# Secrecy in the first-price auction

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**Abstract**

This paper endogenizes bidders' beliefs about their competition in a symmetric first-price auction with independent private values, by allowing bidders to decide whether to participate publicly or secretly. When public participation is more costly, bidders only participate secretly in the unique equilibrium. By contrast, when secret participation is slightly more costly, all symmetric equilibria exhibit a mixture of secret and public participation. In this case, switching to a second-price format increases expected revenue and expected total welfare among all symmetric equilibria.

## 1 Introduction

An often unspoken assumption of auction theory is that the number of bidders is common knowledge before the bidding. However, real-world bidders often take pains to conceal their participation. In the summer of 2010, for example, rumors swirled that both Intel

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and Apple might be in the market to purchase the wireless division of semiconductor manufacturer Infineon.[1] Intel ultimately acquired Infineon, but it remains unknown whether Apple was also quietly pursuing a deal.

This paper endogenizes what bidders know about their competition in the first-price auction, by allowing bidders to choose whether to participate *secretly* or *publicly*. In the "pre-auction stage", bidders observe symmetric independent private values and then simultaneously decide whether to participate publicly at cost $c_P \geq 0$, participate secretly at cost $c_S \geq 0$, or not participate. The set of public participants is then commonly observed. In the "auction stage," all participants simultaneously submit sealed bids in a first-price auction.

Without the possibility of secrecy, each bidder in a symmetric first-price auction bids less and enjoys greater interim expected payoff when faced with fewer competitors. Once secrecy is possible, however, less *public* competition does not necessarily translate into less *total* competition. Indeed, if public bidders are in equilibrium "weaker" than secret bidders, in the sense of having lower values, it is conceivable that some bidders could bid more and/or enjoy lower interim expected payoff when faced with fewer public competitors.

For an intuition why public bidders may differ from secret bidders, consider a sale of real estate in which all bids must be submitted by a deadline. Anticipating that the real-estate agent has an incentive to reveal any interest in the property in order to intensify the bidding competition, the most serious bidders may take steps to conceal their interest until after the deadline. On the other hand, bidders who are only mildly interested in the

[1] "Why Apple Should Buy Infineon: To Own Mobile And Screw Intel" by Steve Cheney, TechCrunch, July 30, 2010. I will return to the Infineon sale at more length in Section 2.1, when discussing some potential sources of public and secret participation costs.

property have little incentive to conceal their intention to participate, since they expect to lose to any serious competitor whether or not their interest is known. Consequently, public bidders in such an auction may tend to have lower values than those who have chosen to participate secretly.

Might bidders even be willing to pay extra in equilibrium to publicize their participation, if doing so convinces others that they have relatively low values? No. Whenever public participation is more costly than secret participation, there is a unique perfect Bayesian equilibrium in weakly undominated strategies (shorthand "PBE") and no bidder ever participates publicly in this equilibrium (Theorem 4). Without public participation, this equilibrium is outcome-equivalent to the unique symmetric equilibrium in the well-known model in which only secret participation is possible; see e.g. Samuelson (1985). In particular, raising the cost of secret participation strictly decreases expected revenue and interim expected bidder surplus (Theorem 6).

On the other hand, when secret participation is slightly more costly than public participation, every symmetric PBE exhibits a mix of public and secret participation (Theorem 5). Intuitively, some participation must be public since the lowest-value bidders who participate only win if no one else participates, and hence prefer to participate in whatever way is least costly. On the other hand, some participation must be secret since secret participation, if unexpected, dampens bidding competition. (If bidders do not expect secret participation, then they will assume that all bidders are public bidders. Participating secretly will then induce others to bid less.) In this case, disallowing secret participation strictly increases expected revenue and expected total welfare, while weakly decreasing interim expected bidder surplus (Theorem 7).

One way to dissuade bidder secrecy is simply to disallow it, as when the seller reveals the set of participants prior to the bidding. Such a step is common in practice, e.g. in

FCC spectrum auctions[2] and government procurement. Another way to dissuade bidder secrecy is to change the auction format. In a second-price or English auction with private values, each participant will bid his value regardless of whether he is secret or public, and regardless of what he believes about others. These auction formats therefore deter bidders from investing in secrecy, to the ultimate benefit of the seller. A simple example shows that losses due to bidder secrecy can amount to more than 25% of expected revenue.

The rest of the paper is organized as follows. The introduction continues with a brief discussion of some related literature. Section 2 presents the model and preliminaries, as well as a discussion why *secret* participation may be more or less costly than *public* participation. Section 3 considers a simple, illustrative example. Section 4 characterizes how bidders participate in equilibrium – only publicly, only secretly, or a mixture of both – as a function of the participation costs $(c_P, c_S)$ and the distribution of bidder values. Section 5 then develops welfare and revenue comparative statics with respect to the cost of secret participation. Section 6 concludes with thoughts on future research directions. Some proofs are in the Appendix.

**Related literature.** The most closely related paper is Samuelson (1985), who considers a first-price auction with costly participation in which participation either must be secret or must be public. Samuelson found that (i) the seller's revenue depends on the cost of participation, but not on whether participation is only-secret or only-public, and (ii) the seller's revenue is decreasing in the cost of participation. When the cost of secret and public participation is the same, these basic findings continue to hold in my setting, although I show that participation is endogenously secret (Theorem 3). If

[2]Another reason why the FCC publicizes the set of participants in spectrum auctions is to help enforce anti-collusion rules. Namely, publicizing the set of participants serves to inform bidders with whom they are lawfully required *not* to communicate about bidding in the auction.

bidders sometimes choose to participate publicly, there must be extra costs associated with secrecy. In this case, disallowing secrecy increases welfare and revenue by dissuading socially wasteful investments in secrecy.

Other papers have compared first-price auctions in which participation must be secret or must be public. Most notably, McAfee and McMillan (1987a) shows that expected revenue in a first-price auction with independent private values is higher when participation must be secret than when it must be public, if (i) the number of bidders is random and (ii) bidders are risk-averse with CARA utility. The key innovation here relative to this prior literature is to allow bidders to *choose* whether to make their participation known. This distinction is significant if the seller cannot completely control what bidders know about their competition.

## 2 Model and preliminaries

*Pre-auction phase.* $n \geq 2$ risk-neutral bidders costlessly observe symmetric independent private values $v_i$. (See comment (a) below.) I assume that bidder values have support $[0, 1]$, c.d.f. $F(.)$, and strictly positive continuously differentiable p.d.f. $f(.)$, and that "virtual values" $v_i - \frac{1-F(v_i)}{f(v_i)}$ are strictly increasing in $v_i$. Each bidder then simultaneously decides whether to incur cost $c_P \geq 0$ to participate publicly, incur cost $c_S \geq 0$ to participate secretly, or not participate. (See comment (b).) I assume that $\max\{c_S, c_P\} > 0$. (See comment (c).)

*Bidding phase.* All participants observe who has chosen to participate publicly, then submit sealed bids in a first-price auction with zero reserve price. Ties are broken in favor of secret bidders,[3] or randomly among like bidders. If no bidder participates, then

[3]Favoring secret bidders ensures that a best response exists off the equilibrium path, for a bidder who

the object is not sold and has no subsequent value to the seller.

**Strategies.** Bidder $i$'s strategy consists of a "participation strategy" specifying probabilities $q_i^P(v)$ and $q_i^S(v)$ of public and secret participation for all $v \in [0,1]$, respectively, and a "bidding strategy" $b_i(v;P)$ specifying bidder $i$'s bid if he participates, as a function of his value and the subset $P \subset \{1,...,n\}$ of bidders who participate publicly. Strategies are *symmetric* if, for all $i$ and all $v$, (i) $q_i^P(v) = q^P(v)$ and $q_i^S(v) = q^S(v)$ and (ii) $b_i(v;P) = b^P(v;\#(P))$ if $i \in P$ and $b_i(v;P) = b^S(v;\#(P))$ if $i \notin P$. That is, each participant's bid depends only on his value, the number of public participants $\#(P)$, and whether he himself participates publicly or secretly.

**Definition 1** (Only-secret participation)**.** Bidder $i$'s participation strategy exhibits *only-secret participation* if $E_{v_i}\left[q_i^P(v_i)\right] = 0$.

**Definition 2** (Only-public participation)**.** Bidder $i$'s participation strategy exhibits *only-public participation* if $E_{v_i}\left[q_i^S(v_i)\right] = 0$.

**Participation thresholds.** In every equilibrium with some public or secret participation, there is a minimal value given which any bidder ever participates publicly or secretly. Such "participation thresholds" will play a central role in the analysis.

**Definition 3** (Participation thresholds)**.** Bidder $i$'s *"secret participation threshold"* $\underline{v}_i^S$ and *"public participation threshold"* $\underline{v}_i^P$ are, respectively, the lowest values given which he participates secretly or publicly:

$$\underline{v}_i^S = \max\left\{v \in [0,1] : \int_0^v q_i^S(v_i)\mathrm{d}v_i = 0\right\}$$
$$\underline{v}_i^P = \max\left\{v \in [0,1] : \int_0^v q_i^P(v_i)\mathrm{d}v_i = 0\right\}.$$

has deviated by participating secretly.

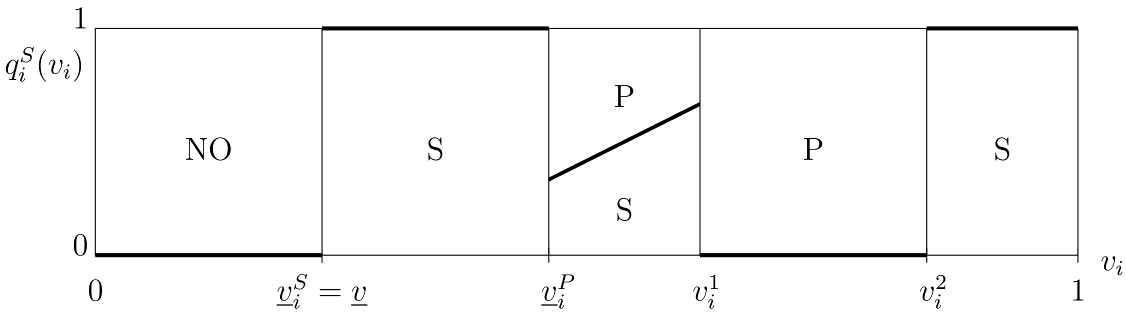


Figure 1: Illustration of a participation strategy for bidder $i$.

Bidder $i$'s *"participation threshold"* $\underline{v}_i = \min\{\underline{v}_i^S, \underline{v}_i^P\}$ is the lowest value given which he participates. Should bidders adopt symmetric strategies, let $\underline{v}^S, \underline{v}^P, \underline{v}$ be shorthand for their (symmetric) participation thresholds.

Figure 1 illustrates a hypothetical participation strategy for bidder $i$. According to this strategy, bidder $i$ does not participate when $v_i \in [0, \underline{v}_i^S)$, only participates secretly when $v_i \in (\underline{v}_i^S, \underline{v}_i^P)$, mixes between secret and public participation when $v \in (\underline{v}_i^P, v_i^1)$, only participates publicly when $v \in (v_i^1, v_i^2)$, and only participates secretly when $v_i \in (v_i^2, 1)$. (Bidder $i$'s probability of secret participation, $q_i^S(v_i)$, is denoted by a thick line.)

**Solution concept.** The solution concept is perfect Bayesian equilibrium in weakly undominated strategies (shorthand "PBE"). Some results restrict attention to symmetric PBE. (See comment (d).)

**Theorem 1** (Existence of symmetric PBE)**.** *There exists a perfect Bayesian equilibrium in symmetric, weakly undominated strategies ("symmetric PBE").*

*Proof.* See the Appendix. □

**Comments on the model.** (a) Bidders observe their private values before deciding whether to participate. Thus, the participation costs considered here can be interpreted as "costs of bidding" rather than "costs of learning." It would be straightforward to overlay the present model with an earlier stage in which bidders decide whether to invest to learn their private values. Indeed, the model can be easily extended in two such ways, whether (i) as in Levin and Smith (1994), one assumes that the set of bidders who invest to learn their values becomes common knowledge prior to the next stage of the game, or (ii) as in McAfee and McMillan (1987b), bidders learn nothing about who else has invested to learn their values.

(b) Participation costs are exogenous. In future work, it could be interesting to endogenize such costs. For example, suppose that an auction intermediary knows who will be participating and can choose whether or not to reveal each participant prior to the bidding. Such an auction intermediary could benefit from charging bidders to stay secret, although doing so lowers the seller's expected revenue. This provides one more reason why sellers may seek to employ *reputable*, incorruptible auction intermediaries.[4]

(c) If $c_S = c_P = 0$, every bidder is certain to participate in the auction, so bidders' participation decisions have no impact on others' beliefs about the set of participants in the auction. Thus, the decision to be "secret" or "public" amounts to cheap talk that may be ignored in equilibrium. Indeed, symmetric PBE exist that can support *any* pattern of public and secret participation but in which bidders ignore the set of public participants when formulating bids. Assuming $\max\{c_P, c_S\} > 0$ allows me to ignore this uninteresting, indeterminate case.

(d) Some results presume that bidders play a symmetric PBE. Such an equilibrium exists (Theorem 1). However, in the special case when only public participation is possible ($c_S = \infty$), Cao and Tian (2010) show that asymmetric equilibria can exist for some

[4] McAdams and Schwarz (2007) shows that a seller prefers to employ a reputable auction intermediary than to conduct the sale herself, when she lacks the commitment power to credibly "close the deal".

distributions of bidder values. More precisely, asymmetric equilibria exist if $F(v) < vf(v)$ for all $v$ but not if $F(v) \geq vf(v)$ for all $v$ (Cao and Tian (2010), Proposition 1). In particular, if values are drawn uniformly on $[0, 1]$, as in the example of Section 3, (i) asymmetric equilibria do *not* exist and (ii) there is a unique symmetric equilibrium when all participation must be public.

## 2.1 Discussion: secret and public participation costs

Participating in an auction can be costly, for a wide variety of reasons. For instance, Samuelson (1985) noted that "competing firms must bear significant bid-preparation and documentation costs." Some of this paper's results are predicated on the notion that the cost of participating in an auction may in fact depend on *how* one participates.[5] In this section, I discuss some reasons why secret or public participation may be more costly in practice.

**Cost of publicity.** There could be costs associated with announcing one's participation in an auction, but such costs seem likely to be quite small. For instance, a firm interested in acquiring a particular target could simply issue a press release. Potentially more substantial are the effects of publicity on a bidder's interactions *outside* of the auction. As an example, consider the 2010 sale of wireless chip-maker Infineon mentioned in the introduction, in which Apple and Intel were potential bidders. At that time, Apple faced a critical decision whether and how much to backward integrate into the design and manufacture of smart-phone components. In 2008, Apple had paid \$278 million

[5] The model allows secret and public participation to be equally costly. In this case, I prove that bidders never participate publicly in all symmetric PBE (Theorem 3). However, other results are only relevant if there are extra costs associated with secrecy or publicity.

to acquire low-power microchip designer P.A. Semi.[6] This acquisition gave them the capability to design the iPad's A4 microchip in-house, released in January 2010.[7] A similar acquisition of Infineon could allow Apple to design and build its own wireless chips, another key potential differentiator in the smartphone market. However, if Apple had an interest in developing its own wireless chips and this became known, it could make actually doing so more costly (perhaps especially if Apple failed to acquire Infineon). For instance, knowing that their talents are sorely needed, chip-designers at or outside Apple might demand a higher salary in order to work at the firm. For this and other reasons, Apple may have been willing to pay a substantial amount to avoid revealing any interest in Infineon. In other words, Apple could have faced a high *cost of publicity*.

**Cost of secrecy.** Sometimes it is impossible to participate in an auction without the seller's knowledge. For example, in a corporate acquisition, the process of due diligence requires extensive interactions between the target and any potential acquirer. Maintaining secrecy therefore requires convincing the target not to reveal one's participation to other bidders. However, providing a sufficient reward or establishing a sufficient threat to induce such silence could entail substantial cost.

More broadly, a bidder who wants to keep his participation secret must actively suppress all signs of his interest in the good being sold. In particular, maintaining secrecy requires that a bidder inefficiently avoid or delay all decisions that signal interest. Such decisions can range from the small – such as whether to schedule a private tour of a home for sale (revealing one's potential interest to the seller's agent and thereby to other buyers' agents) or to attend a more-anonymous open house – to the large. For

[6] "Apple buying microchip designer P.A. Semi" by Scott Hillis, Reuters, April 23, 2008

[7] "A Little Chip Designed by Apple Itself" by Ashlee Vance and Brad Stone, New York Times, February 1, 2010".

example, consider again the sale of wireless chip-maker Infineon in which Apple may have been a bidder. Suppose that, just before the Infineon sale, Apple had to decide whether to (observably) invest in relationships with wireless-chip suppliers other than Infineon. If Apple were interested in Infineon, it might naturally prefer to avoid such investments until after the uncertainty of the Infineon sale is resolved, but prefer to make such investments if uninterested. If so, failure to invest in its other relationships would effectively publicize Apple's intention to bid for Infineon. So, Apple might need to make inefficient relationship-specific investments in order to shroud its intent to bid. The losses due to such inefficient investments would constitute a *cost of secrecy.*

# 3 Illustrative example

Two bidders $i = 1, 2$ have iid private values $v_i \sim U[0, 1]$. Public participation costs $c_P > 0$ while secret participation costs $c_S > c_P$.

**Benchmark: second-price auction.** In the second-price auction, others will bid their values regardless of who participates publicly. Thus, there is no benefit to secrecy and bidders only participate publicly in all PBE. Further, the second-price aucton has a unique *symmetric* PBE, in which each bidder's participation threshold $\underline{v} = \underline{v}^P$ is determined by the the indifference condition $(\underline{v} - 0)F(\underline{v}) = c_P$, or $\underline{v} = \sqrt{c_P}$. (A bidder with value $\underline{v}$ only wins if the other bidder does not participate and, in this case, pays zero for the good.) The seller earns zero revenue unless both bidders have values greater than $\sqrt{c_P}$, in which case she receives revenue equal to the second-highest value. The seller's expected revenue $REV(c_P) = \frac{1+2\sqrt{c_P}}{3}(1 - \sqrt{c_P})^2 = \frac{1}{3} - c_P\left(1 - \frac{2\sqrt{c_P}}{3}\right)$.

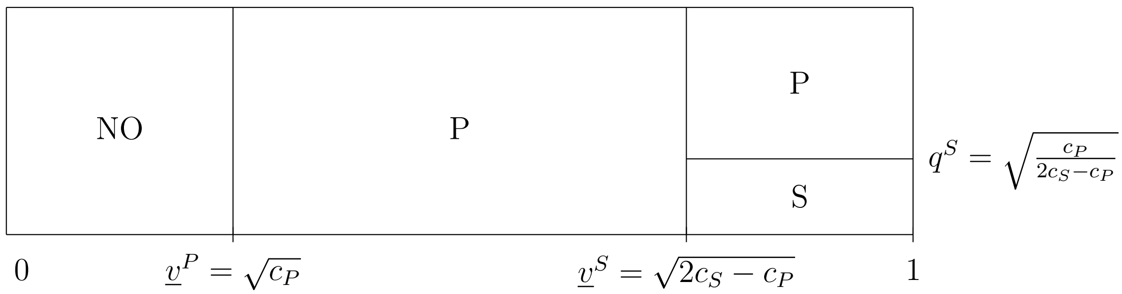


Figure 2: Illustration of participation in a symmetric PBE.

**A symmetric PBE.** I will now construct a symmetric PBE in the first-price auction and compare its expected revenue against that in the second-price auction.

**Claim 1.** *As long as* $c_P < 1$ *and* $c_S \in \left(c_P, \frac{1+c_P}{2}\right)$*, there exists a symmetric PBE in which each bidder's participation strategy is (i)* $q^S(v) = q^P(v) = 0$ *for all* $v < \underline{v}^P$*, (ii)* $q^P(v) = 1$ *for all* $v \in (\underline{v}^P, \underline{v}^S)$*, and (iii)* $q^S(v) = q^S$ *and* $q^P(v) = 1 - q^S$ *for all* $v > \underline{v}^S$*, where*

$$\underline{v}^P = \sqrt{c_P}, \underline{v}^S = \sqrt{2c_S - c_P}, \text{ and } q^S = \frac{\underline{v}^P}{\underline{v}^S} = \sqrt{\frac{c_P}{2c_S - c_P}}.$$

*Proof.* Suppose for the moment that an equilibrium exists with the participation strategies posited in Claim 1. To complete the proof, I will derive subsequent equilibrium bidding strategies in each subgame – when both, one, or none of the bidders participates publicly – and verify that the specified participation strategies are in fact a best response given such subsequent bidding.

*Preliminaries: symmetry above but not below* $\underline{v}^S$*.* Let $F(v|P)$ denote the cdf of bidder values conditional on participating publicly, and let $F(v|S)$ be the corresponding cdf conditional on not participating publicly. Each bidder $i$ participates publicly whenever $v_i \in (\underline{v}^P, \underline{v}^S)$ and with probability $q^P = \frac{\underline{v}^S - \underline{v}^P}{\underline{v}^S}$ when $v_i > \underline{v}^S$. The ex ante probability of public participation is therefore $E[q^P(v))] = (\underline{v}^S - \underline{v}^P) + q^P(1 - \underline{v}^S) = \frac{\underline{v}^S - \underline{v}^P}{\underline{v}^S}$, while the conditional cdf $F(v|P) = \frac{\underline{v}^S - \underline{v}^P + q^P(v - \underline{v}^S)}{E[q^p(v)]}$ simplifies to

$$F(v|P) = v \text{ for all } v > \underline{v}^S.$$

Similarly, bidder $i$ does *not* participate publicly whenever $v_i < \underline{v}^P$ and with probability $q^S = \frac{\underline{v}^P}{\underline{v}^S}$ when $v_i > \underline{v}^S$. Thus, $E[1 - q^P(v))] = \underline{v}^P + q^S(1 - \underline{v}^S) = \frac{\underline{v}^P}{\underline{v}^S}$ and the conditional cdf $F(v|S) = \frac{\underline{v}^P + q^S(v - \underline{v}^S)}{E[1 - q^p(v)]}$ again simplifies to

$$F(v|S) = v \text{ for all } v > \underline{v}^S.$$

On the other hand, at values below $\underline{v}^S$, public bidders and others do not have symmetrically distributed values. Bidders participate publicly given any values $v \in (\underline{v}^P, \underline{v}^S)$, generating conditional cdf

$$F(v|P) = \frac{\underline{v}^S(v - \underline{v}^P)}{\underline{v}^S - \underline{v}^P} \text{ for all } v \in (\underline{v}^P, \underline{v}^S).$$

By contrast, others never participate given values less than $\underline{v}^S$. As a convention, then, I will treat such bidders' values as having an "atom at 0" of magnitude

$$F(v|S) = \frac{\underline{v}^P}{E[1 - q^p(v)]} = \underline{v}^S \text{ for all } v < \underline{v}^S.$$

*Bidding when both public or neither public.* When both bidders participate publicly, subsequent equilibrium bidding proceeds as in a standard first-price auction with costless bidding in which bidders have iid values distributed according to cdf $F(.|P)$. In particular, each bidder $i$ bids $b(v; \{1, 2\}) = E\left[v^{(2)}|v^{(1)} = v, P = \{1, 2\}\right]$, the expected second-highest value conditional on (i) his own value being highest and (ii) both bidders having values distributed according to cdf $F(.|P)$. For future reference, note that $b(v; \{1, 2\}) = \frac{v + \underline{v}^P}{2}$ for all $v \in (\underline{v}^P, \underline{v}^S)$. Similarly, when both bidders participate secretly, each bids $b(v; \emptyset) = E\left[v^{(2)}|v^{(1)} = v, P = \emptyset\right]$, conditioning now that bidders have values distributed according to cdf $F(.|S)$. For future reference, note that $b(\underline{v}^S; \emptyset) = 0$.

*Bidding when one bidder is public and one is not.* Suppose that bidder 1 participates publicly, while bidder 2 does not. Equilibrium bidding now corresponds to that in a standard first-price auction with costless, voluntary bidding in which bidders' values are distributed asymmetrically, as follows. Bidder 1's value has support $[\underline{v}^P, 1]$, with mass $\underline{v}^S$ spread uniformly over $[\underline{v}^P, \underline{v}^S]$ and mass $1 - \underline{v}^S$ spread uniformly over $[\underline{v}^S, 1]$. Bidder 2's value has support $\{-\infty\} \cup [\underline{v}^S, 1]$, with mass $\underline{v}^S$ at $-\infty$ and mass $1 - \underline{v}^S$ again spread uniformly over $[\underline{v}^S, 1]$.

The following is a bidding equilibrium: public bidder 1 bids $b_1(v; \{1\}) = 0$ for all $v \in (\underline{v}^P, \underline{v}^S)$ and $b_1(v; \{1\}) = b(v; \emptyset)$ for all $v > \underline{v}^S$; and secret bidder 2 bids $b_2(v; \{1\}) = b(v; \emptyset)$ for all $v > \underline{v}^S$. To see why, note that each bidder's bid is distributed exactly as in the symmetric equilibrium in the subgame when $P = \emptyset$. Thus, each bidder's best response given any value $v > \underline{v}^S$ must be the same as in that subgame. It remains to check that public bidder 1 prefers to bid zero given any value $v_1 \in (\underline{v}^P, \underline{v}^S)$. But this follows immediately from monotonicity of his best response, since bidder 1 prefers to bid zero given value $v_1 = \underline{v}^S$. (Recall that $b(\underline{v}^S; \emptyset) = 0$.)

*Secret, public, or no participation?* Let $S^S(v), S^P(v)$ denote each bidder's *gross* interim expected surplus when secret or public, and when playing a best response to the equilibrium bidding strategies derived above. To complete the proof, it suffices to show that (i) $S^P(v) - c_P < 0$ and $S^S(v) - c_S < 0$ for all $v < \underline{v}^P$, (ii) $S^P(v) - c_P > 0 > S^S(v) - c_S$ for all $v \in (\underline{v}^P, \underline{v}^S)$, and (iii) $S^P(v) - c_P = S^S(v) - c_S > 0$ for all $v > \underline{v}^S$. (If so, the participation strategy of Claim 1 is a best response given subsequent equilibrium bidding.) Without loss, I will focus on bidder 1.

<u>Case #1: $v_1 < \underline{v}^P = \sqrt{c_P}$.</u> Suppose first that bidder 1 were to participate secretly. By monotonicity of best replies, bidder 1 bids zero in the resulting bidding subgame, whether

bidder 2 participates publicly or not. Thus, bidder 1 only wins when bidder 2 does not participate ($v_2 < \underline{v}^P$), for interim expected gross payoff $S^S(v) = v_1\underline{v}^P < c_P < c_S$. Suppose next that bidder 1 were to participate publicly. If bidder 2 also participates publicly, bidder 1 surely loses since bidder 2 always bids at least $\underline{v}^P > v_1$. If bidder 2 does not participate publicly, however, bidder 1's best response is again to bid zero and only win if $v_2 < \underline{v}^P$, for interim expected gross payoff $S^S(v) = v_1\underline{v}^P < c_P$. Thus, bidder 1 prefers not to participate when $v_1 < \underline{v}^P$.

Case #2: $v_1 \in (\underline{v}^P, \underline{v}^S) = (\sqrt{c_P}, \sqrt{2c_S - c_P})$. Suppose first that bidder 1 were to participate publicly. If bidder 2 is also public, bidder 1 will bid $\frac{v_1+\underline{v}^P}{2}$ and win with (unconditional) probability $v_1 - \underline{v}^P$, for expected payment of $\frac{(v_1)^2-(\underline{v}^P)^2}{2}$. Or, if bidder 2 is not public, bidder 1 will bid zero and win with probability $\underline{v}^P$. Overall, bidder 1's gross interim expected profit is $S^P(v_1) = (v_1)^2 - \frac{(v_1)^2-(\underline{v}^P)^2}{2} = \frac{(v_1)^2+c_P}{2}$. Since $v_1 > \sqrt{c_P}$, $S^P(v_1) > c_P$ and bidder 1 prefers public participation over none.

Suppose next that bidder 1 were to participate secretly. Whether bidder 2 participates secretly or publicly, bidder 1's best response in the resulting bidding subgame will be to bid zero, winning exactly when bidder 2's value is less than $\underline{v}^S$. (If bidder 2 has value in $(\underline{v}^P, \underline{v}^S)$, he will bid zero and lose, since ties are broken in favor of secret bidders.) Thus, bidder 1's gross interim expected profit is $S^S(v_1) = v_1\underline{v}^S$. Note that, when $v_1 = \underline{v}^S = \sqrt{2c_S - c_P}$, we have $S^S(\underline{v}^S) = 2c_S - c_P$ and $S^P(\underline{v}^S) = c_S$. Thus, the incremental benefit of secrecy is $c_S - c_P$, exactly equal to its extra cost. So, bidder 1 is indifferent between participating secretly or publicly given value $v_1 = \underline{v}^S$. Finally, note that $\frac{\mathrm{d}S^S(v)}{\mathrm{d}v} = \underline{v}^S > v = \frac{\mathrm{d}S^P(v)}{\mathrm{d}v}$ for all $v \in (\underline{v}^P, \underline{v}^S)$. Thus, bidder 1 strictly prefers to participate publicly when $v_1 \in (\underline{v}^P, \underline{v}^S)$.

Case #3: $v_1 > \underline{v}^S$. Previously, I showed that (i) $F(v|P) = F(v|S) = v$ for all $v > \underline{v}^S$

and (ii) if $v_1 > \underline{v}^S$, then bidder 1 wins in every bidding subgame – no matter who participates publicly – iff he has the highest value. Thus, by the Envelope Theorem, $S^S(v) = S^S(\underline{v}^S) + \int_{\underline{v}^S}^{v} x \mathrm{d}x$ and $S^P(v) = S^P(\underline{v}^S) + \int_{\underline{v}^S}^{v} x \mathrm{d}x$ for all $v > \underline{v}^S$. In Case #2, I showed that $S^S(\underline{v}^S) - S^P(\underline{v}^S) = c_S - c_P$. Thus, $S^S(v) - S^P(v) = c_S - c_P > 0$ for all $v > \underline{v}^S$ and bidder 1 is indifferent between secret and public participation when $v > \underline{v}^S$. □

**Revenue losses due to bidder secrecy.** The equilibrium of Claim 1 is allocation-equivalent to the unique symmetric PBE of the second-price auction. (The good is sold iff $\max_i v_i > \sqrt{c_P}$, to the bidder with the highest value.) Thus, each bidder's interim expected payoff is the same as in the second-price auction, and all of the extra costs associated with secret participation translate into lower equilibrium expected revenue. Since each bidder $i = 1, 2$ participates secretly with probability $q^S(1 - \underline{v}^S)$ and incurs extra cost $c_S - c_P$ when doing so, this expected revenue loss is

$$LOSS(c_S, c_P) = 2(c_S - c_P)\sqrt{\frac{c_P}{2c_S - c_P}}(1 - \sqrt{2c_S - c_P}). \tag{1}$$

**Claim 2.** *Suppose that $c_S = \frac{8}{27}$ and $c_P = \frac{4}{27}$. Expected revenue in the equilibrium of Claim 1 is about 25% less than in the symmetric PBE of the second-price auction.*

*Proof.* In the second-price auction, expected revenue $REV(c_P) = \frac{1}{3} - c_P\left(1 - \frac{2\sqrt{c_P}}{3}\right) \approx 0.223$ when $c_P = \frac{4}{27}$. However, losses due to secrecy in the equilibrium of Claim 1 are $LOSS\left(\frac{8}{27}, \frac{4}{27}\right) = \frac{8}{81\sqrt{3}} \approx 0.057$, more than 25% of 0.223. □

# 4 Secrecy, publicity, or both?

This section explores how bidders participate in PBE, depending on (i) the costs of secret and public participation $(c_S, c_P)$ and (ii) the distribution of bidder values. The exposition is organized as a case-by-case analysis, considering scenarios in which bidders never participate secretly (Section 4.1), those in which bidders never participate publicly (Section 4.2), and those in which equilibria exhibit a mixture of secret and public participation (Section 4.3). Section 4.4 illustrates these findings in the example of Section 3.

## 4.1 Only-public participation

**Case #1: $\mathbf{c_S > c_P = 0}$.** Suppose that public participation is free but secret participation is costly. In this case, bidders never participate secretly in any PBE.

**Theorem 2.** *Suppose that $c_S > c_P = 0$. Every PBE exhibits only-public participation.*

*Proof.* Suppose that a PBE exists in which some bidder (say bidder 1) sometimes participates secretly, i.e. $\underline{v}_1^S < 1$. I will establish a contradiction by showing that bidder 1 strictly prefers to participate publicly given value $v_1 = \underline{v}_1^S$.[8]

Since bidder 1 can participate publicly at zero cost, he always participates in any PBE. Thus, secrecy does not "fool" others into believing that bidder 1 may not be bidding in the auction. Since secret participation is costly, however, the decision to be secret does affect others' equilibrium beliefs about bidder 1's value. In particular, whenever bidder 1 does not participate publicly, others believe that (i) $v_1 \geq \underline{v}_1^S$ and hence that (ii) bidder 1 is sure to bid *at least* $\hat{b} = b_1(\underline{v}_1^S; X)$ when the realized set of other public bidders is $X$, for all $X \subset \{2, ..., n\}$. In any best response, each bidder $i \neq 1$ will bid at least $\hat{b}$ whenever $v_i > \hat{b}$, allowing bidder 1 to win only when $\max_{i \neq 1} v_i < \hat{b}$, the same *as if* all other bidders always bid their full values. Consequently, bidder 1's expected *gross* payoff conditional

[8] By continuity of interim expected payoffs, bidder 1 strictly prefers to participate publicly given any value $v_1 \in (\underline{v}_1^S - \varepsilon, \underline{v}_1^S + \varepsilon)$ for small enough $\varepsilon > 0$. But then bidder 1's secret participation threshold must be at least $\underline{v}_1^S + \varepsilon$, a contradiction.

on value $v_1 = \underline{v}_1^S$ in the subgame with public bidders $P = X$ will be the same as if all others bid their values:[9]

$$\Pi_1(\underline{v}_1^S; X) + c_S = \max_{b \geq 0} (\underline{v}_1^S - b) \Pr \left( \max_{i \neq 1} v_i < b | P = X \right). \quad (2)$$

Suppose now that bidder 1 were to participate publicly given value $v_1 = \underline{v}_1^S$ and face the same set of public opponents. Since others never bid more than their values in any PBE, his expected gross payoff can be no worse than in (2): $\Pi_1(\underline{v}_1^S; X \cup \{1\}) + c_P \geq \Pi_1(\underline{v}_1^S; X) + c_S$. Since $c_S > c_P = 0$, we conclude that $\Pi_1(\underline{v}_1^S; X \cup \{1\}) > \Pi_1(\underline{v}_1^S; X)$ for all $X \subset \{2, ..., n\}$. Thus, bidder 1 strictly prefers to participate publicly at the interim stage given value $v_1 = \underline{v}_1^S$, a contradiction. □

Theorem 2 establishes that all participation is public in any PBE when $c_S > c_P = 0$. Of course, any such equilibrium remains an equilibrium if one were to raise the cost of secret participation. Any PBE must therefore correspond to an equilibrium in the standard first-price auction in which all bidders always participate and do so publicly. Given symmetric independent private values, it is well-known that the first-price auction has a unique equilibrium (see e.g. Lebrun (2006)). Thus, there is in fact a unique PBE.

**Corollary to Theorem 2**. *Suppose that $c_S > c_P = 0$. There is a unique PBE. In this equilibrium, each bidder always participates publicly.*

## 4.2 Only-secret participation

**Case #2: $\mathbf{c_S = c_P > 0}$.** Suppose that participation is costly but that there are no *extra* costs associated with secrecy or publicity. In this case, bidders never participate publicly in any symmetric PBE.

[9] The argument here establishes that $\Pi_1(\underline{v}_1^S; X) + c_S$ is *less than or equal to* the value of the maximization in (2). It is straightforward to show that these must in fact be equal.

**Theorem 3.** *Suppose that $c_S = c_P > 0$. Every symmetric PBE exhibits only-secret participation.*

*Proof of Theorem 3 with two bidders.* To develop intuition, I will prove Theorem 3 here in the special case with two bidders. Suppose that there exists a symmetric PBE in which bidders sometimes participate publicly, i.e. $\underline{v}^P < 1$. I will establish a contradiction by showing that each bidder $i = 1, 2$ strictly prefers to participate secretly given value $v_i = \underline{v}^P$. Without loss, consider bidder 1.

*Step #1: What if bidder 2 participates publicly?* Suppose first that bidder 2 were to participate publicly. In any bidding equilibrium of the subgame with public bidders $P = \{1, 2\}$, each bidder will always bid at least $\underline{v}^P$ since both bidders' values are sure (on the equilibrium path) to be at least $\underline{v}^P$. Thus, if bidder 1 also participates publicly given value $v_1 = \underline{v}^P$, bidder 1 will never win the object and earn zero gross profit. (Bidder 1's net profit, after accounting for participation costs, is $-c_P$.)

By contrast, if bidder 1 were to participate secretly, bidder 1 would earn positive expected gross payoff given value $v_1 = \underline{v}^P$ in the subgame with public bidders $P = \{2\}$. To see why, note that bidder 2 can earn a positive expected gross profit by bidding zero, since bidder 1 chooses not to participate with positive probability. Consequently, bidder 2 must always strictly shade his bid below value. In particular, bidder 2 bids less than $\underline{v}^P$ with positive probability, allowing bidder 1 to earn positive expected gross payoff given $v_1 = \underline{v}^P$.

*Step #2: What if bidder 2 does not participate publicly?* Suppose next that bidder 2 does not participate publicly. If bidder 1 participates publicly so that $P = \{1\}$, bidder 2 believes that bidder 1's value must be at least $\underline{v}^P$ and hence that bidder 1 will never bid less than $\hat{b} = b_1(\underline{v}^P; \{1\})$. In any best response when participating secretly, then, bidder

2 will (i) bid less than $\hat{b}$ if $v_2 < \hat{b}$ (in order not to win at a price that exceeds his value) and (ii) bid more than $\hat{b}$ if $v_2 > \hat{b}$ (in order to have a chance of winning). In other words, bidder 1 wins the object iff either bidder 2 does not participate or bidder 2 participates secretly but has value $v_2 < \hat{b}$, the same *as if* bidder 2 always bid his full value when participating.

By contrast, if bidder 1 were to participate secretly so that $P = \emptyset$, bidder 2 can earn a positive expected gross profit by bidding zero. Thus, bidder 2 must strictly shade his bid below his value when participating secretly. When playing a best response, then, bidder 1's expected gross payoff must be greater than or equal to his expected gross payoff in the subgame with $P = \{1\}$.

Since secret and public participation are equally costly, we conclude that bidder 1 (i) strictly prefers to participate secretly if bidder 2 participates publicly and (ii) weakly prefers to participate secretly if bidder 2 does not participate publicly, conditional on $v_1 = \underline{v}^P$. Further, since $\underline{v}^P < 1$, bidder 2 participates publicly with positive probability. Thus, bidder 1 strictly prefers to participate secretly at the interim stage when $v_1 = \underline{v}^P$, a contradiction. □

Theorem 3 establishes that all participation is secret in any symmetric PBE when public and secret participation are equally costly, i.e. $c_S = c_P > 0$. Of course, any such equilibrium remains an equilibrium if one were to raise the cost of public participation. Any symmetric PBE must therefore correspond to the unique symmetric equilibrium of the standard first-price auction in which only secret participation is possible, at cost $c_S > 0$. Thus, there is in fact a unique symmetric PBE when $c_S = c_P > 0$.

**Corollary to Theorem 3**. *Suppose that $c_P = c_S > 0$. There is a unique symmetric PBE. In this equilibrium, bidders never participate publicly.*

Theorem 3 is proven for $n \geq 2$ bidders by establishing the following deeper result.

**Theorem 3'.** *Suppose that $c_S = c_P > 0$. In every PBE, at most one bidder participates publicly with positive probability.*[10]

Theorem 3' implies Theorem 3 since, in any symmetric PBE, either all or none of the bidders participate publicly with positive probability.

**Case #3: $\mathbf{c_P > c_S \geq 0}$.** Suppose that public participation is more costly than secret participation. In this case, bidders never participate publicly in all PBE. (Note that Theorem 4 applies to all PBE, not just to all symmetric PBE.)

**Theorem 4.** *Suppose that $c_P > c_S \geq 0$. Every PBE exhibits only-secret participation.*

*Proof of Theorem 4 with two bidders.* To develop intuition, I will prove Theorem 4 here in the special case with two bidders.[11] Without loss, suppose that bidder 1 has the lowest public participation threshold, $\underline{v}_1^P \leq \underline{v}_2^P$. I will establish a contradiction by showing that bidder 1 strictly prefers to participate secretly, conditional on realized value $v_1 = \underline{v}_1^P$.

*Step #1: What if bidder 2 participates publicly?* Suppose first that bidder 2 were to participate publicly. (If $\underline{v}_2^P = 1$, then this occurs with zero probability.) Since $\underline{v}_2^P \geq \underline{v}_1^P$, bidder 2 is certain to bid at least $\underline{v}_1^P$ in any bidding equilibrium of the subgame with public bidders $P = \{1, 2\}$. So, given value $v_1 = \underline{v}_1^P$, bidder 2 is certain to lose and earns payoff $-c_P$ in this subgame. Of course, participating secretly yields a payoff of at least $-c_S$, strictly better since $c_S < c_P$.

[10] I conjecture that no PBE exists in which exactly one bidder participates publicly with positive probabiliy, but this remains an open question.

[11] The argument presented here is distinct from that used to prove Theorem 4' in the Appendix. The proof of Theorem 4' focuses on the bidder with the highest public participation threshold, whereas here I focus on the bidder with the lowest public participation threshold. Both approaches are revealing.

*Step #2: What if bidder 2 does not participate publicly?* Suppose next that bidder 2 does not participate publicly. If bidder 1 participates publicly so that $P = \{1\}$, repeating the argument in the two-bidder proof of Theorem 3, bidder 1's payoff will be the same *as if* bidder 2 always bid his full value whenever participating secretly. Thus, given value $v_1 = \underline{v}_1^P$, bidder 1's expected payoff in the subgame with $P = \{1\}$ is

$$\Pi_1(\underline{v}_1^P; \{1\}) = \max_{b \geq 0}(\underline{v}_1^P - b) \Pr\left(v_2 < b \text{ or } v_2 < \underline{v}_2^S | 2 \notin P\right) - c_P. \tag{3}$$

By contrast, in the subgame with no public bidders, bidder 1's expected payoff given value $v_1 = \underline{v}_1^P$ is

$$\Pi_1(\underline{v}_1^P; \emptyset) = \max_{b \geq 0}(\underline{v}_1^P - b) \Pr\left(b_2(v_2; \emptyset) < b \text{ or } v_2 < \underline{v}_2^S | 2 \notin P\right) - c_S. \tag{4}$$

$\Pi_1(\underline{v}_1^P; \emptyset) > \Pi_1(\underline{v}_1^P; \{1\})$ for a combination of reasons. First, $b_2(v_2; \emptyset) \leq v_2$ since bidder 2 never bids more than his value. Thus, the maximization term of (3) is greater than or equal to that of (4). Second, secret participation is less costly, so that $-c_P < -c_S$.

In summary, bidder 1 strictly prefers to participate secretly given value $v_1 = \underline{v}_1^P$, regardless of whether or not bidder 2 participates publicly. Thus, bidder 1 strictly prefers to be secret at the interim stage given value $v_1 = \underline{v}_1^P$, a contradiction. □

Theorem 4 establishes that all participation is secret in any PBE when public participation is more costly. As discussed after Theorem 3, then, every PBE must correspond to the unique equilibrium[12] of the first-price auction with only-secret participation.

**Corollary to Theorem 4**. *Suppose that $c_P > c_S \geq 0$. There is a unique PBE. In this equilibrium, each bidder $i$ never participates publicly.*

Theorem 4 is proven for $n \geq 2$ bidders by establishing the following, deeper result. First, a definition is needed.

[12]Samuelson (1985) showed that there is a unique symmetric equilibrium. It is straightforward to show that all equilibria must be symmetric. So, there is in fact a unique equilibrium.

**Definition 4** (Bidder $k$)**.** Order bidders by their public participation thresholds in a given PBE, $\underline{v}_1^P \leq ... \leq \underline{v}_n^P$, and define $\underline{v}_0^P = 0$. Let "bidder $k$" be the bidder with the highest public participation threshold, among all who sometimes participate publicly, i.e. $k = \max\{i = 0, 1, ..., n : \underline{v}_i^P < 1\}$. ($k = 0$ iff the PBE exhibits only-secret participation.)

**Theorem 4'.** *In any PBE in which $k \geq 1$, bidder $k$'s interim expected* gross *payoff given value $v_k = \underline{v}_k^P$ is weakly greater if he participates secretly than if he participates publicly.*

Theorem 4' implies Theorem 4 since, when $c_P > c_S$, bidder $k$'s interim expected *net* payoff given value $v_k = \underline{v}_k^P$ is *strictly* greater if he participates secretly than if he participates publicly, a contradiction.

## 4.3 Some public and some secret participation

**Case #4: $\mathbf{c_S > c_P > 0}$.** Consider now the final possibility, that participation is costly but there are extra costs associated with secrecy. The main finding is that, for small enough extra costs of secrecy, all symmetric PBE exhibit some secret and some public participation.

**Theorem 5.** *Suppose that $c_S > c_P > 0$. (i) Every symmetric PBE exhibits some public participation. (ii) As long as $c_P \in (0, 1)$, there exists $c_S(c_P) \in (c_P, 1)$ such that all symmetric PBE exhibit some secret and some public participation iff $c_S \in (c_P, c_S(c_P))$.*

*Proof.* First, suppose for the sake of contradiction that a symmetric PBE exists with only-secret participation. Any such equilibrium must correspond to the unique symmetric equilibrium of a standard first-price auction in which (i) participation costs $c_S$ and each bidder $i$ participates iff $v_i > \underline{v}(c_S)$, (ii) bidders do not observe the number of participants prior to the bidding but, conditional on participation, each bidder's value is drawn according to the cdf $F(v|v > \underline{v}(c_S))$, and (iii) the threshold $\underline{v}(c_S)$ is implicitly determined by the indifference condition $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} = c_S$. In particular, a bidder with value $\underline{v}(c_S)$ bids zero, is sure to lose unless no one else participates, and earns zero net expected profit. But such a bidder could profitably deviate by participating publicly (still bidding zero), for net profit $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} - c_P > 0$. This is a contradiction.

Next, suppose for the moment that a symmetric PBE exists with only-public participation. Any such equilibrium must correspond to the unique symmetric equilibrium of the standard first-price auction in which (i) participation costs $c_P$ and each bidder participates iff $v_i > \underline{v}(c_P)$, (ii) bidders observe the number of participants prior to the bidding where, conditional on participation, each bidder's value is drawn according to the cdf $F(v|v > \underline{v}(c_P))$, and (iii) the threshold $\underline{v}(c_P)$ is implicitly determined by the indifference condition $\underline{v}(c_P)F(\underline{v}(c_P))^{n-1} = c_P$.

In this well-known equilibrium, each bidder's bid is equal to the expected second-highest value for the good, conditional on his own value being the highest. (If there are no other bidders, bidder $i$ bids zero.) In particular, each bidder $i$ bids $b(v_i; m) = E\left[\max_{j=1,...,m} v_j | v_j \in [\underline{v}(c_P), v_i] \text{ for all } j\right]$ when there are $m$ other public bidders.

Without loss, consider bidder $n$ and suppose that only bidders $1, ..., m$ also participate publicly. Let $M$ denote the number of other public bidders, viewed as a random variable. Note that $\Pr(M = m) = (1 - F(\underline{v}(c_P)))^m F(\underline{v}(c_P))^{n-m-1} > 0$ for all $m = 0, ..., n-1$. If bidder $n$ participates publicly and faces $m$ others, his best response to others' strategies will yield expected gross payoff

$$\Pi^P(v_n; m) + c_P = \max_{b \geq 0}(v_n - b) \Pr\left(b > \max_{i=1,...,m} b(v_i; m) | \min_{i=1,...,m} v_i \geq \underline{v}(c_P)\right).$$

By contrast, if bidder $n$ participates secretly and faces the same $m$ opponents, his best response to others' strategies will now yield expected gross payoff

$$\Pi^S(v_n;m) + c_S = \max_{b\geq 0}(v_n - b)\Pr\left(b > \max_{i=1,\ldots,m} b(v_i;m-1) | \min_{i=1,\ldots,m} v_i \geq \underline{v}(c_P)\right).$$

The key difference is that, when bidder $n$ participates secretly, others will bid *as if* there is one fewer bidder. Since $b(v_i;m) > b(v_i;m-1)$ for every $m > 0$ and every $v_i \geq \underline{v}(c_P)$, secret participation yields strictly greater expected gross payoff, $\Pi^S(v_n;m)+c_S > \Pi^P(v_n;m)+c_P$, for all $m \geq 1$. Let $\Delta(v_n;m) = \Pi^S(v_n;m)+c_S-(\Pi^P(v_n;m)+c_P)$ denote this gross-payoff benefit of secrecy. Next, define $\Delta(c_P) = \max_{v_n\in[\underline{v}(c_P),1]} E_M[\Delta(v_n;M)]$ and $v^* = \arg\max_{v_n\in[\underline{v}(c_P),1]} E_M[\Delta(v_n;M)]$.

Define $c_S(c_P) = c_P + \Delta(c_P)$. $c_S(c_P) > c_P$ since $\Delta(c_P) > 0$. Further, it must be that $c_S(c_P) < v^*$. To see why, note that by definition $c_S(c_P)$ is the secrecy cost that makes bidder-type $v^*$ indifferent between participating secretly or publicly. However, since $v^* > \underline{v}(c_P)$, bidder-type $v^*$ earns positive net profit when participating publicly and hence must also earn positive net profit when participating secretly at cost $c_S(c_P)$. So, $c_S(c_P) < v^*$.

If $c_S > c_S(c_P)$, every bidder prefers not to deviate by participating secretly, so a symmetric PBE exists with only-public participation. By contrast, if $c_S < c_S(c_P)$, each bidder strictly prefers to deviate by participating secretly when his value is in a neighborhood of $v^*$. So, every symmetric PBE must exhibit some public and some secret participation in this case. This completes the proof. □

## 4.4 Illustrative example: continued

Consider again the example of Section 3, in which two bidders have iid values $v_i \sim U[0,1]$. How bidders participate in PBE depends on the costs $(c_P, c_S)$, as illustrated in Figure 3:

- If $c_S > c_P = 0$, all PBE exhibit only-public participation (Theorem 2);

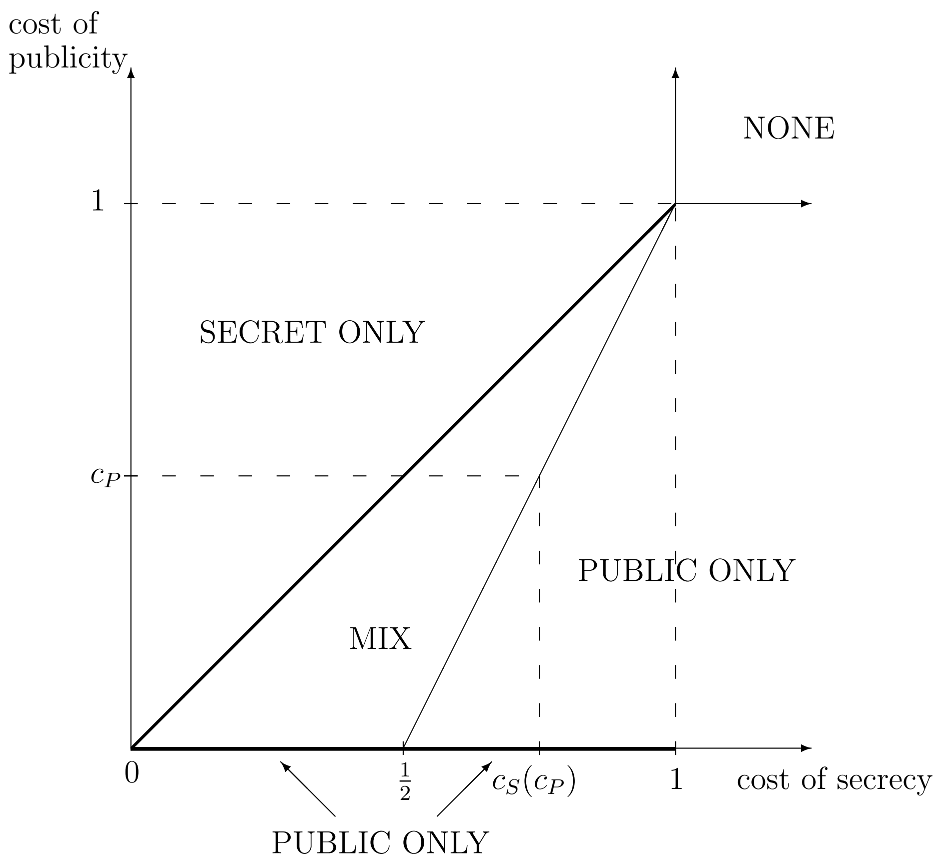


Figure 3: Illustration of how bidders participate in all symmetric PBE.

- If $c_P = c_S > 0$, all symmetric PBE exhibit only-secret participation (Theorem 3);
- If $c_P > c_S \geq 0$, all PBE exhibit only-secret participation (Theorem 4); and
- If $c_S > c_P > 0$, how bidders participate depends on the extra cost of secrecy, $c_S - c_P$. In particular, all symmetric PBE exhibit some secret and some public participation iff $c_S \in (c_S(c_P), 1)$, where $c_S(c_P) \in (c_P, 1)$ (Theorem 5).

It remains to compute the threshold secrecy-cost $c_S(c_P)$ of Theorem 5, below which bidders sometimes participate secretly in all symmetric PBE.

**Claim 3.** $c_S(c_P) = \frac{1+c_P}{2}$ *in the example of Section 3.*

*Proof.* Any symmetric PBE with only-public participation must correspond to the unique symmetric PBE when secrecy is impossible. Namely, each bidder $i$ (i) participates iff $v_i > \sqrt{c_P}$, (ii) bids zero if he is the only public bidder, and (iii) bids $b(v_i; \{1,2\}) = \frac{v_i + \sqrt{c_P}}{2}$ if both are public. If $v_j < \sqrt{c_P}$ so that bidder $j$ does not participate, there is no *ex post* benefit to secrecy since bidder $i$ wins at price zero whether he is secret or public. If $v_i > \sqrt{c_P}$, bidding secretly induces bidder $j$ to bid zero, allowing bidder $i$ to win at price zero for gross payoff $v_i$. By contrast, bidding publicly induces bidder $j$ to bid $b(v_j; \{1,2\}) \geq \sqrt{c_P}$, so that bidder 1's best response is to bid $b(v_i; \{1,2\})$ and earn conditional expected gross payoff $(v_i - b(v_i; \{1,2\})) \Pr(v_j < v_i | v_j \geq \sqrt{c_P}) = \frac{(v_i - \sqrt{c_P})^2}{2(1-\sqrt{c_P})}$. Since bidder $j$ participates with probability $1 - \sqrt{c_P}$, bidder $i$'s expected gross benefit from secrecy is $(1 - \sqrt{c_P}) \left( v_i - \frac{(v_i - \sqrt{c_P})^2}{2(1-\sqrt{c_P})} \right) = v_i - \frac{v_i^2}{2} - \frac{c_P}{2}$.

This secrecy-deviation benefit is increasing in $v_i$, and hence maximized at $\frac{1-c_P}{2}$ when $v_i = 1$. Thus, (i) if $c_S \geq c_P + \frac{1-c_P}{2} = \frac{1+c_S}{2}$, all types at least weakly prefer not to deviate by participating secretly while (ii) if $c_S < \frac{1+c_S}{2}$, each bidder $i$ strictly prefers to deviate secretly given values in a neighborhood of $v_i = 1$. This completes the proof. □

# 5 The cost of secrecy

This section explores how welfare and revenue vary in all symmetric PBE with the cost of secret participation. When secret participation is less costly ($c_S < c_P$), increasing $c_S$ decreases expected bidder surplus and revenue, as one would expect (Theorem 6). However, when secret participation is more costly ($c_S > c_P$), increasing $c_S$ has a non-monotone effect on expected total welfare and revenue. Namely, *either* raising $c_S$ to infinity (Theorem 7) *or* lowering $c_S$ to $c_P$ (corollary to Theorem 7) has the effect of (i) weakly decreasing interim expected bidder surplus, (ii) strictly increasing expected revenue, and (iii) strictly increasing expected total welfare.

**Only-secret participation.** Suppose first that $c_P > c_S \geq 0$ so that all symmetric PBE exhibit only-secret participation (Theorem 4). In this case, both expected bidder surplus and expected seller revenue are strictly decreasing in $c_S$. In fact, the proof of Theorem 6 shows that interim expected bidder surplus and *ex post* seller revenue are each weakly decreasing in $c_S$.

**Theorem 6.** *Suppose that $c_P > c_S \geq 0$ so that the unique symmetric PBE exhibits only-secret participation. Expected bidder surplus and expected seller revenue in this equilibrium are strictly decreasing in $c_S$ over the range $(0, c_P)$.*

*Proof.* Any symmetric PBE with only-secret participation must correspond to the unique symmetric equilibrium of the standard first-price auction in which all participation must be secret. In particular, each bidder participates iff $v_i > \underline{v}(c_S)$, where the threshold $\underline{v}(c_S)$ is determined by $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} = c_S$. Note that $\underline{v}(c_S)$ is increasing in $c_S$.

By the Envelope Theorem, each bidder's interim expected surplus can be characterized as the integral of his probability of winning: $\Pi(v_i) = 0$ for all $v_i \leq \underline{v}(c_S)$ and $\Pi(v_i) = \int_{\underline{v}(c_S)}^{v_i} F(x)^{n-1}\mathrm{d}x$ for all $v_i > \underline{v}(c_S)$. Increasing the secret participation cost from $c_S$ to $c'_S$ leads to an increase in the participation threshold from $\underline{v}(c_S)$ to $\underline{v}(c'_S)$ and therefore decreases interim expected surplus by $\int_{\underline{v}(c_S)}^{\min\{v_i, \underline{v}(c'_S)\}} F(x)^{n-1}\mathrm{d}x > 0$ for all $v_i > \underline{v}(c_S)$. (If $v_i \leq \underline{v}(c_S)$, then bidder $i$ continues to earn zero payoff.) Overall, then, each bidder's ex ante expected surplus is strictly decreasing in $c_S$ over the range $(0, c_P)$.

Since bidders choose not to participate given values less than $\underline{v}(c_S)$, equilibrium bids are the same *as if* in a standard first-price auction with zero participation cost, in which bidders' have iid values with cdf $F(.; c_S)$ defined by $F(v; c_S) = F(\underline{v}(c_S))$ for all $v \leq \underline{v}(c_S)$ and $F(v; c_S) = F(v)$ for all $v > \underline{v}(c_S)$. (Each bidder has an "atom" of mass $F(\underline{v}(c_S))$ at

$-\infty$.) Note that $F(v; c_S') \geq F(v; c_S)$ for all $v$ and all $c_S' > c_S$, since $\underline{v}(c_S) < \underline{v}(c_S')$. Thus, by Lebrun (1998), the distribution of equilibrium bids given cost $c_S$ first-order stochastically dominates the distribution of bids given cost $c_S'$, for all $c_S' > c_S$. Consequently, raising $c_S$ weakly decreases the seller's *ex post* revenue, and strictly decreases ex post revenue conditional on $\max_i v_i > \underline{v}(c_S)$. Thus, expected revenue is strictly decreasing in $c_S$ over the range $(0, c_P)$. □

**Some secret and some public participation.** Suppose next that $c_S \in (c_P, c_S(c_P))$, so that all symmetric PBE exhibit some public and some secret participation (Theorem 5). Theorem 7 shows that *disallowing secrecy* strictly increases both expected total welfare and expected revenue, while weakly decreasing interim expected bidder surplus, when one compares the unique symmetric equilibrium when secrecy is disallowed with *any* symmetric PBE when secrecy is possible.

**Theorem 7.** *Suppose that $c_P \in (0, 1)$ and $c_S \in (c_P, c_S(c_P))$, where $c_S(c_P)$ is the secrecy-cost threshold identified by Theorem 5. Disallowing secrecy (raising $c_S$ to infinity) (i) strictly increases expected total welfare, (ii) strictly increases expected seller revenue, and (iii) weakly decreases interim expected bidder surplus among all symmetric PBE.*

**Corollary to Theorem 7.** *Suppose that $c_P \in (0, 1)$ and $c_S \in (c_P, c_S(c_P))$. Eliminating the extra cost of secrecy (lowering $c_S$ to $c_P$) (i) strictly increases expected total welfare and (ii) strictly increases expected seller revenue, and (iii) weakly decreases interim expected bidder surplus among all symmetric PBE.*

*Proof of the corollary.* If the seller were to eliminate the extra cost of secrecy so that $c_S = c_P = c > 0$, Theorem 3 implies that there would be a unique symmetric PBE that is outcome-equivalent to the unique symmetric equilibrium in the standard first-price auction with only-secret participation (when $c_S = c$ and $c_P = \infty$). As is well-known, this equilibrium generates the same ex post total welfare, expected seller revenue, and interim expected bidder surplus as the unique symmetric equilibrium when only public participation is possible (when $c_P = c$ and $c_S = \infty$). The corollary then follows immediately from Theorem 7. $\square$

*Outline of the proof of Theorem 7.* The rest of this section proves Theorem 7. Part One considers the benchmark in which secrecy is disallowed ($c_S = \infty$). The rest considers the case $c_S \in (c_P, c_S(c_P))$. Parts Two-Three establish that all symmetric PBE share the same participation threshold $\underline{v}(c_P)$ as when $c_S = \infty$. This fact is then used to show that both expected total welfare (Part Four) and expected revenue (Part Five) are strictly higher in the unique symmetric PBE when $c_S = \infty$ than in all symmetric PBE when $c_S \in (c_P, c_S(c_P))$, while interim expected bidder surplus is weakly lower (Part Six).

**Part One: Benchmark when secrecy is disallowed.** If $c_S$ were raised to infinity, any symmetric PBE must correspond to the unique symmetric equilibrium of the standard first-price auction with only-public participation.[13] Namely, (i) each bidder $i$ participates at cost $c_P$ iff $v_i > \underline{v}(c_P)$, where the participation threshold $\underline{v}(c_P)$ is determined implicitly by $\underline{v}(c_P)F(\underline{v}(c_P))^{n-1} = c_P$, and (ii) whenever anyone participates, the highest-value bidder wins.

*Ex post total welfare:* Ex post total welfare equals $\max_i v_i - c_P\#\{i : v_i > \underline{v}(c_P)\}$ if $\max_i v_i > \underline{v}(c_P)$, and zero otherwise.

*Interim expected bidder surplus:* By the Envelope Theorem, bidder $i$'s interim expected surplus for all $v_i > \underline{v}(c_P)$ takes the integral form $\int_{\underline{v}(c_P)}^{v_i} F(x)^{n-1}\mathrm{d}x$, where $F(x)^{n-1}$ is the

[13] The first-price auction with only-public participation may possess multiple equilibria, but has a unique symmetric equilibrium (Cao and Tian (2010)).

probability that a bidder having value $x > \underline{v}(c_P)$ wins the good.

*Expected revenue:* By standard arguments (see e.g. Bulow and Roberts (1989)), the expected value of the good to the winner minus total expected bidder surplus is equal to the expected maximal virtual value, $E\left[\max_i VV(v_i)\right]$, where $VV(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ for all $v_i > \underline{v}(c_P)$ and we set $VV(v_i) = 0$ for all $v_i < \underline{v}(c_P)$. Thus, the seller's expected revenue $E[R] = E\left[\max_i VV(v_i)\right] - nc_P(1 - F(\underline{v}(c_P)))$. (The expected total costs of participation, $nc_P(1 - F(\underline{v}(c_P)))$, are passed through to the seller in equilibrium.)

**Part Two: The lowest-value bidders to participate do so publicly.** Consider any symmetric PBE given costs $c_P < 1$ and $c_S \in (c_P, c_S(c_P))$. The first key step of the proof is to show that the lowest-value bidders who participate must sometimes choose to participate publicly, i.e. $\underline{v} = \underline{v}^P$.

**Lemma 1.** *Suppose that $c_S > c_P$. In every symmetric PBE, $\underline{v}^P = \underline{v}$.*

*Proof.* See the Appendix □

*Discussion of Lemma 1:* In standard models with only-secret or only-public participation, any symmetric equilibrium must have the feature that a bidder whose value equals the participation threshold only wins the object if no one else participates. Since no one else participates with probability $F(\underline{v})^{n-1}$, the participation threshold in such models is therefore simply that determined by $\underline{v}F(\underline{v})^{n-1} = c$, where $c$ is the cost of participation.

When bidders sometimes participate publicly and sometimes secretly, however, a bidder with value $\underline{v}$ might potentially outbid higher-value opponents. For instance, suppose that all bidders adopted a symmetric strategy such as that illustrated earlier in Figure 1, in which $\underline{v} < \underline{v}^P$. Even though secrecy is more costly, a bidder (say bidder 1) with value $\underline{v} > 0$ potentially stands to gain by investing in secrecy. For one thing, if there is exactly one other public bidder, being secret will induce that opponent to bid less and thereby potentially allow bidder 1 to win with positive probability. And even if there are no other public bidders, bidder 1 may prefer to avoid "publicizing" that his value is at least $\underline{v}^P$ (especially since his true value is $\underline{v} < \underline{v}^P$), if doing so leads to more aggressive bidding competition from other *secret* bidders.

Of course, to sustain a symmetric PBE with strategies as depicted in Figure 1, each bidder must also at least weakly prefer *not* to participate secretly given the higher value $\underline{v}^P$. The key step of the proof of Lemma 1 is to show that, if $\underline{v}^S < \underline{v}^P$, then the expected benefit of secrecy is strictly greater when $v_1 = \underline{v}^P$ than when $v_1 = \underline{v}^S$. Thus, if bidders are willing to participate secretly given value $v_1 = \underline{v}^S$, they must *strictly* prefer to participate secretly when $v_1 = \underline{v}^P$, a contradiction.

**Part Three: The lowest-value bidders to participate bid zero when "alone."** Suppose that some bidder (say bidder 1) participates publicly with value $v_1 = \underline{v}^P = \underline{v}$. If any other bidder also participates publicly, both will bid at least $\underline{v}$ since both are sure (on the equilibrium path) to have values greater than or equal to $\underline{v}$. However, what if bidder 1 is the only public bidder? Could *secret* bidders' equilibrium strategies be such that bidder 1's best response is to submit a positive bid, even though he is "alone," or must bidder 1 bid zero as if truly facing no competition? In this step, I will show that he must bid zero.

If bidder 1 participates publicly and faces no public opponent, all others believe that he is certain not to bid less than $\hat{b} = b_1^P(\underline{v}^P; \{1\}) \geq 0$. If $\hat{b} = 0$, we are done. So suppose that $\hat{b} > 0$. Any secret participant having value greater than $\hat{b}$ must bid at least $\hat{b}$, in order to win the good with positive probability. However, since $\underline{v}^S \geq \underline{v}^P$ (by Part One) and $\underline{v}^P > \hat{b}$ (since bidder 1 must shade his bid below his value), any secret participant

$i \neq 1$ must have value $v_i > \hat{b}$ *and bid more than* $\hat{b}$. So, in any bidding equilibrium of the subgame with $P = \{1\}$, bidder 1 only wins the good given value $v_1 = \underline{v}^P$ when facing no competition. In any equilibrium, then, bidder 1 must bid zero given value $v_1 = \underline{v}^P$.

Overall, bidder 1's interim expected payoff when participating publicly with value $v_1 = \underline{v}$ equals $(\underline{v} - 0)F(\underline{v})^{n-1} - c_P$. Since bidder 1 must receive an interim expected payoff of zero at the participation threshold $\underline{v}(c_S, c_P)$, this threshold is characterized by

$$\underline{v}(c_S, c_P)F(\underline{v}(c_S, c_P))^{n-1} = c_P. \tag{5}$$

Note that this participation threshold is the same as that derived in Part One, i.e. $\underline{v}(c_S, c_P) = \underline{v}(c_P)$ for all $c_S \in (c_P, \infty)$.

**Part Four: Total welfare.** The object is sold iff $\max_i v_i > \underline{v}$ whether $c_S = \infty$ or $c_S \in (c_P, c_S(c_P))$, and sold to the highest-value bidder when $c_S = \infty$. Thus, ex post *gross* total welfare is weakly higher when $c_S = \infty$ than when $c_S \in (c_P, c_S(c_P))$. Further, each bidder $i$ participates iff $v_i > \underline{v}$ in either case, but sometimes pays $c_S > c_P$ when $c_S \in (c_P, c_S(c_P))$. Thus, ex post total welfare is weakly higher when $c_S = \infty$ and expected total welfare is strictly higher when $c_S = \infty$ than in any symmetric PBE when $c_S \in (c_P, c_S(c_P))$, by an amount equal to $n(c_S - c_P)E\left[q^S(v_i)\right]$.

**Part Five: Expected revenue.** The seller's expected revenue is equal to the expected virtual value of the winner (or zero if the object is not sold) minus total expected incurred participation costs. Expected incurred participation costs are strictly lower when $c_S = \infty$. Further, since each bidder's virtual value $VV(v_i)$ is increasing in $v_i$ and the highest-value participant always wins when $c_S = \infty$, the winner's expected virtual value is weakly higher when $c_S = \infty$. All together, then, expected revenue is strictly higher when $c_S = \infty$.

**Part Six: Interim expected bidder surplus.** If the highest-value participant always wins in some symmetric PBE when $c_S \in (c_P, c_S(c_P))$, as in the symmetric PBE described in the example of Section 3, then each bidder's interim expected surplus is the same as in the unique symmetric PBE when $c_S = \infty$, by a standard envelope argument. What about symmetric PBE in which the highest-value participant does not always win?[14] To complete the proof, I will show that bidders *benefit* from any such inefficiency of the equilibrium allocation. Intuitively, assigning the good to a bidder with a lower value forces the seller to offer additional information rent to a larger set of bidder types. More formally, let $S_i(v_i) = \int_{\underline{v}}^{v_i} p_i(x)\mathrm{d}x$ denote bidder $i$'s interim expected surplus, where $p_i(v_i)$ is his interim expected probability of winning the good. Let $\overline{S}(v) = \frac{\sum_i S_i(v)}{n}$; in any symmetric mechanism, $S_i(v) = \overline{S}(v)$ for all $v$. Consider any realized values $\mathbf{v} = (v_1, ..., v_n)$ such that $\max_i v_i > \underline{v}$ and the object is sold. Selling the object to bidder $i$ contributes $\mathrm{d}\mathbf{v}$ to bidder $i$'s interim expected surplus $S_i(\hat{v})$ for all $\hat{v} > v_i$, and hence contributes $\mathrm{d}\mathbf{v}/n$ to average interim expected surplus $\overline{S}(\hat{v})$ for all $\hat{v} > v_i$. Selling the good to the highest-value bidder therefore minimizes average interim expected surplus $\overline{S}(v)$ for all $v$ and, in particular, minimizes each bidder's interim expected surplus $S_i(v)$ for all $v$ among all symmetric mechanisms.

[14]It remains an open question whether such "inefficient" symmetric PBE exist. Until I found the example of Section 3, I had (incorrectly) conjectured that *all* symmetric PBE must be inefficient when $c_S \in (c_P, c_S(c_P))$.

# 6 Concluding remarks

The primitives of an auction model – bidders' values,[15] their beliefs, and so on – are typically viewed as exogenous, but they can arise endogenously as the result of strategic interactions prior to the auction. This paper endogenizes one of the most basic features of the standard auction model, bidders' beliefs about their competition, in the context of the first-price auction with independent private values.

When secrecy is more costly but not prohibitively so, every symmetric equilibrium of the first-price auction exhibits a mix of secret and public participation. Thus, it can be inappropriate to model such auctions as if the number of participants is common knowledge before the bidding.[16] This basic observation has implications for the revenue ranking of auction formats, as well as potentially for empirical auction research.

*Revenue ranking.* In any symmetric equilibrium of the second-price or English auction given independent private values, each bidder submits the same (truthful) bid regardless of who participates publicly. So, secret participation has no effect on others' bids and, whenever secrecy is more costly, bidders only participate publicly in equilibrium. The seller then recovers all saved secrecy costs in the form of higher expected revenue. This revenue advantage of the second-price and English auction can be substantial, as large as 25% in a simple example.

*Empirical implications.* Any bidder who participates secretly may be unobserved by the econometrician, especially if he loses. If so, standard empirical auction methods à la Guerre, Perrigne, and Vuong (2000) will mis-identify the distribution of bidder values,

[15] Arozamena and Cantillon (2004) endogenize bidders' values by allowing for observable pre-auction investments, finding that the first-price auction will elicit less investment than a second-price format.

[16] More broadly, it can be inappropriate to model such auctions as having an exogenously random number of symmetric bidders, as in McAfee and McMillan (1987a).

for two reasons. First, the econometrician only sees losing bids from public bidders, and public bidders' values are not drawn from the same conditional distribution as others' values. Second, the econometrician systematically underestimates the intensity of the bidding competition, effectively replacing all losing secret bids with zero bids.

Fortunately, one can test for the presence of secret bids *if* the winning bid is always observed, as well as all losing public bids. If there are no secret bids, then the winning bid is the maximum of all public bids. As such, the distribution of the winning bid (conditional on the number of bids) can be identified from the distribution of losing bids.[17] On the other hand, if bidders sometimes participate secretly, then the winning bid is the maximum of the maximal public bid and the maximal secret bid. Consequently, when bidders sometimes submit secret bids, the winning bid will be drawn from a *higher* distribution than what one would otherwise infer from the observed losing bids.

There are several natural directions for further work that builds upon this paper, three of which I will emphasize here.

**Common values.** Companion paper McAdams (2011) examines another reason why bidders may take pains to conceal their participation in an auction: to induce others to under-estimate the value of the good for sale. McAdams (2011) considers a symmetric second-price auction and a symmetric English auction with *common values.* In a common-value environment, secrecy impacts others' bids by influencing their willingness to pay for the good. By contrast, in this paper's private-value environment, secrecy impacts others' bids by influencing their beliefs about the intensity of competition. Thus, this paper and McAdams (2011) explore distinct channels by which costly secrecy can

[17]Let $G(b|m)$ be the cdf of $m$ iid bids. (Equilibrium bids will vary with the number of bidders.) $G(b|m)$ can be inferred from the cdf of (say) the lowest bid, $G^{(m)}(b|m) = 1 - (1 - G(b|m))^m$, from which one can infer the cdf of the winning bid, $G^{(1)}(b|m) = G(b|m)^m$.

arise in equilibrium. The main finding of McAdams (2011) is that symmetric, common-value bidders have an equilibrium incentive to pay to be secret in the second-price auction but *not* in the English auction. Further, the English auction's ability to deter secrecy translates into greater expected revenue, providing a novel rationale for a seller to choose the English auction over the second-price auction.

**Bid deterrence.** In this paper, I assumed that bidders *simultaneously* decide whether to participate. In future work, it would be interesting to consider an alternative setting in which bidders can publicly pre-commit to bid in the auction and thereby potentially deter others from even submitting a bid. Such "bid deterrence" may be an important factor in real-world bidders' decisions to announce their intention to bid for a good, e.g. in corporate acquisitions.

**Value signaling.** In a first-price auction, each bidder has an incentive to induce others to believe that he has a low value, so as to induce others to bid less. This paper provides an avenue by which to analyze such "value signaling" in auctions. Consider the case in which $c_S > c_P > 0$ so that, in any symmetric equilibrium, each bidder's secret and public participation thresholds satisfy $\underline{v}^S > \underline{v}_P > 0$. A bidder who participates publicly signals that his value is greater than $\underline{v}_P$, while a bidder who participates secretly signals that his value is *either* less than $\underline{v}_P$ *or* greater than $\underline{v}_S$. In future work, it would be interesting to consider more general settings in which bidders can send signals that are (endogenously) informative of values. Such an analysis could shed new light on the process by which bidders' beliefs about each others' values are formed prior to an auction, as well as on how the auction format impacts such belief formation.

# Appendix

## Proof of Theorem 1

*Proof.* **Payoffs.**[18] For all $M \subset N = \{1, ..., n\}$, let $\mathcal{B}(M) = \{(P, S) : P, S \subset M, P \cap S = \emptyset\}$ be the set of disjoint pairs of subsets of $M$. For each profile of realized values $\mathbf{v} = (v_1, ..., v_n)$ and each $(P, S) \in \mathcal{B}(N)$, let $\Pi_i(\mathbf{v}; P, S)$ denote bidder $i$'s *ex post payoff* should the set of bidders $P$ participate publicly, the set of bidders $S$ participate secretly, and bidder $i$ participates one way or the other, i.e. $i \in X \in \{P, S\}$:

$$\begin{aligned}\Pi_i(\mathbf{v}; P, S) &= \frac{v_i - b_i(v_i; P)}{\#\left(\arg\max_{j \in P \cup S} b_j(v_j; P)\right)} - c_X \text{ if } b_i(v_i; P) = \max_{j \in P \cup S} b_j(v_j; P) \\ &= -c_X \text{ if } b_i(v_i; P) < \max_{j \in P \cup S} b_j(v_j; P)\end{aligned}$$

Bidder $i$'s *ex ante payoff* then takes the form

$$\int_{\mathbf{v}} \sum_{(P,S) \in \mathcal{B}(N \setminus i)} \Pi_{j \in P} q_j^P(v_j) \Pi_{k \in S} q_k^S(v_k) \left[q_i^P(v_i) \Pi_i(\mathbf{v}; P \cup i, S) + q_i^S(v_i) \Pi_i(\mathbf{v}; P, S \cup i)\right] \mathrm{d}\mathbf{v}.$$

**Equilibrium existence.** Bidder $i$'s ex ante payoff may be discontinuous with respect to bidders' strategies (in the usual weak-* topology), but only if bidder $i$ ties in the bidding stage with positive ex ante probability. Consequently, each bidder's ex ante expected payoff is better-reply secure.[19] Given bidder symmetry, then, Corollary 5.3 of Reny (1999) establishes existence of a *symmetric* mixed-strategy Nash equilibrium.

[18] I thank Phil Reny for suggesting this proof approach.

[19] See Reny (1999) for a definition of better-reply secure payoffs. Reny (1999, pg. 1046) establishes that payoffs in the mixed extension of a standard first-price auction are better-reply secure. This argument extends to the first-price auction here with secret or public participation. The key idea in either case is that, when making a bid (less than his value) that ties with positive probability, a bidder can secure a better-reply by bidding slightly higher.

Such a Nash equilibrium might fail to constitute a PBE. To avoid this possibility, consider the following perturbation of the game. Independent of his value, each bidder is forced to participate publicly or secretly, each with probability $\delta > 0$, but otherwise free to choose his participation strategy and to choose his bid. Repeating the previous argument, this perturbed game has a symmetric mixed-strategy Nash equilibrium. Any such equilibrium of the perturbed game is an $\varepsilon(\delta)$-equilibrium of the original unperturbed game, where $\lim_{\delta \to 0} \varepsilon(\delta) = 0$. In particular, since (i) each bidder plays a best response with probability $1 - 2\delta$ and (ii) ex post payoffs are bounded above by 1 (values lie in $[0, 1]$) and bounded below by $-\max\{c_P, c_S\}$ (no bidder ever bids more than his value), $\varepsilon(\delta) = 2\delta(1 + \max\{c_P, c_S\})$. By Remark 3.1 of Reny (1999), a convergent subsequence of symmetric equilibria in the $\delta$-perturbed games therefore converges to a symmetric Nash equilibrium of the original game as $\delta \to 0$.

Along this convergent sequence of equilibria, the subgame with public bidders $P$ is reached with positive probability, for all $P \subset N$. Further, each public bidder's value has full support on $[0, 1]$ (since each is "forced" to participate publicly with probability $\delta$, independent of his value). Thus, in any subsequent bidding equilibrium, no bidder ever bids more than his value. We conclude that this equilibrium corresponds to a perfect Bayesian equilibrium in symmetric, weakly undominated strategies ("symmetric PBE"). □

## Proof of Theorem 3' and Theorem 4'

*Proof.* Fix any PBE. Let $\Pi_i^P(v_i; X \cup \{i\})$ and $\Pi_i^S(v_i; X)$ denote bidder $i$'s expected equilibrium payoff when participating publicly and secretly, respectively, conditional on value $v_i$ and public opponents $X$. Let bidder $k$ be the bidder with the highest public participation threshold, among those who sometimes participate publicly: $k = \max\{i = 0, 1, ..., n :$

$\underline{v}_i^P < 1\}$, where $\underline{v}_1^P \leq ... \leq \underline{v}_n^P$ and $\underline{v}_0^P = 0$.

**Proof of Theorem 4'.** Suppose that $k \geq 1$ and let $X \subset \{1, ..., k-1\}$ denote the set of realized public bidders other than bidder $k$. (At the interim stage when bidder $k$ decides whether and how to participate, bidder $k$ only knows the distribution of $X$. However, $X$ is commonly known at the bidding stage.) Recall that bidder $k$ never participates publicly when his value is less than $\underline{v}_k^P$. Since equilibrium bidding strategies are weakly monotone, bidder $k$ never bids less than $\hat{b}(X) = b_k^P(\underline{v}_k^P; X \cup \{k\})$ when public. Thus, any (secret or public) participant $i \neq k$ with value $v_i < \hat{b}(X)$ is certain to lose and any with value $v_i > \hat{b}(X)$ is certain to bid greater than $\hat{b}(X)$ to have a chance of winning.[20]

Bidder $k$'s probability of winning with bid $\hat{b}(X)$ is therefore equal to the probability that all other bidders either do not participate or have values less than $\hat{b}(X)$. Thus, bidder $k$'s equilibrium expected *gross* payoff $\Pi_k^P(\underline{v}_k^P; X \cup \{k\}) + c_P$ given value $\underline{v}_k^P$ in the subgame with public bidders $P = X \cup \{k\}$ is the same *as if* all other bidders always bid their full values whenever participating:

$$\Pi_k^P(\underline{v}_k^P; X) + c_P = \max_{b \geq \max\{\underline{v}_i^P : i \in X\}} \left(\underline{v}_k^P - b\right) \Pi_{i \in X} F(b | i \in X) \Pi_{j \notin X \cup \{k\}} F(\max\{b, \underline{v}_j^S\} | j \notin X). \tag{6}$$

(Each public bidder $i \in X$ is certain to participate and certain to have value greater than or equal to $\underline{v}_i^P \leq \underline{v}_k^P$. Thus, any bid $b < \max\{\underline{v}_i^P : i \in X\}$ is certain to lose.)

Since no bidder ever bids more than his value in any PBE, bidder $k$'s expected

[20] Every bidder $i \neq k$ bids *strictly* greater than $\hat{b}(X)$ when $v_i > \hat{b}(X)$. To see why, note that bidding less than $\hat{b}(X)$ is dominated when $v_i > \hat{b}(X)$ since any such bid always loses. Similarly, bidding exactly $\hat{b}(X)$ always loses unless bidder $k$ bids $\hat{b}(X)$ with positive probability. Further, monotonicity of best replies requires that, if bidder $i$ bids exactly $\hat{b}(X)$ given value $\hat{v}_i > \hat{b}(X)$, then he must bid exactly $\hat{b}(X)$ given all values $v_i \in (\hat{b}(X), \hat{v}_i)$. However, this means that bidders $i, k$ tie with positive probability, an impossibility in equilibrium.

gross payoff $\Pi_k^S(\underline{v}_k^P; X) + c_S$ when secret and bidding a best response to others' equilibrium strategies in the subgame with public bidders $P = X$ must therefore be greater than or equal to the value of the maximization in (6). Finally, $\Pi_k^S(\underline{v}_k^P; X) + c_S \geq \Pi_k^P(\underline{v}_k^P; X) + c_P$ for all $X \subset \{1, ...k-1\}$ implies $E\left[\Pi_k^S(\underline{v}_k^P; X)\right] + c_S \geq E\left[\Pi_k^P(\underline{v}_k^P; X)\right] + c_P$. This completes the proof of Theorem 4'. $\square$

**Proof of Theorem 3'.** Suppose that $c_P = c_S = c > 0$ and, for the sake of contradiction, that there exists a PBE in which $k \geq 2$. Since participation is costly, each bidder sometimes chooses not to participate. Thus, bidder $k$ must face exactly one public opponent, say $X = \{1\}$, with positive probability. As explained in the proof of Theorem 4', bidder $k$'s expected gross payoff $\Pi_k^P(\underline{v}_k^P; \{1, k\}) + c$ in the subgame with public bidders $P = \{1, k\}$ is that given in (6), the same *as if* bidder 1 and all secret participants always bid their values when participating. Suppose instead that bidder $k$ were to participate secretly given value $v_k = \underline{v}_k^P$, leaving bidder 1 as the only public bidder. Since others $j \neq 1, k$ do not bid more than their values, bidder $k$'s gross expected payoff $\Pi_k^S(\underline{v}_k^P; \{1\}) + c$ when playing a best response is now *at least*

$$\max_{b \geq b_1^P(\underline{v}_1^P; \{1\})} \left(\underline{v}_k^P - b\right) \Pr_{v_1}(b > b_1^P(v_1; \{1\}) | 1 \in P) \Pi_{j \neq 1, k} F(\max\{b, \underline{v}_j^S\} | j \notin P). \tag{7}$$

Since $\min\{c_P, c_S\} > 0$, (i) bidders $i \neq 1$ all fail to participate with positive probability and (ii) $\underline{v}_1^P > 0$. Thus, bidder 1 can earn a positive gross expected payoff by bidding zero and hence must always strictly shade his bid below value in the subgame $P = \{1\}$. In particular, (i) $b_1^P(\underline{v}_1^P; \{1\}) < \underline{v}_1^P$ and (ii) $\Pr(b > b_1^P(v_1; \{1\}) | 1 \in P) > F(b | 1 \in P)$ for all $b \geq \underline{v}_1^P$. Thus, the value of the maximization (7) strictly exceeds that of (6).

I have shown that bidder $k$ strictly prefers to participate secretly, conditional on $v_k = \underline{v}_k^P$ and $X = \{1\}$. Since $\Pr(X = \{1\}) > 0$ and bidder k weakly prefers to participate secretly given value $v_k = \underline{v}_k^P$ and any $X \subset \{1, ..., k-1\}$ (by Theorem 4'), bidder $k$ *strictly* prefers to participate secretly at the interium stage given value $v_k = \underline{v}_k^P$, a contradiction. This completes the proof of Theorem 3'. □

## Proof of Lemma 1

*Proof.* Suppose for the sake of contradiction that $\underline{v} < \underline{v}^P$ in some symmetric PBE. By Theorem 5, $\underline{v}^P < 1$ since bidders sometimes participate publicly. Thus, $\underline{v} = \underline{v}^S < \underline{v}^P < 1$. By definition of the thresholds $\underline{v}^S$ and $\underline{v}^P$, each bidder (say bidder 1) must at least weakly prefer to participate secretly given value $v_1 = \underline{v}^S$ and at least weakly prefer to participate publicly given value $v_1 = \underline{v}^P$. That is,

$$E_X\left[\Pi_1^S(\underline{v}^S; X)\right] \geq E_X\left[\Pi_1^P(\underline{v}^S; X \cup \{1\})\right] \text{ and}$$
$$E_X\left[\Pi_1^S(\underline{v}^P; X)\right] \leq E_X\left[\Pi_1^P(\underline{v}^P; X \cup \{1\})\right],$$

where $X \subset \{2, ..., n\}$ is the realized set of other public bidders. I will show that

$$\Pi_1^S(\underline{v}^P; X) - \Pi_1^P(\underline{v}^P; X \cup \{1\}) \geq \Pi_1^S(\underline{v}^S; X) - \Pi_1^P(\underline{v}^S; X \cup \{1\}) \tag{8}$$

for all $X \subset \{2, ..., n\}$, with strict inequality for some set $X$ that is realized with positive probability. That is, bidder 1's *incremental expected return to secrecy* is strictly higher when his value equals $\underline{v}^P$ than when it equals $\underline{v}^S$. Thus, if bidder 1 weakly prefers to participate secretly given value $v_1 = \underline{v}^S$, he must strictly prefer to participate secretly given value $v_1 = \underline{v}^P$, a contradiction.

Table 1 summarizes the rest of the argument, which is broken down into Steps A-D.

*Step A: What if there are two or more public participants?* Suppose that two or more bidders participate publicly, i.e. *either* $\#(X) \geq 2$ *or* $\#(X) = 1$ and bidder 1 participates publicly. Since every public bidder $i$ has value $v_i \geq \underline{v}^P$ on the equilibrium path, all public bidders bid at least $\underline{v}^P$. So, bidder 1 earns zero *gross* expected payoff given any value

| | $\#(X) \geq 2$ | $\#(X) = 1$ | $\#(X) = 0$ |
|---|---|---|---|
| Secret ($v = \underline{v}^P$) | Zero [A] | More [B] | Weakly more [C] |
| Secret ($v = \underline{v}^S$) | Zero [A] | Less [B] | $\underline{v}F(\underline{v})^{n-1}$ [D] |
| Public ($v = \underline{v}^P$) | Zero [A] | Zero [A] | Weakly less [C] |
| Public ($v = \underline{v}^S$) | Zero [A] | Zero [A] | At least $\underline{v}F(\underline{v})^{n-1}$ [D] |

Table 1: *Gross* expected payoff in any (supposed) symmetric PBE with $\underline{v}^P > \underline{v}^S$.

$v_1 \leq \underline{v}^P$:

$$\Pi_1^S(v_1; X) + c_S = 0 \text{ for all } v_1 \leq \underline{v}^P \text{ and } X \subset \{2, ..., n\} \text{ such that } \#(X) \geq 2 \quad (9)$$

$$\Pi_1^P(v_1; X \cup \{1\}) + c_P = 0 \text{ for all } v_1 \leq \underline{v}^P \text{ and } X \subset \{2, ..., n\} \text{ such that } \#(X) \geq 1. \quad (10)$$

*Step B: What if bidder 1 participates secretly and there is one other public bidder?* Suppose now that bidder 1 participates secretly and $X = \{i^*\}$. Clearly, bidder 1's expected payoff is weakly increasing in his value, so that $\Pi_1^S(\underline{v}^P; \{i^*\}) \geq \Pi_1^S(\underline{v}^S; \{i^*\})$. In fact, this inequality is strict. Since all bidders $j \neq i^*$ sometimes fail to participate, bidder $i^*$ can earn positive gross expected payoff (at least $\Delta = \underline{v}^P F(\underline{v}^S)^{n-1}$ since his value is at least $\underline{v}^P$ when public) by bidding zero. Thus, bidder $i^*$ must always shade his bid below value by at least $\Delta$. In particular, bidder $i^*$ bids less than $\underline{v}^P$ with positive probability. So, bidder 1 must earn positive gross expected profit when secret with value $v_1 = \underline{v}^P$, and this expected profit is strictly increasing in $v_1$ in a neighborhood of $\underline{v}^P$. We conclude that

$$\Pi_1^S(\underline{v}^P; \{i^*\}) > \Pi_1^S(\underline{v}^S; \{i^*\}) \text{ for all } i^* \neq 1. \quad (11)$$

*Step C: What if $v_1 = \underline{v}^P$ and there are no other public bidders?* Suppose now that $\#(X) = 0$ and $v_1 = \underline{v}^P$. By Theorem 4', if bidder 1 participates publicly, then his expected gross payoff is the same *as if* all other participants always bid their full value and he plays a best response to such strategies. However, since bidders do not bid more than their values, bidder 1 can earn at least this much expected gross payoff when participating secretly given the same value $v_1 = \underline{v}^P$:

$$\Pi_1^P(\underline{v}^P; \{1\}) + c_P \leq \Pi_1^S(\underline{v}^P; \emptyset) + c_S. \tag{12}$$

*Step D: What if $v_1 = \underline{v}^S$ and there are no other public bidders?* Suppose finally that $\#(X) = 0$ and $v_1 = \underline{v}^S$. If bidder 1 participates secretly, every bidding equilibrium of the subsequent subgame is in strictly monotone, symmetric strategies (see e.g. Lebrun (2006)). Further, a participant with the minimal value $\underline{v}^S$ bids zero and wins the good iff no one else participates, for gross expected payoff $\underline{v}^S F(\underline{v}^S)^{n-1}$. However, if bidder 1 were to participate publicly given value $v_1 = \underline{v}^S$, he could also earn gross expected payoff of $\underline{v}^S F(\underline{v}^S)^{n-1}$ by bidding zero. Thus, when participating publicly and playing a best response to others' strategies in the subgame with public bidders $P = \{1\}$, bidder 1 earns at least as much gross expected payoff:

$$\Pi_1^P(\underline{v}^S; \{1\}) + c_P \geq \Pi_1^S(\underline{v}^S; \emptyset) + c_S = \underline{v}^S F(\underline{v}^S)^{n-1}. \tag{13}$$

Finally, since each bidder sometimes participates publicly and sometimes fails to participate, $X = \{i^*\}$ with positive probability for each $i^* \neq 1$. All together, then, the weak inequalities (9,10,12,13) and the strict inequality (11) imply the desired increasing incremental expected benefit of secrecy (8), with strict inequality for $X = \{i^*\}$. This completes the proof, by contradiction. □

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