# The English Auction is Optimal among Simple Sequential Auctions

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## Abstract

With private and affiliated buyers' values, the English auction maximizes the seller's expected profit within a large family of sequential bidding mechanisms, named "Simple Sequential Auctions." *Journal of Economic Literature* Classification Numbers: D44, D82.

# 1 Introduction

The English auction is often used to determine the terms of trade when the owner of a single object faces a number of potential buyers. In this paper we try to reconcile this evidence with the assumption that the trading mechanism is chosen aiming at maximizing the seller's expected profit.

Myerson [14] and Riley and Samuelson [15] have shown that, if the buyers are risk neutral and have independently and identically distributed private values, then, for a large class of probability distributions, the seller's expected profit is maximized, among all feasible selling procedures, by any mechanism such that i) the object is awarded to one of the buyers with the highest value, and ii) any buyer whose value is lower than an optimally set threshold receives zero expected surplus. These two conditions, however, are satisfied by a large family of trading mechanisms, including each of the "standard" auctions, i.e. the English, the Dutch, the first-price and the second-price auction, all with the same seller's reserve price.[1] Therefore this result provides only a "weak" explanation of the popularity of the English auction in terms of its optimality for the seller, even if the independence and risk neutrality assumptions do not seem too restrictive.

In a more general model, in which the buyers' values are neither necessarily private nor independently distributed, Milgrom and Weber [13] have shown that the English auction (more precisely, the 'irrevocable exit' version of it, where the bidders observe who drops out at what price, and cannot reenter the auction once they drop out[2]) dominates all other standard auctions in terms of seller's expected profit, when the comparison is restricted to symmetric equilibria. In their "general symmetric model," the value of each buyer depends on his privately observed "signal," and possibly on his opponents' private signals. Moreover, the signals' joint probability distribution satisfies the affiliation property which is, essentially, equivalent to non-negative correlation on any subset of the support. Under these assumptions, Milgrom and Weber have shown that the symmetric equilibria of the standard auctions can be ranked in terms of the seller's expected profit: the English auction does at least as well as the second-price auction, which, in turn, does weakly better than the first-price auction.[3]

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[1] The rules of these auctions are as follows. The first-price and the second-price auctions are sealed-bid auctions: the buyers submit their bids simultaneously, and the object is awarded to one of the buyers who has submitted the highest bid. He, the "winner," pays his bid in the first-price auction, and the second highest bid in the second-price auction. In the English auction, the price is raised, either by the auctioneer or by the bidders themselves, up to a level where only one bidder is still willing to buy; he then is awarded the object for this price. In the Dutch auction, the price is set initially at a high level, and is lowered continuously until one bidder "stops the clock" thus buying the object at the current price.

[2] See Bikhchandani and Riley [3] for a classification of various versions of the English auction.

[3] The first inequality is strict if and only if i) the affiliation is strict, ii) there are more than two buyers and iii) the buyers' values are non-private. The second inequality is strict if and only if the affiliation is strict. The Dutch auction is strategically equivalent to the first-price auction.

The English auction, however, *does not* maximize the seller's expected profit among *all* incentive-feasible trading mechanisms, if the buyers' signals are strictly affiliated: there are incentive-compatible mechanisms in which the object is always assigned to one of the buyers with the highest signal, who pays, on average, his full value. Thus, for any degree of positive affiliation, the seller is able to extract, on average, all gains from trade. (This result is due to Crèmer and McLean [4] for the case in which the distribution of the buyers' signals has finite support. McAfee and Reny [10] have established a "near-full extraction" result for the continuous case, under mild regularity conditions.) It has been pointed out that these results hinge critically on the assumption that all buyers are risk neutral, and that this assumption loses its plausibility as the degree of the signals' affiliation approaches zero, because optimal monetary transfers become arbitrarily large. But, with risk averse buyers, the seller can use risk as a screening device: mechanisms in which riskier allocations are assigned to buyers with lower signals generate more seller's expected revenue than the English auction, because they reduce the expected surplus of buyers with higher values (Matthews [8] and Maskin and Riley [7].)

In light of all the above results, it has been argued that "mechanism design has no reasonable explanation of the usual auction forms, and that some other criteria must be invoked," (McAfee and Reny [10].) These criteria should aim at capturing some notion of "robustness," or "simplicity," because, while in any of the standard auctions trade is regulated by a few simple rules, the optimal selling mechanisms are in general sensitive to fine details of the model, such as the shape of the signals' distribution or the curvature of the buyers' utility function.[4]

This paper proposes a notion of simplicity which restricts the set of feasible selling procedures to a class of sequential bidding mechanisms, named *Simple Sequential Auctions* (SSAs). The main result is that, with private and affiliated values, the Perfect Bayesian equilibrium with undominated strategies (UPBE) of the English auction maximizes the seller's expected revenue among the (essentially unique) UPBEs of all SSAs.

The SSAs are open bidding procedures in which the buyers face history-dependent sequences of "bid" and "ask" prices: at any stage during the auction, a buyer has only three options: he can purchase the object for the current ask, remain in the auction by declaring his willingness to pay the current bid, or drop out irrevocably. This class of auctions includes, as special cases, approximations of both the (irrevocable exit) English auction and the Dutch auction. In SSAs that approximate the English (Dutch) auction, all ask (bid) prices are kept outside the range of all buyers' possible values, and each buyer faces the same bid (ask) price at any stage, independent of all previous actions. In generic SSAs however both the ask and the bid price at any stage may depend on the number of buyers that are still in the auction, as well as on the times at which buyers have dropped out.

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[4] "What is perhaps most missed in the theory of optimal auctions is some indication of which institutions for selling an object are robust — that is optimal or nearly so in a range of environments [...] Also missing is some formalization of the idea that auctions are "simple" — for example, in most auctions, all the bids that can be made actually are made with positive probability at equilibrium in several simple environments." Milgrom [12].

From the point of view of a seller aiming at maximizing her expected profit, the crucial restriction of the SSAs is that, at any stage of the auction, the buyers choose their *actual* payment, from a given set, *conditional on being awarded the object*. Hence the payment of any buyer may depend on his opponents' past, but not simultaneous or future actions. This restriction corresponds to a notion of robustness with respect to perturbations in each buyer's beliefs about his opponents' behavior.

The implications for the optimality of the English auction of two other notions of simplicity and robustness have been explored in Lopomo [6]. First, the class of all equilibria of selling mechanisms identified by the sole restriction that "losers do not pay" is taken as the seller's feasible set. This class contains all equilibria of all SSAs, as well as of all standard auctions; and corresponds to a weaker notion of robustness with respect to perturbations in each buyer's beliefs about his opponents' behavior. It is shown, however, that, due to the possibility of making the winner's payment contingent on his opponents' simultaneous bids, any auction in a family of sealed-bid auctions, named "*b*-composite auctions," does strictly better for a risk-neutral seller than the English auction.

Lopomo [6] also considers a stronger notion of robustness which entails restricting the seller's feasible set to all mechanisms in which neither the buyers' strategies nor the trading rules depend on the signals' distribution. It is shown that, in this set, the English auction does maximize the seller's expected profit. The attractive feature of this robustness notion is that the seller can implement any mechanism in this set without any confidence in the assumption that the joint distribution of the buyers' signals is common knowledge. This restriction however may appear too severe, if the common prior assumption is accepted, mainly because it excludes the first-price auction.

This paper proceeds as follows. Section 2 enunciates the assumptions. Section 3 defines the class of all Simple Sequential Auctions. Section 4 shows that, in any SSA, there is essentially a unique perfect Bayesian equilibrium in undominated strategies, and states the optimality result for the English auction. All the proofs are relegated to the appendix.

## 2 The Model

The owner of an indivisible object faces $n$ risk-neutral potential buyers. The utility function of buyer $i \in N \equiv \{1, \ldots, n\}$ is

$$\theta_i \cdot Q_i - M_i,$$

where $\theta_i$ denotes his value for the object, $Q_i$ the probability of being awarded the object, and $M_i$ his expected payment to the seller. The $n$-dimensional vector of buyers' values $\theta = (\theta_1, \ldots, \theta_n)$ (which we also refer to as "signals" or "types") is drawn from a symmetric probability distribution, whose density $f$ is strictly positive in $\Theta^n$, where $\Theta \equiv [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$, and $\underline{\theta} < \overline{\theta}$. In addition, the density $f$ satisfies the following assumptions:

**A1:** *Affiliation*:

$$f(\theta) \cdot f(\theta') \leq f(\theta \vee \theta') \cdot f(\theta \wedge \theta'), \quad \text{all } \theta, \theta' \in \Theta^n,$$

where $\theta \vee \theta'$ and $\theta \wedge \theta'$ denote the component-wise maximum and minimum of $\theta$ and $\theta'$;

**A2:** *Monotonicity of the hazard ratios*:

$$\frac{1 - G(\theta_i|\theta_{-i})}{g(\theta_i|\theta_{-i})} \text{ is nonincreasing in } \theta_i, \quad \text{for all } \theta_{-i} \in \Theta^{n-1},$$

where $G(\cdot \mid \theta_{-i})$ denotes the c.d.f., and $g(\cdot \mid \theta_{-i})$ the density of $\theta_i$, conditional on the $(n-1)$-dimensional vector $\theta_{-i} \equiv (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$; and

**A3:** *No need for reserve prices*: $\underline{\theta} \cdot g(\underline{\theta} \mid \overline{\theta}, ..., \overline{\theta}) \geq 1$.

Assumption A1 is standard in auction theory (Milgrom and Weber [13].) It is used in section 4, both in Proposition 1, which characterizes the equilibrium set of any SSA, and in Proposition 2 which establishes the optimality result for the English auction.

Assumption A2 extends the standard 'regularity' condition of the hazard rate to all conditional distributions, and is used to guarantee that, given the constraints implied by the feasible set, the seller does not benefit from using stochastic selling procedures.

Finally, assumption A3 is made to simplify the derivation of the results, by removing the need for reserve prices: together with A1 and A2, it implies that it is never optimal for the seller to keep the object.[5] Without assumption A3, the seller's expected profit would be maximized by an English auction in which the seller sets the reserve price after $n-1$ bidders have dropped out.[6]

## 3 Simple Sequential Auctions

Formally, a simple sequential auction is a "multi-stage game with observed actions and incomplete information," as defined in Fudenberg and Tirole [5], with affiliated bidders' types. The defining feature is that, at any given stage, the actions taken by all players in all previous stages are common knowledge. Any SSA lasts at most a finite number, $R+1$, of "rounds," with each of the first $R$ rounds consisting of $n$ stages, and the last round consisting of $n-1$ stages. The total number of

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[5] A3, which is equivalent to $\underline{\theta} - \frac{1}{g(\underline{\theta} \mid \overline{\theta}, ..., \overline{\theta})} \geq 0$, together with A1 and A2, implies that, for any type profile $(\theta_1, ..., \theta_n) \in \Theta^n$, both buyer $i$'s "virtual utility" $\theta_i - \frac{1 - G(\theta_i \mid \theta_{-i})}{g(\theta_i \mid \theta_{-i})}$, $i \in N$, is nonnegative, and buyers with the higher values have the higher virtual utilities. As will be shown in the appendix (formula 13) this implies the optimality for the seller of always awarding the object to (one of) the buyer(s) with the highest value.

[6] With nonindependent values, it is optimal to set the reserve price conditional on the realizations of the lowest $n-1$ signals (Lopomo [6]). Ashenfelter [1] reports that significant fractions of all items put up for sale in auctions of various goods are "bought in," i.e., are withdrawn from sale after all buyers have stopped bidding.

stages is thus $T \equiv nR + n - 1$. For each stage $t = 1, ..., T$, the rules of the auction specify an "ask" price $a_t$, a "bid" $b_t$, and the identity $\iota(t)$ of the "active" bidder, that is, the only bidder whose action set is different from the singleton $\{do\ nothing\}$. In the last round, the ask and the bid coincide: $a_{nR+j} = b_{nR+j}$, for $j = 1, ..., n-1$.

The bidders move in rotation. In the $i$-th stage of each round buyer $i$ is the active bidder. Thus $\iota(t) \equiv t - n(r_t - 1) \in N$, where $r_t$ denotes the round to which stage $t$ belongs: $r_{(k-1)n+1} \equiv ... \equiv r_{kn} \equiv k$, for $k = 1, ..., R+1$. The action set of buyer $\iota(t)$ is denoted by $S_t$, and is determined by the history of previous actions $h^t \equiv (s_1, ..., s_{t-1}) \in S_1 \times ... \times S_{t-1}$ as follows. In the first round,

$$S_i = \{\alpha, \beta, x\}, \quad i \in N,$$

where $\alpha$ indicates accepting to buy the object for the current ask, $\beta$ denotes "subscribing the bid," i.e., committing to pay at least the current bid for the object, and $x$ indicates dropping out of the auction. From round 2 to round $R$, i.e. for $t = n+1, ..., nR$,

$$S_t\left(h^t\right) = \begin{cases} \{\alpha, \beta, x\} & \text{if } h^t \in \{\beta, x\}^{t-1}, \text{ and } s_{t-n} = s_{t-j} = \beta \text{ for some } j = 1, ..., n-1; \\ \\ \{x\} & \text{otherwise.} \end{cases}$$

That is, the active bidder $\iota(t)$ can still buy the object for the current ask, subscribe the current bid, or drop out, as long as he has not dropped out in a previous stage, no bidder has accepted the ask in a previous stage, and at least one other bidder has subscribed all his previous bids.

Finally, in the last round (in which the asks and bids coincide)

$$S_t\left(h^t\right) = \begin{cases} \{\alpha, x\} & \text{if } h^t \in \{\beta, x\}^{t-1}, \text{ and } s_{t-n} = s_{nR-j} = \beta \text{ for some } j = 0, ..., T-t; \\ \\ \{x\} & \text{otherwise.} \end{cases}$$

for $t = nR+1, ..., T$.

Thus the auction may end *de facto* in any stage $t = 1, ..., T$: that is, $S_{t'} = \{x\}$ for all $t' = t+1, ..., T$. This happens in stage $i < n$ only if buyer $i$ accepts to buy the object for the ask $a_i$. In stage $t > n$ the auction may end in two ways: either the active bidder accepts the ask $a_t$, or he drops out and only one of the other buyers has not dropped out in a previous stage. In this last case, the object is sold to the buyer who has not dropped out, for the last bid that he has subscribed. Finally, in stage $n$ the auction may end in three ways: buyer $n$ accepts the ask $a_n$; buyer $n$ is the $n$-th buyer to drop out, in which case buyer 1 receives the object and pays $\underline{\theta}$; and buyer $n$ drops out and only one buyer $j < n$ has subscribed his bid, in which case he receives the object and pays his bid $b_j$.

In each stage $t = 2, ..., T$, both the bid $b_t$ and the ask $a_t$ may depend on the history $h^t$. Abusing notation again, as we have done for the action sets $S_t$, we write $b_t\left(h^t\right)$ and $a_t\left(h^t\right)$. Each bidder, however, always faces an increasing sequence of bids and a nonincreasing sequence of asks, i.e.

$$b_{t-n}\left(h^{t-n}\right) < b_t\left(h^t\right), \quad \textit{and} \quad a_{t-n}\left(h^{t-n}\right) \geq a_t\left(h^t\right), \quad h^t \in H^t, \ \ t = n+1, ..., T,$$

where $h^1$ is given, and $H^t \equiv \left\{(s_1, ..., s_{t-1}) \in \{\alpha, \beta, x\}^{t-1} \mid s_{t'} \in S_{t'}(s_1, ..., s_{t'-1}), \ 1 < t' < t\right\}$ is the set of all possible histories after $t$ stages. Also, $\underline{\theta} < b_i < a_i$ for all $i \in N$, and $a_{nR}\left(h^{nR}\right) > b_{nR}\left(h^{nR}\right)$ for all $h^{nR} \in H^{nR}$. These assumptions, together with the fact that bids and asks coincide in the last round, guarantee that $a_t\left(h^t\right) > b_t\left(h^t\right)$, for all $h^t \in H^t$ and $t = 1, ..., nR - 1$. Finally, each ask function $a_t(\cdot)$, $t = 1, ..., T$, is nonincreasing in the order $x < \beta$: that is, $a_t\left(\widehat{h}^t\right) \leq a_t\left(h^t\right)$ for any two histories $h^t \equiv (s_1, ..., s_{t-1})$ and $\widehat{h}^t \equiv (\widehat{s}_1, ..., \widehat{s}_{t-1})$ such that $s_{t'} \neq \widehat{s}_{t'}$, $t' < t$, implies $s_{t'} = x$ and $\widehat{s}_{t'} = \beta$.

Any simple sequential auction in which all bids remain close to the lowest type $\underline{\theta}$, and all buyers $i \in N$ face the same sequence of ask prices

$$a_t\left(h^t\right) = a_{r_t}, \ \text{ all } h^t \in H^t, \quad t = 1, ..., T,$$

approximates the Dutch auction. Any SSA in which all asks are above the highest type $\overline{\theta}$, and all buyers $i \in N$ face the same sequence of bids

$$b_t\left(h^t\right) = b_{r_t}, \ \text{ all } h^t \in H^t, \quad t = 1, ..., T,$$

approximates an (irrevocable exit) English auction with discrete price increments. In generic SSAs however, both the ask $a_t$ and the bid $b_t$ change with the history $h^t$, at each stage $t = 1, ..., T$.

## 4 Results

In any SSA, a *(behavior) strategy profile* $\sigma$ is a function that specifies a conditional probability distribution $\sigma^t\left(\cdot|\tau, h^t\right)$ over the action set $S_t\left(h^t\right)$, for each type $\tau \in \Theta$ of the active bidder $\iota(t)$, each stage $t = 1, ..., T$, and each history $h^t \in H^t$. A *system of beliefs* $\mu$, specifying a probability distribution $\mu^t\left(\theta|h^t\right)$ over $\Theta^n$ for each stage $t = 1, ..., T$, and each history $h^t = (s_1, ..., s_{t-1}) \in H^t$, is consistent with $\sigma$, if, whenever possible, it is determined by $\sigma$ together with the prior distribution $F$, via Bayes' rule. That is, $\mu^1\left(\theta|h^1\right) \equiv F(\theta)$ and, whenever the denominator is positive,

$$\int_A d\mu^{t+1}\left(\theta|h^{t+1}\right) = \frac{\displaystyle\int_A \sigma^t\left(s_t|\theta_{\iota(t)}, h^t\right) \ d\mu^t\left(\theta|h^t\right)}{\displaystyle\int_{\Theta^n} \sigma^t\left(s_t|\theta_{\iota(t)}, h^t\right) \ d\mu^t\left(\theta|h^t\right)}, \qquad t = 1, ..., T-1, \tag{1}$$

for any measurable subset $A \subset \Theta^n$.

For any pair $(\sigma, \mu)$, such that $\mu$ is consistent with $\sigma$, and for any stage $t$ and history $h^t \in H^t$, let $V^t\left(\tau, h^t\right)$ denote the expected payoff of bidder $\iota(t)$ from subscribing his current bid $b_t$, after history

$h^t$, conditional on having type $\tau$. Also let $\underline{h}^t \equiv (s_{t+1}, ..., s_{t+n-1})$ denote the partial history between stages $t+1$ and $t+n$, and, abusing notation, let $V^{t+n}\left(\tau, h^t, \beta, \underline{h}^t\right) \equiv V^{t+n}\left(\tau, h^{t+n}\right)$. If $(\sigma, \mu)$ is a perfect Bayesian equilibrium, then $\mu$ is consistent with $\sigma$, and

$$\begin{aligned} V^t\left(\tau, h^t\right) &= (\tau - b_t) \cdot \Pr\left[\underline{h}_x^t | \tau, h^t\right] \\ &\quad + \sum_{\left\{\underline{h}^t | \left(h^t, \beta, \underline{h}^t\right) \in H^{t+n}\right\}} \max\left\{0,\, \tau - a_{t+n},\, V^{t+n}\left(\tau, h^t, \beta, \underline{h}^t\right)\right\} \cdot \Pr\left[\underline{h}^t | \tau, h^t\right], \end{aligned}$$

for $t = 1, ..., nR$, where $\underline{h}_x^t \equiv (x, ..., x)$, $V^{nR+i}\left(\tau, h^{nR}, \beta, \underline{h}^{nR+i}\right) \equiv 0$ for all $i = 1, ..., n-1$, and

$$\Pr\left[\underline{h}^t | \tau, h^t\right] \equiv \int_{\Theta^{n-1}} \prod_{j=1}^{n-1} \sigma^{t+j}\left(s_{t+j} | \theta_{\iota(t+j)}, h^{t+j}\right) d\mu^t\left(\theta_{-\iota(t)} | \tau, h^t\right).$$

We are now ready to state the first result. Proposition 1 below establishes that, in each SSA, the set of Perfect Bayesian equilibria with undominated strategies (UPBEs) is essentially a singleton: in any UPBE, at any stage $t = 1, ..., nR$, the active bidder drops out if his value is below the bid $b_t\left(h^t\right)$, accepts the ask $a_t$ if his value is above a history-dependent threshold $w_t\left(h^t\right)$, and subscribes the bid if his value is between $b_t\left(h^t\right)$ and $w_t\left(h^t\right)$. Thus, for each $i \in N$, all types $\theta_i$ of buyer $i$ that are not equal to $b_t\left(h^t\right)$ or $w_t\left(h^t\right)$ for all $t \in \iota^{-1}(i)$, have a unique UPBE strategy.

**Proposition 1** *Suppose that $(\sigma, \mu)$ is a UPBE of a Simple Sequential Auction, and that assumption A1 (affiliation) holds. Then, at any stage of the last round $t = nR+1, ..., T$, for any history $h^t \in H^t$,*

$$\sigma^t\left(x | \theta_{\iota(t)}, h^t\right) = 1 \qquad \text{if } \underline{\theta} \leq \theta_{\iota(t)} < a_t\left(h^t\right), \tag{2}$$

*and*

$$\sigma^t\left(\alpha | \theta_{\iota(t)}, h^t\right) = 1 \qquad \text{if } a_t\left(h^t\right) < \theta_{\iota(t)} \leq \overline{\theta}. \tag{3}$$

*Moreover, for any stage $t = 1, ..., nR$, for any history $h^t \in H^t$,*

$$\sigma^t\left(x | \theta_{\iota(t)}, h^t\right) = 1 \qquad \text{if } \underline{\theta} \leq \theta_{\iota(t)} < b_t\left(h^t\right), \tag{4}$$

$$\sigma^t\left(\beta | \theta_{\iota(t)}, h^t\right) = 1 \qquad \text{if } b_t\left(h^t\right) < \theta_{\iota(t)} < w_t\left(h^t\right), \tag{5}$$

$$\sigma^t\left(\alpha | \theta_{\iota(t)}, h^t\right) = 1 \qquad \text{if } w_t\left(h^t\right) < \theta_{\iota(t)} \leq \overline{\theta}, \tag{6}$$

*where $w_t\left(h^t\right) \equiv \inf\left\{\tau \in \left[a_t\left(h^t\right), \overline{\theta}\right] \mid V^t\left(\tau, h^t\right) \leq \tau - a_t\right\}$.*

Claims (2) and (3) in Proposition 1 pertain to the last round. They simply state that, in each stage, all types of the active bidder above the ask price must accept, while all those below must drop out. Claims (4) to (6) pertain to all previous rounds and determine the partition of type set $\left[\underline{\theta}, \overline{\theta}\right]$ in the three intervals, for the active buyer $\iota(t)$, that has been described above. This result follows from a single crossing property: the payoff from accepting the ask increases with slope 1 in the bidder's type, while the expected payoff from subscribing the bid increases at a lower rate, since the bidder may lose the object in the future and, with affiliation, higher types assign a higher probability to higher opponents' types, hence to the possibility of losing the object.

We now turn to the optimality result. Proposition 2 below shows that the *continuous* version of the English auction (i.e., where the price rises continuously with time) maximizes the seller's expected revenue among all SSAs. The proof of this result, given in the appendix, consists of two steps. First, it is shown that the UPBEs of all SSAs share the following three properties: i) any buyer with the lowest value $\underline{\theta}$ earns a nonnegative surplus, ii) the winner's payment in nondecreasing in his value, and iii) if buyer $i$ wins when $\theta = (\theta_i, \theta_{-i})$, he also wins when $\theta = (\theta_i', \theta_{-i})$ with $\theta_i' > \theta_i$, for any $\theta_{-i} \in \Theta^{n-1}$. Second, it is shown that the outcome function of the continuous English auction maximizes the seller's expected revenue among *all* interim incentive compatible outcome functions that satisfy conditions i), ii) and iii) above. The optimality of the continuous English auction is thus established in a set of incentive compatible outcome functions which contains the UPBE outcomes of all SSAs.

Formally, however, the continuous English auction is outside the class of all SSAs. This raises the question of whether any SSA can generate the same seller's expected revenue as the continuous version of the English auction. The next lemma provides the following answer: a sequence of SSAs $\left\{SEA^R\right\}_{R=1}^{\infty}$ exists such that, for $R$ arbitrarily large, the seller's expected revenue generated by $SEA^R$ is arbitrarily close to

$$M^{EA} \equiv n \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta_1} y \, d\Phi\left(y|\theta_1\right) \, dF_1\left(\theta_1\right),$$

where $\Phi$ denotes the c.d.f. of the first-order statistic among the components of $\theta_{-i}$, and $F_1$ denotes the marginal c.d.f. of a buyer's value. Thus the seller's expected revenue can be made arbitrarily close to the $M^{EA}$ with a Simple Sequential Auction. In each $SEA^R$, the ask price sequence can be taken independent of the history, and such that $a_t > \overline{\theta}$; the bid sequence is defined by $b_t = \frac{r_t}{R}\left(\overline{\theta} - \underline{\theta}\right)$, for all $t = 1, ..., T$, where $r_t$ denotes the round to which stage $t$ belongs, as defined in section 3, and $R$ satisfies the identity $T \equiv nR + n - 1$.

**Lemma 1** *As $R$ grows to infinity, the seller's expected revenue generated by $SEA^R$ converges to the revenue generated by the continuous version of the English auction $M^{EA}$.*

We conclude this section by stating the optimality result.

**Proposition 2** *Under assumptions A1-A3, the symmetric equilibrium of the continuous version of the English auction, maximizes the seller's expected profit among all UPBEs of all Simple Sequential Auctions.*

## 5 Conclusion

One can interpret the optimality result of Proposition 2 as characterizing an equilibrium of a game in which the seller moves first by choosing the trading procedure among all SSAs. This suggests that a connection between auction theory and bargaining theory may be established: the SSAs share essential properties of dynamic bargaining games, in which a process of offers and counteroffers determines the terms of trade.

To investigate this connection further, one should consider a larger class of "bargaining-like" auctions, in which the buyers are allowed to do more than simply buying at the ask price, subscribing a pre-specified bid, or dropping out irreversibly. For example, the buyers could be allowed to propose a trading price, say by "jump-bidding," during the course of the auction. A natural question, which this paper leaves open, is whether the seller can do better with an auction in this class than with the English auction.

The difficulty in answering this question comes from the fact that auctions in this larger class typically have multiple equilibria, due to the indeterminacy of beliefs at information sets following unexpected actions. It is known, however, that, with common values, allowing the buyers to engage in some signaling activity does not increase the seller's expected profit (Avery [2]). This suggest then that, possibly under appropriate restrictions of beliefs off the equilibrium path, the arguments in the proof of Proposition 2 can be used to obtain a more general optimality result for the English auction.

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# A Appendix

**Proposition 1** *Suppose that $(\sigma, \mu)$ is a UPBE of a Simple Sequential Auction, and that assumption A1 (affiliation) holds. Then, at any stage of the last round $t = nR+1, ..., T$, for any history $h^t \in H^t$,*

$$\sigma^t\left(x|\theta_{\iota(t)}, h^t\right) = 1 \quad \text{if } \underline{\theta} \leq \theta_{\iota(t)} < a_t\left(h^t\right), \tag{2}$$

*and*

$$\sigma^t\left(\alpha|\theta_{\iota(t)}, h^t\right) = 1 \quad \text{if } a_t\left(h^t\right) < \theta_{\iota(t)} \leq \overline{\theta}. \tag{3}$$

*Moreover, for any stage $t = 1, ..., nR$, for any history $h^t \in H^t$,*

$$\sigma^t\left(x|\theta_{\iota(t)}, h^t\right) = 1 \quad \text{if } \underline{\theta} \leq \theta_{\iota(t)} < b_t\left(h^t\right), \tag{4}$$

$$\sigma^t\left(\beta|\theta_{\iota(t)}, h^t\right) = 1 \quad \text{if } b_t\left(h^t\right) < \theta_{\iota(t)} < w_t\left(h^t\right), \tag{5}$$

$$\sigma^t\left(\alpha|\theta_{\iota(t)}, h^t\right) = 1 \quad \text{if } w_t\left(h^t\right) < \theta_{\iota(t)} \leq \overline{\theta}, \tag{6}$$

*where $w_t\left(h^t\right) \equiv \inf\left\{\tau \in \left[a_t\left(h^t\right), \overline{\theta}\right] \mid V^t\left(\tau, h^t\right) \leq \tau - a_t\right\}$.*

**Proof.** Claims (2) and (3) in Proposition 1, concerning all stages in the last round, are obvious. The proof of (4), (5) and (6) is broken into five Lemmas. First, Lemma 2 proves claim (4): in any stage of the first $R$ rounds, the active bidder drops out if his value is below his current bid. Lemma 2 also shows that the active buyer does not drop out if his value is above his current bid (claim (7)) and subscribes the bid if his value is between the bid and the ask (claim (8)).

**Lemma 2** *If $\sigma$ is a UPBE strategy profile then, for all $t = 1, ..., nR$, and all $h^t \in H^t$, claim (4) in Proposition 1 holds. Moreover,*

$$\sigma^t\left(x|\tau, h^t\right) = 0 \quad \text{if } b_t\left(h^t\right) < \tau \leq \overline{\theta}, \tag{7}$$

*and*

$$\sigma^t\left(\beta|\tau, h^t\right) = 1 \quad \text{if } b_t\left(h^t\right) < \tau < a_t\left(h^t\right). \tag{8}$$

Proof. If $\tau < b_t\left(h^t\right)$, accepting the ask $a_t\left(h^t\right)$ yields $\tau - a_t\left(h^t\right) < 0$, since $b_t\left(h^t\right) < a_t\left(h^t\right)$. Subscribing the bid $b_t\left(h^t\right)$ yields $\tau - b_t\left(h^t\right) < 0$ if $\Pr\left[\underline{h}_x^t|\tau, h^t\right] > 0$, and at most zero otherwise, since all future bids and asks are higher than $b_t\left(h^t\right)$. Thus subscribing the bid is part of a strategy that is weakly dominated and (4) holds.

If $\tau > b_t\left(h^t\right)$, subscribing $b_t\left(h^t\right)$ and quitting in the next round yields a positive payoff if $\Pr\left[\underline{h}_x^t|\tau, h^t\right] > 0$ and zero otherwise. Thus dropping out is part of a dominated strategy and (7) must hold. If $b_t\left(h^t\right) < \tau < a_t\left(h^t\right)$, then accepting the ask $a_t\left(h^t\right)$ yields $\tau - a_t\left(h^t\right) < 0$. Hence (8) holds. □

Next, Lemma 3 shows that, in any system of beliefs $\mu$ that is consistent with an UPBE strategy profile, affiliation is preserved. The proof uses the following two facts. First, each sequence of bids for each buyer is strictly increasing; hence, by Lemma 2, any history $h^t \in H^t$ has positive probability in any UPBE, and each bidder's conditional beliefs, at any stage, are determined by Bayes' rule. Second, the bidders choose their strategies independently: hence Bayes' rule preserves the affiliation inequality.

**Lemma 3** *If $(\sigma, \mu)$ is an UPBE, then $\mu^t\left(\cdot|h^t\right)$ satisfies affiliation for any history $h^t \in H^t$ and any stage $t = 1, ..., nR$.*

Proof. By Lemma 2, for each stage $t = 1, ..., nR$ and each history $h^t \in H^t$, we have

$$\sigma^{\iota(t)}\left(\beta|\tau, h^{\iota(t)}\right) = ... = \sigma^{t-n}\left(\beta|\tau, h^{t-n}\right) = \sigma^t\left(x|\tau, h^t\right) = 1$$

if $b_{t-n}\left(h^{t-n}\right) < \tau < b_t\left(h^t\right)$, and

$$\sigma^{\iota(t)}\left(\beta|\tau, h^{\iota(t)}\right) = ... = \sigma^{t-n}\left(\beta|\tau, h^{t-n}\right) = \sigma^t\left(\beta|\tau, h^t\right) = 1$$

if $b_t\left(h^t\right) < \tau < a_t\left(h^t\right)$. Thus, for each $h^{t+1} \in H^{t+1}$, the denominator in (1) is positive and Bayes' rule determines $\mu^{t+1}\left(\cdot|h^{t+1}\right)$. Then $\mu^{t+1}\left(\cdot|h^{t+1}\right)$ has a density $\nu^{t+1}\left(\cdot|h^{t+1}\right)$.

By assumption $\mu^1\left(\theta|h^1\right) \equiv F(\theta)$ satisfies affiliation. Proceeding by induction, we have that, for any $\theta = (\theta_1, ..., \theta_n)$ and $\widehat{\theta} = \left(\widehat{\theta}_1, ..., \widehat{\theta}_n\right)$ in $\Theta^n$, if

$$\nu^t\left(\widehat{\theta} \vee \theta|h^t\right) \cdot \nu^t\left(\widehat{\theta} \wedge \theta|h^t\right) \geq \nu^t\left(\widehat{\theta}|h^t\right) \cdot \nu^t\left(\theta|h^t\right),$$

then

$$\begin{aligned} & \nu^{t+1}\left(\widehat{\theta} \vee \theta|h^{t+1}\right) \cdot \nu^{t+1}\left(\widehat{\theta} \wedge \theta|h^{t+1}\right) - \nu^{t+1}\left(\widehat{\theta}|h^{t+1}\right) \cdot \nu^{t+1}\left(\theta|h^{t+1}\right) \\ = & \frac{\sigma^t\left(s_t|\theta_{\iota(t)}, h^t\right) \cdot \sigma^t\left(s_t|\widehat{\theta}_{\iota(t)}, h^t\right)}{\left(\Pr\left[s_t|h^t\right]\right)^2}\left[\nu^t\left(\widehat{\theta} \vee \theta|h^t\right) \cdot \nu^t\left(\widehat{\theta} \wedge \theta|h^t\right) - \nu^t\left(\widehat{\theta}|h^t\right) \cdot \nu^t\left(\theta|h^t\right)\right] \\ \geq & \ 0, \end{aligned}$$

where $\Pr\left[s_t|h^t\right] \equiv \int_{\Theta^n} \sigma^t\left(s_t|\theta_{\iota(t)}, h^t\right) \nu^t\left(\theta|h^t\right) d\theta$. □

The proof of claims (5) and (6) in Proposition 1 is by (backward) induction, starting from stage $nR$. Define $\Upsilon^{nR} \equiv \left[\underline{\theta}, a_{nR+1}\left(h^{nR+1}\right)\right] \times \left[\underline{\theta}, a_{nR+2}\left(h^{nR+2}\right)\right] \times ... \times \left[\underline{\theta}, a_T\left(h^T\right)\right]$ and

$$\begin{aligned}\phi^{nR}\left(\tau_1, \tau_2, h^{nR}\right) &\equiv E\left[\left(\tau_1 - b_{nR}\right) \cdot \mathbf{1}_{\{\theta_{-n} \in \Upsilon^{nR}\}} \,|\, \theta_n = \tau_2, h^{nR}\right] \\ &= \left(\tau_1 - b_{nR}\right) \cdot \mu^{nR}\left(\Upsilon^{nR} \,|\, \theta_n = \tau_2, h^{nR}\right),\end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, that is, $\mathbf{1}_{\{\chi\}}$ equals one if $\chi$ is true and zero otherwise. Since the function inside the expectation is nonincreasing in $\theta_{-n}$ and, by Lemma 3, $\mu^{nR}\left(\cdot|h^{nR}\right)$ satisfies affiliation, $\phi^{nR}$ is nonincreasing in $\tau_2$. Moreover, $V^{nR}\left(\tau, h^{nR}\right) = \phi^{nR}\left(\tau, \tau, h^{nR}\right)$, since the support of $\mu^{nR}\left(\cdot \,|\, \theta_n = \tau_2, h^{nR}\right)$ is included in $\Upsilon^{nR}$. Therefore, for any $\tau', \tau \in \Theta$, $\tau' \neq \tau$, we have

$$\begin{aligned}\frac{V^{nR}\left(\tau', h^{nR}\right) - V^{nR}\left(\tau, h^{nR}\right)}{\tau' - \tau} &= \frac{\phi^{nR}\left(\tau', \tau', h^{nR}\right) - \phi^{nR}\left(\tau, \tau, h^{nR}\right)}{\tau' - \tau} \\ &= \frac{\phi^{nR}\left(\tau', \tau', h^{nR}\right) - \phi^{nR}\left(\tau', \tau, h^{nR}\right)}{\tau' - \tau} \\ &\quad + \frac{\phi^{nR}\left(\tau', \tau, h^{nR}\right) - \phi^{nR}\left(\tau, \tau, h^{nR}\right)}{\tau' - \tau} \qquad (9) \\ &\leq \frac{\phi^{nR}\left(\tau', \tau, h^{nR}\right) - \phi^{nR}\left(\tau, \tau, h^{nR}\right)}{\tau' - \tau} \\ &= \mu^{nR}\left(\Upsilon^{nR} \,|\, \theta_n = \tau, h^{nR}\right) \\ &\leq 1.\end{aligned}$$

Now, if $V^{nR}\left(\tau, h^{nR}\right) < \tau - a_{nR}$ (hence $\sigma^{nR}\left(\alpha|\tau, h^{nR}\right) = 1$) for all $\tau \geq a_{nR}\left(h^{nR}\right)$, then $w_{nR}\left(h^{nR}\right) = a_{nR}\left(h^{nR}\right)$, hence claim (6) holds and claim (5) follows immediately from (8) in Lemma 2. If $V^{nR}\left(\tau, h^{nR}\right) > \tau - a_{nR}\left(h^{nR}\right)$ (hence $\sigma^{nR}\left(\beta|\tau, h^{nR}\right) = 1$) for all $\tau \geq a_{nR}\left(h^{nR}\right)$, then $w_{nR}\left(h^{nR}\right) = \overline{\theta}$, claim (6) holds trivially, and claim (5) again follows from (8) in Lemma 2. If neither of the above is true, then, since $V^{nR}$ is continuous, there exists a $\tau_0 \geq a_{nR}\left(h^{nR}\right)$ such that

$$\left(\tau_0 - b_{nR}\right) \cdot \mu^{nR}\left(\Upsilon^{nR} \,|\, \theta_n = \tau_0, h^{nR}\right) = \tau_0 - a_{nR}, \qquad (10)$$

This implies $\mu^{nR}\left(\Upsilon^{nR} \,|\, \theta_n = \tau_0, h^{nR}\right) < 1$, since $b_{nR}\left(h^{nR}\right) < a_{nR}\left(h^{nR}\right)$. Thus, by (9),

$$\frac{V^{nR}\left(\tau, h^{nR}\right) - V^{nR}\left(\tau_0, h^{nR}\right)}{\tau - \tau_0} < 1,$$

for all $\tau \in \left[a_{nR}\left(h^{nR}\right), \tau_0\right) \cup \left(\tau_0, \overline{\theta}\right]$. Therefore $a_{nR}\left(h^{nR}\right) \leq \tau' < \tau_0$ implies $V^{nR}\left(\tau', h^{nR}\right) > \tau' - a_{nR}\left(h^{nR}\right)$, hence $\sigma^{nR}\left(\beta|\tau', h^{nR}\right) = 1$; and $\tau_0 < \tau'' \leq \overline{\theta}$ implies $V^{nR}\left(\tau'', h^{nR}\right) < \tau'' - a_{nR}\left(h^{nR}\right)$, hence $\sigma^{nR}\left(\alpha|\tau'', h^{nR}\right) = 1$. We thus have $w_{nR}\left(h^{nR}\right) = \tau_0$. Together with (8) in Lemma 2, this establishes (5) and (6) for $t = nR$.

For the inductive step, assume that, the equivalent of (9) holds for in all stages $t' = t+1, ..., nR$. That is, assume that, for all $k = 1, ..., nR - t$, $\tau < \tau'$ implies $V^t\left(\tau', h^{t+k}\right) - V^t\left(\tau, h^{t+k}\right) \leq \tau' - \tau$. Together with claim (4), this implies both claim (5) and claim (6) for all stages $t' = t+1, ..., nR$. For any $j = 1, ..., n-1$, and any history $h^{t+j}$ in which $s_t = \beta$ and buyer $\iota(t+j)$ has subscribed all his previous bids, define the correspondence

$$\Upsilon_1^{t+j}\left(h^{t+j}, s_{t+j}\right) \equiv \begin{cases} \left\{\tau \in \Theta | b_{t+j}\left(h^{t+j}\right) < \tau < w_{t+j}\left(h^{t+j}\right)\right\} & \text{if } s_{t+j} = \beta, \\ \\ \left\{\tau \in \Theta | \underline{\theta} < \tau < b_{t+j}\left(h^{t+j}\right)\right\} & \text{if } s_{t+j} = x; \end{cases}$$

for all $j = 1, ..., n$, and let

$$\Upsilon^t\left(h^t, \beta, \underline{h}^t\right) \equiv \Upsilon_1^{t+1}\left(h^t, \beta, s_{t+1}\right) \times ... \times \Upsilon_1^{t+j}\left(h^t, \beta, s_{t+1}, ..., s_{t+j}\right) \times ... \times \Upsilon_1^{t+n-1}\left(h^t, \beta, \underline{h}^t\right).$$

By the inductive hypothesis and Lemma 2, only types of bidder $\iota(t+j)$ inside the interval $\Upsilon_1^{t+j}\left(h^t, \beta, s_{t+1}, ..., s_{t+j-1}\right)$ can choose action $s_{t+j}$; hence the support of $\mu^t\left(\cdot|\theta_{\iota(t)}, h^t\right)$ is included in $\Upsilon^t\left(h^t, \beta, \underline{h}^t\right)$, and we can write

$$V^t\left(\theta_{\iota(t)}, h^t\right) = E\left[\pi^t\left(\theta_{\iota(t)}, \theta_{-\iota(t)}, h^t\right) | \theta_{\iota(t)}, h^t\right],$$

where

$$\begin{aligned} \pi^t\left(\theta_{\iota(t)}, \theta_{-\iota(t)}, h^t\right) &\equiv \left(\theta_{\iota(t)} - b_t\left(h^t\right)\right) \cdot \mathbf{1}_{\left\{\theta_{-\iota(t)} \in \Upsilon^t\left(h^t, \beta, \underline{h}_x^t\right)\right\}} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (11) \\ + \sum_{\left\{\underline{h}^t | \left(h^t, \beta, \underline{h}^t\right) \in H^{t+n}\right\}} & \max\left\{0,\ \theta_{\iota(t)} - a_{t+n}\left(h^{t+n}\right),\ V^{t+n}\left(\theta_{\iota(t)}, h^t, \beta, \underline{h}^t\right)\right\} \cdot \mathbf{1}_{\left\{\theta_{-\iota(t)} \in \Upsilon^t\left(h^t, \beta, \underline{h}^t\right)\right\}}. \end{aligned}$$

If we can show that the function $\pi^t$ defined in (11) is nondecreasing in $\theta_{-\iota(t)}$, then we can complete the proof by mimicking the steps of the proof for stage $nR$. That is, defining

$$\begin{aligned} \phi^t\left(\tau_1, \tau_2, h^t\right) &\equiv E\left[\pi^t\left(\tau_1, \theta_{-\iota(t)}, h^t\right) | \tau_2, h^t\right] \\ \\ &\equiv \left(\tau_1 - b_t\left(h^t\right)\right) \cdot \mu^t\left(\Upsilon^t\left(h^t, \beta, \underline{h}_x^t\right) | \tau_2, h^t\right) \end{aligned}$$

$$+ \sum_{\left\{\underline{h}^t | \left(h^t, \beta, \underline{h}^t\right) \in H^{t+n}\right\}} \max\left\{0, \tau_1 - a_{t+n}\left(h^{t+n}\right), V^{t+n}\left(\tau_1, h^t, \beta, \underline{h}^t\right)\right\} \cdot \mu^t\left(\Upsilon^t\left(h^t, \beta, \underline{h}^t\right) | \tau_2, h^t\right),$$

and proceeding as in (9) yields

$$\frac{V^t\left(\tau', h^t\right) - V^t\left(\tau, h^t\right)}{\tau' - \tau} \leq 1,$$

for any $\tau, \tau' \in \Theta$, $\tau \neq \tau'$, since $V^{t+n}\left(\tau', h^t, \beta, \underline{h}^t\right) - V^{t+n}\left(\tau, h^t, \beta, \underline{h}^t\right) \leq \tau' - \tau$ by the inductive assumption.

If there is no $\tau_0$ such that $V^t\left(\tau_0, h^t\right) = \tau_0 - a_t\left(h^t\right)$, then either $w_t\left(h^t\right) = a_t\left(h^t\right)$ or $w_t\left(h^t\right) = \overline{\theta}$. If such $\tau_0$ exists, then the probability that bidder $\iota(t)$ is awarded the object, given $\tau_0$ and the history $h^t$, must be strictly less than 1 because $b_t\left(h^t\right) < a_t\left(h^t\right)$, no future bid or ask for buyer $\iota(t)$ is higher than $a_t\left(h^t\right)$, and, by Lemma 2, $\Pr\left[\underline{h}^t_x | \tau, h^t\right] > 0$. Thus

$$\frac{V^t\left(\tau, h^t\right) - V^t\left(\tau_0, h^t\right)}{\tau - \tau_0} < 1,$$

for all $\tau \in \left[a_t\left(h^t\right), \tau_0\right) \cup \left(\tau_0, 1\right]$. Hence $\tau \in \left[a_t\left(h^t\right), \tau_0\right)$ implies $V^t\left(\tau, h^t\right) > \tau - a_t\left(h^t\right)$, hence $\sigma^t\left(\beta | \tau, h^t\right) = 1$, and $\tau > \tau_0$ implies $V^t\left(\tau, h^t\right) < \tau - a_t\left(h^t\right)$, hence $\sigma^t\left(\alpha | \tau, h^t\right) = 1$. Thus we have $w_t\left(h^t\right) = \tau_0$. Together with (8) in Lemma 2, this establishes (5) and (6).

Thus, to complete the proof, it is sufficient to show that $\pi^t\left(\theta_{\iota(t)}, \theta_{-\iota(t)}, h^t\right)$ is nonincreasing in $\theta_{-\iota(t)}$. The next three Lemmas are devoted to this end. The key property will be that the expected payoff of the active bidder from subscribing his bid, when an opponent has dropped out in a previous stage, is higher than the expected payoff that he earns when the same opponent has subscribed her bid. Formally, $V^t\left(\tau, h^t\right)$ is nonincreasing in $h^t$ in the order where $\beta > x$. Recall that we are working under the inductive hypothesis: that is, we are assuming that *claims (5) and (6) hold for all stages* $t' = t+1, ..., nR$.

**Lemma 4** *Fix* $\xi \in \{t, ..., nR\}$. *Suppose that, in the order where* $\beta > x$,

1. $V^{\xi+n}\left(\theta_{\iota(\xi)}, h^\xi, \beta, \underline{h}_\xi\right)$ *in nonincreasing in* $\underline{h}_\xi$; *and*
2. *for all* $\zeta = \xi + 2, ..., \xi + n - 1$, $w_\zeta\left(h^\xi, \beta, s_{\xi+1}, ..., s_{\zeta-1}\right)$ *is nonincreasing in* $\left(s_{\xi+1}, ..., s_{\zeta-1}\right)$.

*Then, under the inductive hypothesis,* $\pi^\xi\left(\theta_{\iota(\xi)}, \theta_{-\iota(\xi)}, h^\xi\right)$ *is nonincreasing in* $\theta_{-\iota(\xi)}$.

PROOF. Consider two vectors $\theta_{-\iota(\xi)}$ and $\theta'_{-\iota(\xi)}$ that only differ in one element, $\theta'_{\iota(\xi+j)} > \theta_{\iota(\xi+j)}$, for some $j \in N \backslash \{\iota(\xi)\}$. There are three exhaustive, and mutually exclusive, possibilities:

i) $\pi^\xi\left(\theta_{\iota(\xi)}, \theta_{-\iota(\xi)}, h^\xi\right) = \theta_{\iota(\xi)} - b_\xi$. In this case, all other buyers choose $x$ in the next $n-1$ stages, hence $\theta_{\iota(\xi+j)} \leq b_{\xi+j}$, for all $j = 1, ..., n-1$. By the inductive hypothesis, there exist a $w_{\xi+j}\left(h^{\xi+j}\right)$

such that buyer $\iota\left(\xi+j\right)$ with type $\theta'_j$ subscribes $b_{\xi+j}$, if $b_{\xi+j} < \theta'_{\iota(\xi+j)} < w_{\xi+j}\left(h^{\xi+j}\right)$, and accepts $a_{\xi+j}$ if $w_{\xi+j}\left(h^{\xi+j}\right) < \theta'_{\iota(\xi+j)}$. Thus, if $b_{\xi+j} < \theta'_{\iota(\xi+j)} < w_{\xi+j}\left(h^{\xi+j}\right)$, then $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta'_{-\iota(\xi)},h^{\xi}\right) = V^{\xi+n}\left(\theta_{\iota(\xi)},h^{\xi},\beta,\underline{h}'_{\xi}\right)$, ( $\underline{h}'_{\xi}$ denotes the new partial history) which cannot be larger than $\theta_{\iota(\xi)} - b_{\xi}$, since all future asks and bids that bidder $\iota\left(\xi\right)$ may face are higher than $b_{\xi}$. If $w_{\xi+j}\left(h^{\xi+j}\right) < \theta'_{\iota(\xi+j)}$, then $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta'_{-\iota(\xi)},h^{\xi}\right) = 0$.

ii) $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta_{-\iota(\xi)},h^{\xi}\right) = V^{\xi+n}\left(\theta_{\iota(\xi)},h^{\xi},\beta,...,\beta\right)$. In this case, all other buyers subscribe their bid in the next $n-1$ stages and, by the inductive hypothesis, $b_{\xi+j} < \theta_{\iota(\xi+j)} < w_{\xi+j}\left(h^{\xi+j}\right)$, for all $j = 1,...,n-1$. If $\theta'_{\iota(\xi+j)} > w_{\xi+j}\left(h^{\xi+j}\right)$, then $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta'_{-\iota(\xi)},h^{\xi}\right) = 0$.

Assumptions 1 and 2 are used in the third case:

iii) $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta_{-\iota(\xi)},h^{\xi}\right) = V^{\xi+n}\left(\theta_{\iota(\xi)},h^{\xi},\beta,\underline{h}_{\xi}\right)$, where at least one element of $\underline{h}_{\xi}$ is equal to $x$. If $b_{\xi+j} < \theta_{\iota(\xi+j)}$ the proof is as in case ii) above. If $\theta_{\iota(\xi+j)} < b_{\xi+j} < \theta'_{\iota(\xi+j)}$, and bidder $\iota\left(\xi+j\right)$ with type $\theta'_{\iota(\xi+j)}$ chooses $\beta$ instead of $x$, (hence changing the histories from $h^{\zeta}$ to $\widehat{h}^{\zeta}$ for any $\zeta = \xi+j+1,...,\xi+n-1$) then, by assumption 2, $w_{\zeta}\left(\widehat{h}^{\zeta}\right) \leq w_{\zeta}\left(h^{\zeta}\right)$. Some other buyer may then choose $\alpha$ instead of $\beta$. If this happens, then $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta'_{-\iota(\xi)},h^{\xi}\right) = 0$. Otherwise $\pi^{\xi}\left(\theta_{\iota(\xi)},\theta'_{-\iota(\xi)},h^{\xi}\right) = V^{\xi+n}\left(\theta_{\iota(\xi)},h^{\xi},\beta,\widehat{\underline{h}}^{\xi}\right)$, which is lower that $V^{\xi+n}\left(\theta_{\iota(\xi)},h^{\xi},\beta,\underline{h}^{\xi}\right)$, by assumption 1. $\square$

For $\xi = t$, Lemma 4 establishes the desired monotonicity property for $\pi^{t}\left(\theta_{\iota(t)},\theta_{-\iota(t)},h^{t}\right)$ under the assumptions that:

(1) $V^{t+n}\left(\theta_{\iota(t)},h^{t},\beta,\underline{h}^{t}\right)$ in nonincreasing in $\underline{h}^{t}$, and,

(2) for all $\zeta = t+2,...,t+n-1$, $w_{\zeta}\left(h^{t},\beta,s_{t+1},...,s_{\zeta-1}\right)$ is nonincreasing in $\left(s_{t+1},...,s_{\zeta-1}\right)$.

The next two Lemmas show that these assumptions are implied by the inductive hypothesis and Lemmas 3 and 4 above.

**Lemma 5** *Under the inductive hypothesis, for each $\zeta = t+1,...,nR$, $w_{\zeta}\left(h^{\zeta}\right)$ is nonincreasing (in the order where $\beta > x$), if $V^{\zeta}\left(\theta_{\iota(\zeta)},h^{\zeta}\right)$ is nonincreasing in $h^{\zeta}$.*

PROOF. We want to show that $w_{\zeta}\left(\widehat{h}^{\zeta}\right) \leq w_{\zeta}\left(h^{\zeta}\right)$ if $\widehat{h}^{\zeta} > h^{\zeta}$. If $w_{\zeta}\left(h^{\zeta}\right) = a_{\zeta}\left(h^{\zeta}\right)$, there is nothing to prove, since $w_{t}\left(h^{t}\right) \geq a_{t}\left(h^{t}\right)$ for all $t = 1,...,nR$. Suppose that $w_{\zeta}\left(h^{\zeta}\right) > a_{\zeta}\left(h^{\zeta}\right)$. Then

$$V^{\zeta}\left(w_{\zeta}\left(h^{\zeta}\right),\widehat{h}^{\zeta}\right) \quad \leq \quad V^{\zeta}\left(w_{\zeta}\left(h^{\zeta}\right),h^{\zeta}\right)$$

$$
\begin{aligned}
&= w_\zeta\left(h^\zeta\right) - a_\zeta\left(h^\zeta\right) \\
&\leq w_\zeta\left(h^\zeta\right) - \widehat{a}_\zeta\left(h^\zeta\right)
\end{aligned}
$$

since $a_\zeta$ is nonincreasing. Under the inductive hypothesis, $V^\zeta\left(\tau, \widehat{h}^\zeta\right) \leq \tau - a_\zeta\left(\widehat{h}^\zeta\right)$ implies $w_\zeta\left(\widehat{h}^\zeta\right) \leq \tau$; therefore $w_\zeta\left(\widehat{h}^\zeta\right) \leq w_\zeta\left(h^\zeta\right)$. □

**Lemma 6** *Under the inductive hypothesis, the function* $V^\xi\left(\theta_{\iota(\xi)}, h^t, \beta, s_{t+2}, ..., s_{\xi-1}\right)$ *is nonincreasing in* $(s_{t+1}, ..., s_{\xi-1})$ *in the order where* $\beta > x$*, for any history* $h^t$*, and any* $\xi = t+n, ..., nR$*,*

PROOF. By induction. For the last stage, affiliation (Lemma 3) and the inductive hypothesis imply that

$$V^{nR}\left(\theta_n, h^t, s_{t+1}, ..., s_{nR-1}\right) = \left(\theta_n - b_{nR}\right) \cdot \mu^{nR}\left(\Upsilon^{nR} \,|\, \theta_n, h^{nR}\right)$$

is nonincreasing in $(s_{t+1}, ..., s_{nR-1})$.

Now assume that $V^\zeta\left(\theta_{\iota(\zeta)}, h^t, s_{t+1}, ..., s_{\zeta-1}\right)$ is nonincreasing in $(s_{t+1}, ..., s_{\zeta-1})$ for all $\zeta = \xi + 1, ..., nR$. Then, by Lemma 5, $w_\zeta\left(h^t, \beta, s_{t+1}, ..., s_{\zeta-1}\right)$ is nonincreasing in $(s_{t+1}, ..., s_{\zeta-1})$ for all $\zeta = \xi + 1, ..., nR$. In particular, $V^\xi\left(\theta_{\iota(\xi)}, h^\xi, \beta, \underline{h}_\xi\right)$ is increasing in $\underline{h}_\xi$ and $w_\zeta\left(h^\xi, \beta, s_{\xi+2}, ..., s_{\zeta-1}\right)$ is nonincreasing in $(s_{\xi+2}, ..., s_{\zeta-1})$ for all $\zeta = \xi+2, .., \xi+n-1$. Thus, by Lemma 4, $\pi^\xi\left(\theta_{\iota(\xi)}, \theta_{-\iota(\xi)}, h^\xi\right)$ is nonincreasing in $\theta_{-\iota(\xi)}$, and, by affiliation (Lemma 3)

$$V^\xi\left(\theta_{\iota(\xi)}, h^t, \beta, s_{t+1}, ..., s_{\xi-1}\right) = E\left[\pi^\xi\left(\theta_{\iota(\xi)}, \theta_{-\iota(\xi)}, h^\xi\right) | h^t, \beta, s_{t+1}, ..., s_{\xi-1}\right]$$

is nonincreasing in $(s_{t+1}, ..., s_{\xi-1})$. □

This ends the proof of Proposition 1. □

**Lemma 1** *As* $R$ *grows to infinity, the seller's expected revenue generated by* SEA$^R$ *converges to the revenue generated by the continuous version of the English auction* $M^{EA}$*.*

**Proof.** For each $i \in N$, let $y_i$ denote the highest among the components of $\theta_{-i}$, and let $p_i^R(\theta_i, y_i)$ denote bidder $i$'s equilibrium payment in SEA$^R$. (By Proposition 1, $p_i^R$ is well defined almost everywhere.) Fix $\theta_i$ in $\Theta$, and, for any integer $R$, let $\overline{r}(\theta_i, R)$ denote the unique integer such that $\frac{1}{R}\overline{r}(\theta_i, R) \leq \theta_i < \frac{1}{R}\overline{r}(\theta_i, R) + \frac{1}{R}$. By Lemma 2, $p_i^R(\theta_i, y_i) = \frac{r}{R}\mathbf{1}_{\left\{\frac{r-1}{R} < y_i < \frac{r}{R}\right\}}$, for all

$r = 2, ..., \overline{r}(\theta, R)$, and $p_i^R(\theta_i, y_i) \in \left\{0, \frac{1}{R}\overline{r}(\theta_i, R)\right\}$ if $\frac{\overline{r}(\theta_i,R)}{R} < y_i < \frac{\overline{r}(\theta_i,R)}{R} + 1$. Therefore, for each integer $R$,

$$\int_{\underline{\theta}}^{\theta_i} \left|y_i - p_i^R(\theta_i, y_i)\right| d\Phi(y_i|\theta_i) \le \frac{1}{R}\int_{\underline{\theta}}^{\frac{1}{R}\overline{r}(\theta_i,R)} d\Phi(y_i|\theta_i) + \int_{\frac{1}{R}\overline{r}(\theta_i,R)}^{\frac{1}{R}\overline{r}(\theta_i,R)+\frac{1}{R}} y_i \, d\Phi(y_i|\theta_i).$$

As $R$ grows to infinity, since $\frac{1}{R}\overline{r}(\theta_i, R)$ tends to $\theta_i$, both terms on the right-hand side tend to zero. Thus bidder $i$'s interim expected payment $m_i^R(\theta_i) \equiv \int_{\underline{\theta}}^{\theta_i} p_i^R(\theta_i, y_i)\, d\Phi(y_i|\theta_i)$ converges pointwise to $m^{EA}(\theta_i) \equiv \int_{\underline{\theta}}^{\theta_i} y_i \, d\Phi(y_i|\theta_i)$, almost everywhere; and this, together with the inequality $\left|p_i^R(\theta_i, y_i)\right| \le \overline{\theta}$, implies, by Lebesgue's dominated convergence theorem, that $n\int_{\underline{\theta}}^{\overline{\theta}} m_i^R(\theta_i)\, dF(\theta_i)$ converges to $M^{EA}$. □

**Proposition 2** *Under assumptions A1-A3, the symmetric equilibrium of the continuous version of the English auction, maximizes the seller's expected profit among all UPBEs of all Simple Sequential Auctions.*

**Proof.** Let $\left\{q^i(\theta_i, \theta_{-i}), m^i(\theta_i, \theta_{-i});\ (\theta_i, \theta_{-i}) \in \Theta^n,\ i \in N\right\}$ be an UPBE outcome of a SSA. The next two Lemmas follow from Proposition 1.

**Lemma 7** *Any UPBE outcome function* $q^i(\theta_i, \theta_{-i})$, $i \in N$, *satisfies*

$$q^i(\theta_i, \theta_{-i}) = \mathbf{1}_{\{\theta_i > z^i(\theta_{-i})\}}, \quad \textit{for almost all } \theta \in \Theta^n, \tag{12}$$

*for some function* $z^i : \Theta^{n-1} \to \Theta$.

Proof: By Proposition 1, for almost all given type profiles $\theta \in \Theta^n$, any UPBE is in pure strategies. Therefore, the auction ends at one stage $t_*(\theta)$ with certainty, and $q^i(\theta_i, \theta_{-i}) \in \{0, 1\}$ for all $i \in N$. It is then sufficient to show that, for almost all $\theta \in \Theta^n$, $q^i(\theta_i, \theta_{-i}) = 0$ and $\theta_i' < \theta_i$ imply $q^i(\theta_i', \theta_{-i}) = 0$.

If $q^i(\theta_i, \theta_{-i}) = 0$, then either $\sigma^t(x|\theta_i, h^t) = 1$ for some $t \le t_*(\theta)$, hence $b_{t-n}(h^{t-n}) < \theta_i \le b_t(h^t)$; or $\sigma^t(\beta|\theta_i, h^t) = 1$, for $t$ such that $t < t_*(\theta) < t + n$, hence $b_t(h^t) \le \theta_i \le w_t(h^t)$, and the auction ends with buyer $\iota(t_*(\theta)) \ne i$ accepting the ask $a_{t_*(\theta)}$. In the first case, all types $\theta_i' \in [\underline{\theta}, \theta_i)$ also drop out at or before stage $t$. In the second case, all types $\theta_i' \in (b_t(h^t), \theta_i)$ follow the same strategy as type $\theta_i$; and all types $\theta_i' \in [\underline{\theta}, b_t(h^t))$ drop out before stage $t$. Thus, in both cases, almost all types $\theta_i' \in [\underline{\theta}, \theta_i)$ lose the auction. □

**Lemma 8** *There exists a restriction of* $m^i(\theta_i, \theta_{-i})$ *to a subset of* $\Theta^n$ *with full measure which is nondecreasing in* $\theta_i$.

Proof: By Lemma 7, buyer $i$'s payment is zero if $\theta_i < z^i(\theta_{-i})$. Thus it is sufficient to show that $z^i(\theta_{-i}) < \theta_i < \theta_i' \leq \overline{\theta}$ implies $m^i(\theta_i, \theta_{-i}) \leq m^i(\theta_i', \theta_{-i})$, for all $\theta \in \Theta^n$ such that $q^i(\theta_i, \theta_{-i}) \in \{0, 1\}$. As in proof of Lemma 7 $t_*(\theta)$ denotes the stage at which the auctions ends with type profile $\theta$. Buyer $i$ with type $\theta_i$ can win the auction in two ways: either he accepts the ask $a_{t_*(\theta)}\left(h^{t_*(\theta)}\right)$, or the active buyer $\iota(t_*(\theta)) \neq i$ in stage $t_*$ drops out and buyer $i$ pays $b_t\left(h^t\right)$, for some $t$ such that $t < t_*(\theta) < t+n$. In the first case, any type $\theta_i' > \theta_i$ accepts his ask no later than stage $t_*(\theta)$, thus paying at least $a_{t_*(\theta)}\left(h^{t_*(\theta)}\right)$. In the second case, any type $\theta_i' > \theta_i$ either follows the same strategy as type $\theta_i$ or accepts the an ask no later that stage $t_*(\theta)$, hence pays more than $b_t\left(h^t\right)$. ☐

By Proposition 1, any buyer with value $\underline{\theta}$ is awarded the object with probability zero and pays nothing. Thus, given Lemmas 7 and 8, it is sufficient to show that the equilibrium outcome of the English auction

$$z^i_{ea}(\theta_{-i}) \equiv y_i \quad \text{and} \quad m^i_{ea}(\theta_i, \theta_{-i}) \equiv y_i \cdot \mathbf{1}_{\{\theta_i > y_i\}}, \ \ \theta \in \Theta^n, \ i \in N,$$

where, $q^i_{ea}(\theta_i, \theta_{-i}) \equiv \mathbf{1}_{\{\theta_i > z^i_{ea}(\theta_{-i})\}}$, and, as in the proof of Lemma 1, $y_i \equiv \max\{\theta_j; j \in N \backslash \{i\}\}$, solves the program $(P)$:

$$\max_{\{z^i(\theta),\, m^i(\theta),\, \theta \in \Theta^n, i \in N\}} \ \sum_{i \in N} \int_{\Theta^{n-1}} \int_{\underline{\theta}}^{\overline{\theta}} m^i(\theta_i, \theta_{-i}) \ \ dF(\theta_i, \theta_{-i}),$$

subject to

$$\begin{aligned} &\int_{\Theta^{n-1}} \left[\theta_i \cdot \mathbf{1}_{\{\theta_i > z^i(\theta_{-i})\}} - m^i(\theta_i, \theta_{-i})\right] f(\theta_{-i}, \theta_i) \, d\theta_{-i} \\ &\leq \int_{\Theta^{n-1}} \left[\theta_i \cdot \mathbf{1}_{\{\widehat{\theta}_i > z^i(\theta_{-i})\}} - m^i\left(\widehat{\theta}_i, \theta_{-i}\right)\right] f(\theta_{-i}, \theta_i) \, d\theta_{-i}, \end{aligned} \qquad \text{for all } \theta_i, \widehat{\theta}_i \in \Theta; \qquad (\mathrm{IC}_i)$$

$$m^i(\theta_i, \theta_{-i}) \ \text{ is nondecreasing in } \theta_i, \ \text{for all } \theta_{-i} \in \Theta^{n-1}; \qquad (\mathrm{M}_i)$$

$$m^i(\underline{\theta}, \theta_{-i}) \leq 0, \ \text{ for all } \theta_{-i} \in \Theta^{n-1}; \qquad (\mathrm{L}_i)$$

and

$$\sum_{i \in N} \mathbf{1}_{\{\theta_i > z^i(\theta_{-i})\}} \leq 1, \ \text{ for all } \theta_{-i} \in \Theta^{n-1}, \qquad (\mathrm{U})$$

for all $i \in N$. The main step consists in establishing the following

**Lemma 9** *For any given n-tuple of functions $z^i$, $i \in N$, the seller's expected profit is maximized, subject to (IC$_i$), (M$_i$), (L$_i$), $i \in N$, and (U), by*

$$m^i_o(\theta_i, \theta_{-i}; z^i) \equiv z^i(\theta_{-i}) \cdot \mathbf{1}_{\{\theta_i \geq z^i(\theta_{-i})\}}, \ \ i \in N.$$

In fact, if the Lemma 9 holds, we can substitute $m_o^i\left(\theta_i, \theta_{-i}; z^i\right),\ i \in N$, into the objective function and obtain

$$\sum_{i \in N} \int_{\Theta^{n-1}} z^i\left(\theta_{-i}\right) \int_{z^i\left(\theta_{-i}\right)}^{\overline{\theta}} f\left(\theta_i, \theta_{-i}\right) d\theta_i \, d\theta_{-i}$$

$$= \int_{\Theta^n} \sum_{i \in N} \left[\theta_i - \frac{1 - G\left(\theta_i | \theta_{-i}\right)}{g\left(\theta_i | \theta_{-i}\right)}\right] \cdot \mathbf{1}_{\left\{\theta_i \geq z^i\left(\theta_{-i}\right)\right\}} \, dF\left(\theta\right), \tag{13}$$

($G$ and $g$ were defined in Assumption A2) which, under assumptions A1 to A3, is maximized subject to (U) by $z_{ea}^i\left(\theta_i, \theta_{-i}\right) \equiv y_i$, i.e., the object should always be assigned to the buyer with highest value. To see this, fix $\theta = (\theta_1, ..., \theta_n)$, pick two elements two elements $\theta_i$ and $\theta_j$, with $\theta_i > \theta_j$, and let $\theta_{-ij} \in \Theta^{n-2}$ denote the vector of the remaining $n - 2$ types. Then we have

$$\begin{aligned} \theta_i - \frac{1 - G(\theta_i | \theta_j, \theta_{-ij})}{g(\theta_i | \theta_j, \theta_{-ij})} &\geq \theta_j - \frac{1 - G(\theta_j | \theta_j, \theta_{-ij})}{g(\theta_j | \theta_j, \theta_{-ij})} && \text{(by A2)} \\ &\geq \theta_j - \frac{1 - G(\theta_j | \theta_i, \theta_{-ij})}{g(\theta_j | \theta_i, \theta_{-ij})} && \text{(by A1.)} \end{aligned}$$

The "virtual utility" $\theta_i - \dfrac{1 - G\left(\theta_i | \theta_{-i}\right)}{g\left(\theta_i | \theta_{-i}\right)}$ is always nonnegative, because

$$\begin{aligned} \theta_i - \frac{1 - G\left(\theta_i | \theta_{-i}\right)}{g\left(\theta_i | \theta_{-i}\right)} &\geq \theta_i - \frac{1 - G\left(\theta_i | \overline{\theta}, ..., \overline{\theta}\right)}{g\left(\theta_i | \overline{\theta}, ..., \overline{\theta}\right)}, && \text{(by A1)} \\ &\geq \underline{\theta} - \frac{1 - G\left(\underline{\theta} | \overline{\theta}, ..., \overline{\theta}\right)}{g\left(\underline{\theta} | \overline{\theta}, ..., \overline{\theta}\right)} && \text{(by A2)} \\ &= \underline{\theta} - \frac{1}{g\left(\underline{\theta} | \overline{\theta}, ..., \overline{\theta}\right)} \\ &\geq 0; && \text{(by A3)} \end{aligned}$$

and, by the Lemma 9 above, the optimal payment is $m_o^i(\theta_i, \theta_{-i}; z_{ea}^i) = y_i \, \mathbf{1}_{\left\{\theta_i \geq z^i\left(\theta_{-i}\right)\right\}} \equiv m_{ea}^i\left(\theta_i, \theta_{-i}\right),\ i \in N$; hence the program is solved by the equilibrium outcome of the English auction.

To establish Lemma 9, define

$$s^i\left(\theta_i, \theta_{-i}\right) \equiv \theta_i \, q^i\left(\theta_i, \theta_{-i}\right) - m^i\left(\theta_i, \theta_{-i}\right), \quad \theta \in \Theta^n, \ i \in N,$$

so that

$$\theta_i\, q^i(\widehat{\theta}_i, \theta_{-i}) - m^i(\widehat{\theta}_i, \theta_{-i}) = s^i(\widehat{\theta}_i, \theta_{-i}) + \left(\theta_i - \widehat{\theta}_i\right)\ q^i(\widehat{\theta}_i, \theta_{-i});$$

and, using the equality $q^i(\widehat{\theta}_i, \theta_{-i}) = \mathbf{1}_{\left\{\widehat{\theta} > z^i(\theta_{-i})\right\}}$, rewrite the (IC$_i$) constraints as

$$\int_{\Theta^{n-1}} \left[s^i\left(\theta_i, \theta_{-i}\right) - s^i(\widehat{\theta}_i, \theta_{-i})\right]\ f\left(\theta_i, \theta_{-i}\right)\ d\,\theta_{-i}\ \geq\ \int_{z^i(\theta_{-i}) \leq \widehat{\theta}_i} \left[\theta_i - \widehat{\theta}_i\right]\ f\left(\theta_i, \theta_{-i}\right)\ d\,\theta_{-i}, \qquad \theta_i, \widehat{\theta}_i \in \Theta; \tag{IC$_i$}$$

the (L$_i$) constraints as

$$s^i\left(\underline{\theta}, \theta_{-i}\right) \geq 0, \text{ for all } \theta_{-i} \in \Theta^{n-1}; \tag{L$_i$}$$

and the (M) constraints as: for all $\theta_{-i} \in \Theta^{n-1}$,

$$s^i\left(\theta_i, \theta_{-i}\right) - s^i\left(\theta_i', \theta_{-i}\right) \geq \theta_i - \theta_i', \qquad \text{for } z^i(\theta_{-i}) \leq \theta_i < \theta_i' \leq 1, \tag{WMP$_i$}$$

$$s^i\left(\theta_i, \theta_{-i}\right) - s^i\left(\theta_i', \theta_{-i}\right) \geq 0, \qquad \text{for } 0 \leq \theta_i < \theta_i' \leq\ z^i\left(\theta_{-i}\right). \tag{LMP$_i$}$$

Finally, rewrite the objective function as

$$\sum_{i=1}^{n} \int_{\Theta^n} m^i\left(\theta_i, \theta_{-i}\right)\ f(\theta)\, d\theta = \sum_{i=1}^{n} \int_{\Theta^n} \theta_i\, q^i\left(\theta_i, \theta_{-i}\right) dF(\theta) - \sum_{i=1}^{n} \int_{\Theta^n} s^i\left(\theta_i, \theta_{-i}\right)\ dF(\theta).$$

Since the functions $q^i\left(\theta_i, \theta_{-i}\right) = \mathbf{1}_{\{\theta_i > z^i(\theta_{-i})\}}$, $i \in N$, are fixed, the program $(P)$ becomes separable in $n$ subprograms, each formally identical to the following program $(S)$

$$\min_{s:\Theta^n \to \mathbb{R}} \int_{\Theta^n} s\left(x, y\right)\ \ dF\left(x, y\right),$$

subject to:

$$s\left(\underline{\theta}, y\right) \geq 0, \qquad \underline{\theta} \leq x \leq z(y); \tag{L}$$

$$\int_{\Theta^{n-1}} \left[s\left(x, y\right) - s\left(\widehat{x}, y\right)\right]\ f\left(x, y\right)\ dy\ \geq \int_{z(y) \leq \widehat{x}} \left[x - \widehat{x}\right]\ f\left(x, y\right)\ dy; \qquad x, \widehat{x} \in \Theta; \tag{IC}$$

and,

$$-\left[s\left(x', y\right) - s\left(x, y\right)\right] \geq -\left[x' - x\right]; \quad z(y) \leq x < x' \leq 1, \tag{WMP}$$

$$-\left[s\left(x', y\right) - s\left(x, y\right)\right] \geq 0; \quad \underline{\theta} \leq x < x' \leq\ z(y), \tag{LMP}$$

for all $y \in \Theta^{n-1}$.

Thus we can complete the proof by showing that $s_o(x, y; z) \equiv (x - z(y)) \cdot \mathbf{1}_{\{x > z(y)\}}$ solves program $(S)$. Define

$$\begin{aligned} \lambda(x) &\equiv \frac{1 - G(x|\overline{\theta}, ..., \overline{\theta})}{g(x|\overline{\theta}, ..., \overline{\theta})}, & x \in \Theta, \\ \psi(x, y) &\equiv f(x, y) \left[ \frac{1 - G(x|\overline{\theta}, ..., \overline{\theta})}{g(x|\overline{\theta}, ..., \overline{\theta})} - \frac{1 - G(x|y)}{g(x|y)} \right] & (x, y) \in \Theta^n. \end{aligned}$$

The IIC constraints imply

$$\int_{\Theta^{n-1}} \left[ \int_{\underline{\theta}}^{\overline{\theta}} \lambda(x) f(x, y) \, d\overline{s}(x|y) \right] dy \geq \int_{\Theta^{n-1}} \int_{z(y)}^{\overline{\theta}} \lambda(x) f(x, y) \, dx \, dy.$$

where $\int_{\underline{\theta}}^{\overline{\theta}} \gamma(x, y) \, d\overline{s}(x|y)$ denotes the integral of $\gamma$ with respect to $s$ on $\Theta^n$ for a fixed value of $y$. This integral exists since $\gamma(x, y) \equiv \lambda(x) f(x, y)$ is continuous and $s$ is nondecreasing in $x$. Similarly, the constraints (WMP) and (LMP) imply

$$-\int_{\Theta^{n-1}} \int_{\underline{\theta}}^{\overline{\theta}} \psi(x, y) \, d\overline{s}(x|y) \, dy \;\geq\; -\int_{\Theta^{n-1}} \int_{z(y)}^{\overline{\theta}} \psi(x, y) \, dx \, dy,$$

since $\psi(x, y) \geq 0$, by affiliation. Summing these two inequalities yields

$$\int_{\Theta^{n-1}} \int_{\underline{\theta}}^{\overline{\theta}} [1 - F(x|y)] \, d\overline{s}(x, y) \, dF^{(-1)}(y) \geq \int_{\Theta^{n-1}} \int_{z(y)}^{\overline{\theta}} [1 - F(x|y)] \, dx \, dF^{(-1)}(y),$$

which, integrating each term by parts, can be written as

$$\int_{\Theta^n} [s(x, y) - s(\underline{\theta}, y)] \, dF(x, y) \geq \int_{\Theta^{n-1}} \int_{z(y)}^{\overline{\theta}} [x - z(y)] \, dF(x|y) \, dF^{(-1)}(y).$$

Thus the last inequality is implied by the constraints of program $(S)$, and $s_o(x, y; z)$ minimizes the objective function subject to it, and (L). Then $s_o(x, y; z)$ is also a solution of program $(S)$. □