot)$ for type t_i.

Bids and types are endowed with the product order. A pure strategy $b_i(\cdot)$ is *monotone* iff $b_i(t_i') \geq b_i(t_i)$ whenever $t_i' > t_i$.

The following definitions are made for pure strategy profiles to simplify the exposition, but the extension to mixed strategy profiles should be clear.

Definition 1 (Ex post allocation-equivalence ('AE')). Two pure strategy profiles $\mathbf{b}'(\cdot)$, $\mathbf{b}(\cdot)$ are *ex post allocation-equivalent* (shorthand 'AE') if the induced allocation is the same with probability one:

$$\Pr{}_{\mathbf{t}}\left(\mathbf{q}(\mathbf{b}'(\mathbf{t})) = \mathbf{q}(\mathbf{b}(\mathbf{t}))\right) = 1$$

Similarly, two *bids* b_i', b_i are ex post allocation-equivalent when

$$\Pr{}_{\mathbf{t}_{-i}}\left(\mathbf{q}(b_i', \mathbf{b}_{-i}(\mathbf{t}_{-i})) = \mathbf{q}(b_i, \mathbf{b}_{-i}(\mathbf{t}_{-i}))\right) = 1$$

Definition 2 (Interim expected payment-equivalence ('PE')). Two pure strategy profiles $\mathbf{b}'(\cdot)$, $\mathbf{b}(\cdot)$ are *interim expected payment-equivalent* (shorthand 'PE') if each bidder i makes the same interim expected payment with probability one:

$$\Pr{}_{t_i}\left[E_{\mathbf{t}_{-i}}\left(Z_i(b_i'(t_i), \mathbf{b}_{-i}'(\mathbf{t}_{-i}))\right) = E_{\mathbf{t}_{-i}}\left(Z_i(b_i(t_i), \mathbf{b}_{-i}(\mathbf{t}_{-i}))\right)\right] = 1$$

3 MONOTONICITY OF EQUILIBRIA

This section proves the two main results. A monotone pure strategy equilibrium ('MPSE') exists (Theorem 1) and every mixed strategy equilibrium is ex post allocation- and interim expected payment-equivalent to some MPSE (Theorem 2). Theorem 1 is an extension of Theorem 2 in McAdams (2003a) to the case of uniform $(S+\alpha)$-th price auctions with $\alpha \notin \{0,1\}$. Theorem 2, on the other hand, is unlike anything in the existing literature.

3.1 Existence

Theorem 1. *Monotone pure strategy equilibrium (MPSE) exists in the uniform $(S+\alpha)$-th price ($\alpha \in [0,1]$) auction.*[7]

Proof. The proof has two parts: (A) Bidders' interim expected payoffs are modular in own bid. (B) MPSE exists.

(A) Interim expected payoffs modular in own bid: Without loss I will focus on bidder 1.

Definition 3 (Modularity in x). Let $(X, \geq, \vee, \wedge)$ be a lattice. $g : X \to \mathfrak{R}$ is *modular* in x iff

$$g(x' \vee x) + g(x' \wedge x) = g(x') + g(x)$$

for all $x', x \in X$.

In our context, X is the set of bidder 1's permissible bids, g is bidder 1's interim expected payoff given own type t_1 and others' strategies $\lambda_{-1}(\cdot;\cdot)$, and modularity in own bid is equivalent to additive separability in own bid. That is to say, the incremental return to varying one's bid on any unit q does not depend on the level of bids on other units.

Since modularity is preserved under addition and scaling (and hence under integration), it suffices to prove that bidder 1's *ex post* payoff is modular in own bid. Given risk-neutrality, this follows whenever ex post *valuation* and ex post *payment* are modular in own bid. In the following, fix the profile of others' bids $\mathbf{b}_{-1}$ and the state $\mathbf{t}$. The analysis focuses on the realized allocation and payments when bidder 1 submits one of two bids b_1^1 or b_1^2 or their join $b_1^{1\vee 2} \equiv b_1^1 \vee b_1^2$ or meet $b_1^{1\wedge 2} \equiv b_1^1 \wedge b_1^2$.

Shorthand notation: $q_j^1 \equiv q_j\left(b_1^1, \mathbf{b}_{-1}\right)$ and so on for the other bids b_1^2, $b_1^{1\vee 2}$, and $b_1^{1\wedge 2}$. (Note that while bidder 1's bid varies, others' bids are held fixed.) Similarly, define shorthand $b^{S,1} \equiv b^S\left(b_1^1, \mathbf{b}_{-1}\right)$, $b^{S+1,1} \equiv b^{S+1}\left(b_1^1, \mathbf{b}_{-1}\right)$ and so on. (Recall that $b^S(\mathbf{b})$ and $b^{S+1}(\mathbf{b})$ are the S-th and $(S+1)$-st highest unit-bids given the profile of bids $\mathbf{b}$.)

[7] It is straightforward to apply techniques developed by Jackson and Swinkels (2005) to show that MPSE exists with positive probability of trade. See the working-paper version for details.

Modularity of ex post valuation: A bidder's valuation for what he wins does not depend on the auction's payment rule. Consequently, McAdams (2003a)'s proof of modularity in the uniform S-th and $(S+1)$-st price auctions works for this part of our proof. The key fact that McAdams proves and uses is that

$$\left\{q_1^1, q_1^2\right\} = \left\{q_1^{1\vee 2}, q_1^{1\wedge 2}\right\}$$

Modularity of ex post payment: Recall that bidder 1's ex post payment takes the form $q_1(\alpha b^{S+1} + (1-\alpha)b^S)$. For modularity, then, we need

$$\begin{aligned} &q_1^1(\alpha b^{S+1,1} + (1-\alpha)b^{S,1}) + q_1^2(\alpha b^{S+1,2} + (1-\alpha)b^{S,2}) \\ &\quad = q_1^{1\wedge 2}(\alpha b^{S+1,1\wedge 2} + (1-\alpha)b^{S,1\wedge 2}) + q_1^{1\vee 2}(\alpha b^{S+1,1\vee 2} + (1-\alpha)b^{S,1\vee 2}) \end{aligned} \tag{1}$$

Since $\{q_1^1, q_1^2\} = \{q_1^{1\vee 2}, q_1^{1\wedge 2}\}$, it suffices to show *either*

$$q_1^1 = q_1^2 \textit{ or} \tag{2}$$

$$\begin{aligned} &\left\{\left(b^{S,1}, b^{S+1,1}, q_1^1\right), \left(b^{S,2}, b^{S+1,2}, q_1^2\right)\right\} \\ &\quad = \left\{\left(b^{S,1\wedge 2}, b^{S+1,1\wedge 2}, q_1^{1\wedge 2}\right), \left(b^{S,1\vee 2}, b^{S+1,1\vee 2}, q_1^{1\vee 2}\right)\right\} \end{aligned} \tag{3}$$

(By definition of the $\vee, \wedge$ operators,

$$\left\{b^{S,1}, b^{S,2}\right\} = \left\{b^{S,1\vee 2}, b^{S,1\wedge 2}\right\} \text{ and } \left\{b^{S+1,1}, b^{S+1,2}\right\} = \left\{b^{S+1,1\vee 2}, b^{S+1,1\wedge 2}\right\}$$

In particular, $b^{S,1}+b^{S,2} = b^{S,1\vee 2}+b^{S,1\wedge 2}$ and $b^{S+1,1}+b^{S+1,2} = b^{S+1,1\vee 2}+b^{S+1,1\wedge 2}$ so that (1) follows from (2). Given (3), similarly, (1) holds since the terms on its left- and right-hand sides are identical.)

Without loss, suppose that $b^{S,1} < b^{S,2}$ or $b^{S,1} = b^{S,2}$ and $b^{S+1,1} \leq b^{S+1,2}$. If $b^{S,1} = b^{S,2}$ and $b^{S+1,1} = b^{S+1,2}$ or if $b^{S,1} \leq b^{S,2}$ and $b^{S+1,1} \leq b^{S+1,2}$, then we are done by (3): $q_1^1 \leq q_1^2 = q_1^{1\vee 2}$ and $(b^{S,2}, b^{S+1,2}) = (b^{S,1\vee 2}, b^{S+1,1\vee 2})$.

As the final (and most difficult) case, suppose that $b^{S,2} > b^{S,1} \geq b^{S+1,1} > b^{S+1,2}$. I will show that $q_1^1 = q_1^2$. First, observe that $b^{S,2}$ and $b^{S+1,2}$ are *consecutive* unit-bids given the profile of bids $(b_1^2, \mathbf{b}_{-1})$, so no bid in this profile includes any unit-bids equal to $b^{S,1}$ or $b^{S+1,1}$. Since others' bids are held fixed, this means that b_1^1 must include consecutive unit-bids equal to $b^{S,1}$ and $b^{S+1,1}$, indeed that $b_{1,q_1^1} = b^{S,1}$ and $b_{1,q_1^1+1} = b^{S+1,1}$. Now suppose that $q_1^2 > q_1^1$. This requires that $b_1^2(q_1^1+1) > b_1^1(q_1^1+1)$ in which case $b^{S+1,2} > b^{S+1,1}$, a contradiction. Similarly, suppose $q_1^2 < q_1^1$. This requires that $b_1^2(q_1^1) < b_1^1(q_1^1)$ in which case $b^{S,2} < b^{S,1}$, a contradiction. So, $q_1^1 = q_1^2$ and we are done with part (A).

(B) MPSE exists: This part of the proof is straightforward given the analysis of McAdams (2003a), but some definitions are needed.

Definition 4 (Single-crossing in (x, t)). Let X, T be partially ordered sets. The function $g : X \times T \to R$ has single-crossing in (x, t) if, for all $x' > x$ and $t' > t$, $g(x', t) > (\geq) g(x, t)$ implies $g(x', t') > (\geq) g(x, t')$, i.e. strict inequality implies strict, weak inequality implies weak.

Definition 5 (Single-crossing condition ('SCC')). Following Athey (2001), I will say that the 'single-crossing condition' (SCC) is satisfied if each bidder's interim expected payoff has single-crossing in own bid and own type for any given monotone strategies by others.

Theorem (corollary to McAdams (2003a), Theorem 1). *Let G be a game in which each player's action a_i is chosen from a finite lattice, each player's type $t_i \in [0, 1]^h$, and each player's interim expected payoff function is modular in a_i and satisfies SCC. Then G has a monotone pure strategy equilibrium.*

In our context, actions are permissible bids and modularity in own bid was proven in (A). Now I show that expected payoffs satisfy non-decreasing differences in own bid and type (NDD) for any given strategies by others, a (much) stronger property than SCC.

Definition 6 (Non-decreasing differences in $(x; t)$). Let X, T be partially ordered sets. The function $f : X \times T \to R$ has non-decreasing differences in (x, t) iff $f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$ for all $x' > x$, $t' > t$.

No matter what others bid, submitting a higher bid causes one to win (weakly) more quantity. Since incremental values are assumed to be non-decreasing in own type, each bidder's ex post surplus thus satisfies non-decreasing differences in own bid and own type ('NDD'). By risk-neutrality, then, ex post *payoff* satisfies NDD and, by independence, interim *expected* payoff satisfies NDD. This completes the proof of Theorem 1. □

3.2 Characterization

Theorem 2. *In the two-sided uniform $(S + \alpha)$-th price ($\alpha \in [0, 1]$) auction, each mixed strategy equilibrium is ex post allocation- and interim expected payment-equivalent to some MPSE.*

Corollary to Theorem 2. *In the two-sided uniform* $(S+\alpha)$*-th price* ($\alpha \in [0,1]$) *auction, auctioneer expected revenue and bidder interim expected surplus are the same in any given mixed strategy equilibrium as in some MPSE.*

The corollary follows immediately from the definitions of ex post allocation- and interim expected payment-equivalence.

The rest of this section lays the groundwork for, and proves, Theorem 2. For any given mixed strategy equilibrium, we will construct a MPSE that is ex post allocation- and interim expected payment-equivalent to it. Crucial to this construction is a procedure to 'monotonize' a mixed strategy.

Procedure to 'monotonize' strategies. *Preliminaries.* Fix any $k \in \{1, ..., h\}$. We can partition the space of bidder i's types into one-dimensional subsets that vary only along the k-th dimension: $[0,1]^h = \cup_{t_{i,-k}} \Gamma_i^k(t_{i,-k})$, where $\Gamma_i^k(t_{i,-k}) \equiv \{(x, t_{i,-k}) : x \in [0,1]\}$.

Fix any mixed strategy $\lambda_i(\cdot;\cdot)$ for bidder i. For each $t_{i,-k} \in [0,1]^{h-1}$, quantity $1 \leq q \leq S$, and unit-bid $b_{i,q} \in \mathfrak{p}_i$, let $p_{i,q}^k(b_{i,q}|t_{i,-k})$ denote the probability that bidder i submits a unit-bid of $b_{i,q}$ on quantity q, conditional on having type in $\Gamma_i^k(t_{i,-k})$:

$$p_{i,q}^k(x|t_{i,-k}) \equiv \sum_{b_i : b_{i,q} = x} \int_0^1 \lambda_i(b_i; t_{i,k}, t_{i,-k}) f_i(t_{i,k}|t_{i,-k}) \mathrm{d}t_{i,k} \tag{4}$$

Let $F_i(t_{i,k}|t_{i,-k}) \equiv \int_0^{t_{i,k}} f_i(x|t_{i,-k})\mathrm{d}x$ be the conditional c.d.f. of bidder i's type when restricted to $\Gamma_i^k(t_{i,-k})$. For every quantity $1 \leq q \leq S$ and $\alpha \in [0,1]$, define

$$\mathfrak{b}_{i,q}^k(\alpha) \equiv \max \left\{ \tilde{b}_{i,q} : \sum_{b_{i,q} < \tilde{b}_{i,q}} p_{i,q}^k(b_{i,q}|t_{i,-k}) \leq \alpha \right\}$$

Monotonizing a strategy along the k-th dimension. Define the *pure* strategy $b_i(\cdot; \lambda_i(\cdot;\cdot); k)$ that results from 'monotonizing $\lambda_i(\cdot;\cdot)$ along the k-th dimension', as follows: for all $t_i \in [0,1]^h$,

$$b_i(t_i; \lambda_i(\cdot;\cdot); k) \equiv \left(\mathfrak{b}_{i,1}^k(F_i(t_{i,k}|t_{i,-k})), ..., \mathfrak{b}_{i,S}^k(F_i(t_{i,k}|t_{i,-k})) \right) \tag{5}$$

I will use simpler notation $b_i(\cdot; k)$ when the original strategy $\lambda_i(\cdot;\cdot)$ is clear from context.

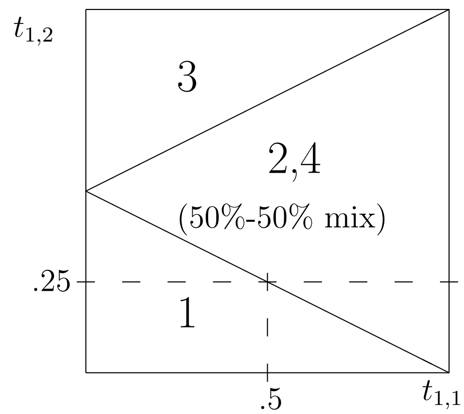

Figure 1: Graphical representation of a mixed strategy

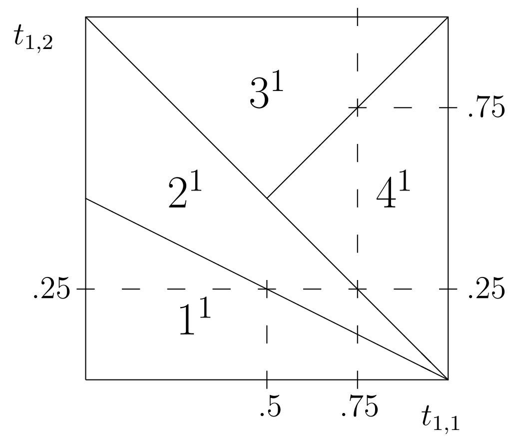

Figure 2: Monotonizing the strategy of Figure 1 along the first dimension.

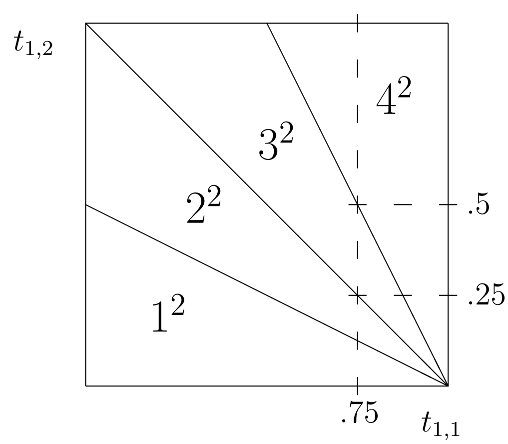

Figure 3: Monotonizing the strategy of Figure 2 along the second dimension.

Example. Figures 1, 2, 3 illustrate two iterations of this procedure in a simple example. Consider a single-object auction in which bidder 1 has a two-dimensional type uniformly distributed over $[0,1]^2$ and the only permissible bids are \$1, \$2, \$3, and \$4. Figure 1 describes a (mixed) strategy graphically. Bidder 1 bids \$1 and \$3 given types in the regions labeled '1' and '3', respectively, and randomizes equally between \$2 and \$4 in the region labelled '2,4 (50%-50% mix)'.

Monotonizing the mixed strategy of Figure 1 along the first dimension leads to the *pure* strategy depicted in Figure 2. Consider for instance the one-dimensional subset of the type-space in which $t_{1,2} = .25$ (dashed in Figures 1,2). Under the original mixed strategy, bidder 1 bids \$1 when $t_{1,1} < .5$ and bids $\$2, \$4$ each half of the time when $t_{1,1} > .5$. Overall this gives probabilities $(.5, .25, 0, .25)$ of bidding \$1,\$2,\$3,\$4, respectively. Under the pure strategy of Figure 2, these probabilities are preserved but lower types make lower bids. Bidder 1 still bids \$1 when $t_{1,1} < .5$ but now bids \$2 when $t_{1,1} \in (.5, .75)$ and \$4 when $t_{1,1} > .75$.

Monotonizing the pure strategy of Figure 2 along the second dimension leads to the *monotone* pure strategy depicted in Figure 3. Consider for instance the one-dimensional subset of the type-space in which $t_{1,1} = .75$ (dashed in Figures 2,3). Under the strategy of Figure 2, bidder 1 bids \$1 when $t_{1,2} < .125$, \$2 when $t_{1,2} \in (.125, .25)$, \$4 when $t_{1,2} \in (.25, .75)$, and \$3 when $t_{1,2} > .75$. Overall this gives probabilities $(.125, .125, .25, .5)$ of bidding \$1,\$2,\$3,\$4, respectively. Under the monotone pure strategy of Figure 3, these probabilities are again preserved but lower types make lower bids. Now bidder 1 bids \$3 when $t_{1,1} \in (.25, .5)$ and \$4 when $t_{1,1} > .5$.

A few other features of this example are worth noting as they generalize and play an important role in the analysis. (i) Each bid $\{\$1, ..., \$4\}$ is made with the same ex ante probability in all of the strategies illustrated in Figures 1, 2, and 3. (ii) Once the set of types that bid less than or equal to b is a decreasing set, that set remains fixed thereafter. (X is decreasing when $x \in X, y < x$ implies $y \in X$.) For instance, '1' is decreasing in the initial mixed strategy, so '1' = '1^1' = '1^2'. Similarly, '1^1' $\cup$ '2^1' is decreasing, so '1^1' $\cup$ '2^1' = '1^2' $\cup$ '2^2'. (iii) Conversely, if the set of types that bid less than or equal to b is not a decreasing set for some strategy, then monotonizing that strategy along *some* dimension 'moves' this less-than-set toward the origin. More precisely, the less-than-sets decrease with respect to the metric $|X| \equiv \int_{t_1 \in X} f_1(t_1) \sum_{k=1}^{h} t_{1,k} \mathrm{d}t_1$ on subsets $X \subset [0,1]^h$. For instance, monotonizing the strategy of Figure 2 along the second dimension moves some of the region

'3^1' downward so that $|`1^1\text{'} \cup `2^1\text{'} \cup `3^1\text{'}| > |`1^2\text{'} \cup `2^2\text{'} \cup `3^2\text{'}|$. (iv) Finally, the process of repeatedly monotonizing a given strategy along each of the various dimensions (perhaps many times) eventually leads to a monotone pure strategy. What monotone pure strategy we end up with depends on which dimension we monotonize along first, etc. The arguments in the proof of Theorem 2, however, will apply to all such monotone pure strategies. Point (iv) is stated as Lemma 1. Points (i,ii,iii) are proven along the way to proving Lemma 1.

Lemma 1. *For any mixed strategy $\lambda_1(\cdot;\cdot)$, define a sequence $\{b_1^m(\cdot)\}^{m=0,1,2,...}$ of pure strategies as follows:*[8]

$$b_1^0(\cdot) \equiv b_1(\cdot; \lambda_1(\cdot;\cdot); 1)$$
$$b_1^m(\cdot) \equiv b_1(\cdot; b_1^{m-1}(\cdot); m(\text{mod } h)+1) \textit{ for all } m = 1, 2, ...$$

This sequence converges to a monotone pure strategy $b_1^\infty(\cdot)$, in the sense that for all $\varepsilon > 0$ there exists M^* such that $\Pr_{t_1}(b_1^m(t_1) = b_1^\infty(t_1)) > 1 - \varepsilon$ for all $m > M^*$.

Proof. In the Appendix. □

Restrictions on bids imposed by revealed preference. Since bidders' expected payoffs are modular in own bid and have non-decreasing differences in own bid and own type (see proof of Theorem 1), Milgrom and Shannon (1994)'s Monotonicity Theorem implies that each bidder i's set of best response bids is a lattice for all types t_i and that bidder i's best response correspondence is increasing in the strong set order.

Definition 7 (Increasing in strong set order (ISSO)). Let X be a lattice, T a partially ordered set, and $\phi : T \to X$ a correspondence. ϕ is *increasing in the strong set order* if $t' > t$, $x \in \phi(t)$, and $x' \in \phi(t')$ implies that $x \wedge x' \in \phi(t)$ and $x \vee x' \in \phi(t')$.

This ISSO structure puts strong restrictions on what bids any two comparable types $t_i' > t_i$ can find to be best responses. Let $\lambda^*(\cdot;\cdot)$ be a mixed strategy equilibrium. Without loss of generality, I will focus on bidder 1. Note that mixed strategy $\lambda_1^*(\cdot;\cdot)$ is not a monotone pure strategy iff there exists bids $b_1' \ngeq b_1$ and types $t_1' > t_1$ such that $\lambda_1^*(b_1'; t_1') > 0$ and $\lambda_1^*(b_1; t_1) > 0$. When $y \geq x$, $[x, y]$ refers to the order-interval $\{z : x \leq z \leq y\}$.

[8] $m(mod\ h)$ ("m modulo h") refers to the remainder when m is divided by h.

Lemma 2. *Suppose that* $\lambda_1^*(b_1'; t_1') > 0$ *and* $\lambda_1^*(b_1; t_1) > 0$ *for some* $b_1' \ngeq b_1$ *and types* $t_1' > t_1$*. Then all bids in the order interval* $[b_1' \wedge b_1, b_1' \vee b_1]$ *are ex post allocation-equivalent.*

Proof. In the Appendix. □

Now we are finally ready to prove the characterization result.

Proof of Theorem 2. Let $b_1^0(\cdot) = b_1(\cdot; \lambda_1^*(\cdot;\cdot); 1)$ denote the pure strategy gotten by monotonizing the equilibrium mixed strategy $\lambda_1^*(\cdot;\cdot)$ along the first dimension. I will prove that (a) the new profile of strategies $\left(b_1^0(\cdot), \lambda_{-1}^*(\cdot;\cdot)\right)$ is an equilibrium that is (b) ex post allocation-equivalent ('AE') and (c) interim expected payment-equivalent ('PE') to the equilibrium $\left(\lambda_1^*(\cdot;\cdot), \lambda_{-1}^*(\cdot;\cdot)\right)$. Proving these three facts will complete the proof of Theorem 2. To see why, note that repeating the argument shows that $\left(b_1^m(\cdot), \lambda_{-1}^*(\cdot;\cdot)\right)$ is an equilibrium that is AE and PE to the original equilibrium for all m. Thus, the limiting strategy profile $\left(b_1^\infty(\cdot), \lambda_{-1}^*(\cdot;\cdot)\right)$ of Lemma 1 is also AE and PE to the original equilibrium and, because other bidders' expected payoffs are continuous in bidder 1's strategy, an equilibrium itself. Finally, repeating this argument for all bidders yields a MPSE $(b_1^\infty(\cdot), ..., b_n^\infty(\cdot))$ that is AE and PE to the original equilibrium.

Allocation equivalence and payment equivalence. Lemma 2 implies that $\left(b_1^0(\cdot), \lambda_{-1}^*(\cdot;\cdot)\right)$ is AE to $(\lambda_1^*(\cdot;\cdot), ..., \lambda_n^*(\cdot;\cdot))$. To see why, suppose that some type t_1 exists who does not always make the same bid as in the original equilibrium, i.e. $b_1^0(t_1) = b'$ but that $\lambda_1^*(b, t_1) > 0$ for some $b' \neq b$. We want to show that b', b each lead to the same allocation with probability one. Fix any quantity $1 \leq q \leq S$. By definition of the monotonization procedure, $b_q' < b_q$ (respectively, $b_q' > b_q$) implies that there must be some type $t_1' > t_1$ (respectively, $t_1' < t_1$) that bids b_q' on unit q. (Type t_1''s bid may differ from b' at other quantities.) Let $\hat{b}^q$ denote this bid, so that by definition $\hat{b}_q^q = b_q'$. Lemma 2 implies that all bids in the order interval $[b \wedge b^q, b \vee b^q]$ are AE, for each q. It is easy to show that any set of AE bids must form a lattice. Thus, all bids in $\left[b \bigwedge_{q=1}^S \hat{b}^q, b \bigvee_{q=1}^S \hat{b}^q\right] \supset [b \wedge b', b \vee b']$ are AE. In particular, b', b are AE.

To see that the new strategy profile is also PE to the original equilibrium, consider first bidders $i \neq 1$. Bidder i's interim expected payment 'only' depends on the distribution of all others' unit-bids on every unit, i.e. on the probabilities that each bidder $j \neq i$ bids each bid level b_q on each unit

q. But monotonizing bidder 1's strategy does not change the probability that bidder 1 submits unit-bid b_q on unit q for all b_q, q. Thus, bidder i's interim expected payment from making any given bid stays the same as in the original equilibrium.

Now consider bidder 1. Suppose again that some type t_1 exists who does not make the same bid as in the original equilibrium, i.e. $b_1^0(t_1) = b'$ but that $\lambda_1^*(b, t_1) > 0$ for some $b' \neq b$. We have shown that these two bids always lead to the same allocation. If either of these bids leads to a greater expected payment for bidder 1, it will be strictly dominated by the other. By presumption, each of these bids is a best response for some type, so the bids must lead to the same expected payment for bidder 1 as well. Thus, the new strategy profile must be PE to the original equilibrium.

Preserving equilibrium. Since the strategies of bidders $2, ..., n$ are held fixed and the new strategy profile is AE and PE to the original equilibrium, it is clear that bidder 1 still plays a best response. Similarly, every other bidder gets the same expected payoff as before *if* it chooses to continue to follow its old equilibrium strategy. Lastly, since the distribution of bidder 1's unit-bid on each unit stays the same, every other bidder (say bidder 2) wins at least q units after making any deviation $\tilde{b}$ with the same probability given either the old strategies $\lambda_{-2}^*(\cdot;\cdot)$ or the new strategies $(b_1^0(\cdot), \lambda_{-12}^*(\cdot;\cdot))$.

The only thing that could change is bidder 2's expected value conditional on winning after some deviation. Lemma 3 shows that, if anything, bidder 2's expected value conditional on winning *falls* once bidder 1's strategy is monotonized. Intuitively, bidder 2 suffers more of a 'winner's curse' when bidder 1 plays a more monotone strategy because, when bidder 2 wins, bidder 1 is more likely than before to have a lower type.

Lemma 3 **(Monotonizing strategies worsens the 'winner's curse').** *Consider any bid $\tilde{b}_2$, any quantity q, and any profile of others' strategies $\lambda_{-2}(\cdot;\cdot)$. Bidder 2's expected marginal value for a q-th unit conditional on winning at least q units with bid $\tilde{b}_2$ is weakly lower when bidder 1 changes its strategy from $\lambda_1(\cdot;\cdot)$ to $b_1^0(\cdot) \equiv b_1(\cdot;\lambda_1(\cdot;\cdot);1)$, holding $\lambda_{-12}(\cdot;\cdot)$ fixed.*

Proof. In the Appendix. □

Since the original strategy profile is an equilibrium, bidder 2 has no profitable deviation when others adopt strategies $\lambda_{-2}^*(\cdot;\cdot)$. By Lemma 3, all deviations become weakly less profitable once bidder 1's strategy is changed to $b_1^0(\cdot)$. Consequently, bidder 2 still has no profitable deviation. This completes the proof of Theorem 2. □

While the analysis here has focused on uniform price auctions, the results hold as well for other commonly studied multi-unit auctions.[9] (See the working-paper version for the proof of Theorems 3,4.)

Theorem 3. *MPSE exists in the discriminatory, Vickrey, and all-pay auctions.*

Theorem 4. *All mixed strategy equilibria are ex post allocation- and interim expected payment-equivalent to a MPSE in the discriminatory, Vickrey, and all-pay auctions.*

3.3 Discussion of assumptions

Three assumptions play an important role in the analysis: (i) ties can not occur since each bidder has a different set of prices from which to bid, (ii) types are independent across bidders, and (iii) bidders are risk-neutral.

Rationing rule. When bidders bid from the same finite set of prices, there is no way to avoid ties and the rationing rule can be important. For instance, in a setting with two prices and two bidders, Reny and Perry (2003) provide an example in which a monotone best reply does not exist. Similarly, McAdams (2002a) provides an example with three prices and two bidders in which no MPSE exists but a non-monotone pure strategy equilibrium does exist. As Reny and Perry (2003) show, however, the existence of such counterexamples depends crucially on the fact that there are few possible prices and/or few bidders. In particular, they prove that MPSE always exists given *any* rationing rule if there are many bidders and a very fine grid of prices.

Independence. The literature on monotone equilibria in the first-price auction does not require independence but only affiliation of bidder types to prove that MPSE exists and that all equilibria are equivalent to MPSE [Athey (2001), Reny and Zamir (2004), McAdams (2003b)]. The reason for this is that each bidder's ex post payoff satisfies a weak form of complementarity

[9] The existence result (Theorems 1, 3) also applies in some settings with allocative externalities, specifically when bidders have valuation functions of the form $V_i(\mathbf{q}, \mathbf{t})$ where $V_i(\mathbf{q}', \mathbf{t}) - V_i(\mathbf{q}, \mathbf{t})$ is non-decreasing in $\mathbf{t}$ whenever $q_i' \geq q_i$ and $q_j' \leq q_j$ for all $j \neq i$. See the working-paper version for details.

across bids in the first-price auction: if bidding higher increases my payoff when others bid lower, then bidding higher can not decrease my payoff when others bid higher. When bidder signals are affiliated, this complementarity across bids (loosely speaking) *increases* bidders tendency to adopt monotone strategies in response to monotone strategies by their opponents: when I receive a higher signal, you are more likely to have received a higher signal and hence be bidding higher, which in turn makes me want to bid no lower.

In multi-unit auctions, however, complementarity across bids can fail. For example, consider a uniform $(S+1)$-st price auction example with $S=2$. Suppose that bidder 1 has value 100 for both units and compares two bids $(50,50)$ vs. $(50,20)$. If bidder 2 bids $(25,0)$, then bidder 1 prefers $(50,50)$ over $(50,20)$ since the former gives him payoff 100 whereas the latter only 80. But if bidder 2 increases his bid to $(100,0)$, bidder 1 now prefer $(50,20)$ over $(50,50)$ since the former gives him 80 whereas the latter only 50. Thus, when bidders receive positively correlated types, the virtuous cycle is broken: my having a higher type implies that I am more likely to face high bids if my opponents are following monotone strategies, but this may lead me to prefer to bid lower and hence not adopt a monotone strategy myself. Indeed, this feature of the uniform $(S+1)$-st price auction can lead *all* equilibria in weakly undominated strategies to be non-monotone when bidder signals are (one-dimensional) and affiliated. See the working-paper version for more details.

Risk-neutrality. For the sake of contrast, consider a first-price auction with independent private values (IPV) and bidders having utility of the form $u(v_i - b)$ from winning at price p. Let $p(b)$ be bidder i's probability of winning given bid b and others' strategies. Bidder i's expected utility $U_i(b,v_i) = p(b)u(v_i-b)$ and $\frac{\mathrm{d}U_i}{\mathrm{d}b,\mathrm{d}v_i} = p'(b)u'(v_i-b) - p(b)u''(v_i-b) \geq 0$ as long as $u' \geq 0$ and $u'' \leq 0$. Because of this 'increasing differences' property, risk-averse bidders always have a monotone pure strategy best response in the first-price auction. In some multi-unit auctions, however, risk-aversion can lead bidders only to have non-monotone best responses. For example, consider a discriminatory auction given IPV and just two bidders competing for two units. (See McAdams (2002a) for a complete example along these lines.) Here the discriminatory auction is akin to two separate first-price auctions, where my first-unit bid competes against your second-unit bid and vice versa, with the caveat that second-unit bids can not exceed first-unit bids. Holding our bids fixed, suppose that my value for the first unit increases. If I am winning the first unit, this increases my 'wealth' in the auction for the second unit. Consequently, *if* I have decreasing absolute risk aversion (DARA), then I will tend to be 'less risk averse' and hence bid less in the auction for the second unit. Of course, this logic depends crucially on my having DARA utility and applies only to the discriminatory auction. It remains an open question whether bidders always have monotone best replies in uniform-price auctions given risk-averse bidders. It seems plausible to me that bidders may always have monotone best replies in the uniform $(S+1)$-st price auction given DARA bidders.

4 CONCLUDING REMARKS

This paper introduces a new style of analysis for multi-unit auctions that exploits the lattice structure of bids.[10] This lattice-theoretical approach leverages the fact that, in these auctions, each bidder's expected surplus is modular (i.e. additively separable) in own bid. This powerful property allows us to 'reduce' the issue of monotone pure strategy equilibrium (MPSE) existence in these multi-unit auctions to that in single-object auctions: the sufficient condition for MPSE existence in single-object auctions, Athey (2001)'s 'single-crossing condition', is also sufficient in multi-unit auctions.[11] Furthemore, modularity is the key to my proof that *all* mixed strategy equilibria are equivalent to MPSE.

The analysis here employs a discrete bid-space and an atomless type-space, but the results extend to other settings as well. First, suppose that there is a discrete bid-space and an atomic type-space, i.e. each bidder has finitely many types forming a lattice within $[0,1]^h$. In this setting, it is straightforward to extend all arguments to prove that a monotone *mixed* strategy equilibrium ('MMSE') exists and that all mixed strategy equilibria are ex post allocation- and interim expected payment-equivalent to some MMSE. (A monotone mixed strategy is one in which $t_i' > t_i$ implies $b_i' \geq b_i$

[10] There are several sorts of auctions of multiple identical objects to which this paper's analysis does *not* apply. Most prominently among them are (i) menu auctions in which bidders may demand (say) 'q units or nothing' [Bernheim and Whinston (1986)], (ii) sequential auctions in which the sale of units is spread out over several rounds, and (iii) auctions in which some of the bidders choose their bids after observing others' bids.

[11] More precisely, Reny and Zamir (2004)'s weaker version of this condition, so-called 'best-reply single-crossing condition' (BR-SCC) is sufficient for existence in both single-object and multi-unit auctions.

for *all* bids b_i', b_i made with positive probability by types t_i', t_i, respectively.) See the working-paper version for details. Next, building on an earlier draft of this paper, Kazumori (2002) has studied the case in which bidders have atomless types and there is a continuum of prices. Kazumori shows that a sequence of the equilibria that I construct, as the price grids converge to a continuum, converges to a monotone pure strategy equilibrium when there is a continuum of prices. Thus, Theorem 1 extends to the continuum price case. It remains an open question whether the characterization result Theorem 2 also extends to a price continuum, but this seems likely. Furthermore, it seems likely that a combination of these arguments can apply to the case with both atomic types and continuum prices.

Appendix

Proof of Lemma 1

A strategy is monotone iff for every unit $q \in \mathfrak{q}$ and unit-bid level $b \in \mathfrak{p}_1$, the set of types that bid less than or equal to b on that unit is a decreasing set. Let $\{b_1^m(\cdot)\}^{m=0,1,2,\cdots}$ be the sequence of pure strategies defined in the text. For each $m = 0, 1, ...,$ $1 \leq q \leq S$, and $b \in \mathfrak{p}_1$, define the less-than-set

$$L^m(b, q) \equiv \{t_1 : b_{1,q}^m(t_1) \leq b\}$$

Let $D^m(b, q)$ be its *largest decreasing subset* (possibly empty)

$$D^m(b, q) \equiv \{t_1 \in L^m(b, q) : \tilde{t}_1 \in L^m(b, q) \text{ for all } \tilde{t}_1 \leq t_1\}$$

and let $E^m(b, q) \equiv L^m(b, q) \backslash D^m(b, q)$. For instance, in the example on page 13, $b_1^0(\cdot)$ and $b_1^2(\cdot)$ are illustrated in Figures 2 and 3, respectively, and $L^m(\$3, 1)$ is the union of the sets labeled '1^m', '2^m', and '3^m'. For $m = 1$ this is not a decreasing set, however, so $D^1(\$3, 1) \subsetneq L^1(\$3, 1)$, as illustrated in Figure 4. ($D^1(\$3, 1)$ is traced by heavy lines.)

Now, as m increases, it is straightforward to verify that the sets $L^m(b, q)$ and $D^m(b, q)$ evolve in three important ways: (i) The less-than-sets $L^m(b, q)$ 'become more like decreasing sets': $D^m(b, q) \subseteq D^{m+1}(b, q)$ for all m, b, q. (ii) The probability of a unit-bid less than or equal to b on each unit q stays the same: $\Pr_{t_1}(L^m(b, q)) = \Pr_{t_1}(L^{m+1}(b, q))$ for all m, b, q. (iii) The less-than-sets 'move toward the origin'. In particular, for any $X \subset [0, 1]^h$, define the

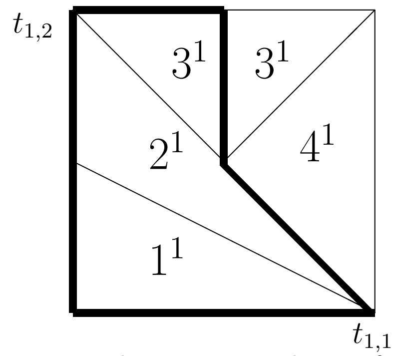

Figure 4: Largest decreasing subset of a less-than-set

metric $|X| \equiv \int_{t_1 \in X} f_1(t_1) \sum_{j=1}^h t_{1,j} \mathrm{d}t_1$ where by convention $|\emptyset| = 0$.[12] Then $|L^m(b,q)| \geq |L^{m+1}(b,q)| \geq 0$. Together, these facts imply also that

$$|E^m(b,q)| \geq |E^{m+1}(b,q)| \geq 0$$

so that $\lim_{m\to\infty} |E^m(b,q)|$ exists.

Before proceeding, I prove that $\lim_{m\to\infty} E^m(b,q) \equiv E^\infty(b,q)$ exists. Suppose not. Then there exists $\varepsilon > 0$ such that for all m^* there exists $m > m^*$ with $\Pr_{t_1}\left(E^m(b,q)\backslash E^{m+1}(b,q) \cup E^{m+1}(b,q)\backslash E^m(b,q)\right) > 2\varepsilon$. By (i,ii), $\Pr_{t_1}\left(E^m(b,q)\right) \geq \Pr_{t_1}\left(E^{m+1}(b,q)\right)$ so $\Pr_{t_1}\left(E^m(b,q)\backslash E^{m+1}(b,q)\right) > \varepsilon$. Suppose now that $b^{m+1}(\cdot)$ is gotten from $b^m(\cdot)$ by monotonizing along (say) the first dimension and for the moment fix $t_{1,-1}$. Define shorthand $E_1^m(b,q) \equiv E^m(b,q) \cap \Gamma^1(t_{1,-1})$ and similarly $E_1^{m+1}(b,q)$. By constuction

$$\check{t}_{1,1} \equiv \sup E_1^{m+1}(b,q) \leq \inf \left(E_1^m(b,q)\backslash E_1^{m+1}(b,q)\right) \equiv \hat{t}_{1,1}$$

Since by assumption the joint density $f_1(t_{1,1}|t_{1,-1})$ is uniformly bounded above by $\overline{f}$, *on average* the types in $\left(E_1^m(b,q)\backslash E_1^{m+1}(b,q)\right)$ 'move left' at

[12] I will say that $\lim_{m\to\infty} X^m = Y$ when $|X^m\backslash Y| \to 0$, which implies convergence with respect to the usual metric, i.e. $\int_{t_1 \in X^m\backslash Y \cup Y\backslash X^m} f(t_1)\mathrm{d}(t_1) \to 0$.

least $\Pr_{t_{1,1}|t_{1,-1}}\left(E_1^m(b,q)\backslash E_1^{m+1}(b,q)|t_{1,-1}\right)/2\overline{f}$, so that

$$\begin{aligned}
\int_{E_1^m(b,q)}&\sum_{k=1}^h t_{1,k}f_1(t_{1,1}|t_{1,-1})\mathrm{d}t_{1,1}-\int_{E^{m+1}(b,q)}\sum_{k=1}^h t_{1,k}f_1(t_{1,1}|t_{1,-1})\mathrm{d}t_{1,1}\\
&=\int_{E_1^m(b,q)}t_{1,1}f_1(t_{1,1}|t_{1,-1})\mathrm{d}t_{1,1}-\int_{E^{m+1}(b,q)}t_{1,1}f_1(t_{1,1}|t_{1,-1})\mathrm{d}t_{1,1}\\
&\geq\left(\Pr_{t_{1,1}|t_{1,-1}}\left(E_1^m(b,q)\backslash E_1^{m+1}(b,q)\right)|t_{1,-1}\right)/2\overline{f} \qquad (6)
\end{aligned}$$

(The right-hand-side of inequality (6) is the shortest possible length for an interval in $\Gamma^1(t_{1,-1})$ that contain (conditional) probability mass equal to $\Pr_{t_{1,1}|t_{1,-1}}\left(E_1^m(b,q)\backslash E_1^{m+1}(b,q)\right)$.) Integrating over all $t_{1,-1}$, finally, gives $|E^m(b,q)|-|E^{m+1}(b,q)|\geq\varepsilon^2/\overline{f}$, contradicting the fact that $\lim_{m\to\infty}|E^m(b,q)|$ exists.

Let $L^\infty\equiv D^\infty\cup E^\infty$ exist. If $E^\infty(b,q)=\emptyset$ for all b,q then we are done. So suppose otherwise that $E^\infty(b,q)\neq\emptyset$ for some b,q. Clearly, no elements of $E^\infty(b,q)$ 'move down' when we monotonize $L^\infty(b,q)$ along *any* dimension since otherwise $E^\infty(b,q)$ would not be a limit of $\{E^m(b,q)\}^{m=1,2,...}$. So $\widehat{t}_1\in E^\infty(b,q)$ implies $(\tilde{t}_{1,k},\widehat{t}_{1,-k})\in L^\infty(b,q)$ for all $k=1,...,h$ and all $\tilde{t}_{1,k}<\widehat{t}_{1,k}$. But since any such type $(\tilde{t}_{1,k},\widehat{t}_{1,-k})\notin E^\infty(b,q)$ must belong to $D^\infty(b,q)$, we know that all types less than $(\tilde{t}_{1,k},\widehat{t}_{1,-k})$ must also belong to $L^\infty(b,q)$. So, indeed, *all* types less than $\widehat{t}_1$ belong to $L^\infty(b,q)$ so that $\widehat{t}_1\in D^\infty(b,q)$, a contradiction. □

Proof of Lemma 2

Let $Q'_*(b'_1,b_1)\equiv\{q\in\mathfrak{q}:b'_{1,q}>b_{1,q}\}$ (shorthand Q'_*) be the set of units on which bid b'_1 specifies a strictly higher unit-bid than b_1, and similarly define $Q_*(b'_1,b_1)$ (shorthand Q_*) to be those units on which b'_1 specifies a weakly lower unit-bid than b_1. Define $b^+\equiv b'_1\vee b_1=(b'_{1,Q'_*},b_{1,Q_*})$ and $b^-\equiv b'_1\wedge b_1=(b_{1,Q'_*},b'_{1,Q_*})$.

ISSO (see Definition 7) implies that the lower type t_1 must find both b_1,b^- to be best responses while the higher type t'_1 must find both b'_1,b^+ to be best responses. In particular, type t_1 is indifferent to raising its unit-bids on units in Q_* from b'_{Q_*} to b_{Q_*} when its other unit-bids are $b_{Q'_*}$ and type t'_1 is indifferent to raising its unit-bids on units in Q_* from b'_{Q_*} to b_{Q_*} when its other unit-bids are $b'_{Q'_*}$. By modularity in own bid, however, we can express bidder 1's interim expected payoff as the sum of functions that each depend only on an individual unit-bid, the bidder's type, and others' strategies:

$$\Pi_1\left(b, t_1; \lambda_{-1}(\cdot;\cdot)\right) = \sum_{q \in \mathfrak{q}} \Pi_{1,q}\left(b_q, t_1; \lambda_{-1}(\cdot;\cdot)\right)$$

For every $q \in Q_*$ consider the two unit-bid levels $b_q \geq b'_q$. If bidder 1 wins the q-th unit more often with the higher unit-bid b_q, then the incremental return to bidding b_q versus b'_q is *strictly increasing* in own type. This is impossible, however, since two different types $t'_1 > t_1$ are indifferent to raising these unit-bids (albeit with their other unit-bids at different levels). Thus, either $b_q = b'_q$ or bidder 1's likelihood of winning unit $q \in Q_*$ does not change as he raises his unit-bid from b'_q to b_q. Furthermore, changing one's own bid can only affect others' allocations if doing so affects one's own allocation. We may conclude that, for each $q \in Q_*$, if bidder 1 were to modify its strategy by sometimes submitting unit-bid b_q when he originally had bid b'_q on that unit or vice versa, then the allocation must remain the same with probability one. By the same logic, changing one's bid to any bid level *in between* b_q and b'_q will not change the allocation. All together, this implies that all bids $b \in [b^+, b^-]$ are ex post allocation-equivalent. □

Proof of Lemma 3

Consider a fixed bid $\tilde{b}_2$ for a fixed type $\hat{t}_2$ of bidder 2. When others play strategies $\lambda_{-2}(\cdot;\cdot)$, bidder 2's expected marginal value for a q-th unit conditional on winning at least q units is

$$\frac{\sum\limits_{\mathbf{b}_{-2}: q_2(\tilde{b}_2, \mathbf{b}_{-2}) \geq q} \int_{\mathbf{t}_{-2}} MV_2(q, \hat{t}_2, \mathbf{t}_{-2}) \Pi_{i \neq 2}\left(\lambda_i(b_i; t_i) f(t_i)\right) \mathrm{d}\mathbf{t}_{-2}}{\sum\limits_{\mathbf{b}_{-2}: q_2(\tilde{b}_2, \mathbf{b}_{-2}) \geq q} \int_{\mathbf{t}_{-2}} \Pi_{i \neq 2}\left(\lambda_i(b_i; t_i) f(t_i)\right) \mathrm{d}\mathbf{t}_{-2}} \tag{7}$$

where $MV_2(q, \mathbf{t}) \equiv V_2(q, \mathbf{t}) - V_2(q-1, \mathbf{t})$ is bidder 2's *ex post* marginal value. If bidder 1 instead adopts the monotonized strategy $b_1^0(\cdot)$, bidder 2's expected marginal value becomes

$$\frac{\sum\limits_{\mathbf{b}_{-2}: q_2(\tilde{b}_2, \mathbf{b}_{-2}) \geq q} \int_{\mathbf{t}_{-2}} MV_2(q, \hat{t}_2, \mathbf{t}_{-2}) \Pi_{i \neq 1,2}\left(\lambda_i(b_i; t_i) f(t_i)\right) 1_{\{b_1^0(t_1) = b_1\}} f(t_1) \mathrm{d}\mathbf{t}_{-2}}{\sum\limits_{\mathbf{b}_{-2}: q_2(\tilde{b}_2, \mathbf{b}_{-2}) \geq q} \int_{\mathbf{t}_{-2}} \Pi_{i \neq 1,2}\left(\lambda_i(b_i; t_i) f(t_i)\right) 1_{\{b_1^0(t_1) = b_1\}} f(t_1) \mathrm{d}\mathbf{t}_{-2}} \tag{8}$$

where $1_{\{X\}} = 1$ if X is true, and zero otherwise. We need to prove that (7) $\geq$ (8). To do so, first partition the space of type profiles $\mathbf{t}_{-2}$ into one-dimensional subsets of the form $\{\mathbf{t}_{-2} : \mathbf{t}_{-12} = \hat{\mathbf{t}}_{-12}, t_{1,-1} = \hat{t}_{1,-1}\}$ for some fixed $\hat{\mathbf{t}}_{-12}, \hat{t}_{1,-1}$. Next, consider any fixed profile of bids $\hat{\mathbf{b}}_{-12}$ made by bidders $i \neq 1, 2$ given these types, i.e. $\lambda_i(\hat{b}_i, \hat{t}_i) > 0$ for all $i \neq 1, 2$.

Now, whether bidder 2 wins a q-th unit with bid $\tilde{b}_2$ only depends on bidder 1's bid. For every price p and profile of others' bids $\mathbf{b}_{-12}$, let $RS_{12}(p; \mathbf{b}_{-12})$ denote the 'residual supply' available to bidders in $1, 2$ at price p:

$$RS_{12}(p; \mathbf{b}_{-12}) \equiv S - \#\{(j, q) : j \neq 1, 2, b_{j,q} > p\}$$

where $\#X$ denotes the number of elements in X. Bidder 2 wins a q-th unit iff $\tilde{b}_{2,q} > b_{1,RS_{12}(\tilde{b}_{2,q},\hat{\mathbf{b}}_{-12})-q+1}$.[13] To shorten notation, define $Q_{12} \equiv RS_{12}(\tilde{b}_{2,q}, \hat{\mathbf{b}}_{-12}) - q + 1$.

To prove that (7) $\geq$ (8), it suffices to prove the related inequality (9) $\geq$ (10) for *every* fixed $(\hat{\mathbf{t}}_{-12}, \hat{t}_{1,-1}, \hat{\mathbf{b}}_{-12})$.

$$\frac{\sum\limits_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^1 MV_2(q, t_{1,1}, \hat{t}_{1,-1}, \hat{\mathbf{t}}_{-1}) \lambda_1(b_1, t_{1,1}, \hat{t}_{1,-1}) f_1(t_{1,1}|\hat{t}_{1,-1}) \mathrm{d}t_{1,1}}{\sum\limits_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^1 \lambda_1(b_1, t_{1,1}, \hat{t}_{1,-1}) f_1(t_{1,1}|\hat{t}_{1,-1}) \mathrm{d}t_{1,1}} \tag{9}$$

$$\frac{\sum\limits_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^1 MV_2(q, t_{1,1}, \hat{t}_{1,-1}, \hat{\mathbf{t}}_{-1}) 1_{\{b_1^0(t_{1,1},\hat{t}_{1,-1}) = b_1\}} f_1(t_{1,1}|\hat{t}_{1,-1}) \mathrm{d}t_{1,1}}{\sum\limits_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^1 1_{\{b_1^0(t_{1,1},\hat{t}_{1,-1}) = b_1\}} f_1(t_{1,1}|\hat{t}_{1,-1}) \mathrm{d}t_{1,1}} \tag{10}$$

To compare these ratios, define

$$z(t_{1,1}) \equiv MV_2(q, t_{1,1}, \hat{t}_{1,-1}, \hat{t}_{-1}) \tag{11}$$

$$g(t_{1,1}) \equiv \sum_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \lambda_1(b_1, t_{1,1}, \hat{t}_{1,-1}) f_1(t_{1,1}|\hat{t}_{1,-1}) \tag{12}$$

$$h(t_{1,1}) \equiv \sum_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} 1_{\{b_1^0(t_{1,1},\hat{t}_{1,-1}) = b_1\}} f_1(t_{1,1}|\hat{t}_{1,-1}) \tag{13}$$

[13] At price $\tilde{b}_{2,q}$, there are only $RS_{12}(\tilde{b}_{2,q}, \hat{\mathbf{b}}_{-12})$ units available to bidders 1,2. So, bidder 2 wins a q-th unit iff bidder 1 bids less on the $(RS_{12}(\tilde{b}_{2,q}, \hat{\mathbf{b}}_{-12}) - q + 1)$-st unit than bidder 2 bids on the q-th unit.

To complete the proof, it suffices to show that $\int_0^1 g(t_{1,1})\mathrm{d}t_{1,1} = \int_0^1 h(t_{1,1})\mathrm{d}t_{1,1}$ (denominators of (9,10) equal) and $\int_0^1 z(t_{1,1})g(t_{1,1})\mathrm{d}t_{1,1} \geq \int_0^1 z(t_{1,1})h(t_{1,1})\mathrm{d}t_{1,1}$ (numerator of (9) $\geq$ numerator of (10)).

Lemma 4. *Let $z, g, h : [0,1] \to R$ be any measurable functions where z is non-decreasing and g, h are non-negative. If (i) $\int_0^1 g(x)\mathrm{d}x = \int_0^1 h(x)\mathrm{d}x$ and (ii) $\int_0^y g(x)\mathrm{d}x \leq \int_0^y h(x)\mathrm{d}x$ for all $y \in [0,1]$, then $\int_0^1 z(x)g(x)\mathrm{d}x \geq \int_0^1 z(x)h(x)\mathrm{d}x$.*

g, h defined by (12,13) are clearly non-negative, while z defined by (11) is non-decreasing: $V_2(q', \mathbf{t}) - V_2(q, \mathbf{t})$ is non-decreasing in $\mathbf{t}_{-2}$ whenever $q' > q$, so in particular $MV_2(q, \mathbf{t})$ is non-decreasing in $t_{1,1}$. Lemma 5 completes the proof of Lemma 3, by establishing conditions (i,ii) of Lemma 4.

Lemma 5. *Consider g, h defined by (12,13). Then (i) $\int_0^1 g(t_{1,1})\mathrm{d}t_{1,1} = \int_0^1 h(t_{1,1})\mathrm{d}t_{1,1}$ and (ii) $\int_0^y g(t_{1,1})\mathrm{d}t_{1,1} \leq \int_0^y h(t_{1,1})\mathrm{d}t_{1,1}$ for all $y \in [0,1]$.*

Proof of Lemma 4

Define $G(y) \equiv \int_0^y g(x)\mathrm{d}x$ and $H(y) \equiv \int_0^y h(x)\mathrm{d}x$. As long as g, h are positive functions,[14] inverses G^{-1}, H^{-1} exist with well-defined right-derivatives: $\frac{\mathrm{d}^+ G^{-1}(\alpha)}{\mathrm{d}\alpha} = \frac{1}{g^+(G^{-1}(\alpha))}$ and $\frac{\mathrm{d}^+ H(y)}{\mathrm{d}y} = \frac{1}{h^+(H^{-1}(\alpha))}$. (The notation $f^+(x) \equiv \lim_{\varepsilon \to 0} f(x+\varepsilon)$ refers to the right-limit, as usual.) By (i), $G^{-1}(G(1)) = H^{-1}(G(1)) = 1$. Thus, it suffices to show

$$\int_0^{G^{-1}(\alpha)} z(x)g(x)\mathrm{d}x \geq \int_0^{H^{-1}(\alpha)} z(x)h(x)\mathrm{d}x \tag{14}$$

for all $\alpha \in [0, G(1)]$. This inequality holds trivially when $\alpha = 0$. To complete the proof, it suffices to show that $\int_0^{G^{-1}(\alpha)} z(x)g(x)\mathrm{d}x - \int_0^{H^{-1}(\alpha)} z(x)h(x)\mathrm{d}x$ is

[14] Without loss of generality, we may restrict attention to positive g, h. To see why, suppose that g, h are non-negative functions and define positive functions $\hat{g}(x) = g(x) + \varepsilon$, $\hat{h}(x) = h(x) + \varepsilon$ where $\varepsilon > 0$. As long as (i),(ii) hold for g, h, they also hold for $\hat{g}, \hat{h}$. As long as this implies $\int_0^1 z(x)\hat{g}(x)\mathrm{d}x \geq \int_0^1 z(x)\hat{h}(x)\mathrm{d}x$ for all non-decreasing z, it also implies $\int_0^1 z(x)g(x)\mathrm{d}x \geq \int_0^1 z(x)h(x)\mathrm{d}x$ for all non-decreasing z.

non-decreasing in α.

$$\frac{\mathrm{d}^+\left[\int_0^{G^{-1}(\alpha)} z(x)g(x)\mathrm{d}x\right]}{\mathrm{d}\alpha} = \frac{\mathrm{d}^+G^{-1}(\alpha)}{\mathrm{d}\alpha} z^+(G^{-1}(\alpha))g^+(G^{-1}(\alpha)) = z^+(G^{-1}(\alpha))$$
$$\frac{\mathrm{d}^+\left[\int_0^{H^{-1}(\alpha)} z(x)h(x)\mathrm{d}x\right]}{\mathrm{d}\alpha} = \frac{\mathrm{d}^+H^{-1}(\alpha)}{\mathrm{d}\alpha} z^+(H^{-1}(\alpha))h^+(H^{-1}(\alpha)) = z^+(H^{-1}(\alpha))$$

By (ii), $G(y) \leq H(y)$ for all y. In particular, $G(H^{-1}(\alpha)) \leq H(H^{-1}(\alpha)) = \alpha = G(G^{-1}(\alpha))$. Since G is increasing, this implies $H^{-1}(\alpha) \leq G^{-1}(\alpha)$ for all $\alpha \in [0, G(1)]$. Since z is non-decreasing, $z^+(G^{-1}(\alpha)) \geq z^+(H^{-1}(\alpha))$ and we are done. □

Proof of Lemma 5

Define

$$G(y) \equiv \int_0^y g(t_{1,1})\mathrm{d}t_{1,1} = \sum_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^y \lambda_1(b_1, t_{1,1}, \hat{t}_{1,-1}) f_1(t_{1,1}|\hat{t}_{1,-1})\mathrm{d}t_{1,1} \quad (15)$$
$$H(y) \equiv \int_0^y h(t_{1,1})\mathrm{d}t_{1,1} = \sum_{b_1 : b_{1,Q_{12}} < \tilde{b}_{2,q}} \int_0^y 1_{\{b_1^0(t_{1,1},\hat{t}_{1,-1}) = b_1\}} f_1(t_{1,1}|\hat{t}_{1,-1})\mathrm{d}t_{1,1} \quad (16)$$

We need to show that $G(1) = H(1)$ and that $G(y) \leq H(y)$ for all $y \in [0,1]$. $G(1)$ (respectively $H(1)$) is the probability that bidder 1 bids less than $\tilde{b}_{2,q}$ on unit Q_{12} in the strategy $\lambda_1(\cdot;\cdot)$ (respectively $b_1^0(\cdot)$), conditional on $t_{1,-1} = \hat{t}_{1,-1}$. By definition, $G(1) = \sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(b_{1,Q_{12}}|\hat{t}_{1,-1})$ (for the definition, see (4) in the text). By construction of $b_1^0(\cdot)$, furthermore, $b^0_{1,Q_{12}}(t_{1,1}, \hat{t}_{1,-1}) < \tilde{b}_{2,q}$ iff $F_1(t_{1,1}|\hat{t}_{1,-1}) < \sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(b_{1,Q_{12}}|\hat{t}_{1,-1})$. Thus, $H(1) = \sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(b_{1,Q_{12}}|\hat{t}_{1,-1})$ as well.

$G(y)$ (respectively $H(y)$) is the probability that $t_{1,1} \leq y$ *and* bidder 1 bids less than $\tilde{b}_{2,q}$ on unit Q_{12} in the strategy $\lambda_1(\cdot;\cdot)$ (respectively $b_1^0(\cdot)$), conditional on $t_{1,-1} = \hat{t}_{1,-1}$. $G(y)$ and $H(y)$ are each bounded above by $F_1(y|\hat{t}_{1,-1})$, the mass of all types such that $t_{1,1} < y$, *and* bounded above by $\sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(b_{1,Q_{12}}|\hat{t}_{1,-1})$, the mass of all types that bid less than $\tilde{b}_{2,q}$. We conclude the proof by showing that *either* $H(y) = F_1(y|\hat{t}_{1,-1})$ *or* $H(y) = \sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(b_{1,Q_{12}}|\hat{t}_{1,-1})$, for all $y \in [0,1]$.

Suppose that $\tilde{b}_{2,q} > \mathfrak{b}_{1,Q_{12}}(F_1(y|\hat{t}_{1,-1}))$. In this case, $b^0_{1,Q_{12}}(t_{1,1}, \hat{t}_{1,-1}) < \tilde{b}_{2,q}$ for all $t_{1,1} \leq y$, so $H(y) = F_1(y|\hat{t}_{1,-1})$. Otherwise suppose that $\tilde{b}_{2,q} \leq \mathfrak{b}_{1,Q_{12}}(F_1(y|\hat{t}_{1,-1}))$. In this case, $b^0_{1,Q_{12}}(y, \hat{t}_{1,-1}) \geq \tilde{b}_{2,q}$, so that $b^0_{1,Q_{12}}(t_{1,1}, \hat{t}_{1,-1}) \geq \tilde{b}_{2,q}$ for all $t_{1,1} \geq y$. In particular, $b^0_{1,Q_{12}}(t_{1,1}, \hat{t}_{1,-1}) < \tilde{b}_{2,q}$ implies $t_{1,1} < y$, i.e. the entire mass of bidder 1 types who bid less than $\tilde{b}_{2,q}$ on unit Q_{12} have $t_{1,1} < y$. So, $H(y) = \sum_{b_{1,Q_{12}} < \tilde{b}_{2,q}} p^1_{1,Q_{12}}(\tilde{b}_{1,Q_{12}}|\hat{t}_{1,-1})$. $\square$

References

Athey, S. (2001), "Single Crossing Properties and the Existence of Pure-Strategy Equilibria in Games of Incomplete Information", *Econometrica*, **69**, 861–889.

Back, K. and Zender, J. (1993), "Auctions of Divisible Goods: On the Rationale for the Treasury Experiment", *Review of Financial Studies*, **6**, 733–764.

Bernheim, D. and Whinston, M. (1986), "Menu Auctions, Resource Allocation, and Economic Influence", *Quarterly Journal of Economics*, **101**, 1–31.

Bresky, M. (2000), "Equilibria in Multi-Unit Auctions" (Mimeo, CERGE-EI).

Cripps, M. and Swinkels, J. (2005), "Efficiency of Large Double Auctions", *Econometrica*, forthcoming.

Fudenberg, D., Mobius, M. and Sziedl, A. (2003), "Existence of Equilibrium in Large Double Auctions" (Mimeo, Harvard).

Jackson, M. and Swinkels, J. (2005), "Existence of Equilibria in Single and Double Auctions", *Econometrica*, **73**, 93–139.

Jehiel, P., Moldovanu, B. and Stacchetti, E. (1996), "How (Not) to Sell Nuclear Weapons", *American Economic Review*, **86**, 814–29.

Kazumori, E. (2002), "Toward a Theory of Strategic Markets with Incomplete Information: Existence of Isotone Equilibrium" (Mimeo, CalTech).

McAdams, D. (2002a), "Bidding Lower with Higher Values in Multi-Object Auctions", MIT Sloan Working Paper # 4249-02.

McAdams, D. (2002b), "Monotone Equilibrium in Multi-Unit Auctions", MIT Sloan Working Paper #4254-02.

McAdams, D. (2003a), "Isotone Equilibrium in Games of Incomplete Information", *Econometrica*, **71**, 1191–1214.

McAdams, D. (2003b), "Monotonicity in Asymmetric First-Price Auctions with Affiliation" (Mimeo, MIT).

McAdams, D. (2005), "Identification and Testable Restrictions in Private Value Multi-Unit Auctions" (Mimeo, MIT).

Milgrom, P. and Shannon, C. (1994), "Monotone Comparative Statics", *Econometrica*, **62**, 157–180.

Reny, P. (1999), "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games", *Econometrica*, **67**, 1029–1056.

Reny, P. and Perry, M. (2003), "Toward a Strategic Foundation for Rational Expectations Equilibrium" (Mimeo, U. Chicago).

Reny, P. and Zamir, S. (2004), "On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions", *Econometrica*, **72**, 1105–25.

Wilson, R. (1979), "Auctions of Shares", *Quarterly Journal of Economics*, **94**, 675–689.