

# Resource and Revenue Management in Nonprofit Operations <sup>1</sup>

Francis de Véricourt<sup>2</sup>

Miguel Sousa Lobo<sup>3</sup>

April 2005 <sup>4 5</sup>

<sup>1</sup>Keywords: **Capacity Allocation, Revenue Management, Dynamic Pricing, Nonprofit.**

<sup>2</sup>Fuqua School of Business, Duke University, [fdv1@duke.edu](mailto:fdv1@duke.edu)

<sup>3</sup>Fuqua School of Business, Duke University, [mlobo@duke.edu](mailto:mlobo@duke.edu)

<sup>4</sup>First draft September 2003.

<sup>5</sup>Submitted to *Operations Research*, Manufacturing, Service and Supply Chain Operations area.

## Abstract

Nonprofit firms sometimes engage in for-profit activities with the sole purpose of generating revenue to subsidize their mission activities. The organization is then confronted with a consumption vs. investment trade-off. Investment corresponds to the allocation of capacity for revenue customers, while consumption corresponds to serving mission customers. We model this problem as a multi-period stochastic dynamic program. In each period, the organization must decide how much of the current assets should be invested in revenue-customer service capacity, and at what price the service should be sold. We provide sufficient conditions under which the optimal capacity allocation and pricing decisions are of threshold type. Similar results are derived when the selling price is fixed but banking of assets is allowed. We compare the performance of the optimal threshold policies with heuristics that may be more appealing to managers of non-profit organizations. Numerical experiments indicate that, while banking appears to only have a marginal effect, dynamic pricing can provide a significant benefit.

# 1 Introduction

Many nonprofit organizations engage in for-profit activities, with the purpose of generating revenue to subsidize their mission activities. In recent years, the decrease in the number of grants and donations has put increased pressure on these organizations to become self-sustaining (Dees 1998; Elstrodt, Schindler, and Waslander 2004). A recent study by Bradley, Jansen, and Silverman (2003) estimates that the nonprofit sector could earn an additional USD 100 billion from an improved management process in its for-profit activities. In this paper, we explore how a nonprofit organization should dynamically allocate its assets over time between its revenue generating activities and its mission in order to maximize the organization's social impact, which differs from the profit-maximization objective in commercial ventures. To our knowledge, we provide the first analytical approach to examining resource management issues in the context of nonprofit operations.

A well-documented success story of an organization raising cash from a commercial activity in order to fund a social mission is the Aravind eye hospital in India (Pande 1998). This organization provides quality eye care in free hospitals by generating revenue from distinct paying hospitals. Another, more recent example described by Elstrodt, Schindler, and Waslander (2004) concerns the Brazilian organization CIPÓ Productions, which provides impoverished young people with training in photography, video production, and Web design. CIPÓ Productions also uses part of its capacity (video production and computer equipment) to sell services to other organizations<sup>1</sup>.

Weisbrod *et al.* (Steinberg and Weisbrod 1997; Weisbrod 1998; Sinitsyn and Weisbrod 2002) have proposed a two-goods model to formalize the analysis of the joint operation of nonprofit and for-profit activities. This model considers two kinds of goods or services, addressing two types of customers, *mission customers* (M-customers) and *revenue customers* (R-customers). A nonprofit organization seeks to maximize the servicing of mission customers. In order to finance that operation, the organization may choose to serve revenue customers. Sinitsyn and Weisbrod (2002) find the behavior of nonprofit firms in six industries to be consistent with this two-good model. We model the organization's objective to be the maximization of the expected discounted number of mission customers served over time, at some given social discounting rate. The organization is confronted with a resource management problem, in the form of a consumption vs. investment trade-off. Investment corresponds to the allocation of capacity for revenue customers, while consumption corresponds to serving mission customers. If too many resources are consumed for the organization's mission today, revenue that would allow serving more mission customers in the future is lost. On the other hand, investing too much in revenue customers can result in ineffective use of service capacity which could have

---

<sup>1</sup>CIPÓ Productions won *The Social Entrepreneur Award* (SEA) in 2001; the SEA was launched by McKinsey in partnership with Ashoka, a global nonprofit body in 1999.

served mission customers.

Our focus here is on operational decisions, not infrastructure investment or other strategic decisions. This paper is concerned solely with the operating budget that determines service capacities (such as staffing levels) for both types of customers. It is assumed that the organization has already invested in its infrastructure and does not perform any further strategic investments over the problem's time horizon. We also assume ample demand of M-customers, that is, that the service capacity allocated to this group will always be consumed. This makes sense in a nonprofit context, where social needs are often very high compared to the service capacity of the organization. On the other hand, the R-customer demand is assumed to be random with a known distribution.

We first consider the case where a risk-free investment exists, so that the organization can bank part of its assets at a fixed return rate. The risk-free investment may be attractive to avoid the possibility of having to restart building capital from a low position, which would delay spending in the social mission of the organization. Relative to banking, capacity for R-customers can be interpreted as a higher-risk investment, with higher expected return. We model this multi-period resource allocation problem as a stochastic dynamic program. In each period, the organization must decide how much of the current assets should be invested in revenue-customer service capacity and what fraction of the assets to bank. The remainder of the assets are then spent on mission customers. After these decisions have been made, demand for the for-profit activities is then realized and determines, along with the return on the risk-free investment, the assets available in the next period. The objective is to maximize the expected total discounted social return, measured as the number of mission customers served.

We provide sufficient conditions under which the optimal capacity allocation and pricing decisions are of threshold type. Under this policy, assets are split only between banking and R-customer service capacity up to a given threshold. The surplus is allocated to the mission customers. When this threshold is not reached, we show that the banked fraction of the assets (the rest being allocated to the R-customers), is increasing in the current asset-level. We also provide sufficient conditions on the number of periods to go under which banking is never optimal. That is, banking is suboptimal toward the end of the time horizon. Our numerical results also suggest that there are only small expected gains from including the option of banking in problems with reasonable parameters.

We then turn our attention to the case where the organization competes in a market as a price-setter, and observes an R-customer demand which is a function of that price. In each period the organization is confronted with the decision of choosing the price at which to offer its services to revenue customers, which affects the potential return on the investment in revenue-customer capacity. For instance, the organization can decide to “consume” more of its assets, leaving less capacity for the revenue customers and, by increasing

the selling price, adjust demand to a lower level while maintaining expected revenue. The organization needs to develop a pricing strategy jointly with its capacity allocation and banking decisions. Characterizing the optimal policy for these joint decisions for the general model with banking appears, however, to be far more challenging. To be able to provide some insight into the pricing problem, we assume that the organization never banks, or that the banked amount is negligible compared to the assets allocated to capacity for the M- or R-customers. The previous results suggest that this is a reasonable assumption.

This allows us to explore the structure of the optimal pricing and capacity allocation decisions in more detail. More precisely, we present sufficient conditions ensuring that the optimal allocation strategy is also of threshold type: assets are invested in R-customer service capacity up to a given threshold and the surplus is allocated to the mission customers. If the current asset level is above that threshold, the service is sold to the revenue customers at a fixed price (i.e. the price does not depend on the current asset-level). If the asset level is below the threshold, the selling price decreases with the asset level.

Finally, when the price is fixed and banking is not allowed we also show, perhaps surprisingly, that the optimal service-capacity allocation is myopic. The optimal threshold is not time-dependent and can be analytically derived.

Based on these results we provide numerical examples, and discuss managerial insights. In particular we investigate how the optimal policies perform compared to some heuristics. One of the main downsides of threshold policies is that they can eventually jeopardize the culture of the organization as non-profit oriented. When the asset-level is below the threshold, the organization will not act on its mission at all which, in the long run, can affect its ability to recruit volunteers, and to provide high service quality to the M-customers, among other concerns (Dees 1998; Dees and Anderson 2003). Strategies based on proportional allocation may therefore be more appealing to managers. Under such policies, a fixed percentage of the current assets is always allocated to M-customers, so that the organization always contributes by some degree to its mission. Our numerical studies show however that under reasonable parameters, proportional allocation heuristics perform poorly compared to the optimal threshold policies. This difference in how efficiently the organization fulfills its mission may justify the adoption of a threshold policy to increase the expected social impact.

Resource management concerns the compromise between consuming a resource right away or investing for future growth. While our work is motivated by an interest in the management of nonprofit organizations, our model is also relevant to pricing and financial decisions in other resource management problems. In a for-profit firm, the investment/consumption tradeoff corresponds to the dividend payment problem where a firm must decide whether to issue dividends or to re-invest in order to collect future returns.

The dynamics in our model, which determine the cash constraints in the following periods, are hence similar to interactions between financial and operational decisions explored in recent work. The large literatures

on cash flow management or corporate finance do not consider the interaction of operational and financial decisions. When firms operate in a perfect market Modigliani and Miller (1958) show that these decisions should be made independently. Recent work, however, has pointed out that many firms do not operate in a perfect market and has focused on integrating operational and financial decisions, arguing that the cash constraints determined by a firm's financial policy substantially interact with its operations policy. This is especially true for small firms that do not have access to reasonably priced capital (such as through the bond market). Buzacott and Zhang (2001) consider a manufacturer with limited funds in need of obtaining an asset-based loan from a bank to finance future growth. The objective is then to maximize the retained earnings at the end of the time horizon. (Babich and Sobel 2004) study production, sales, and loan size decisions to optimize the expected discounted proceeds from an initial public offering of stock.

In this stream of research, the control of dividends problem studied by Li, Shubik, and Sobel (2003), which includes inventory-replenishment decisions, seems to be the most closely related to our model. In particular, they establish that the optimal policy is myopic for linear holding and backorder costs. Their model is similar to our problem with banking, but with the organization having unlimited access to borrowing. Demand is also backordered, while we assume lost-sale, which is more suited to the problem faced by many firms, especially in service industries.

More generally we consider marketing strategies, and pricing decisions in particular, that have generally not been addressed in the literature on joint operational and financial decisions, as the prices set by a firm determine its revenue which in turns affects the cash constraints and future revenue and growth.

Our model is primarily linked to the vast literature on dynamic inventory models with pricing and stochastic demand. Petruzzi and Dada (1999) and more recently Elmaghraby and Keskinocak (2003) present comprehensive reviews of this literature. In this stream of research, the pricing strategy and the inventory stock replenishment decisions determine the inventory levels available for the following periods. This differs from the cash constraints that follow from the capacity allocation decisions in our model.

For the single period case, our model can be shown to be equivalent to a newsvendor problem with pricing and with a constraint on the capacity decision. For the classical newsvendor problem with pricing (without a limit on the capacity), there exist conditions on the hazard rate of the demand distribution for the additive and multiplicative pricing models which guarantee the uniqueness of the optimal price and stock level (Petruzzi and Dada 1999). More recent articles have proposed other conditions on the generalized hazard rate of the demand distribution for there to exist a unique optimal price and stock level. Wang, Jiang, and Shen (2004) show that if the generalized hazard rate of the demand distribution increases the expected profit is unimodal. Bernstein and Federgruen (2005) prove that if the generalized hazard rate is bounded below by one half, the expected profit is log-concave. However, stronger conditions than unimodality (or

log-concavity) are usually required when analyzing a multi-period dynamic problem. The condition we derive for an optimal threshold policy to exist bounds the generalized failure rate below by the inverse of the price elasticity. This in turns allows us to propagate concavity of the optimal value function in the multi-period case.

The rest of the paper is organized as follows. Section 2 addresses the problem of optimal capacity and banking decisions, with a fixed selling price to R-customers. We also discuss conditions under which it is optimal not to bank. Section 3 then considers the problem without banking where the organization makes a pricing decision in each period. Section 4 presents numerical studies. We first investigate the relative merit of the optimal capacity policy relative to a proportional-allocation policy. We then explore the expected value of including banking and pricing decisions in the problem specification. Section 5 discusses directions for future research, and Section 6 concludes.

## 2 Resource Allocation with Banking

### 2.1 The Model

Consider a non-profit organization whose mission is to deliver a service to a specific group of people, which we refer to as its mission customers, or M-customers. Serving one M-customer increases the social impact of the organization by  $s$  (in social return units) but does not generate revenue. In order to fund this non-profit activity, the organization offers its services to a market of revenue customers, or R-customers. The unit service capacity costs for M- and R-customers are equal to  $c_M$  and  $c_R$ , respectively. Without loss of generality, we assume  $c_R = 1$  (this is simply a change of asset units).

We assume ample demand of M-customers, that is, service capacity allocated to this group will always be consumed. On the other hand, the R-customer demand is random. Specifically, demand is a random variable  $\Theta$  with a finite mean. We represent by  $f(\cdot) : [\underline{\theta}, \bar{\theta}] \mapsto \mathbf{R}^+$  the probability density function of  $\Theta$ . We will also consider the corresponding cumulative distribution  $F(\cdot)$ , and the tail distribution  $G(\cdot) = 1 - F(\cdot)$ .

In this section, we assume the price is fixed, that is the organization is a price-taker, so that there is no dependence of the R-customer demand on the unit selling price  $p$ . This price may be fixed by the market in which the organization is competing for R-customers or, due to marketing or organizational constraints, the organization may decide to propose its service at a fixed price for R-customers.

Also for this section, we assume that the organization has alternative financial application for its assets in a risk-free investment with return  $\beta$ . The organization needs to determine how much of its assets should be ‘banked’, how much should be invested in capacity for R-customers, and how much should be allocated to capacity for M-customers.

In addition to shedding some light into efficient capacity allocation and banking strategies for this problem, we will briefly discuss in this section the relative importance of including banking in the problem formulation. In particular, we show that if the time horizon is short enough, banking is never optimal. (We conjecture that similar results exist for the price-dependent demand models of Section 3, where we will ignore banking.)

We begin by considering discrete-time problems, with a finite horizon of  $T$  periods. This may arise, for instance, if the organization is the beneficiary of a grant that specifies a duration of time to achieve its objectives. The social discount factor for delaying service to an M-customer to the next period is  $\alpha$ , with  $0 \leq \alpha < 1$ . This discount factor measures the urgency of the social need that the organization addresses. The overall social return from the organization's activities is the total discounted number of M-customers served. The objective is then to determine a banking and capacity allocation policy that maximizes the expected social return over the time horizon  $T$ . We formulate the problem of jointly determining the best banking and capacity allocation decisions as a finite-horizon Markov decision process.

The system state is the assets  $a_t$  held by the organization at the beginning of period  $t \in [1, \dots, T]$ . At the beginning of each period, the organization decides how much service capacity  $y_t$  to provide for R-customers and sets the amount of current assets which are banked,  $z_t \geq 0$ . The choices of  $y_t$  and  $z_t$  are limited by the current resources, which corresponds to the constraint  $y_t + z_t \leq a_t$ . The remaining resources,  $x_t = a_t - y_t - z_t$ , are allocated to serving M-customers. Demand  $\Theta$  is then realized, and the number of R-customers served by the organization is the smallest of demand and capacity, which we write  $y_t \wedge \theta$ . The service is perishable so that any unused capacity is lost. The resources available to the organization at the beginning of the following period are

$$a_{t+1} = p(y_t \wedge \theta) + \beta z_t.$$

The organization contributes to its mission by serving  $(a_t - y_t - z_t)/c_M$  M-customers, yielding a social return of  $s/c_M(a_t - y_t - z_t)$ . Without loss of generality, we set in the rest of this paper  $s/c_M = 1$  (this is simply a change in social return units).

Denote by  $v_t(a)$  the maximum social-impact-to-go at period  $t$ , given the current resources  $a \geq 0$ . For the last period, all assets are allocated to the service of M-customers, so that  $v_T(a) = a$ . For  $t < T$ ,  $v_t(a)$  can be shown to satisfy the optimality equations (see, for instance, Heyman and Sobel 1984)

$$v_t(a) = \max_{\substack{0 \leq y \quad 0 \leq z \\ y + z \leq a}} (a - y - z) + \alpha H^{v_{t+1}}(y, z) \tag{1}$$

where the operator  $H^{v(y,z)}$  is defined for any real-valued function  $v$  as

$$H^v(y, z) = E_{\Theta} v(p(y \wedge \Theta) + \beta z). \tag{2}$$

where  $E_\Theta$  is the expectation over  $\Theta$ .

We denote by  $(y_t^*(a), z_t^*(a))$  the optimal capacity allocation and banking decision at period  $t$  given the current assets  $a$ . The optimal policy  $(y_t^*(a), z_t^*(a))$  corresponds to the maximizer of  $\alpha H^{v_t}(y, z) - y - z$  subject to the constraints  $0 \leq y + z \leq a$ . (In case of multiple optima  $(y_t^*(a), z_t^*(a))$  designates the minimum optimal decision using, say, the lexicographic order.)

As a side result, note that if  $\alpha\beta \geq 1$  the organization always prefers to bank rather than to serve M-customers (except for period  $T$ ). Likewise, conditions on  $p$  can also ensure that serving R-customer is never optimal.

**Proposition 1**

- If  $\alpha\beta > 1$ , allocating capacity to mission customers only in the last time period is always optimal.
- If  $p < \beta$  or  $\alpha p < 1$ , serving R-customers is never optimal.
- If  $\alpha\beta < 1$  and  $(p < \beta$  or  $\alpha p < 1)$ , using all assets to serve M-customers in the first period is optimal.

**Proof.** This is shown from a sample-path argument. Suppose a policy is optimal, and it is such that  $x_t = a_t - y_t - z_t$  M-customers are served at time  $t$ , with  $x_t > 0$ . Denote by  $x_s$  and  $z_s$  the optimal decision for  $s = 1, \dots, T$ . Consider now an alternative policy, with decisions  $\bar{x}_s, \bar{z}_s$  such that  $\bar{x}_s = x_s$  for  $s \neq t, T$  and  $\bar{x}_t = 0, \bar{x}_T = x_T + \beta^T x_t$ . This corresponds to  $\bar{z}_s = z_s$  for  $s = 1, \dots, t - 1$ , and  $\bar{z}_s = z_s + \beta^{t-s} x_t$  for  $s = t, \dots, T - 1$ . Note that before time  $t$ , the policies are identical. With the new policy, for any realization of the  $\Theta_s, s = 1, \dots, T$ , the objective is improved by  $x_t((\alpha\beta)^{T-t} - 1) > 0$ . The second part of the property can be shown using a similar argument. The third part is implied by the first two.

■

The interesting case is then  $\alpha\beta < 1$  and  $\alpha p > 1$ , which we assume for the rest of this section. This also implies  $p > \beta$ . The next result states that the optimal policy for R-customer capacity plus banking is of threshold type.

**Theorem 1** *For each time period  $t$ , there exists a threshold  $a_t^*$  such that the optimal capacity decision  $y_t^*(a)$  and banking decision  $z_t^*(a)$  satisfy  $y_t^*(a) + z_t^*(a) = a_t^* \wedge a$ . Furthermore, the optimal banking decision  $z_t^*(a)$  is increasing in  $a$ .*

**Proof.** Assume that  $v_{t+1}$  is differentiable, non-decreasing and concave. Then  $v_{t+1}(p(y \wedge \theta) + \beta z)$  and hence  $H^{v_{t+1}}(y, z)$  are also non-decreasing and jointly concave in  $y$  and  $z$ . It follows that  $v_t(\cdot)$  is the sum of a linear function and the maximum of a differentiable non-increasing concave function subject to convex constraints. As a result,  $v_t(\cdot)$  is non-increasing, concave (see Heyman and Sobel (1984) for instance), and differentiable

(see Theorem 4.16 of Bonnans and Shapiro (2000) for instance). Since  $v_T(a) = a$ ,  $v_t$  is differentiable, non-decreasing, concave for all  $t \leq T$ , and the first part of the theorem follows directly.

For the second part, assume that  $a < a_t^*$  such that the optimal decisions verify  $y_t^* = a - z_t^*$  and the social impact-to-go  $v_t(a)$  is equal to  $\max_{0 \leq z \leq a} \alpha H^{v_{t+1}}(a - z, z)$ . Defining  $h(z) = H^{v_{t+1}}(a - z, z)$ , the first order condition  $h(z)' = 0$  is equivalent to

$$\beta \int_0^{a-z} v'_{t+1}(p\theta + \beta z) f(\theta) d\theta + (\beta - p) v'_{t+1}((\beta - p)z + pa) G(a - z) = 0.$$

The first term of Equation (3) increases in  $a$  and decreases in  $z$  since  $v'_{t+1}(\cdot)$  is a positive decreasing function. From the concavity of  $v_{t+1}(\cdot)$  and for  $p > \beta$ , the second term is also increases in  $a$  and decrease in  $z$ , so that  $z_t^*(a)$  increases in  $a$ .

■

As a side note, the equivalent first order condition for  $y$  with  $H^{v_{t+1}}(y, a - y)$  yields

$$-\beta \int_0^y v'(p\theta + \beta(a - y)) f(\theta) d\theta + (p - \beta) v'((p - \beta)y + \beta a) G(y) = 0.$$

The first term increases in  $a$ , while the second one decreases, so that the monotonicity of  $y_t^*(\cdot)$  remains indeterminate.

This result extends to the infinite horizon case by letting  $T$  approach  $+\infty$ , as stated in the following result.

**Corollary 1** *The optimal policy for the infinite horizon case is of threshold type. Further, the optimal banking decision  $z^*(\cdot)$  is non-increasing in the current assets.*

**Proof.**

Consider the time period  $T - 1$ . Note then that  $H^{v_T}(y, z) \leq pE_\Theta \Theta + \beta z$  and

$$\max_{y \geq 0, z \geq 0} \alpha H^{v_T}(y, z) - y - z \leq \alpha p E_\Theta \Theta + \max_{y \geq 0, z \geq 0} (\alpha \beta - 1) z - y$$

Since  $\alpha \beta < 1$ ,  $\max_{y \geq 0, z \geq 0} \alpha H^{v_T}(y, z) - y - z$  is then bounded from above by the finite value  $M = \alpha p E_\Theta \Theta$ .

It follows that  $v_{T-1}(a) \leq a + M$ . For the time period  $T - 2$  we have then

$$\begin{aligned} \max_{y \geq 0, z \geq 0} \alpha H^{v_{T-1}}(y, z) - y - z &\leq \alpha \max_{y \geq 0, z \geq 0} E_\Theta (py \wedge p\Theta + \beta z + M) - y - z \\ &\leq (1 + \alpha)M \end{aligned}$$

where the first equality comes from  $v_{T-1}(a) \leq a + M$  and the second from the definition of  $M$ . As a result,  $v_{T-2}(a) \leq a + (1 + \alpha)M$ . Using a backward iteration on  $t$ , we conclude

$$v_t(a) \leq a + \frac{1 - \alpha^{T-1}}{1 - \alpha} M, \quad T \geq 2 \tag{3}$$

and, since  $v_t$  is increasing in  $T$ ,  $v_t$  has a finite limit  $v^*(a)$  as  $T$  approaches  $+\infty$  where  $v^*(a) \leq a + M/(1 - \alpha)$ . The result then follows then directly from Theorem 1 and the application of Theorem 8-15 in Heyman and Sobel (1984).

■

When  $\beta = 0$ , which also corresponds to the case considered in Section 3, the organization should never bank. The optimal policy is then myopic, that is, the optimal threshold does not depend on  $t$  and corresponds to the optimal threshold of the single period problem, as shown by the following result,

**Corollary 2** *When the price is fixed and no banking is possible, the myopic policy is optimal. At time  $t$ , the optimal capacity allocation is  $y_t^*(a) = a^* \wedge a$  where*

$$a^* = \begin{cases} F^{-1}\left(1 - \frac{1}{\alpha p}\right) & \text{if } \alpha p > 1 \\ 0 & \text{if } \alpha p \leq 1 \end{cases} \quad (4)$$

**Proof.** Assume  $a^*$  is the optimal threshold at  $t+1$ . The threshold at time  $t$  is the maximand of  $H^{v_{t+1}}(y, 0) - y$  and from Theorem 1,  $v_{t+1}$  is differentiable and the first order condition yields

$$\alpha p v'_{t+1}(pa)G(a) = 1 \quad (5)$$

For  $\alpha p > 1$ , we have that  $pa^* > a^*$ , and therefore  $v'_{t+1}(pa^*) = 1$  (since  $a^*$  is the optimal threshold at time  $t+1$ ). It follows that  $a^*$  as defined in (4) satisfies equation (5). Similarly, since  $v'_T(a) = 1$ ,  $a^*$  is the optimal threshold at time  $T - 1$  and the result follows by backward iteration.

■

Note that Sobel (1984) derived very general sufficient conditions for myopic policies to be optimal. In particular, these conditions would require in our model that the set  $S(a^*) = [a^*, +\infty[$  be consistent, that is, for all  $\theta \in [\underline{\theta}, \bar{\theta}]$ ,  $p(y \wedge \theta) \in S(a^*)$ . This is not the case here for  $\theta < a^*/p$ . The organization should never invest in capacity for R-customers when  $\alpha p \leq 1$ . When  $\alpha p > 1$ , the optimal threshold corresponds to the optimal capacity level of a newsvendor model with overage unit cost equal to 1 (corresponding to an M-customer), and underage unit cost  $\alpha p$ . For general values of  $c_R$  and  $s/c_M$ , the previous results directly extend to

$$a^* = F^{-1}\left(1 - \frac{c_R s}{c_M \alpha p}\right),$$

with  $\alpha p/c_R \geq 1$ . The optimal policy for  $y_t^*(\cdot)$  will also be overall increasing (since  $y_t^*(0) = 0$  and  $y_t^*(a) = y^*$  for  $a \geq a^*$ ), but note that, for  $\beta > 0$ , Theorem 1 does not address how  $y_t^*(\cdot)$  evolves with  $a$ .

Corollary 2 allow us to derive conditions for banking never to be optimal when the price is fixed. We first give sufficient conditions to propagate an upper bound on the derivative of the social-impact-to-go. With

$n = T - t$  the number of periods to go this upper bound is equal to

$$u(n) = (\alpha p)^n.$$

**Lemma 1** *Assume that the following three conditions hold at time  $t$ ,*

- *the optimal banking decision yields  $z_t^*(a) = 0$  for all  $a \geq 0$ , (C1)*
- *the optimal social impact to go  $v_t$  is differentiable concave (C2)*
- *$v_t'(a) \leq u(T - t)$  (C3)*

*If in addition*

$$u(T - t) \leq (\alpha\beta)^{-1}, \tag{6}$$

*then (C1), (C2) and (C3) also hold at time  $t - 1$ .*

**Proof.** To show (C1), fix  $y$  and consider the derivative of  $\varphi(y, z) = a - y - z + \alpha H^{vt}(y, z)$  with respect to the second variable,

$$\begin{aligned} \frac{d\varphi}{dz}(y, z) &= -1 + \alpha\beta E_{\Theta} v_t'((py \wedge p\Theta) + \beta z) \\ &\leq 1 - \alpha\beta u(T - t) \\ &< 0, \end{aligned}$$

where the inequalities follow from (C3) and from (6). As a result, for any given  $y$ ,  $\varphi(y, z)$  is maximized  $z = 0$ , and  $v_{t-1}(a)$  satisfies (C1). Since  $v_t(\cdot)$  is differentiable and concave,  $v_{t-1}(\cdot)$  also satisfies (C2) (see the proof of Theorem 1). Finally, using the chain rule at the optimal decision and (C3), we obtain for  $y = a$  (below the threshold)

$$v'_{t-1}(a) = \alpha p v'_t(a) G(a) \leq \alpha p u(T - t) = u(T - t + 1)$$

Since  $v'_{t-1}$  is non-increasing, the inequality in (C3) also holds for  $v'_{t-1}$  at every  $a$ .

■

We can now show that there exists a time period starting from which the organization should stop banking and follow the threshold policy deccribed in Corollary 2.

**Proposition 2** *Define  $n_0$  as*

$$n_0 = \left\lfloor -\frac{\ln(\alpha\beta)}{\ln(\alpha p)} \right\rfloor.$$

*For  $T - n_0 \leq t \leq T$ , the optimal policy never involves banking, i.e.,  $z_t^*(a) = 0$ . The optimal capacity allocation is  $y_t^*(a) = a^* \wedge a$  where  $a^*$  is given by (4).*

**Proof.** For  $n = 0$ ,  $v_T(a) = a$  satisfies (C1), (C2) and (C3) of Lemma 1. We can then recursively apply Lemma 1 as long as  $n = T - t$  satisfies (6) which is equivalent to  $n \leq n_0$ . The proposition holds from Theorem 1.

■

In particular, if the time horizon is short enough ( $T \leq n_0$ ) the optimal policy is myopic and banking is never optimal.

Clearly, with only one time period, the organization should never bank. Under the stated conditions, the risk-free asset has the worst return of all applications for the assets. It is of use as a hedge against uncertain demand, as a low realization may lead to having to rebuild capital from a very low position and to having to go through many periods unable to serve M-customers. As the value-function becomes more concave due to there being more time periods to go, the optimal policy becomes more ‘risk-averse’, as there is a higher opportunity cost of having to start the next period with a low cash position, and banking becomes more attractive.

### 3 Resource Allocation with Pricing

#### 3.1 The Model

We have assumed above that the organization is a price-taker, that is that it proposes its service at a fixed price to the revenue customers. Non-profit organizations do however often have market power in their revenue-generating activities. In such a setting, the organization competes in a market as a price-setter, and observes a demand which is a function of that price. The organization needs to develop a pricing strategy jointly with its capacity allocation and banking strategy. Characterizing the optimal policy for these joint decisions for the general model appears however to be far more challenging. To be able to provide some insight into the pricing problem, we assume that the organization never banks, or the banked amount is negligible compared to the assets allocated to capacity for the M- or R-customers. This allows us to explore the structure of the optimal pricing and capacity allocation decisions in more detail. Although we do not formally show an equivalent to Proposition 2 for the price-dependent case, we expect that similar conditions on the time horizon exist for which banking is never optimal.

We consider a demand function with multiplicative uncertainty, that is  $D = \gamma(p)\Theta$ , where  $\gamma(\cdot) : \mathbf{R} \mapsto \mathbf{R}$  is the price response function, and  $\Theta$  is a random variable with finite mean. We will also refer to the revenue function, defined as  $r(p) = p\gamma(p)$ . As before,  $f(\cdot)$  denotes the probability density function of  $\Theta$ , and  $F(\cdot)$  and  $G(\cdot)$  the corresponding cumulative and tail distributions. We also assume  $F(\cdot)$  to be infinitely differentiable.

We also introduce the generalized failed rate (GFR), defined as  $g(\cdot) : [\underline{\theta}, \bar{\theta}] \mapsto \mathbf{R}^+$  such that

$$g(\theta) = \theta \frac{f(\theta)}{G(\theta)}. \quad (7)$$

In this section we restrict our analysis to the price response function

$$\gamma(p) = cp^{-b},$$

with  $c > 0$  and  $b > 1$ , and where  $b$  is the price-elasticity index of demand. This restriction simplifies the exposition and is commonly used in the literature (see Petruzzi and Dada (1999), for instance). Similar results can be derived for other demand functions, as briefly discussed in Section 5.1.

At the beginning of each period, the organization sets the selling price  $p_t$  and decides how much service capacity  $y_t$  to provide for R-customers. The choice of  $y_t$  is limited by the current resources, which corresponds to the constraints  $0 \leq y_t \leq a_t$ . The remaining resources,  $a_t - y_t$ , are allocated to serving M-customers. Demand  $D$  is then realized, and the number of R-customers served by the organization is  $y_t \wedge \gamma(p)\theta$ . The resources available to the organization at the beginning of the following period are

$$a_{t+1} = p_t(y_t \wedge \gamma(p_t)\theta).$$

The goal is then to determine a pricing and capacity allocation policy that maximizes the social return over the time horizon  $T$ .

For  $t < T$ , social-impact-to-go at period  $t$  can be shown to satisfy the optimality equations

$$v_t(a) = \max_{\substack{0 < p \\ 0 \leq y \leq a}} a - y + \alpha H^{v_{t+1}}(p, y), \quad (8)$$

where the operator  $H^v(p, y)$  is defined for any real-valued function  $v$  as

$$H^v(p, y) = E_{\Theta} v(py \wedge r(p)\Theta). \quad (9)$$

We denote by  $(p_t^*(a), y_t^*(a))$  the optimal pricing and capacity allocation decision at period  $t$  given the current assets  $a$ . The optimal policy  $(p_t^*(a), y_t^*(a))$  corresponds to the maximizer of  $\alpha H^{v_t}(p, y) - y$  subject to the constraints  $p > 0, 0 \leq y \leq a$ .

Solving (8) however can be challenging, even for the single period case. When  $T = 1$ , the objective can be written as

$$v(a) = a + \max_{\substack{0 < p \\ 0 \leq y \leq a}} \alpha p E_{\Theta} (y \wedge \gamma(p)\Theta) - y \quad (10)$$

and the optimization problem is equivalent to a newsvendor problem with pricing and with capacity constraint. The unconstrained newsvendor problem with pricing has been the subject of much research (see Petruzzi and Dada (1999) for a complete survey). More recently, Wang, Jiang, and Shen (2004) provide sufficient conditions for the uniqueness of the optimal decisions  $y^*$  and  $p^*$ . They show that if  $\Theta$  has increasing generalized failure rate (i.e.,  $g(\cdot)$  is increasing over  $[\underline{\theta}, \bar{\theta}]$ ), then  $y^*$  and  $p^*$  exist and are unique. To that end, they introduce the “stocking factor”  $y/\gamma(p)$  in Equation (10). They show that for any given  $z$ , an optimal  $p(z)$  exists and the function  $H(z)$  defined as

$$H(z) = \alpha E_{\Theta} r(p)(z \wedge \Theta) - \gamma(p)z$$

is unimodal in  $z$ . Nothing is said however about the concavity of the objective function, even along the optimal price  $p^*(z)$ . Similarly, Bernstein and Federgruen (2005) propose a sufficient condition on  $g(\cdot)$  which guaranties that the previous function  $H(\cdot)$  is log-concave and hence unimodal.

The analysis of multi-period dynamic problems however typically requires stronger properties than unimodality or log-concavity. In particular, most of the previous approaches to address dynamic joint pricing-inventory problems (for general probability distributions) rely on the joint concavity of  $\alpha H^v(p, y) - y$  in  $(p, y)$  for any concave value function  $v$  (see, for instance, Federgruen and Heching (1999)). However, even for the single period case, the  $py$  term in the definition of  $H^v$  in (9) is not jointly concave. Changes of variable such as  $z = y/\gamma(p)$ , or the condition on the generalized failure rate  $g(\cdot)$  proposed by Wang et al. (2004) or Bernstein and Federgruen (2005) do not circumvent this problem.

In the following, we assume that the generalized failure rate is not smaller than the inverse of the price elasticity, that is  $\forall \theta \in [\underline{\theta}, \bar{\theta}]$ ,

$$g(\theta) \geq \frac{1}{b}. \tag{11}$$

For the single period case, Condition (11) is very similar to the condition proposed by Bernstein and Federgruen (2005) for which the generalized hazard rate is bounded below by one half. In our case however, the bound depends on the price elasticity which makes Condition (11) more restrictive. Condition (11) is also neither more general nor more restrictive than the increasing generalized failure rate condition proposed by Wang et al. (2004). On the one hand, condition (11) allows for non-monotonic generalized failure rates. On the other hand, it imposes conditions on the distribution parameters and the price elasticity. For instance, if  $\Theta$  is uniformly distributed over  $[\underline{\theta}, \bar{\theta}]$ , condition (11) is equivalent to  $b + 1 > \bar{\theta}/\underline{\theta}$ . More generally, for any distribution defined over  $[0, +\infty[$  with an increasing generalized failure rate, condition (11) holds if the distribution is truncated at  $\underline{\theta}$  such that  $g(\underline{\theta}) > b^{-1}$ , and scaled accordingly.<sup>2</sup>

---

<sup>2</sup>When the organization is required to sell the service at a minimum price, it can be shown that this condition can be relaxed as follows. If  $p$  must be larger than or equal to  $\underline{p}$ , our results hold with the condition  $g(\theta) \geq b^{-1}$  for  $\theta \geq F^{-1}(1 - 1/\alpha/\underline{p})$ . For the exponential distribution with rate  $\lambda$  for instance, this condition is equivalent to  $\alpha \underline{p} \geq e^{1/b}$ .

Under condition (11), the concavity property can be propagated across time periods. More precisely, we will show that if condition (11) holds then  $H^v(p, y)$  is concave along the optimal price, that is,

$$\tilde{H}^v(y) = \max_{0 < p} H^v(p, y) \tag{12}$$

is concave in  $y$ .  $\tilde{H}^v$  is well defined for any continuous function  $v$  since

$$H^v(0, y) = \lim_{p \rightarrow +\infty} H^v(p, y) = 0$$

for any  $y \in \mathfrak{R}^+$ . Concavity of  $\tilde{H}^v$  guaranties in turn that the social-impact-to-go function  $v_t(\cdot)$  is concave and that the optimal policy is of threshold type.

A threshold policy in our context is characterized by  $T$  thresholds  $(\hat{p}_t, \hat{a}_t)$ ,  $t \in [1 \dots T]$ , such that the decisions  $(p(a), y(a))$  given assets  $a$  at the beginning of period  $t$  are

$$\begin{aligned} y(a) &= a \wedge \hat{a}_t \\ p(a) &= \hat{p}_t \quad \text{if } a \geq \hat{a}_t. \end{aligned}$$

Under a threshold policy, the organization tries to guarantee a particular service capacity for R-customers. Only when this threshold is assured are M-customers served.

Note that the previous definition does not specify how the organization should set the price when  $a < \tilde{a}_t$ . Finding the optimal price is not a trivial problem, and solutions can be counter-intuitive. One could indeed reasonably expect the optimal price to be decreasing in the capacity allocated to R-customers. However, we can see from a simple counter-example that this is not always the case. For the model described here and with a distribution  $F$  with two mass points. the results of a numerical simulation are presented in Figure 1 (parameters for the example are: probability mass 1/2 at  $\theta = 0.1$  and 2,  $c = 10$ ,  $b = 2$ ). The optimal price is not monotonic. An informal way of interpreting this result is as follows. If the price is set with full information about demand, the optimal price for the high-demand outcome is higher than the optimal price for the low-demand outcome. The uncertain demand case is a compromise between the two. At some point, as capacity increases more “high-demand customers” are expected to be served, and the optimal price moves towards the higher price. Similar examples can be constructed with a continuous  $F$ . We will see, however, that with condition (11) the optimal price is non-increasing.

### 3.2 Optimal capacity allocation and pricing strategy

In the following, we explore the optimal capacity allocation and pricing strategy. To that end, we make use of the first and second order derivatives of  $\tilde{H}^v(y)$ . However,  $\tilde{H}^v(y)$  may not be twice differentiable due to the constraint  $y \leq a$  in its definition. For this reason, we introduce a family of unconstrained dynamic problems parametrized by  $\epsilon > 0$ . We show that the corresponding operators  $\tilde{H}^{v^\epsilon}(y)$  are concave, and likewise for the

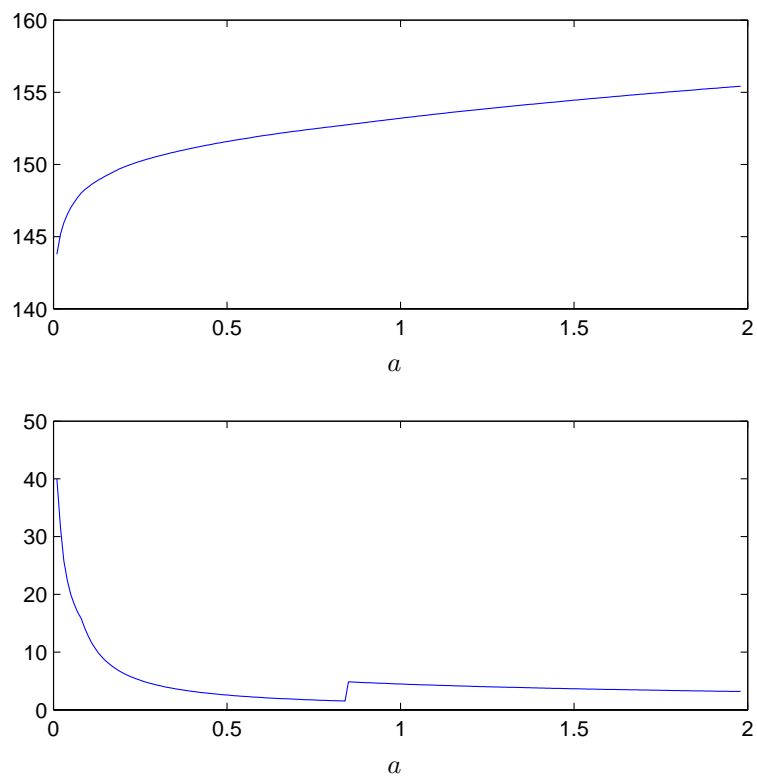


Figure 1: Example of non-monotonic price: value function and optimal price.

optimal value functions  $v_t^\epsilon$ . We then show that  $v_t^\epsilon \rightarrow v_t$  pointwise when  $\epsilon \rightarrow 0$ , where the  $v_t$  satisfy the optimality equations.

Consider  $\mathbf{R}_{-\infty} = \mathbf{R} \cup \{-\infty\}$  and the extension of any function  $\varphi(\cdot) : ]\underline{x}, \bar{x}[ \rightarrow \mathbf{R}$  such that  $\varphi(x) = -\infty$  when  $x \notin ]\underline{x}, \bar{x}[$ . (In the following we use the same notation for a given function and its extension.) For instance, we will consider the extension of the logarithm function such that  $\varphi(x) = -\infty$  when  $x \leq 0$ . The family of unconstrained problems is obtained by omitting the constraints  $0 \leq y \leq a$ , and introducing logarithmic barrier functions in the objective function.

More precisely, for any  $\epsilon > 0$  we consider the dynamic problem,

$$v_t^\epsilon(a) = \max_y \left\{ a - y + \epsilon \log[(a - y)y] + \alpha \tilde{H}^{v_{t+1}^\epsilon}(y) \right\} \quad (13)$$

$$v_T^\epsilon(a) = a. \quad (14)$$

The optimal capacity decision never equals the bounds (that is,  $0 < \tilde{y}^\epsilon(a) < a$ ). Backward iteration then shows that  $v_t^\epsilon(a)$  is infinitely differentiable.

Using condition (11) on the generalized failure rate, we can show that  $\tilde{H}^{v_t^\epsilon}$  is concave if  $v_t^\epsilon$  is also concave.

**Lemma 2** *Assume that condition (11) holds. If for any  $\epsilon > 0$ ,  $v_{t+1}^\epsilon$  is concave, then  $\tilde{H}^{v_t^\epsilon}$  is also concave. Furthermore, the pricing decision  $\tilde{p}^\epsilon(y)$  maximizing  $H^{v_{t+1}^\epsilon}(p, y)$  is non-increasing in  $y$ .*

**Proof.** For clarity, we drop the subscript of  $v_{t+1}^\epsilon$  in the following, writing simply  $v^\epsilon$ .

We first check the conditions for joint strict concavity of  $H^{v^\epsilon}$ , locally at the optimal decision  $(\tilde{p}^\epsilon(y), y)$ . Since  $H^{v^\epsilon}$  is concave in each of the variables, we only need to show

$$H_{pp}^{v^\epsilon}(p, y)H_{yy}^{v^\epsilon}(p, y) - [H_{py}^{v^\epsilon}(p, y)]^2 > 0$$

where  $H_{pp}^{v^\epsilon}$ ,  $H_{yy}^{v^\epsilon}$ , and  $H_{py}^{v^\epsilon}$  represent the second derivatives of  $H^{v^\epsilon}$ . Likewise, we denote by  $H_p^{v^\epsilon}$  and  $H_y^{v^\epsilon}$  the first derivative of  $H^{v^\epsilon}$  with respect to  $p$  and  $y$ . Define  $z(y, p) = y/\gamma(p)$ , which we sometimes denote simply by  $z$ . For any given  $y$ ,  $H^v(0, y) = 0$ , and  $\lim_{p \rightarrow +\infty} H^v(p, y) = 0$ , so that the corresponding optimal price  $\tilde{p}(y)$  is an interior point. Therefore, the first-order optimality condition for  $p$  applies, that is

$$H_p^{v^\epsilon}(p, y) = 0, \quad (15)$$

where

$$H_p^{v^\epsilon}(p, y) = r'(p) \int_0^z v^{\epsilon t}(r(p)\theta)\theta f(\theta) d\theta + yv^{\epsilon t}(py)G(z(y, p)). \quad (16)$$

Since we are concerned with concavity at the optimal  $\tilde{p}$ , we may use the optimality conditions (15-16) to obtain

$$H_{pp}^{v^\epsilon}(p, y) = -y \left( \frac{r''(p)}{r'(p)} G(z) + \frac{\gamma'^2(p)}{\gamma^3(p)} py f(z) \right) v^{\epsilon t}(py) + y^2 G(z) v^{\epsilon t t}(py) +$$

$$\begin{aligned}
& (r'(p))^2 \int_0^z v^{\epsilon''}(r(p)\theta) \theta^2 f(\theta) d\theta \\
&= y \frac{b}{p} [1 - bg(z)] G(z) v^{\epsilon'}(py) + y^2 G(z) v^{\epsilon''}(py) + \\
& (r'(p))^2 \int_0^z v^{\epsilon''}(r(p)\theta) \theta^2 f(\theta) d\theta,
\end{aligned} \tag{17}$$

where  $\gamma'(p) = -b/p\gamma(p)$ ,  $r'(p) = (1+b)\gamma(p)$  and  $r''(p) = -b/pr'(p)$  with  $\gamma(p) = cp^{-b}$  and  $r(p) = p\gamma(p)$ . Note that  $H_{pp}^{v^\epsilon}$  is negative from condition (11) and from the concavity of  $v$ .

Similarly, the other derivatives can be written as

$$H_y^{v^\epsilon}(p, y) = pv^{\epsilon'}(py)G(z) \tag{18}$$

$$H_{yy}^{v^\epsilon}(p, y) = p^2 v^{\epsilon''}(py)G(z) - \frac{p}{\gamma(p)} v^{\epsilon'}(py)f(z) \tag{19}$$

$$H_{py}^{v^\epsilon}(p, y) = v^{\epsilon'}(py) [1 - bg(z)] G(z) + pyv^{\epsilon''}(py)G(z) \tag{20}$$

Note also that  $H_{py}^{v^\epsilon} < 0$  from condition (11) and from the concavity of  $v$ . This implies that the derivative of  $\tilde{p}(\cdot)$  is negative with  $\tilde{p}'(\cdot) = -H_{py}^{v^\epsilon}/H_{pp}^{v^\epsilon}$ .

From these derivatives we obtain, after algebraic simplification,

$$\begin{aligned}
& H_{pp}^{v^\epsilon}(p, y)H_{yy}^{v^\epsilon}(p, y) - [H_{py}^{v^\epsilon}(p, y)]^2 = (v^{\epsilon'}(py))^2 [bg(z) - 1] G(z) \\
& + v^{\epsilon'}(py) v^{\epsilon''}(py) py G(z)^2 [b - 2 - (b-1)^2 g(z)] \\
& + p(r'(p))^2 \int_0^z v^{\epsilon''}(r(p)\theta) \theta^2 f(\theta) d\theta \left( pv^{\epsilon''}(py)G(z) - \frac{1}{\gamma(p)} v^{\epsilon'}(py)f(z) \right)
\end{aligned}$$

The first and third terms are positive from condition (11) and from the concavity of  $v$ . To see that the second term is non negative note that, also from condition (11),

$$\begin{aligned}
v^{\epsilon''}(py)(b - 2 - (b-1)^2 g(z)) &\geq v^{\epsilon''}(py) \left( b - 2 - \frac{(b-1)^2}{b} \right) \\
&= -v^{\epsilon''}(py) \frac{1}{b},
\end{aligned}$$

where  $-v^{\epsilon''}$  is non-negative.

$H_{pp}^{v^\epsilon} < 0$  and  $H_{pp}^{v^\epsilon} H_{yy}^{v^\epsilon} - (H_{py}^{v^\epsilon})^2 > 0$  implies that  $H^{v^\epsilon}(p, y)$  is locally concave at  $(\tilde{p}^\epsilon(y), y)$ . From Lemma 5 (in the Appendix),  $\tilde{H}^{v^\epsilon}(y)$  is infinitely differentiable and concave.

■

It follows that the dynamic equations in (13) preserve concavity as stated by the following result.

**Lemma 3** *Assume that condition (11) holds. If for any  $\epsilon > 0$ ,  $v_{t+1}^\epsilon$  is concave, then  $v_t^\epsilon$  is also concave. Furthermore, the optimal pricing decision  $p_t^{*\epsilon}(a)$  is non-increasing in the current assets  $a$ .*

**Proof.** From Lemma 2 and using  $\varphi(y, a) = a - y + \epsilon \log((a - y)y) + \tilde{H}^{v^\epsilon}(y)$  in (13), we conclude that  $v_t$  is concave as the maximum of a concave function subject to a linear constraint. Note then that  $p_t^{*\epsilon}(a) = \tilde{p}^\epsilon(y^{*\epsilon}(a))$ . The second part then follows from Lemma 2 and the fact that  $y^{*\epsilon}(a)$  increases in  $a$ .

■

Since  $v_T^\epsilon = a$  is concave, Lemma 3 actually ensures that  $v_t^\epsilon$  is concave for all  $t \in [0, T]$ . By letting  $\epsilon \rightarrow 0$ , we obtain the same result for  $v_t$  of the original problem:

**Lemma 4** *If condition (11) holds then for  $\epsilon \rightarrow 0$ , the  $v_t^\epsilon$  converge pointwise to the  $v_t$  which solve the optimality equations (8). Further, the  $v_t(\cdot)$  are concave and the optimal pricing decision  $\tilde{p}_t(\cdot)$  is non-increasing.*

**Proof.** We show this result by iterating on  $t$ . Assume that at the time period  $t$ ,  $v_t^\epsilon$  converge point-wise to the optimal value function  $v_t(\cdot)$  of the original problem. By Lebesgue's dominated convergence theorem (see, e.g., Billingsley 1995, p.209, thm. 16.4), this implies that  $H^{v_t^\epsilon}$  converges point-wise. Note also that since  $v_t(\cdot)$  satisfies the optimality equations,  $v_t(\cdot)$  can be shown to be differentiable so that  $v_t^{\epsilon'}$ , the derivative of  $v_t^\epsilon$ , also converges point-wise. At the time period  $t-1$ ,  $\tilde{p}_{t-1}^\epsilon$  converges pointwise in  $y$ , because of strict concavity in  $p$  of  $H^{v_t^\epsilon}$ . Denote by  $\tilde{p}$  the limit which is non-increasing (non-increasing property is preserved through point-wise convergence). Also, recall that  $0 < \tilde{p}(a) < \infty$ . Similarly, the  $\tilde{y}^\epsilon$  converge pointwise and we denote by  $\tilde{y}$  the limit. Then  $v_{t-1}^\epsilon$  converges point-wise to a limit  $v_{t-1}$  (see, for instance, Bonnans and Shapiro (2000), proposition 4.6). Also,  $v_{t-1}$  is concave (concavity is preserved through point-wise convergence).

We show next that  $v_t$  satisfies the optimality equations (8). To that end, we need to show that for any  $a$ ,

$$\lim_{\epsilon \rightarrow 0} \epsilon \log [(a - \tilde{y}^\epsilon)\tilde{y}^\epsilon] = 0 \quad (21)$$

which is immediately seen to be true when  $0 < \tilde{y}^\epsilon(a) < a$ . Assume that  $\tilde{y}^\epsilon(a) = 0$  or  $\tilde{y}^\epsilon(a) = a$ . The optimal capacity of the  $\epsilon$ -problem satisfies the first order condition,

$$\frac{\epsilon}{(a - \tilde{y}^\epsilon)\tilde{y}^\epsilon} = \frac{1 - \alpha(\tilde{H}^{v_t^\epsilon})'(\tilde{y}^\epsilon)}{a - 2\tilde{y}^\epsilon}, \quad (22)$$

where  $(\tilde{H}^{v_t^\epsilon})'(y)$  is the derivative of  $\tilde{H}^{v_t^\epsilon} = H^{v_t^\epsilon}(\tilde{p}^\epsilon(y), y)$  which is equal to using the chain rule and the first order condition  $H_p^{v_t^\epsilon}(\tilde{p}^\epsilon(y), y) = 0$ ,

$$\begin{aligned} (\tilde{H}^{v_t^\epsilon})'(y) &= H_y^{v_t^\epsilon}(\tilde{p}^\epsilon(y), y) + H_p^{v_t^\epsilon}(\tilde{p}^\epsilon(y), y) \frac{d\tilde{p}^\epsilon}{dy} \\ &= \tilde{p}^\epsilon(y) v_t^{\epsilon'}(\tilde{p}^\epsilon(y), y) G\left(\frac{y}{\gamma(\tilde{p}^\epsilon(y))}\right) \end{aligned}$$

But  $v_t^{\epsilon'}$  converges point-wise and since  $\tilde{p}(a) < \infty$ , we have that  $\gamma(\tilde{p}(a)) > 0$  and  $H^{v_t^{\epsilon'}}$  also converge point-wise. It follows that  $\epsilon/[(a - \tilde{y}^\epsilon)\tilde{y}^\epsilon]$  converges to a finite value as  $\epsilon$  approaches zero when  $\tilde{y}$  is equal to 0 or  $a$ . As a result

$$\lim_{\epsilon \rightarrow 0} \epsilon \log [(a - \tilde{y}^\epsilon)\tilde{y}^\epsilon] = 0.$$

Since  $v_T^\epsilon(a) = v_T(a)$ , the result holds by backward iteration.

■

We are now ready to characterize the optimal capacity allocation and pricing decisions.

**Theorem 2** *If condition (11) holds, the optimal policies are of threshold type and the optimal pricing decisions are non-increasing in the current assets.*

**Proof.** Immediate from Lemma 3 and the concavity of the  $\tilde{H}^{v_t}(\cdot)$ .

■

A threshold policy may appear somewhat counter-intuitive to many organizations. To our knowledge, no empirical studies exist on how resources are actually allocated in practice between mission and revenue customers in non-profit organizations. However, strategies based on proportional allocations may be more appealing to managers. Under such policies, a fixed percentage of the current assets is allocated to the M-customers, so that the organization always contributes to its mission. In contrast, under a thresholds policy, no M-customers are served if  $a_t \leq a^*$ . This may, in the long run, jeopardize the culture of the organization as non-profit oriented, which in turn can affect its ability to recruit volunteers and to provide high service quality to the M-customers, among other concerns (see Dees 1998; Dees and Anderson 2003). From Theorem 2, the optimal threshold policy has a higher social impact than proportional allocation strategies. If this gain is significant, the organization may benefit from obtaining buy-in from its stakeholders on the benefit of a threshold policy, while acting in other ways to maintain a non-profit oriented culture. This is discussed further in Section 4.

Theorem 2 also shows that the optimal price is decreasing in the current assets. However, the theorem does not give any more information regarding the shape of  $p_t^*(\cdot)$ . The following proposition states properties satisfied by  $ap_t^*(a)$ . This quantity represents the total revenue that the organization would generate if all its current assets were allocated to and fully demanded by R-customers. These properties can be of use in developing effective heuristics.

**Proposition 3** *If condition (11) holds,  $ap_t^*(a)$  is increasing in  $a$ . Further,  $\lim_{a \rightarrow 0} ap_t^*(a) = 0$ .*

**Proof.** To show the first part, recall that the derivative of optimal price for the  $\epsilon$ -problems is equal to  $\tilde{p}'(\cdot) = -H_{py}^{v_\epsilon}/H_{pp}^{v_\epsilon}$ . It follows that, using (17), (20) and the fact that  $H_{pp}^{v_\epsilon} \leq 0$ ,

$$\begin{aligned} (y\tilde{p}(y))' \geq 0 &\Leftrightarrow y\tilde{p}'(y) + p(y) \geq 0 \\ &\Leftrightarrow -H_{py}^{v_\epsilon}y + H_{pp}^{v_\epsilon}\tilde{p}(y) \leq 0 \\ &\Leftrightarrow v^{\epsilon l}(py)G(z)(1 - bg(z)) \leq 0 \end{aligned}$$

where the last inequality holds from condition (11) and the result follows from Lemma 4.

To show the second part of the proposition, assume that  $\lim_{a \rightarrow 0} ap_t^*(a) = 2l$  for some positive  $l$ . We can then choose  $a$  small enough such that  $ap_t^*(a) > l$ . Note also that for  $a$  sufficiently small,

$$v_t(a) = \alpha E_\Theta v_{t+1}(p_t^*(a)a \wedge r(p_t^*(a))\Theta)$$

$$\begin{aligned}
&\geq \alpha E_{\Theta} v_{t+1} (l \wedge r(p_t^*(a))\Theta) \\
&\geq \alpha v_{t+1}(l)G(l) \\
&> 0
\end{aligned}$$

It follows that  $\lim_{a \rightarrow 0} v_t(a) > 0$  which leads to a contradiction.

■

Proposition 3 can help develop heuristics by restricting the choice of the price strategy  $p(a)$ . For instance, if the pricing decision is of the form  $ka^{-1/b}$  then  $ap(a) = ka^{(b-1)/b}$  satisfies the properties of Proposition 3 (recall that  $b > 1$ ). The choice of  $k/a^{-1/b}$  as a heuristic is motivated by choosing the price which scales the demand such that the assets correspond to a constant fractile of the distribution. More precisely, a class of heuristics can be defined by the set of price thresholds  $\hat{p}_t, t \in [1..T]$  such that the pricing decision at time  $t$  is given by

$$p_t(a) = \hat{p}_t \left( \frac{\hat{a}_t}{a} \right)^{1/b}, \text{ for } a < \hat{a}_t$$

where  $\hat{a}_t$  can be taken equal to

$$\hat{a}_t = F^{-1}[1 - 1/(\alpha \hat{p}_t)]. \quad (23)$$

This last equation comes from the first order condition at the optimal threshold

$$v'_t(a) = 0 \Leftrightarrow pv'_{t+1}(pa)G(a/\gamma(p)) = 1$$

If we assume that  $\hat{p}_t \hat{a}_t$  is larger than the optimal threshold of period  $t + 1$ , then  $v'_{t+1}(\hat{p}_t \hat{a}_t) = 1$ , and we can deduce (23).

We conclude this section by exploring the infinite horizon case. We extend our results by letting  $T$  approach  $+\infty$ .

**Corollary 3** *If condition (11) holds, the optimal policy for the infinite horizon case is of threshold type. Further, the optimal pricing decision  $p^*(.)$  is non-increasing in the current assets*

**Proof.**

Consider the last period  $T$ . Define  $M$  as  $M = \max_{y \geq 0} \alpha \tilde{H}^{v_T}(y) - y$ . We first show that  $M < +\infty$ . Note that from Theorem 2 the optimal price  $\tilde{p}(y)$  is decreasing and hence converge to a non-negative limit  $l$ . Assume first that  $l > 0$  so that

$$\lim_{y \rightarrow +\infty} \alpha \tilde{H}^{v_T}(y) - y = \alpha r(l) E_{\Theta}[\Theta] - \lim_{y \rightarrow +\infty} y = -\infty.$$

Assume now that  $l = 0$ . We have

$$\lim_{y \rightarrow \infty} \alpha \tilde{H}^{v_T}(y) - y \leq \lim_{y \rightarrow +\infty} (\alpha \tilde{p}(y) - 1)y = -\infty.$$

Furthermore,  $\tilde{H}^{v_T}(y) \leq y\tilde{p}(y)$  with  $\lim_{y \rightarrow 0} y\tilde{p}(y) = 0$  from Proposition 3 so that  $\alpha\tilde{H}^{v_T}(y) - y$  is null for  $y = 0$ . Thus,  $M$  is finite and  $v_{T-1}(a) = a + \tilde{H}^{v_T}(y) - y \leq a + M$ . It follows that

$$\begin{aligned} \alpha\tilde{H}^{v_{T-1}}(y) - y &\leq \alpha E_{\Theta}(py \wedge r(p)\Theta + M) - y \\ &\leq (1 + \alpha)M \end{aligned}$$

where the first equality comes from  $v_{T-1}(a) \leq a + M$  and the second from the definition of  $M$ . The result follows then from a backward iteration on  $t$  (see Theorem 8-15 in Heyman and Sobel (1984)).

■

## 4 Numerical studies

### 4.1 The Value of Optimal Capacity Decisions

We begin by studying the value of optimal capacity decisions, by considering the case where a banking option is not available and the price is fixed in a typical example. The parameters for this example are  $T = 8$ ,  $\alpha = 0.85$ ,  $\theta$  uniform in  $[1, 2]$  and  $p = 8.76$ . The distribution satisfies the conditions for theorem 2, and the price was selected as the price with is optimal above the threshold when a pricing decision is included with the same problem parameters.

The optimal policy for this problem is of threshold type, as shown in corollary 2. Nevertheless, a proportional allocation policy may be appealing because of its simplicity. Such a policy, where the same fraction of assets is allocated to mission customers independently of the asset, is also advantageous in other respects. Serving mission customers in all periods may be important in maintaining the mission culture of the organization, especially if it also includes volunteer staff. We compare the performance of the optimal policy with a proportional allocation policy. This trade-off should be taken in consideration in deciding whether an otherwise sub-optimal policy is to be followed.

In practice, it may be difficult to find the best fraction for the proportional allocation policy, that is, what fraction of  $a$  is allocated to  $y$ . The best value  $y/a$ , in term of expected discounted social impact, will depend on the initial endowment. Figure 2 plots the value function for three different  $y/a$  proportions (where, by definition of fixed-proportion policy, each proportion remains constant throughout all periods). Also plotted is the maximum of the value functions associated with every possible choice for the fixed proportion. This corresponds to the organization making an initial decision on the proportion optimally based on the asset level at beginning of the first period.

Figure 3 compares the value function associated with the optimal policy (where the  $y/a$  ratio can change in each period) with the maximum over different ratios of the value functions for the proportional allocation

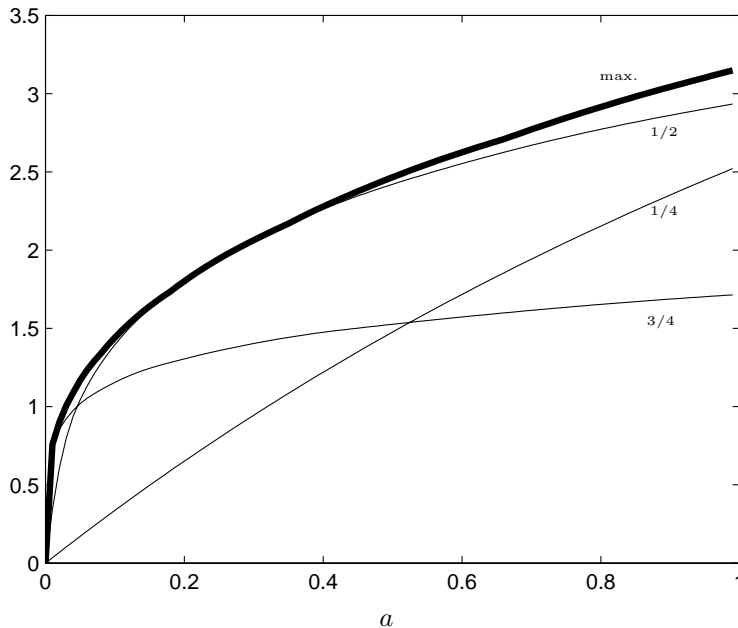


Figure 2: Capacity allocation to each class of customers. Value functions for fixed-proportion policy with different allocation ratios, and maximum over all ratios.

policy. Note that relative difference between the two curves is substantial. For higher values of  $a$  (e.g.,  $a = 1$ ) it is on the order of 30%. Near zero it becomes extremely large (e.g., higher than 100% for  $a < 0.02$ ). The value of the optimal policy relative to the best fixed proportion is very high if the organization starts with few assets. In this case, the optimal policy allows the organization to grow more aggressively and more quickly reach a substantial asset level. This allows the organization to then fully exploit the demand from revenue customers, and use the resulting resources to serve mission customers.

## 4.2 The Value of Banking

We begin by considering the problem with a fixed price and banking decision. Figure 4 shows the optimal policy, for a typical case. The parameters are  $T = 8$ ,  $\beta = 1.15$ ,  $\alpha = 0.85$ ,  $p = 5.56$ , and  $\theta$  uniform in  $[0, 1]$ . In agreement with theorem 1, the optimal policy is of threshold type for  $z + y$ , and  $z$  is non-decreasing. In our numerical examples we have also always found  $y$  to be non-decreasing.

Note that banking becomes attractive when the organization has more assets. For a low asset level, the risk of unmet demand is lower. The organization's focus is then on growing the available cash, and all assets are allocated to servicing mission customers which generates higher expected profit. At a higher asset level, the organization is at a higher risk from realizations with very low demand, and banking is then used to hedge the outcome and avoid transitioning to a very low cash positions.

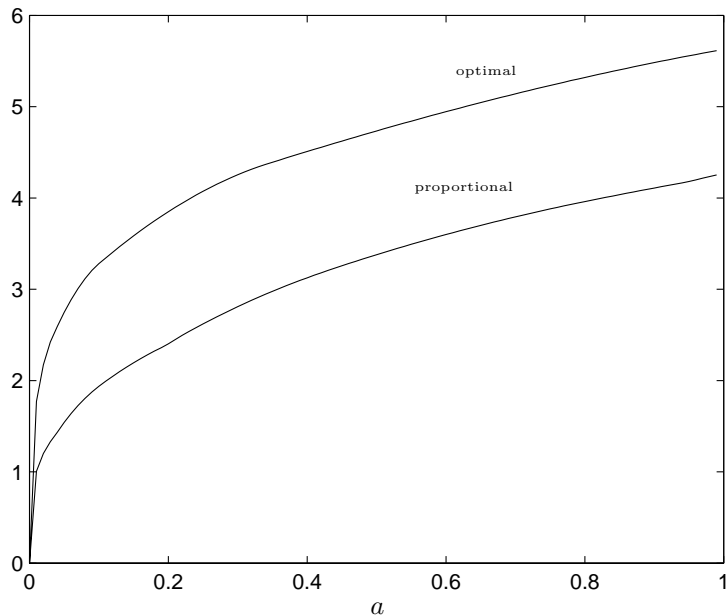


Figure 3: Capacity allocation to each class of customers. Max. of value functions for proportional-allocation policies, and value function for optimal policy.

The value of banking depends on the price being charged. To investigate this we compute the value function for the same return on the banked assets and different prices. We also compute, for the same set of prices, the value function without banking (or, equivalently, with zero return on banking). For each price, we compute the ratio of the two value functions, and the maximum of this ratio over all possible states at the initial period (i.e., the initial asset level). Figure 5 plots this maximum ratio as a function of price, with  $T$ ,  $\beta$  and  $\alpha$  as above. The main observation is that banking is of relatively limited value, providing for a gain of about 4% or less (and, again, note that this is the highest gain over all initial asset levels).

The expected gain from have the banking decision available is not monotonic in the price. Banking is more valuable when allocating capacity to R-customers is profitable, but the profit margins are small. For small prices, R-customers are never profitable and it is optimal to spend all the capital on M-customers in the first period. The option of banking is therefore of no value. At the other end of the plot, when the profit margin is very large, there is also less value in having the banking option available. For higher prices (and same demand distribution) the return rate from banking is relatively less attractive. The organization has less of an incentive to hedge the outcome by banking as, given the high profit margin, it is easier to recover quickly from a period with unusually low demand. (For very large prices, we found the percentage gain to converge to a non-zero value.)

Finally, note that, on the other hand, the value of the option of taking a loan, would be substantial for

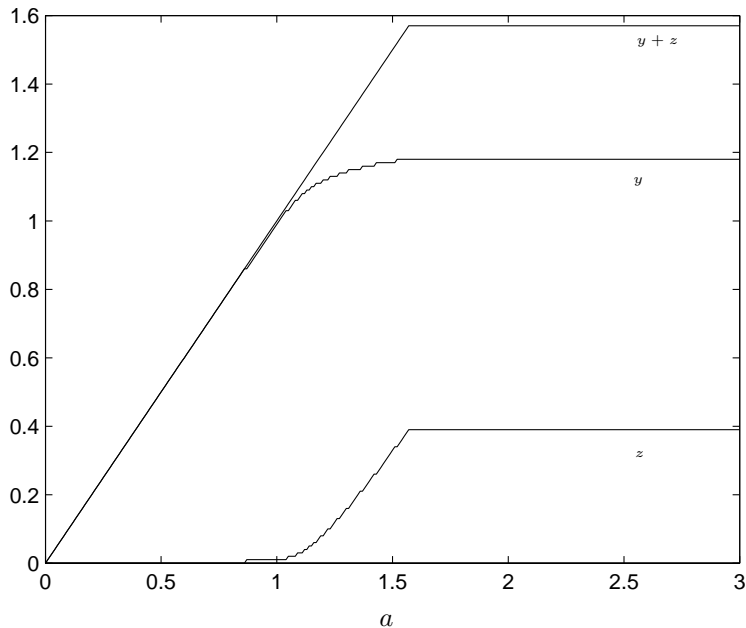


Figure 4: Optimal policy with banking.

low asset-levels (given a reasonable rate). In particular, the value function would no longer be zero at zero net capital.

### 4.3 The Value of Pricing Decisions

We now investigate the value of optimal pricing decision. The model used is that of section 3, that is without the option of banking.

Figure 6 shows a typical example of the optimal policy. The parameters are  $T = 8$ ,  $\beta = 1.15$ ,  $\alpha = 0.85$ , and  $\theta$  uniform in  $[1, 2]$  (which satisfies conditions for theorem 2). The structure of the optimal policy is consistent with theorem 2. The optimal capacity policy is of threshold type, and the optimal pricing policy is non-increasing.

We compare the optimal policy with a fixed-price policy. For the fixed-price policy, we use the price which is optimal above the threshold in the optimal price policy. A number of recent articles investigating different models in for-profit firms have found that addition of dynamic pricing provides only small percentual gains in expected revenue (usually 5% or less; see for instance Gallego and van Ryzin 1994; Chen, Wu, and Yao 2004). However, in this case, and as seen in Figure 7, the difference in expected value can be substantial. This is driven by the fact that the capacity constraint is different from period to period. When the asset level is small, there will be a large amount of unmet demand if the threshold price is charged. Note that the difference in expected value between the two policies is larger for a small initial endowment. This is when

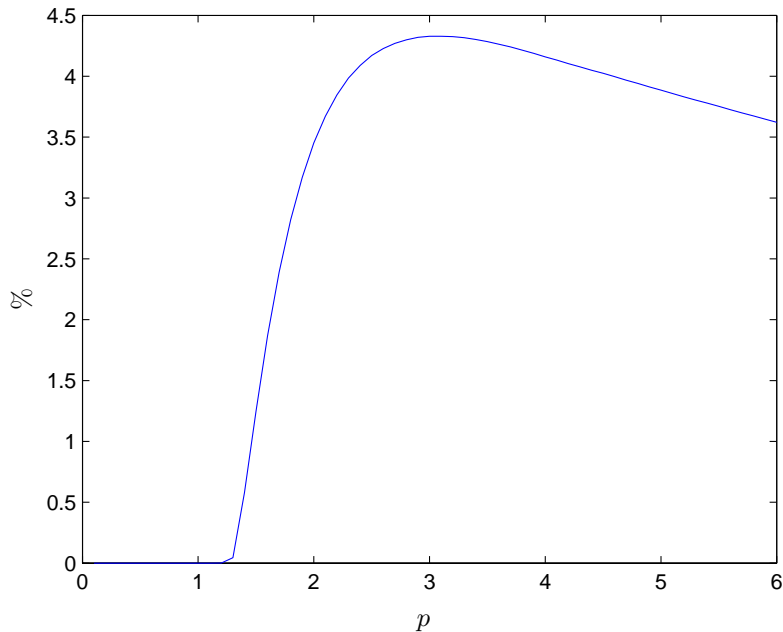


Figure 5: Maximum percentage expected gain from banking. Maximum ratio of value functions over different prices (and same demand distributions).

the organization is sure to benefit from charging a significantly higher price in the initial periods.

Note, however, that if loans are available at a reasonable rate there will be little or no value in dynamic pricing. Even when the asset-level is low, the optimal policy will be to borrow capital in order to be able to provide optimal capacity for the given demand distribution.

## 5 Extensions and Future Research

### 5.1 Other Demand Functions

The previous results are based on Condition (11), which imposes conditions on the distribution  $f$  of the random component  $\Theta$ . It is worth noting however that Condition (11) is used in the proofs only *at the optimal price*. That is, if we assume that for all  $a$

$$g\left(\frac{a}{\gamma(p^*(a))}\right) \geq \frac{1}{b} \tag{24}$$

then Theorem 2 and Proposition 2 still hold.

One approach to extend our results would then be to find sufficient conditions on  $f$  guarantying (24). Note however that showing (24) for more general  $f$  would require propagating properties stronger than concavity on the social impact-to-go  $v_t$ . This has proven to be challenging so far, but we believe that this

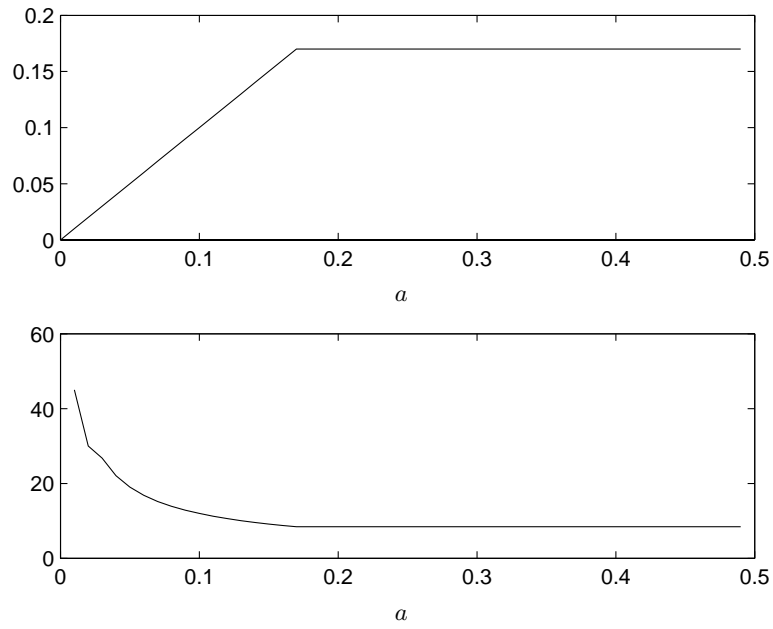


Figure 6: Optimal policy with dynamic pricing ( $y(a)$  and  $p(a)$  for first period).

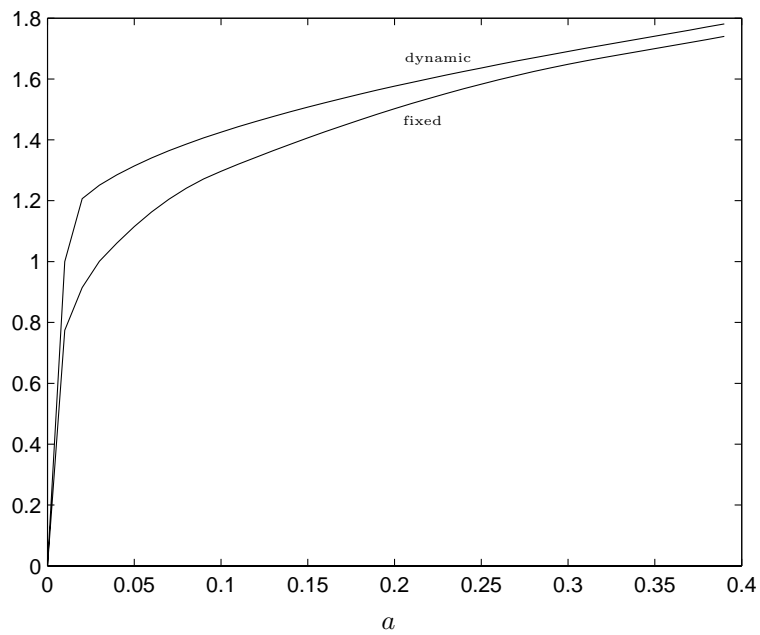


Figure 7: Value functions with dynamic price and with fixed price.

constitutes an interesting direction for future research.

Other natural extensions of the demand function involve the price response  $\gamma(\cdot)$ . We have assumed that  $\gamma(p) = cp^{-b}$ ,  $b > 1$ , but other functions can be considered. If  $\gamma(\cdot) : [\underline{p}, \bar{p}] \mapsto \mathbf{R}$  is differentiable and decreasing ( $\gamma'(\cdot) < 0$ ), and if  $r(p) = p\gamma(p)$  is concave, results similar to the ones of the previous sections can be obtained. More precisely, if the following condition holds,

$$g\left(\frac{a}{\gamma(p^*(a))}\right) \geq -\frac{\gamma(p^*(a))}{p^*(a)\gamma'(p^*(a))}. \quad (25)$$

then Theorem 2 and Proposition 2 can also be shown to hold. As we introduced Condition (11) (which implied (24)), it is possible to impose conditions on  $f$  for (25) to hold. For instance, if  $\gamma(\cdot)$  is linear, i.e.,  $\gamma(p) = q - p$  for  $p \in [0, q]$  and  $\Theta$  is uniformly distributed over  $[\underline{\theta}, \bar{\theta}]$ , then  $\bar{\theta}/\underline{\theta} \leq 2$  implies (25) and the optimal policy is of threshold type with a price decreasing in the current assets. Again, it may be possible to show that (25) holds for more general distributions by propagating other properties of the social impact-to-go.

## 5.2 Objective Functions

Our model assumes that only M-customers contribute to the objective of the organization. In practice however serving R-customers may have an impact on the mission of the organization. This may for instance be the case when the organization's mission involves education- or health-related objectives. We can easily consider this case in our model by introducing a social return  $s_R$  generated by a served R-customer. For instance, for the model with fixed price and banking, the organization contributes to its mission in Period  $t$  by, given the decisions  $(p, y)$ ,

$$(a - y - z) + s_R E_{\Theta} (py \wedge p\Theta). \quad (26)$$

Note that the second term of (26) is concave in  $z$  and all our results can be extended to this case. In particular, the optimal capacity decision and banking decision are of threshold type as in Theorem 1. Furthermore, when  $\beta = 0$ , the optimal policy is still myopic with threshold  $a^* = F^{-1}[1 - 1/(s_R + \alpha p)]$  or, when  $s/c_M$  and  $c_R$  are not normalized to 1,

$$a^* = F^{-1}\left(1 - \frac{c_R s}{c_M s_R + \alpha p}\right).$$

For the model with dynamic pricing of Section 3, when  $T = 2$ , the social impact-to-go becomes  $v_1(a) = \max_{0 \leq y \leq a} a - y + (s_R + \alpha)\tilde{H}^{vT}(y)$  and Theorem 2 can be extended to this case. On the other hand, when  $T > 2$ , the term  $s_R E_{\Theta} (py \wedge r(p)\Theta)$  may not be concave along the optimal price  $p_t^*(\cdot)$ , and the analysis becomes significantly more difficult.

Finally, we have assumed so far a linear social return  $s x_t$ , where  $x_t$  is the capacity allocated to M-customers in Period  $t$ . A natural extension of our model is to the case of decreasing marginal social return, where the impact of serving  $x$  mission customers is described by a concave function  $s(\cdot)$ . A special case

of particular interest corresponds to the situation where demand for M-customers is also random. If the demand from M-customers is described by a random variable  $\Phi$ , the organization contributes to its mission by  $s(x) = E_{\Phi} (x \wedge \Phi)$ . The optimal policy is however no longer of threshold type. Nonetheless, it should be possible to derive simple and efficient heuristics for this case, based on the results presented in this paper.

## 6 Conclusion

The problem faced by a non-profit organization that also runs a for-profit operation to generate resources differs in fundamental ways from the profit-maximization problem of commercial ventures. To our knowledge, we provide the first analytical approach for examining resource management issues in this context. This is likely to become a more topical question, as a decrease in the number of grants and donations has put increased pressure on non-profit organizations to become self-sustaining. As noted in the introduction, a recent study (Bradley, Jansen, and Silverman 2003) estimates that the nonprofit sector could earn an additional USD 100 billion from improved management processes in for-profit activities. There has been, as far as are aware, little or no research on this subject in the field of operations management, and we believe the approach proposed in this paper to be a productive initial framework to pursue such work.

We have investigated how a nonprofit organization should dynamically allocate its assets over time, between its revenue generating activities and its mission, in order to maximize the organization's social impact. Theoretical analysis and numerical studies suggest that there is limited value in the option of banking assets from one period to the next. Dynamic pricing in the for-profit side of the operation, on the other hand, can in some circumstances be of significant value.

This paper does not address strategic infrastructure investment decisions. Our concern is to provide insight into operational decisions. We show that the optimal capacity allocation policy is of threshold type. In practical terms this means that, should an adverse cash situation arise in a given period which limits the organization's ability to provide services, it is best for the organization not to compromise its revenue source. If the asset-level is low, the organization should first re-build its asset-base by servicing exclusively revenue customers, and only then act towards its mission. Over all periods, this policy will allow the organization to have a higher expected social impact, that is to serve more mission customers. While this needs to be balanced with other considerations such as maintaining the organization's mission culture (which will likely become an issue if the organization only services revenue customers for several consecutive periods), the analysis allows us to quantify the trade-off.

## A Appendix

**Lemma 5** Consider  $\varphi(\cdot, \cdot) : ]\underline{x}, \bar{x}[ \times ]\underline{y}, \bar{y}[ \mapsto \mathbf{R}$ . Assume that  $\varphi$  is infinitely differentiable and that  $\varphi(\cdot, y)$  is strictly concave for all  $y$ .

Then, for all  $y \in ]\underline{y}, \bar{y}[$ ,  $\tilde{x}(y) = \operatorname{argsup}_x \varphi(x, y)$  exists and is unique.

Further if locally at  $(\tilde{x}(y), y)$  for all  $y \in ]\underline{y}, \bar{y}[$ ,  $\varphi$  is infinitely differentiable and strictly concave, then  $\tilde{\varphi}(x) = \sup_y \varphi(x, y) = \varphi(\tilde{x}(y), y)$  is infinitely differentiable and strictly concave.

**Proof.** Note that  $\tilde{x}$  is infinitely-differentiable (using the derivative of the implicit function on the first-order condition,  $\tilde{x}' = -\varphi_{xy}/\varphi_{xx}$ ). The derivatives of  $\tilde{\varphi}$  are

$$\tilde{\varphi}'(y) = \tilde{x}'(y)\varphi_x(\tilde{x}(y), y) + \varphi_y(\tilde{x}(y), y) = \varphi_y(\tilde{x}(y), y),$$

where we use the first order condition in  $\tilde{x}$  that is  $\tilde{x}'(y) = 0$ ,

and

$$\begin{aligned} \tilde{\varphi}''(y) &= \tilde{x}''(y)\varphi_x(\tilde{x}(y), y) + (\tilde{x}'(y))^2\varphi_{xx}(\tilde{x}(y), y) + 2\tilde{x}'(y)\varphi_{xy}(\tilde{x}(y), y) + \varphi_{yy}(\tilde{x}(y), y) \\ &= (\tilde{x}'(y))^2\varphi_{xx}(\tilde{x}(y), y) + 2\tilde{x}'(y)\varphi_{xy}(\tilde{x}(y), y) + \varphi_{yy}(\tilde{x}(y), y) \\ &< -\left(\tilde{x}'\sqrt{-\varphi_{xx}(\tilde{x}(y), y)} - \sqrt{-\varphi_{yy}(\tilde{x}(y), y)}\right)^2 \\ &< 0, \end{aligned}$$

where we used strict concavity at  $(\tilde{x}(y), y)$ , that is

$$\begin{aligned} \varphi_{xx}(\tilde{x}(y), y)\varphi_{yy}(\tilde{x}(y), y) - \varphi_{xy}^2(\tilde{x}(y), y) &> 0, \varphi_{xx}(\tilde{x}(y), y) < 0, \varphi_{yy}(\tilde{x}(y), y) < 0 \\ \Rightarrow \varphi_{xy}(\tilde{x}(y), y) &< \sqrt{-\varphi_{xx}(\tilde{x}(y), y)}\sqrt{-\varphi_{yy}(\tilde{x}(y), y)}. \end{aligned}$$

■

## Acknowledgments

This work was supported in part by Fuqua's Center for the Advancement of Social Entrepreneurship. The authors are grateful to Dinah Vernik for her work on the numerical examples. We also thank Gregory Dees, Beth Anderson, James Smith, Gregory DeCroix, and the participants at Fuqua's Operations Management Seminar for useful comments.

## References

- Babich, V. and M. J. Sobel (2004). Pre-ipo operational and financial decisions. *Management Science* 50(7).
- Bernstein, F. and A. Federgruen (2005). Decentralized supply chains with competing retailers under demand uncertainty. *Management Science* 51(1), 18–29.
- Billingsley, P. (1995). *Probability and Measure* (3 ed.). John Wiley & Sons.
- Bonnans, F. and A. Shapiro (2000). *Perturbation analysis of optimization problems*. Springer.
- Bradley, B., P. Jansen, and L. Silverman (2003). The nonprofit sector’s \$100 billion opportunity. *HBR May*.
- Buzacott, J. and R. Zhang (2001). Inventory management with asset-based financing. *Management Science* 50(9).
- Chen, H., O. Wu, and D. D. Yao (2004). Optimal pricing and replenishment in a single-product inventory system. *Working paper*.
- Dees, G. (1998). Entreprising non-profit. *Harvard Business Review, January-February*.
- Dees, G. and B. Anderson (2003). Sector-bending: Blurring the lines between nonprofit and for-profit. *Society* 40(4).
- Elmaghraby, W. and P. Keskinocak (2003). Dynamic pricing in the presence of inventory considerations: Research overview, current practices and future directions. *Management Science* 10(7).
- Elstrodt, H.-P., A. Schindler, and A. C. Waslander (2004). Helping nonprofits do business. *The McKinsey Quarterly, Web exclusive December*.
- Federgruen, A. and A. Heching (1999). Combined pricing and inventory control under uncertainty. *Operations Research* 47(3).
- Gallego, G. and G. van Ryzin (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science* 40, 999–1020.
- Heyman, D. and M. Sobel (1984). *Stochastic Models In Operations Research (II)*. Dover Publications.
- Li, L., M. Shubik, and M. J. Sobel (2003). Control of dividends, capital subscriptions, and physical inventories. Technical report, Yale University and Case Western Reserve University.
- Modigliani, F. and H. Miller (1958). The cost of capital, corporation finance, and the theory of investment. *American Economic Review* 48, 261–267.
- Pande, T. (1998). A passion for restoring lost vision. *India Abroad (New York edition)* XXIX(1).
- Petruzzi, N. C. and M. Dada (1999). Pricing and the newsvendor problem: A review with extensions. *Operations Research* 47(2), 183–194.
- Sinitsyn, M. and B. Weisbrod (2002). Behavior of nonprofit organizations in for-profit markets. Technical report. working paper, Northwestern University, Department of Economics.
- Sobel, M. (1984). Myopic solutions of markovian decision processes and stochastic games. *Operations Research* 26.
- Steinberg, R. and B. Weisbrod (1997). Price discrimination and rationing by nonprofit organizations with distributional objectives. Technical report. working paper, Northwestern University, Department of Economics.
- Wang, Y., L. Jiang, and Z. Shen (2004). Channel performance under consignment contract with revenue sharing. *Management Science* 50(1).

Weisbrod, B. (1998). *To Profit or Not to Profit: the Commercial Transformation of the Nonprofit Sector*. Cambridge University Press.