

# Demand Management and Inventory Control for Substitutable Products

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## Abstract

This paper studies dynamic inventory and pricing decisions for a set of substitutable products over a finite planning horizon. We present a general stochastic, price-dependent demand model that unifies many commonly used demand models in the literature. Unsatisfied demands are backlogged. There are linear purchasing, inventory-holding, and backordering costs. The objective is to maximize the total expected discounted profit. The original formulation is not jointly concave in the decision variables and is therefore intractable. One key observation here is that the problem becomes jointly concave if we work with the inverse of the price vector – the market share vector. We characterize the optimal policy and develop algorithms to compute it. We establish conditions under which the optimal policy demonstrates certain monotonicity property, which, in turn, can greatly enhance computation. We also analyze the myopic policy and its optimality, and present a numerical study to illustrate the interplay of the pricing and inventory decisions.

**Keywords:** Inventory; Dynamic Pricing; Substitution; Consumer Choice; Market Share; M-matrices, Inverse M-matrices.

## 1 Introduction

Enabled by information technology, demand and revenue management has allowed many companies to gain competitive advantage. Hence, it is no surprising that the subject has received tremendous attention in the last decade or so, from both industry and academia. Numerous researchers have studied optimal joint pricing and inventory decisions. However, most of these studies assume either a single product or multiple products but a single period. (See Elmaghraby and Keskinocak 2003, Chan et al. 2004 and Section 2 for reviews.) This paper analyzes a multi-period and multi-product model, which extends both of these streams of research.

In particular, we consider a finite-horizon, periodic-review inventory system of a set of substitutable products with price-dependent random demand. At the beginning of each period, based on the current inventory status, we decide the price and order quantity for each product. The replenishments arrive immediately, before demand occurs. The pricing decision influences the demand vector in the current period. Unsatisfied demands are backlogged subject to linear penalty costs. Inventories incur linear holding costs. The objective is to maximize the total expected discounted profit.

A motivating example of this model is the "demand conditioning" practice by Dell Computer. Dell constantly changes the product offerings (configuration and pricing) on its web site to influence consumer demand. The driving forces for these changes are the current inventory levels of various components. (Reference) The underlying philosophy is to ensure high "inventory velocity" (Dell 1998).

The setting is also applicable to any online retailer selling a set of products differentiated in either quality or style, such as laptop computers, digital cameras, or running shoes. Demands arise randomly and are price sensitive, so the products are price-driven substitutable. Due to fast-changing technology and consumer tastes, the wholesale prices of the products may change over time. To meet the challenge, in addition to the regular inventory replenishment decisions, the retailer may set prices based on the current inventory status to influence future demands. This approach can help facilitate the sales of slowly moving items so to reduce potential losses due to obsolescence. By diverting some demands for products with low inventory to those with high inventory, it can also help reduce backlogs for some products and at the same time generate earlier realized revenues from other products. But, exactly how to make such pricing decisions? Which products should be replenished and how much? These are the questions we aim to address in this paper.

In Section 3, we introduce a general nonlinear stochastic demand function. It encompasses both additive and multiplicative uncertainty models. It also includes the well-known linear demand model and the multinomial logit (MNL) and locational choice models as special cases.

In Section 4, we formulate the problem as a dynamic program. The original formulation is not jointly concave in the decision variables and is therefore intractable. However, the problem becomes jointly concave if we work with the inverse of the price vector – the market share vector. With this decision variable transformation, we are able to characterize the optimal policy and develop algorithms to compute it. For each period, the optimal policy consists of three components: 1) the not-to-order list, 2) the order-up-to levels, and 3) the target market shares. The not-to-order list records the products that need not to order. The order-up-to levels determine the ordering amounts. The target market shares provide the optimal market share assigned to each product, which, in turn, determines the optimal price for each product. The optimal order-up-to levels and

target market shares are functions of the inventory levels of products on the not-to-order list.

In Section 5, we establish conditions under which the optimal policy demonstrates certain monotonicity property – the Property D. That is, the optimal base-stock levels of any subset of the products are decreasing in the initial inventory levels of the products in the complementary set. This property, in turn, greatly enhances computation of the optimal policy (see the Pooling Algorithm).

Section 6 studies the myopic policy and its optimality. There, we also report a numerical study to illustrate the interplay of the pricing and inventory decisions. Section 7 summarizes the main findings and discusses future research directions. For expositional purposes, all proofs are provided in Appendix.

## 2 Related Literature

Most papers on multi-period joint pricing and replenishment decisions concern single-product, periodic-review inventory systems. These works differ in their modeling assumptions on (1) the treatment of unsatisfied demand (backordering or lost-sales), (2) the ordering cost structure (with or without a fixed ordering cost), and (3) the form of the demand function (additive or multiplicative). Assuming backordering and a linear ordering cost, Federgruen and Heching (1999) showed that a base-stock list-price policy is optimal. They also demonstrated that dynamic pricing can result in significant benefit compared to static pricing. Chen and Simchi-Levi (2004 a,b) considered both variable and fixed ordering costs. They proved that an  $(s, S, p)$  policy is optimal for additive demand, and an  $(s, S, A, p)$  policy is optimal for multiplicative demand, assuming the unsatisfied demand is fully backordered. With the same cost structure but assuming lost-sales and an additive demand function, Chen, et al. (2003) showed that an  $(s, S, p)$  policy is optimal. Song et al. (2005) extended this result to multiplicative demands. Finally, Huh and Janakiraman (2005) streamlined the above results by using a sample path argument. In this paper, we consider multiple products, backordering, linear ordering costs, and both additive and multiplicative demands.

There is a rich literature on inventory planning with substitutable products. But almost all those models assume predetermined prices and a single period. Some models allow hierarchy substitution, in which a product of higher quality can be used to fulfill the demand for products of lower quality. See, for example, Bassok, et al. (1999) and Chen, et al. (1999). Other authors focused on the stock-out based substitution, where consumers might switch to some other available products when their first choice is out of stock. See, for example, Parlar and Goyal (1984), Ernst and Kouvelis (1999), Smith and Agrawal (2000), Rajaram and Tang (2001), and Honhon et al. (2006). Still other papers study assortment-based substitution, assuming a consumer may switch the most preferred product when the composition of a category changes. This is static substitution.

That is, consumers do not look for a substitute in a given category, if their first choice is stocked out. Thus, in these papers the optimal size and composition of a category is of special interest. See, for example, van Ryzin and Mahajan (1999), Mahajan and van Ryzin (2001), Gaur and Honhon (2006). We refer the reader to Kök et al. (2006) for a complete review of inventory management of substitutable products and assortment management.

Our model differs from this literature in that we do not assume fixed prices and we focus on price-driven substitutions, i.e., a consumer's favorite within a category may change when prices change. We do not consider assortment-based substitution, since we study a certain category. We also ignore stock-out based substitution and assume that the consumers will delay their purchases when their favorite products are out of stock.

Aydin and Porteus (2005) studied joint pricing and inventory decisions for an assortment in a single-period setting. Tomlin and Wang (2007) investigated a single-period, random yield production system of two products with downward substitution. They consider joint decisions on production quantity, product prices and the downconversion quantity. We are interested in the *dynamic* pricing and inventory decisions in a multi-period setting.

We are aware of only two previous studies that investigate dynamic pricing and inventory policies for multiple products within a category. Hall et al. (2003) examined a setting with (1) non-stationary procurement costs, (2) joint setup costs of ordering, (3) both own-price and cross-price effects of all products within a category, and (4) deterministic demand. In this paper, we assume demand is uncertain. Zhu and Thonemann (2005) investigated combined pricing and inventory decisions across *two* products, assuming (1) the cross-price effect is linear, and (2) the random demand has additive form. They segment the state space into four regions and characterize the optimal policy in each region. Our model and results include theirs as a special case. Our analytical approach is very different from theirs, which appears to be the key enabler to study a much more general model.

### 3 Demand Model and Preliminaries

Throughout the paper, all vectors are *column* vectors and are expressed by boldface lowercase letters, such as  $\mathbf{u} = (u_i)_{i=1}^N$ . If two vectors  $\mathbf{u}$  and  $\mathbf{v}$  have the same dimension  $N$ , their inner product is denoted by  $\mathbf{u}\mathbf{v} = \sum_{i=1}^N u_i v_i$ . Matrices are expressed by boldface uppercase letters, such as  $\mathbf{U} = (u_{ij})$ . Let  $\mathbf{g} : \mathbb{R}^N \rightarrow \mathbb{R}^M$ . For any  $\mathbf{x} \in \mathbb{R}^N$ , we denote by  $\nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x})$  the  $M \times N$  matrix whose  $(m, n)^{th}$  entry is  $\partial g_m(\mathbf{x})/\partial x_n$ . For any function  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  and any  $\mathbf{x} \in \mathbb{R}^N$ , let  $\nabla_{\mathbf{x}}^2 g(\mathbf{x})$  be its Hessian matrix, i.e., the  $N \times N$  matrix whose  $(i, j)^{th}$  entry is  $\partial^2 g/\partial x_i \partial x_j$ .

In general, we use  $F_X$ ,  $f_X$ , and  $\mu_X$  to denote the c.d.f., p.d.f. and mean of a random variable  $X$ .

### 3.1 General Demand Model

Consider a retailer managing a category of substitutable products, indexed by  $\mathcal{J} = \{1, \dots, J\}$ , which are differentiated in their attributes. We now present a general demand model to describe consumers' choices among different products in response to prices. In later subsections, we illustrate a few special cases.

Assume the planning horizon contains of  $T$  periods. Consider any period  $t \in \{1, \dots, T\}$ . Let  $p_{jt}$  be the price for product  $j$  at the beginning of  $t$ , and  $\mathcal{P}_t$  be the control space of  $\mathbf{p}_t = (p_{jt})_{j \in \mathcal{J}}$ . Define  $q_j(\mathbf{p}_t)$  be the probability that an arriving customer in period  $t$  selects product  $j$  in response to  $\mathbf{p}_t$ . We also call  $q_j(\mathbf{p}_t)$  the *market share* of product  $j$  in period  $t$ , a term coined in the literature of MNL models (see Anderson et al. 1992). Here, “ $q$ ” is used to imply the *quota* of the market assigned to each variant in the category. Note that the total market share  $\sum_{j \in \mathcal{J}} q_j(\mathbf{p}_t) \leq 1$ .

Given any price vector  $\mathbf{p} \in \mathcal{P}_t$ , we assume the demand in that period,  $\mathbf{d}_t(\mathbf{p}) = (d_{jt}(\mathbf{p}))_{j \in \mathcal{J}}$ , is of the following form:

$$\mathbf{d}_t(\mathbf{p}) = \mathbf{q}(\mathbf{p})\Lambda_t + \mathbf{L}(\mathbf{q}(\mathbf{p}))\varepsilon_t. \quad (1)$$

Here,

- $\mathbf{q}(\mathbf{p}) = (q_j(\mathbf{p}))_{j \in \mathcal{J}}$  is the market-share function defined above. It satisfies  $\partial q_j / \partial p_j < 0$  and  $\partial q_j / \partial p_k > 0$  for  $j \neq k$ , so the products are price-driven substitutable.
- $\Lambda_t$  is a random variable with finite mean and variance, representing the volume of potential customers in period  $t$ .
- $\varepsilon_t = (\varepsilon_{jt})_{j \in \mathcal{J}}$  is the vector of random error terms, in which  $\varepsilon_{jt}$  has zero mean and finite variance, representing the uncertainty of the demand of product  $j$ . The random errors  $\varepsilon_{jt}$  can be correlated cross  $j$ , either positively or negatively.
- $\mathbf{L}(\mathbf{q})$  is a  $J \times J$ -matrix.

The demand function consists of two parts – a controllable part  $\mathbf{q}$ , which is sensitive to prices, and an uncontrollable, stochastic part  $(\Lambda_t, \varepsilon_t)$ , which are independent across time periods. Note that the expected demand of product  $j$  is proportional to the market share  $q_j(\mathbf{p})$ .

This demand model unifies several commonly used, but often separately considered random demand models in the literature:

- (1) *Additive-I* demand:  $\mathbf{d}_t(\mathbf{p}) = \mathbf{q}(\mathbf{p})\mu_{\Lambda_t} + \varepsilon_t$ . Here,  $\Lambda_t = \mu_{\Lambda_t}$ , and  $\mathbf{L}(\mathbf{q}(\mathbf{p})) = \mathbf{I}$ , a  $J \times J$  identity matrix.
- (2) *Additive-Diag*( $\mathbf{q}(\mathbf{p})$ ) demand,  $\mathbf{d}_t(\mathbf{p}) = \mathbf{q}(\mathbf{p})\mu_{\Lambda_t} + (q_j(\mathbf{p})\varepsilon_{jt})_{j \in \mathcal{J}}$ . Here,  $\Lambda_t = \mu_{\Lambda_t}$  and  $\mathbf{L}(\mathbf{q}(\mathbf{p})) = \mathbf{Diag}(\mathbf{q}(\mathbf{p}))$ , a diagonal matrix with the  $(j, j)^{th}$  entry equal to  $q_j(\mathbf{p})$ .

(3) *Multiplicative* demand:  $\mathbf{d}_t(\mathbf{p}) = \mathbf{q}(\mathbf{p})\Lambda_t$ . Here,  $\varepsilon_t = \mathbf{0}$ .

### Variable Transformation: The Market-Share Vector

It turns out that the expected one-period revenue function is not jointly concave in the inventory vector and the price vector  $\mathbf{p}$ . This fact makes it intractable to use dynamic programming to characterize the optimal policy. We observe, however, that in several commonly used models, if we use the *inverse* of the market-share function, i.e., to express the price vector  $\mathbf{p}$  as a function of the market-share vector  $\mathbf{q}$ , then the revenue function becomes jointly concave, allowing us to invoke dynamic programming. Thus, in the rest of this paper, we shall use  $\mathbf{q}$  as the control variable instead of original price vector  $\mathbf{p}$ . Correspondingly, we now make this variable change in the demand function. Let

$$\mathbf{p}(\mathbf{q}) = (p_j(\mathbf{q}))_{j \in \mathcal{J}}, \text{ where } \mathbf{q} \in \mathcal{Q}_t = \{\mathbf{q}(\mathbf{p}) \mid \mathbf{p} \in \mathcal{P}_t\}$$

be the inverse of the market-share function  $\mathbf{q}(\mathbf{p})$ , and term it *the price function*. Then, the demand function can be expressed as

$$\mathbf{d}_t(\mathbf{q}) = \mathbf{q}\Lambda_t + \mathbf{L}(\mathbf{q})\varepsilon_t. \quad (2)$$

### The Revenue Rate Function

For any period  $t$ , let  $\hat{R}_t(\mathbf{q}) = \mathbf{p}(\mathbf{q})\mathbf{d}_t(\mathbf{q})$ . Then the expected revenue in period  $t$  is  $R_t(\mathbf{q}) = E\hat{R}_t(\mathbf{q}) = \mu_{\Lambda_t}\mathbf{p}(\mathbf{q})\mathbf{q} = \mu_{\Lambda_t}r(\mathbf{q})$ . Here,  $r(\mathbf{q}) = \mathbf{p}(\mathbf{q})\mathbf{q}$  is the *revenue rate* function, which is the ratio of the expected revenue to the expected volume of potential customers. Throughout the paper, we impose the following assumption about  $r(\mathbf{q})$ .

**Assumption 1** *The revenue rate function  $r(\mathbf{q})$  is concave in  $\mathbf{q}$ , for  $\mathbf{q} \in \mathcal{Q}_t$ , and  $\mathcal{Q}_t$  is a compact and convex set.*

In the multi-product revenue management literature, similar to our approach, several authors used demand rates as decision variables and made concavity assumption on the revenue rate function. See, for example, Talluri and van Ryzin (2005), Section 7.3.2, Maglaras and Meissner (2006), Dong et al. (2007). In this literature, there is no inventory replenishment decisions. We refer the reader to Elmaghraby and Keskinocak (2003) and Talluri and van Ryzin (2005) for more details.

In the rest of this section, we present several popular market-share models as special cases of our general model. For each case, we derive the price function as the inverse market-share function and demonstrate that Assumption 1 is valid.

### 3.2 Special Case 1: Linear Demand Model

One common model of price-substitutable demand can be represented by the following linear function:

$$q_j(\mathbf{p}) = a_j - u_{jj}p_j - \sum_{j \neq k} u_{jk}p_k, \quad \text{where } \sum_{j \in \mathcal{J}} a_j = 1, \quad u_{jj} > 0, \quad u_{jk} \leq 0, \quad k \neq j, \quad k, j \in \mathcal{J}.$$

Here,  $u_{jj}$  and  $u_{jk}$  are price and cross-price sensitivity parameters. The price range is  $\mathcal{P} = \{\mathbf{p} : q_j(\mathbf{p}) \geq 0, j \in \mathcal{J}\}$ . Let  $\mathbf{a} = (a_j)_{j \in \mathcal{J}}$  and  $\mathbf{U} = [u_{jk}]_{J \times J}$ . Then the above market-share function can be written as  $\mathbf{q}(\mathbf{p}) = \mathbf{a} - \mathbf{U}\mathbf{p}$ .

Assume  $u_{jj} > \sum_{k \neq j} |u_{jk}|$  or  $u_{jj} > \sum_{k \neq j} |u_{kj}|$ , for all  $j \in \mathcal{J}$ . In other words, the aggregate cross-price effect is less than the self-price effect. Then  $\mathbf{U}$  is invertible and positive definite (Horn and Johnson 1994, Theorem 6.1.10).

**Proposition 1** *Consider the linear demand model above. The price function  $\mathbf{p} = \mathbf{U}^{-1}(\mathbf{a} - \mathbf{q})$  is then well defined. Also, the revenue rate function  $r(\mathbf{q}) = \mathbf{q}\mathbf{U}^{-1}(\mathbf{a} - \mathbf{q})$  is strictly concave.*

The two-product linear demands with additive- $\mathbf{I}$  randomness considered by Zhu and Thonemann (2005) is a special case of this model.

### 3.3 Special Case 2: Multinomial Logit (MNL) Choice Model

An MNL (logit) model is often used to study a set of products differentiated in either quality or style, such as digital cameras. A product is differentiated from others by the average value of consumers' overall perception on the product. An individual customer  $i$ 's utility of product  $j$  is characterized by  $U(\theta_{ij}) = \Theta_j - p_j + \theta_{ij}$ , where  $\Theta_j$  is the average perceptive value on product  $j$ ,  $\theta_{ij}$  is customer  $i$ 's deviation from  $\Theta_j$ . Let  $\Theta_0 \geq 0$  and  $p_0 \geq 0$ , represent the average value and price offered by the competitors. Let  $\theta_{i0}$  be customer  $i$ 's deviation from  $\Theta_0$ . Without loss of generality, we assume  $\Theta_0 = 0$  and  $p_0 = 0$ . Then, a customer will always purchase one product with the highest utility among these  $J + 1$  options, that is, customer  $i$  selects product  $j$ , whenever  $j \in \arg \max_{l \in \mathcal{J} \cup \{0\}} \{\Theta_l - p_l + \theta_{il}\}$ . Assume  $(\theta_{i0}, \theta_{i1}, \dots, \theta_{iJ})$  is a realization of the random variables,  $(\vartheta_0, \vartheta_1, \dots, \vartheta_J)$ , which describe the distribution of consumers' deviation from the average perceptive value. We further assume the  $\vartheta_j$ , for  $j \in \mathcal{J} \cup \{0\}$ , is i.i.d distributed with a double exponential distribution, that is,  $Pr(\vartheta_j \leq \theta) = e^{-e^{-\theta}}$ ,  $-\infty < \theta < \infty$ . See Guadagni and Little (1983) and Anderson et al. (1992).

It can be shown (Theil 1969, McFadden 1974, Anderson et al. 1992) that a consumer's choice probability has a simple form:

$$q_j(\mathbf{p}) = \frac{\exp(\Theta_j - p_j)}{1 + \sum_{l \in \mathcal{J}} \exp(\Theta_l - p_l)}, \quad j \in \mathcal{J}. \quad (3)$$

This function satisfies the property of our general demand model. We have:

**Proposition 2** *Under the logit model, the price function is*

$$p_j = \Theta_j + \ln(1 - \sum_{l \in \mathcal{J}} q_l) - \ln(q_j), \quad \forall j \in \mathcal{J}. \quad (4)$$

The corresponding market-share space  $\mathcal{Q} = \{\mathbf{q} : \mathbf{p}(\mathbf{q}) \geq 0\}$  can be expressed as

$$\mathcal{Q} = \{\mathbf{q} : 0 \leq \sum_{l \in \mathcal{J}_s} q_l \leq \frac{\sum_{l \in \mathcal{J}_s} \exp(\Theta_l)}{1 + \sum_{l \in \mathcal{J}_s} \exp(\Theta_l)}, \quad \forall \mathcal{J}_s \subseteq \mathcal{J},\} \quad (5)$$

which is convex and compact. The revenue rate function

$$r(\mathbf{q}) = \mathbf{p}(\mathbf{q})\mathbf{q} = \sum_{l \in \mathcal{J}} [\Theta_l q_l + q_l \ln(1 - \sum_{j \in \mathcal{J}} q_j) - q_l \ln(q_l)] \quad (6)$$

$r(\mathbf{q})$  is strictly concave in  $\mathbf{q} \in \mathcal{Q}$ .

### 3.4 Special Case 3: Locational Choice Model

A locational (spacial) model concerns a set of products that provides similar quality but various styles, such as a garment category consists of Polo T-shirts having various colors and patterns. These products locate in a space that describes consumers' taste. Each consumer's taste can be represented as a point in the space, called *ideal point*. If the ideal point is not at the same position of a product, the consumer feels unsatisfactory due to the taste mismatch. Compared to the logit model, locational model provides a more tractable framework to analyze the horizontal differentiation, and avoid the independence of irrelevant alternatives (IIA) property, which is the shortcoming of logit model, see Guadagni and Little (1983), and Talluri and van Ryzin (2005).

Assume the taste space is a horizontal line (Hotelling 1929). The position of product  $j$  is  $\tau_j$ . Assume all products have the same perceptive quality  $\Psi$ . A customer's utility of product  $j$  is characterized by

$$U(\tau, \tau_j) = \Psi - \kappa|\tau - \tau_j| - p_j, \text{ for } j \in \{1, \dots, J\}, \quad (7)$$

in which  $\tau$  is the consumer's ideal point, and  $\kappa$  is the transportation cost, measuring the loyalty of the consumer. The quantity  $\kappa|\tau - \tau_j|$  measures the disutility of taste mismatch. Consumers self-select a product in the category, which gives them the highest positive utility. Assume  $\tau$  is uniformly distributed on the unit line.

In the following we derive the market share function of this model and verify the concavity of its revenue rate function. For simplicity, we focus on the simplest case of two products. The results hold true in general.

According to (7), if a product is priced at  $\Psi$ , it does not provide positive utility to any buyer, and therefore does not intervene the pricing of other products. So  $p_{max} = \Psi$  can be regarded

as the null price, see Karlin and Carr (1962) and Gallego and van Ryzin (1994). For any given  $0 \leq p_j, p_k \leq p_{max}$ , define

$$\begin{aligned}\tau_j^l &= \tau_j - \frac{\Psi - p_j}{\kappa}, \\ \tau_j^r &= \tau_j + \frac{\Psi - p_j}{\kappa}, \\ \tau_{jk}^m &= \frac{1}{2} (\tau_j^r + \tau_k^l), \quad \text{for } j < k.\end{aligned}$$

Then  $\tau_j^l$  is the location of the left most customer who feels nonnegative utility on product  $j$ ,  $\tau_j^r$  the location of right most customer who feels nonnegative utility on product  $j$ , and  $\tau_{jk}^m$  the location of the customer who feels indifferent between products  $j$  and  $k$ .

To establish a continuous market share function, we impose an assumption on the price gap so that the price space is given by

$$\mathcal{P} = [0, \Psi] \times [0, \Psi] \cup \{(p_1, p_2) \mid \kappa(\tau_2 - \tau_1) \geq |p_1 - p_2|\}.$$

That is, the prices will be set in a way so that the most loyal customer will not be turned away from his/her favorite product and not a single product can dominate the entire market.

**Proposition 3** *Consider the two-product logit model. Assume  $\mathbf{p} \in \mathcal{P}$ .*

(1) *The market share function  $q_j(\mathbf{p})$  is concave in  $\mathbf{p}$ , for  $j \in \mathcal{J}$ , where*

$$\begin{cases} q_1(\mathbf{p}) = \min\{\tau_{12}^m, \tau_1^r\} - \max\{0, \tau_1^l\}, \\ q_2(\mathbf{p}) = \min\{1, \tau_2^r\} - \max\{\tau_{12}^m, \tau_2^l\}, \end{cases}$$

(2) *(Symmetric positioning) Assume  $\tau_1 = 0$  and  $\tau_2 = 1$ , and  $\mathcal{Q} = \{\mathbf{q} \mid 0 \leq q_1, q_2 \leq 1; q_1 + q_2 \leq 1\}$ .*

*The price functions can be expressed as  $p_i(\mathbf{q}) = \Psi - \kappa q_i$ ,  $i = 1, 2$ , which are concave in  $\mathbf{q}$ .*

*The revenue function  $r(\mathbf{q})$  is also concave in  $\mathbf{q}$ .*

(3) *(Asymmetric positioning) Assume  $0 < \tau_1 < 1$  and  $\tau_2 = 1$ , and  $\mathcal{Q} = \{\mathbf{q} \mid 0 \leq q_1 \leq 1; 0 \leq q_2 \leq 1 - \tau_1; q_1 + q_2 \leq 1\}$ . The price functions can be expressed as*

$$\begin{cases} p_1(\mathbf{q}) = \min\{\Psi - \frac{\kappa q_1}{2}, \Psi - \kappa(q_1 - \tau_1), \Psi + \kappa(1 - \tau_1 - q_1 - q_2)\}, \\ p_2(\mathbf{q}) = \min\{\Psi - \kappa q_2, \Psi + \kappa(\frac{3}{2}(1 - \tau_1) - q_1 - 2q_2)\}, \end{cases}$$

*which are concave in  $\mathbf{q}$ . The revenue function  $r(\mathbf{q})$  can be expressed as*

$$r(\mathbf{q}) = \min \left\{ \begin{array}{l} q_1[\Psi - \frac{\kappa q_1}{2}] + q_2[\Psi - \kappa q_2], \\ q_1[\Psi - \kappa(q_1 - \tau_1)] + q_2[\Psi - \kappa q_2], \\ q_1[\Psi + \kappa(1 - \tau_1 - q_1 - q_2)] + q_2[\Psi + \kappa(\frac{3}{2}(1 - \tau_1) - q_1 - 2q_2)], \end{array} \right\}$$

*which is concave in  $\mathbf{q}$ .*

In the symmetric case, the market shares of two products are still correlated, since  $\mathcal{Q}$  requires  $q_1 + q_2 \leq 1$ . The proposition shows the optimal policy can also be applied to the locational model.

## 4 Joint Inventory and Pricing Decisions

In this section, we formulate the problem as a dynamic program and characterize the optimal policy. We also develop algorithms to compute the optimal policy.

### 4.1 Dynamic Programming Formulation

We consider a  $T$ -period problem. At the beginning of period  $t$ ,  $t \in \{1, \dots, T\}$ , we review the current inventory, set product prices and place replenishment orders. The replenishment orders arrive immediately, before the demand unfolds. The demand for the current period depends on the newly set product prices. Unsatisfied demands are backlogged.

Define

$x_{jt}$  =inventory level of product  $j$  before ordering at the beginning of period  $t$ ,  $\mathbf{x}_t = (x_{jt})_{j \in \mathcal{J}}$ .

$y_{jt}$  =inventory level of product  $j$  after ordering at the beginning of period  $t$ ,  $\mathbf{y}_t = (y_{jt})_{j \in \mathcal{J}}$ .

$q_{jt}$  =market share for product  $j$  in period  $t$ ,  $\mathbf{q}_t = (q_{jt})_{j \in \mathcal{J}}$ ,  $\mathbf{q}_t \in \mathcal{Q}_t$ .

$c_{jt}$  =unit procurement cost of product  $j$ ,  $\mathbf{c}_t = (c_{jt})_{j \in \mathcal{J}}$ .

$h_{jt}$  =holding cost for unit excess of product  $j$ ,  $\mathbf{h}_t = (h_{jt})_{j \in \mathcal{J}}$ .

$b_{jt}$  =penalty cost for unit shortage of product  $j$ ,  $\mathbf{b}_t = (b_{jt})_{j \in \mathcal{J}}$ .

Let  $\hat{H}_t(\mathbf{z}) = \mathbf{h}_t[\mathbf{z}]^+ + \mathbf{b}_t[\mathbf{z}]^-$ . Then  $H_t(\mathbf{y}, \mathbf{q}) = E\hat{H}_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q}))$  is the expected inventory cost in period  $t$ .

**Lemma 1**  $H_t(\mathbf{y}, \mathbf{q})$  is jointly concave in  $(\mathbf{y}, \mathbf{q})$  for all  $t$ .

Let  $v_t(\mathbf{x})$  be the maximum total expected discounted profit from period  $t$  to  $T$ , with a discount factor  $\beta$ . Recall that  $R_t(\mathbf{q})$  is the expected revenue in period  $t$  have

$$\begin{aligned} v_t(\mathbf{x}) &= \max_{\mathbf{y} \geq \mathbf{x}, \mathbf{q} \in \mathcal{Q}_t} \{ \beta R_t(\mathbf{q}) - \mathbf{c}_t(\mathbf{y} - \mathbf{x}) - H_t(\mathbf{y}, \mathbf{q}) + \beta E v_{t+1}(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) \}, \\ v_{T+1}(\mathbf{x}) &= \mathbf{c}_{T+1} \mathbf{x}. \end{aligned} \quad (8)$$

Define  $V_t(\mathbf{x}) = v_t(\mathbf{x}) - \mathbf{c}_t \mathbf{x}$ , for  $t \in \{1, \dots, T+1\}$ . We have

$$V_t(\mathbf{x}) = \max_{\mathbf{y} \geq \mathbf{x}, \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}), \text{ for } t \in \{1, \dots, T\}, \quad (9)$$

where

$$G_t(\mathbf{y}, \mathbf{q}) = \beta \mu_{\Lambda_t} r(\mathbf{q}) - \beta \mu_{\Lambda_t} \mathbf{c}_{t+1} \mathbf{q} + (\beta \mathbf{c}_{t+1} - \mathbf{c}_t) \mathbf{y} - H_t(\mathbf{y}, \mathbf{q}) + \beta E v_{t+1}(\mathbf{y} - \mathbf{d}_t(\mathbf{q})). \quad (10)$$

## 4.2 Optimal Policy and Algorithm

Our first result is:

**Theorem 1** *Consider any time period  $t$  with initial inventory  $\mathbf{x}$ . We have*

(a)  $V_{t+1}$  is concave and non-increasing.

(b)  $G_t(\mathbf{y}, \mathbf{q})$  is jointly concave in  $\mathbf{y}$  and  $\mathbf{q}$ , and has at least one maximizer,

$$(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J, \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}),$$

where  $\emptyset$  means no constraint is on the ordering decision  $\mathbf{y}$ , while maximizing  $G_t(\mathbf{y}, \mathbf{q})$ .

(c) If  $\mathbf{x} \leq \mathbf{y}_t^*(\emptyset)$ , then it is optimal to order up to  $\mathbf{y}_t^*(\emptyset)$ .

The theorem indicates that there is a set of threshold inventory levels that achieves the global maximum of  $G_t(\mathbf{y}, \mathbf{q})$ , therefore these thresholds are the desired (optimal) inventory levels for period  $t$ . Consequently, if the initial inventory levels are all below the thresholds, it is optimal to order these products up to the thresholds.

However, what should one do if the initial inventory levels of some products are *higher* than the thresholds  $\mathbf{y}_t^*(\emptyset)$ ? To ease understanding, we first analyze a two-products case, and then generalize to  $J$  products.

### Two Products System

Now suppose  $J = 2$  and the initial inventory vector  $\mathbf{x}$  is not below  $\mathbf{y}_t^*(\emptyset)$ . There are two possibilities of interest: (i) the initial inventory level of only one of the products is above the threshold; (ii) the initial inventory levels of both products are above the thresholds. Propositions 4 and 5 below concern these two scenarios separately.

**Proposition 4** *In any period  $t$  with initial inventory  $\mathbf{x}$ , if  $x_1 > y_{1t}^*(\emptyset)$ , and  $x_2 \leq y_{2t}^*(\emptyset)$ , then:*

(a) *It is optimal not to order product 1. Therefore, the optimization problem reduces to:*

$$V_t(\mathbf{x}) = \max_{y_1=x_1, y_2 \geq x_2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}) = \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1).$$

(b) *Relax  $y_2 \geq x_2$  and obtain a new set of thresholds:*

$$(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1).$$

(Note  $y_{1t}^*(x_1) = x_1$ ). If  $x_2 \leq y_{2t}^*(x_1)$ , then it is optimal to order product 2 up to  $y_{2t}^*(x_1)$  and set market share at  $\mathbf{q}_t^*(x_1)$ . Otherwise, it is optimal not to order product 2; the optimal market share can be obtained by restricting  $\mathbf{y} = \mathbf{x}$  in the optimization problem.

$$\mathbf{q}_t^*(\mathbf{x}) \in \arg \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q}).$$

It is interesting to observe that it is possible not to order product 2, even though  $x_2 \leq y_{2t}^*(\emptyset)$ . This is different from the single-product system. There, the list price base-stock policy implies it is optimal to order a product if its initial inventory level is lower than the base-stock level  $y^*(\emptyset)$ . Here, with two products, if product 1 has a higher inventory level than the base-stock level, then it is optimal not to order it and possibly lower its price to quickly deplete the inventory. Because the products are substitutable, the expected demand of product 2 is likely to decrease. Therefore, the newly set base-stock level for product 2, conditional on the inventory of product 1, is possibly lower than the initial inventory level of product 2, rendering not to order product 2 as well.

For the second scenario, we have:

**Proposition 5** Consider any time period  $t$  with initial inventory  $\mathbf{x} > \mathbf{y}_t^*(\emptyset)$ . We have:

(a) It is optimal not to order at least one product. The original problem can be decomposed into two subproblems:  $V_t(\mathbf{x}) = \max \left\{ V_t^{\{1\}}(\mathbf{x}), V_t^{\{2\}}(\mathbf{x}) \right\}$ , where

$$V_t^{\{k\}}(\mathbf{x}) = \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_k = x_k), \quad k \in \{1, 2\}.$$

(The subproblem  $V_t^{\{k\}}$  assumes that it is optimal not to order product  $k$ .) Let  $\mathcal{J}^+$  be the set of all products for which it is optimal not to order. Then  $\mathcal{J}^+ \neq \emptyset$ , and  $\mathcal{J}^+$  can be determined by Algorithm 1 below.

(b) If  $\mathcal{J}^+ = \{k^*\}$ , then it is optimal not to order product  $k^*$  but order product  $j = 3 - k^*$  up to  $y_j^*(x_{k^*})$ , and set the market share at  $\mathbf{q}_t^*(x_{k^*})$ , where

$$(\mathbf{y}_t^*(x_{k^*}), \mathbf{q}_t^*(x_{k^*})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_{k^*} = x_{k^*}).$$

(c) If  $\mathcal{J}^+ = \mathcal{J}$ , then it is optimal to order neither product and set the market share at  $\mathbf{q}_t^*(\mathbf{x}) \in \arg \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q})$ .

The following algorithm determines the optimal not-to-order list  $\mathcal{J}^+$ ,

**Algorithm 1** (Branching Algorithm for Two Products)

Consider any time period  $t$  with initial inventory  $\mathbf{x} > \mathbf{y}_t^*(\emptyset)$ .

Step 1. Set  $V_t^{\mathcal{J}}(\mathbf{x}) = V_t^{\{k\}}(\mathbf{x}) = -\infty$ ,  $k = 1, 2$ .

Step 2. For each  $k \in \{1, 2\}$ , relax the constraint  $\mathbf{y} \geq \mathbf{x}$  in subproblem  $V_t^{\{k\}}$  and obtain a new set of thresholds

$$(\mathbf{y}_t^*(x_k), \mathbf{q}_t^*(x_k)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_k = x_k). \quad (11)$$

Step 3. For each  $k \in \{1, 2\}$ , let  $j = 3 - k$ . If  $x_j \leq y_{jt}^*(x_k)$ , compute  $V_t^{\{k\}}(\mathbf{x}) = \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_k = x_k, y_j = y_{jt}^*(x_k))$ . Otherwise, compute  $V_t^{\mathcal{J}}(\mathbf{x}) = \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q})$ .

Step 4.  $\mathcal{J}^+ = \arg \max\{V_t^{\{1\}}(\mathbf{x}), V_t^{\{2\}}(\mathbf{x}), V_t^{\mathcal{J}}(\mathbf{x})\}$ .

Note that  $\mathbf{x} > \mathbf{y}_t^*(\emptyset)$  implies that  $\mathbf{y}_t^*(\emptyset)$  is infeasible. Due to the joint concavity of  $G_t(\mathbf{y}, \mathbf{q})$ , we know that the optimal solution must be located on the boundary of the feasible region. The above branching algorithm is to simply search on the boundaries, which means the lines  $\{y_k = x_k\}$  (corresponding to solving  $V_t^{\{k\}}(\mathbf{x})$ ),  $k = 1, 2$ .

In contrast to the single-product case, Proposition 5 implies that it is not necessarily optimal to order no product, when  $\mathbf{x} > \mathbf{y}_t^*(\emptyset)$ . This is not very intuitive. As discussed after Proposition 4, when it is optimal not to order one product, it is possible to simultaneously lower that product's price. Due to substitutability, the demand for the other product is likely to decrease, hence its base-stock level should decrease, so it is unlikely optimal to order the second product. The following property ensures the new threshold for the second product is no larger than its initial base-stock level.

**Proposition 6** Consider two products. Take any time period  $t$  with initial inventory  $\mathbf{x} > \mathbf{y}_t^*(\emptyset)$ . Suppose the new thresholds  $y_{jt}^*(x_k)$  given by (11) satisfy the following **decreasing property**:

$$y_j^*(x_k) \leq y_j^*(\emptyset), \text{ for } j \neq k, k = 1, 2.$$

Then it is optimal not to order either product in period  $t$ , and set the market share at  $\mathbf{q}_t^*(\mathbf{x}) \in \arg \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q})$ . In particular, the decreasing property holds for Additive-I demand.

The decreasing property is a special case of Property D described later in Assumption 2 for the general system and further studied in Sections 5 and 6.

In the two-product case, our characterization of the optimal policy is consistent with that of Zhu and Thonemann (2005), who focused on the linear, additive-I demand. For any initial inventory levels, the not-to-order list  $\mathcal{J}^+$  has four possible outcomes:  $\emptyset$ ,  $\{1\}$ ,  $\{2\}$  or  $\{1, 2\}$ . This is

essentially the same as their four-part optimal policy that divides the state space in four regions by using Kuhn-Tucker conditions. Our approach, which constructs the not-to-order list by iteratively solving a sequences of equality constrained optimization problems, allows us to extend our analysis to the general  $J$ -product system.

### General System

For any vector  $\mathbf{z} = (z_j)_{j \in \mathcal{J}}$  and any set  $\mathcal{L} \subseteq \mathcal{J}$ , we denote  $\mathbf{z}_{\mathcal{L}} = (z_j)_{j \in \mathcal{L}}$ . To make the notation consistent, we define  $(\mathbf{y}_t^*(\mathbf{x}_{\emptyset}), \mathbf{q}_t^*(\mathbf{x}_{\emptyset}))$  to be  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ . We have:

**Theorem 2** (*Structure of Optimal Policy*) Consider a  $J$ -product system at any time period  $t$ , with initial inventory  $\mathbf{x}$ . The optimal policy for period  $t$  consists of three parts:

- (1) the not-to-order list  $\mathcal{J}^+$ , which contains the indices of the products for not ordering,
- (2) the order-up-to levels  $\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}^+})$ ,
- (3) the market share values  $\mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}^+})$ , where

$$(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}_{\mathcal{J}^+}).$$

$\mathcal{J}^+$  can be obtained from Algorithm 2 below.

For intuitive reasons, we term  $\mathbf{x}_{\mathcal{J}^+}$  the *overstocking levels* in period  $t$ . Note that if  $\mathbf{x} \leq \mathbf{y}_t^*(\emptyset)$  then  $\mathcal{J}^+ = \emptyset$  and there is no product *overstocked*, so we place an order for each product. In this case Theorem 2 reduces to Theorem 1, part (c).

Similarly to Algorithm 1 for the two-product case, we construct the following branching algorithm to generate the optimal not-to-order list  $\mathcal{J}^+$ . Here, again, when the solution for a relaxed problem is infeasible, we search on the boundaries of the feasible region. We call  $\mathcal{F}^+$  a *feasible list*, if  $y_j^*(\mathbf{x}_{\mathcal{F}^+}) \geq x_j$ , for  $j \in \mathcal{F}^- = \mathcal{J} \setminus \mathcal{F}^+$ . This is because  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}^+}))$  is a feasible solution. In Algorithm 2, we identify all feasible lists. The feasible list that provides the highest value function is the optimal not-to-order list.

### Algorithm 2 (*Branching Algorithm*)

Consider a  $J$ -product system at any time period  $t$ , with initial inventory  $\mathbf{x}$ .

*Step 0:* Let  $\mathcal{F}_0^+ = \emptyset$ ,  $V_t(\mathbf{x}) = -\infty$ . Set  $m = 1$  and go to Step  $m$ .

*Step  $m$ :* Solve the thresholds,  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{F}_{m-1}^+} = \mathbf{x}_{\mathcal{F}_{m-1}^+})$ .

If  $\mathbf{x} \leq \mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})$ , let  $\mathcal{F}^+ = \mathcal{F}_{m-1}^+$ , stop. Compute  $V_t^{\mathcal{F}^+}(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{F}^+} = \mathbf{x}_{\mathcal{F}^+})$ .

Update  $V_t(\mathbf{x})$  to  $\max\{V_t^{\mathcal{F}^+}(\mathbf{x}), V_t(\mathbf{x})\}$ .

Otherwise, define  $\mathcal{A}_m(\mathcal{F}_{m-1}^+) = \{j \mid x_j > y_{jt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})\}$ , a non-empty set. For each  $k_m \in \mathcal{A}_m(\mathcal{F}_{m-1}^+)$ . Let  $\mathcal{F}_m^+ = \mathcal{F}_{m-1}^+ \cup \{k_m\}$ . If  $\mathcal{F}_m^+ = \emptyset$ , stop. Otherwise, set  $m = m + 1$  and go to step  $m$ .

We now provide a three-product example to illustrate the algorithm.

**Example 1** Assume  $J = 3$  and in period  $t$  the initial inventory levels  $\mathbf{x} = (14, 4, 20)$ .

*Step 0:* Let  $\mathcal{F}_0^+ = \emptyset$ ,  $V_t(\mathbf{x}) = -\infty$ . Set  $m = 1$  and go to Step  $m$ .

*Step 1:* Suppose  $\mathbf{y}^*(\mathbf{x}_0) = (13, 14, 10)$  and compare it to  $\mathbf{x}$ . Because  $x_1 > y_1^*(\mathbf{x}_0)$  and  $x_3 > y_3^*(\mathbf{x}_0)$ , we have  $\mathcal{A}_1(\emptyset) = \{1, 3\}$ . We know that it is optimal not to order at least one of the two products, thereby we create two branches -  $\mathcal{F}_1^+ = \{1\}$  and  $\mathcal{F}_1^+ = \{3\}$ , respectively. In the following, we construct a feasible list along each branch.

*Step 2.1:* Start from  $\mathcal{F}_1^+ = \{1\}$ , we compute  $\mathbf{y}^*(x_1) = (14, 12, 11)$ , and have  $x_3 > y_3^*(x_1)$ . Then  $\mathcal{A}_2(1) = \{3\}$ , and we will continue to construct a feasible list along  $\mathcal{F}_2^+ = \{1, 3\}$ .

*Step 2.2* Start from  $\mathcal{F}_1^+ = \{3\}$ , we compute  $\mathbf{y}^*(x_3) = (15, 7, 20)$ , and have  $\mathbf{x} \leq \mathbf{y}^*(x_3)$ . Then we stop searching along  $\{3\}$ , record a feasible list  $\mathcal{F}^+ = \mathcal{F}_1^+ = \{3\}$ , compute  $V_t^{\{3\}}(\mathbf{x})$ , compare it to  $V_t(\mathbf{x})$ , and update it if  $V_t^{\{3\}}(\mathbf{x}) > V_t(\mathbf{x})$ .

*Step 3.1* Continue to search a feasible solution along  $\mathcal{F}_2^+ = \{1, 3\}$ . We compute  $\mathbf{y}^*(x_1, x_3) = (14, 8, 20)$  and have  $x_2 \leq y_2^*(x_1, x_3)$ . Then we stop searching along  $\{1, 3\}$ , record a feasible list  $\mathcal{F}^+ = \mathcal{F}_2^+ = \{1, 3\}$ , compute  $V_t^{\{1,3\}}(\mathbf{x})$  and update  $V_t(\mathbf{x})$  if  $V_t^{\{1,3\}}(\mathbf{x}) > V_t(\mathbf{x})$ .

Note that  $\mathcal{J}^+$  is an *optimal not-to-order list*, if  $V_t(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}_{\mathcal{J}^+})$ . We call  $\mathcal{J}_m^+$  a *partial list*, if  $V_t(\mathbf{x}) = \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}_m^+} = \mathbf{x}_{\mathcal{J}_m^+})$ . This is because we know it is optimal not to order products in  $\mathcal{J}_m^+$ . However, it is still possible not to order some products outside  $\mathcal{J}_m^+$  at optimality. A partial list is a subset of the optimal not-to-order list. For example, if the optimal not-to-order list  $\mathcal{J}^+ = \{1, 3\}$ , then  $\mathcal{J}_1^+ = \{1\}$  is a partial list. A list is an optimal not-to-order list, if it is both feasible and partial. To prove the existence of the optimal policy and the effectiveness of the branching algorithm, we need to show that there exists a *good path* that always updates a partial list and finally makes it feasible.

Recall that in the branching algorithm, each feasible list  $\mathcal{F}^+$  is a mapping to a feasible solution of the original problem. However, the solution might be suboptimal, since  $\{\mathcal{F}_m^+\}$  generated by the algorithm might not be a sequence of partial lists. Therefore, the branching algorithm can be time consuming. Below we present Assumption 2, under which we can avoid any unnecessary path that generates a feasible but suboptimal list.

**Assumption 2** *At any period  $t$  with initial inventory  $\mathbf{x}$ . Let  $\mathcal{L}^+ \subset \mathcal{J}$ ,  $\mathcal{L}^- = \mathcal{J} \setminus \mathcal{L}^+$ . The inventory thresholds derived from  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{L}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{L}^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{L}^+} = \mathbf{x}_{\mathcal{L}^+})$  possess the following decreasing property:*

(Property D) *If  $\mathbf{x}_{\mathcal{L}^+} \leq \mathbf{x}'_{\mathcal{L}^+}$ , then  $y_{jt}^*(\mathbf{x}_{\mathcal{L}^+}) \geq y_{jt}^*(\mathbf{x}'_{\mathcal{L}^+})$ , for  $j \in \mathcal{L}^-$ .*

We will discuss sufficient conditions for Property D in Section 5. Under Assumption 2, the next algorithm always updates a sequence of partial lists and finally reaches the optimal one.

**Algorithm 3** (*Pooling Algorithm*)

*Suppose Assumption 2 holds. We design two sequences of lists  $\mathcal{J}_m^+$  and  $\mathcal{J}_m^-$ , where  $\mathcal{J}_m^+$  is a sequence of partial lists, and  $\mathcal{J}_m^- = \mathcal{J} \setminus \mathcal{J}_m^+$ . The optimal list  $\mathcal{J}^+ = \mathcal{J}_M^+$ , which can always be made recursively in  $M$  steps, where  $M \leq J$ .*

*Step 0: Let  $\mathcal{J}_0^+ = \emptyset$ . Set  $m = 1$  and go to Step  $m$ .*

*Step  $m$ : Obtain  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}_{m-1}^+} = \mathbf{x}_{\mathcal{J}_{m-1}^+})$ .*

*If  $\mathbf{x} \leq \mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})$ , let  $\mathcal{J}_m^+ = \mathcal{J}_{m-1}^+ \cup \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}) = x_j\}$ , set  $M = m - 1$ , and stop.*

*Otherwise, define  $\mathcal{A}_m = \{j \mid x_j > y_{jt}^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})\}$ , a non-empty set. Let  $\mathcal{J}_m^+ = \mathcal{J}_{m-1}^+ \cup \mathcal{A}_m$ . If  $\mathcal{J}_m^- = \emptyset$ , let  $M = m$ , stop ordering and set market share at  $\mathbf{q}_t^*(\mathbf{x})$ . Otherwise, set  $m = m + 1$  and go to Step  $m$ .*

Now, we use a three-product example to illustrate the pooling algorithm. We first compute  $\mathbf{y}^*(\mathbf{x}_\emptyset)$  and compare it to  $\mathbf{x}$ . Suppose we have  $x_1 > y_1^*(\mathbf{x}_\emptyset)$  and  $x_3 > y_3^*(\mathbf{x}_\emptyset)$ , then  $\mathcal{A}_1 = \{1, 3\}$ , and  $\mathcal{J}_1^+ = \{1, 3\}$ . We compute  $\mathbf{y}^*(\mathbf{x}_{\mathcal{J}_1^+})$  and compare it to  $\mathbf{x}$ . Suppose  $x_2 \leq y_2^*(\mathbf{x}_{\mathcal{J}_1^+})$ , then we stop the algorithm and obtain the optimal not-to-order list  $\mathcal{J}^+ = \mathcal{J}_1^+$ .

To make the optimal solution unique, we require  $r(\mathbf{q})$  to be strictly concave. That is,

**Theorem 3** *Assume  $r(\mathbf{q})$  is strictly concave in  $\mathbf{q}$ , for  $\mathbf{q} \in \mathcal{Q}_t$ . Then (1)  $G_t(\mathbf{y}, \mathbf{q})$  is strictly concave in  $(\mathbf{y}, \mathbf{q})$ ; (2) the optimal solution generated by Algorithm 2 or Algorithm 3 is unique.*

## 5 Monotonicity Property of the Optimal Policy

In this section, we focus on identifying sufficient condition for Property D. This property implies that the optimal order-up-to levels for products that need to be ordered are nonincreasing in the overstocking levels. Even stronger, when the initial inventories of some products are higher, the optimal order-up-to levels for the other products stay the same or become lower.

Because this property concerns the monotonicity of the optimal solution of a dynamic program, a natural approach would be to investigate the supermodularity of the functions involved. However, in our setting, the market share space  $\mathcal{Q}_t$  is usually not a sublattice, the theory of supermodularity cannot be applied. Therefore, we resort to a direct approach by examining the Hessian matrices of the functions. To proceed, we first provide some preliminaries on the theory of M-matrices and inverse M-matrices, which is still an active area in the field of linear algebra.

### Preliminary

Let  $\mathbf{A}$  be an  $N \times N$  matrix. It is a *Z-matrix* if all the off-diagonal entries are non-positive.  $\mathbf{A}$  is an *M-matrix* if it is a Z-matrix and  $\mathbf{A}^{-1}$  is a nonnegative matrix.  $\mathbf{A}$  is an M-matrix, if and only if it is a positive definite Z-matrix. (See Horn and Johnson Definition 2.5.2, 1991.)  $\mathbf{A}$  is an *inverse M-matrix* if  $\mathbf{A}^{-1}$  is an M-matrix.

Given  $\mathbf{A}$ , for any non-empty index set  $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{N} = \{1, \dots, N\}$ . Let  $\mathbf{A}(\mathcal{N}_1, \mathcal{N}_2)$  be the submatrix of  $\mathbf{A}$  that lies in the rows indicated by  $\mathcal{N}_1$  and the columns indicated by  $\mathcal{N}_2$ . The principal submatrix  $\mathbf{A}(\mathcal{N}_1, \mathcal{N}_1)$  can be abbreviated to  $\mathbf{A}(\mathcal{N}_1)$ . For brevity,  $\mathbf{A}(\mathcal{N}_1, \{j\})$  and  $\mathbf{A}(\{i\}, \mathcal{N}_2)$  are denoted by  $\mathbf{A}(\mathcal{N}_1, j)$  and  $\mathbf{A}(i, \mathcal{N}_2)$  respectively.

Let  $\mathcal{N}^+ \subset \mathcal{N}$ , and  $\mathcal{N}^- = \mathcal{N} \setminus \mathcal{N}^+$ , both arranged in increasing order. Then we call  $\mathbf{A}/\mathcal{N}^+$  the Schur complement with respect to  $\mathbf{A}(\mathcal{N}^+)$ , defined as follows:

$$\mathbf{A}/\mathcal{N}^+ = \mathbf{A}/\mathbf{A}(\mathcal{N}^+) = \mathbf{A}(\mathcal{N}^-) - \mathbf{A}(\mathcal{N}^-, \mathcal{N}^+) [\mathbf{A}(\mathcal{N}^+)]^{-1} \mathbf{A}(\mathcal{N}^+, \mathcal{N}^-).$$

In our research, the following properties of *symmetric* M- and inverse M- matrices are of interest. (See Appendix for Lemma 3 and 4 on properties of general M- and inverse M- matrices.)

**Definition 1** A matrix  $\mathbf{A} = [a_{i,j}]$  in  $\mathbb{R}^{N \times N}$  is a *Stieltjes matrix* if

- (a)  $\mathbf{A}$  is a real symmetric M-matrix.
- (b)  $\mathbf{A}^{-1}$  is a real nonsingular and symmetric matrix with all entries nonnegative.

**Definition 2** A matrix  $\mathbf{A} = [a_{i,j}]$  in  $\mathbb{R}^{N \times N}$  is *strictly ultrametric* if

- (a)  $\mathbf{A}$  is symmetric with non-negative entries;
- (b)  $a_{i,j} \geq \min\{a_{i,k}; a_{k,j}\}$ , for all  $j, k, l \in \mathcal{N}$ ;
- (c)  $a_{i,i} > \max\{a_{i,k} : k \in \mathcal{N} \setminus \{i\}\}$  for all  $i \in \mathcal{N}$ .

The following result is from Martinez, et al. (1994), Nabben and Varga (1994):

**Lemma 2** *If  $\mathbf{A} = [a_{i,j}]$  is strictly ultrametric, then  $\mathbf{A}$  is nonsingular, and  $\mathbf{A}^{-1} = \mathbf{B} = [b_{i,j}]$  is a strictly diagonally dominant Stieltjes matrix, in which  $b_{i,j} \leq 0$ , for all  $i \neq j$ , and  $b_{i,i} > \sum_{k=1, k \neq i}^N |b_{i,k}|$ , for all  $1 \leq i, j \leq N$ .*

**Corollary 1** *A  $2 \times 2$  symmetric matrix  $\mathbf{A}$  is an inverse M-matrix, if and only if  $\mathbf{A}$  is a positive definite and nonnegative matrix.*

However, the above sufficient condition breaks down for matrices of dimensions greater than 2. For instance, in the following example,  $\mathbf{A}$  is a positive definite nonnegative matrix. However  $\mathbf{A}^{-1}$  is not a Z-matrix, so is not an M-matrix.

$$\mathbf{A} = \begin{bmatrix} 9 & 7 & 7 \\ 7 & 6 & 5 \\ 7 & 5 & 6 \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} 11 & -7 & -7 \\ -7 & 5 & 4 \\ -7 & 4 & 5 \end{bmatrix}.$$

### Inverse M-structure and Property D of Market Shares

We observe that Property D of the inventory thresholds is related to a similar property of the market shares. When the overstocking levels of some products increase, we should set higher market shares for these products to help deplete the inventory. This implies the market shares of other products should decrease, yielding lower expected demands for these products and their base-stock levels. Based on this observation, we introduce Property D of the market shares.

Consider a simple profit maximization problem at period  $t$ :  $\max \pi_t(\mathbf{q}) = \beta r(\mathbf{q}) - \mathbf{c}_t \mathbf{q}$ , under the constraints  $\mathbf{q}_{\mathcal{L}^+} = \hat{\mathbf{q}}_{\mathcal{L}^+}$ , for  $\mathcal{L}^+ \subset \mathcal{J}$ , and  $\mathcal{L}^- = \mathcal{J} \setminus \mathcal{L}^+$ . Let

$$\mathbf{q}_t^*(\hat{\mathbf{q}}_{\mathcal{L}^+}) \in \arg \max_{\mathbf{q} \in \mathcal{Q}_t} \pi_t(\mathbf{q} | \mathbf{q}_{\mathcal{L}^+} = \hat{\mathbf{q}}_{\mathcal{L}^+}).$$

Then Property D of the market shares is defined as follows:  $\mathbf{q}_{t\mathcal{L}^-}^*(\hat{\mathbf{q}}_{\mathcal{L}^+})$  is non-increasing in  $\hat{\mathbf{q}}_{\mathcal{L}^+}$ .

It turns out that to establish the above property, we need the following assumption:

**Assumption 3**  *$r(\mathbf{q})$  is twice differentiable, and the negative Hessian Matrix,  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q})]$ , is an inverse M-matrix.*

We have:

**Corollary 2** *Assumption 3 holds for the following models:*

- (a) *The linear demand model, in which  $\mathbf{U}$  is a Z-matrix and positive definite.*
- (b) *The MNL model.*

In the linear market share model, we have  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q})] = \mathbf{U}^{-1}$ . Therefore, Assumption 3 holds, if and only if  $\mathbf{U}$  is an M-matrix. However,  $\mathbf{U}$  is an M-matrix, if and only if it is a positive definite Z-matrix. The linear demand function  $\mathbf{q} = \mathbf{a} - \mathbf{U}\mathbf{p}$  describes substitutability ( $\partial q_j / \partial p_k > 0$ , for  $j \neq k$ ), if  $\mathbf{U}$  is a Z-matrix. Therefore, linear market share model for substitutable products satisfies Assumption 3, if and only if it satisfies Assumption 1.

In the MNL model, we can show that  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q})]$  is strictly ultrametric (see the proof of Proposition 2). In this model, we have  $\left| \frac{\partial^2 r(\mathbf{q})}{\partial q_j^2} \right| > \left| \frac{\partial^2 r(\mathbf{q})}{\partial q_j \partial q_k} \right|$ , for  $j \neq k$ . It means the marginal revenue rate of product  $j$ ,  $\frac{\partial r(\mathbf{q})}{\partial q_j}$ , is more responsive to its own market share change than to that of other products.

We can show:

**Corollary 3** *Under Assumption 3,  $\mathbf{q}_{t\mathcal{L}^-}^*(\hat{\mathbf{q}}_{\mathcal{L}^+})$  is non-increasing in  $\hat{\mathbf{q}}_{\mathcal{L}^+}$ , for  $\mathcal{L}^+ \subset \mathcal{J}$ , and  $\mathcal{L}^- = \mathcal{J} \setminus \mathcal{L}^+$ .*

### Property D of Inventory Threshold

We first consider Additive-I demand. Let  $\mathbf{s} = \mathbf{y} - \mu_{\Lambda_t} \mathbf{q}$ , be the safety stock. Now we use  $\mathbf{s}$  and  $\mathbf{q}$  as the control variables.  $G_t(\mathbf{y}, \mathbf{q})$  is transformed to  $\tilde{G}_t(\mathbf{s}, \mathbf{q})$ , which is separable in  $\mathbf{s}$  and  $\mathbf{q}$ . Let  $\tilde{H}_t(\mathbf{s}) = E\hat{H}_t(\mathbf{s} - \varepsilon_t)$

$$\tilde{G}_t(\mathbf{s}, \mathbf{q}) = \beta \mu_{\Lambda_t} r(\mathbf{q}) - \mu_{\Lambda_t} \mathbf{c}_t \mathbf{q} + (\beta \mathbf{c}_{t+1} - \mathbf{c}_t) \mathbf{s} - \tilde{H}_t(\mathbf{s}) + \beta EV_{t+1}(\mathbf{s} - \varepsilon_t).$$

We define the Lagrangian vector,  $\gamma = (\gamma_j)_{j \in \mathcal{J}}$ , and the Lagrangian function,  $\tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma) = \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma(\mathbf{s} + \mu_{\Lambda_t} \mathbf{q} - \mathbf{x})$ . We have the following results.

**Proposition 7** *For  $J = 2$ , under Additive-I demand, suppose Assumption 3 holds, then*

- (a)  $V_t(\mathbf{x})$  is twice differentiable everywhere in  $\mathbb{R}^2$ ;
- (b)  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is either a nonnegative diagonal matrix or an inverse M-matrix, thereby  $V_t(\mathbf{x})$  is submodular;
- (c)  $\left[ \nabla_{\mathbf{s}}^2 \left( \tilde{H}_{t-1}(\mathbf{s}) - \beta EV_t(\mathbf{s} - \varepsilon_{t-1}) \right) \right]$  is an inverse M-matrix, therefore  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_{t-1}(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix.

Now consider general  $J$ . Let  $\mathcal{L}^+$  be any subset of  $\mathcal{J}$  and  $\mathcal{L}^- = \mathcal{J} \setminus \mathcal{L}^+$ . Define

$$\mathbf{A}_t = - \left[ \left[ \nabla_{\mathbf{s}}^2 \tilde{G}_t \left( \mathbf{s}_t^*(\mathbf{x}_{\mathcal{L}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{L}^+}) \right) \right]^{-1} + \frac{\mu_{\Lambda_t}}{\beta} \left[ \nabla_{\mathbf{q}}^2 r \left( \mathbf{q}_t^*(\mathbf{x}_{\mathcal{L}^+}) \right) \right]^{-1} \right],$$

which is an M-matrix for  $J = 2$  (by Proposition 7 (c)). For  $\mathcal{L}^+ = \emptyset$ , we define

$$\mathbf{A}_t^* = - \left[ \left[ \nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \right]^{-1} + \frac{\mu_{\Lambda_t}}{\beta} \left[ \nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*(\emptyset)) \right]^{-1} \right].$$

We have

**Proposition 8** *For any  $J$ , at any period  $t$  with initial inventory  $\mathbf{x}$ , consider Additive-I demand. Suppose Assumption 3 holds.*

(a) *Property D holds, iff  $\mathbf{A}_t(\mathcal{L}^-, \mathcal{L}^+) \mathbf{A}_t(\mathcal{L}^+)^{-1}$  is non-postive.*

(b) *Property D holds, then  $\mathbf{A}_t^*$  is an M-matrix.*

(c) *Property D holds, if  $\mathbf{A}_t$  is an M-matrix.*

(d) *Property D holds, if  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix.*

Next, we consider the general random demand model. Let the Lagrangian vector,  $\gamma = (\gamma_j)_{j \in \mathcal{J}}$ , and the Lagrangian function,  $L_t(\mathbf{y}, \mathbf{q}, \gamma) = G_t(\mathbf{y}, \mathbf{q}) + \gamma(\mathbf{y} - \mathbf{x})$ . The Hessian matrix  $[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G_t(\mathbf{y}, \mathbf{q})]$  is a  $2J \times 2J$  matrix. We can partition into four blocks, each with the size  $J \times J$ . Let  $\mathbf{B}$  be a  $2J \times 2J$  matrix, such that

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}. \quad (12)$$

We define the generalized Schur complement  $\mathbf{B}/\mathbf{B}_{22}$ ,

$$\mathbf{B}/\mathbf{B}_{22} = \mathbf{B}_{11} - \mathbf{B}_{12} \mathbf{B}_{22}^{-1} \mathbf{B}_{21}. \quad (13)$$

We can express  $\mathbf{C} = \mathbf{B}^{-1}$  by using the Schur complement (see Zhang 2005 p13) as follows:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \mathbf{B}^{-1} = \begin{bmatrix} (\mathbf{B}/\mathbf{B}_{22})^{-1} & -(\mathbf{B}/\mathbf{B}_{22})^{-1} \mathbf{B}_{12} \mathbf{B}_{22}^{-1} \\ -\mathbf{B}_{22}^{-1} \mathbf{B}_{21} (\mathbf{B}/\mathbf{B}_{22})^{-1} & \mathbf{B}_{22}^{-1} + \mathbf{B}_{22}^{-1} \mathbf{B}_{21} (\mathbf{B}/\mathbf{B}_{22})^{-1} \mathbf{B}_{12} \mathbf{B}_{22}^{-1} \end{bmatrix} \quad (14)$$

**Proposition 9** *Let  $[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G_t(\mathbf{y}, \mathbf{q})]$  be partitioned as in (12), and its Schur complement is defined as in (13). If the Schur complement is an inverse M-matrix, then (a)  $V_t(\mathbf{x})$  is submodular, and (b) Property D holds.*

To our knowledge, little is known on the closure property of inverse M-matrices under addition (see Fallat, and Johnson 1999, Bru et al. 2005). Therefore, it is difficult to show the inverse M-structure to be preserved during the iteration of dynamic programming, for  $J > 2$ . However, we can use the above results to show Property D of the myopic policy, which will be discussed in Corollary 4 in the next section.

## 6 Stationary Analysis

In this section, we assume the demand process the cost parameters are stationary. We therefore suppress the index  $t$  from these parameters. In particular, the demand function now is of the following form:

$$\mathbf{d}_t(\mathbf{q}) = \mathbf{d}(\mathbf{q}) = \Lambda \mathbf{q} + \mathbf{L}(\mathbf{q})\varepsilon. \quad (15)$$

### 6.1 Myopic Policy

The total expected discount profit during  $T$  periods can be expressed as

$$V_1(\mathbf{x}_1) = E \left\{ \sum_{t=1}^T \beta^{t-1} [\beta \mathbf{p}(\mathbf{q}_t) \mathbf{d}(\mathbf{q}_t) - \mathbf{c}(\mathbf{y}_t - \mathbf{x}_t) - \hat{H}(\mathbf{y}_t - \mathbf{d}(\mathbf{q}_t))] + \beta^T \mathbf{c}(\mathbf{y}_T - \mathbf{d}(\mathbf{q}_T)) \right\}, \quad (16)$$

where  $\mathbf{c}(\mathbf{y}_T - \mathbf{d}(\mathbf{q}_T))$  is the salvage value, based on the assumption that the supplier is willing to buy back the remaining products at the wholesales price at the end of the planning horizon (e.g., a business season), or a discount store is willing to buy the remains at the price equal to the procurement cost. Substitute  $\mathbf{y}_{t-1} - \mathbf{d}(\mathbf{q}_{t-1})$  for  $\mathbf{x}_t$ , if  $t > 1$ , and rewrite  $V_1(\mathbf{x}_1)$ , we obtain:

$$\begin{aligned} V_1(\mathbf{x}_1) &= \mathbf{c}\mathbf{x}_1 + E \left\{ \sum_{t=1}^T \beta^{t-1} [\beta \mathbf{p}(\mathbf{q}_t) \mathbf{d}(\mathbf{q}_t) - \mathbf{c}\mathbf{y}_t + \beta \mathbf{c}(\mathbf{y}_t - \mathbf{d}(\mathbf{q}_t)) - \hat{H}(\mathbf{y}_t - \mathbf{d}(\mathbf{q}_t))] \right\} \\ &= \mathbf{c}\mathbf{x}_1 + \sum_{t=1}^T \beta^{t-1} G(\mathbf{y}_t, \mathbf{q}_t), \end{aligned}$$

where

$$G(\mathbf{y}_t, \mathbf{q}_t) = \beta \mu_\Lambda r(\mathbf{q}_t) - (1 - \beta) \mathbf{c}\mathbf{y}_t - \beta \mu_\Lambda \mathbf{c}\mathbf{q}_t - H(\mathbf{y}_t, \mathbf{q}_t). \quad (17)$$

The following theorem states the existence of the myopic policy, which is optimal for a single period. It shows that the myopic policy is optimal for multiple periods in a stationary system, if the initial inventory levels are under the myopic thresholds.

**Theorem 4** *Consider any period  $t$ , with the initial inventory level  $\mathbf{x}$  in a stationary system. Define the myopic policy to maximize the constrained expected periodical profit function,*

$$(\mathbf{y}^m, \mathbf{q}^m) \in \arg \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G(\mathbf{y}, \mathbf{q}).$$

(1)  $G(\mathbf{y}, \mathbf{q})$  is jointly concave in  $(\mathbf{y}, \mathbf{q})$ , and admits a global maximizer  $(\mathbf{y}^m(\emptyset), \mathbf{q}^m(\emptyset))$ .

$$(\mathbf{y}^m(\emptyset), \mathbf{q}^m(\emptyset)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G(\mathbf{y}, \mathbf{q})$$

*If  $r(\mathbf{q})$  is strictly concave in  $\mathbf{q}$ , then  $G(\mathbf{y}, \mathbf{q})$  is strictly jointly concave in  $(\mathbf{y}, \mathbf{q})$ , and admits a unique maximizer.*

(2) If  $\mathbf{x} \leq \mathbf{y}^m(\emptyset)$ , then  $(\mathbf{y}^m(\emptyset), \mathbf{q}^m(\emptyset))$  is an optimal policy from then on.

(3) Otherwise, there exists a myopic policy that consists of three parts: (1) the myopic not-to-order list,  $\mathcal{M}^+$ , (2) the myopic order-up-to levels,  $\mathbf{y}^m(\mathbf{x}_{\mathcal{M}^+})$ , and (3) the myopic market shares,  $\mathbf{q}^m(\mathbf{x}_{\mathcal{M}^+})$ .

The proof is similar to that of Theorem 1 and 3. The optimality of the myopic policy abides by the analysis by Heyman and Sobel (1984). The computation of the myopic policy is the same as in Algorithm 2 and 3. If the initial inventory levels are zeros, our results are compatible with those of Aydin and Porteus (2005).

If the initial inventory levels of some products are higher than  $\mathbf{y}^m(\emptyset)$ , we say these products are *overstocked*, and term the situation *overstocking*. In general, we can use the myopic policy as a heuristic policy for the entire planning horizon. For any given period, if there is overstocking, we employ the policy described in part (c).

## 6.2 Interplay of Marketing and Operations Decisions

The myopic policy depends on the specific form of randomness of demand. We study three kinds of demand function, and show the degree of interaction between pricing and replenishment. These can be described as marketing and operations decisions. The demand models of interest are:

- Additive-**I** demand:  $\mu_\Lambda \mathbf{q} + \varepsilon$ , where  $\varepsilon = (\varepsilon_j)_{j \in \mathcal{J}}$ , and  $\varepsilon_j$  are i.i.d..
- Additive-**Diag**( $\mathbf{q}$ ) demand:  $\mu_\Lambda \mathbf{q} + (q_j \varepsilon_j)_{j \in \mathcal{J}}$ , and  $\varepsilon_j$  are i.i.d..
- Multiplicative demand:  $\Lambda \mathbf{q}$ .

Define

$$\xi_j = \frac{b_j - c_j(1 - \beta)}{b_j + h_j}, \quad \omega_j = F_{\varepsilon_j}^{-1}[\xi_j].$$

We have

**Proposition 10** Consider any period  $t$  with the initial inventory  $\mathbf{x}$ . We can obtain  $(\mathbf{y}^m(\emptyset), \mathbf{q}^m(\emptyset))$  by solving the following equations:

$$\begin{array}{l} \text{Additive-**I** demand} \\ \text{Additive-**Diag**( $\mathbf{q}$ ) demand} \\ \text{Multiplicative demand} \end{array} \left\{ \begin{array}{l} y_j = \omega_j + \mu_\Lambda q_j ; \\ \beta \frac{\partial r(\mathbf{q})}{\partial q_j} = c_j , \\ y_j = (\omega_j + \mu_\Lambda) q_j \\ \beta \frac{\partial r(\mathbf{q})}{\partial q_j} = c_j + (b_j + h_j) \frac{1}{\mu_\Lambda} \int_{\omega_j}^{\infty} \epsilon_j f_{\varepsilon_j}(\epsilon_j) d\epsilon_j, \\ y_j = q_j \omega_j ; \\ \beta \frac{\partial r(\mathbf{q})}{\partial q_j} = c_j + (b_j + h_j) \frac{1}{\mu_\Lambda} \int_0^{\omega_j} (\mu_\Lambda - \lambda) f_\Lambda(\lambda) d\lambda. \end{array} \right.$$

Thus, under Additive-**I** demand, if none of the product is overstocked, the marketing and operations decisions can be made separately. The optimal market share is obtained by equate the marginal revenue to the marginal procurement cost, regardless of the inventory decision. If some products are overstocked, however, both marketing and inventory decisions must integrate.

Unlike in the Additive-*I* demand case, under the Additive-**Diag**( $\mathbf{q}$ ) demand, even if there is no overstocked product, optimal prices depends on the inventory decisions, and vice versa. The operations cost is measured by  $(b_j + h_j) \frac{1}{\mu_\Lambda} \int_{\omega_j}^{\infty} \epsilon_j f_{\epsilon_j}(\epsilon_j) d\epsilon_j$ . The marketing decision is not only based on the controllable part  $r(\mathbf{q})$ , but also on the random part  $\epsilon$ . Moreover, the decision should concern not only the procurement cost  $\mathbf{c}$ , but also the inventory costs. Such interaction is static, however, since it does not depend on the realtime operations information (e.g. inventory level). For this reason, we call it *static* interaction between operations and marketing management. On the other hand, if there is some products overstocked, the optimal prices will depend on the not-to-order list as well. Consequently, we call this type of interaction *dynamic* interaction.

Under the multiplicative demand, when there is no overstocking, the pricing decisions need to take into account an additional operations cost,  $(b_j + h_j) \frac{1}{\mu_\Lambda} \int_0^{\omega_j} (\mu_\Lambda - \lambda) f_\Lambda(\lambda) d\lambda$ . If overstocking takes place, the pricing decisions need to depend on the not-to-order list. Therefore, again, we see dynamic interaction.

We further have:

**Corollary 4** *Suppose the demand function is of the Additive-**I**, or the Additive-**Diag**( $\mathbf{q}$ ), or the multiplicative form. If Assumption 3 holds, then  $y_j^m(\mathbf{x}_{\mathcal{M}^+})$  satisfy Property D, for  $j \in \mathcal{M}^- = \mathcal{J} \setminus \mathcal{M}^+$ . Hence, Algorithm 3 applies.*

### 6.3 Numerical Study

We now conduct a numerical study of the myopic policy,  $(\mathbf{y}^m, \mathbf{q}^m) \in \arg \max_{\mathbf{y} \geq \mathbf{x}, \mathbf{q} \in \mathcal{Q}_t} G(\mathbf{y}, \mathbf{q})$ , for two substitutable products. The demand follows the Additive-**Diag**( $\mathbf{q}$ ) logit model with i.i.d. error terms  $\epsilon_j \sim \mathbf{U}(-50, 50)$ , for  $j = 1, 2$ . We assume  $(\Theta_1, \Theta_2) = (13.2, 13)$ , so product 1 has a higher average perceptive value than product 2. The cost structure is symmetric:  $\mathbf{h} = (0.5, 0.5)^T$ ,  $\mathbf{b} = (4.5, 4.5)^T$ , and  $\mathbf{c} = (10, 10)^T$ . The discount rate is  $\beta = 0.95$ .

With these settings,  $\mathbf{y}^m(\emptyset) = (43, 35)$  and  $\mathbf{q}^m(\emptyset) = (32.7\%, 26.8\%)$ . Interestingly, both products have the same price, therefore the same profit margin, which is consistent with a finding in Aydin and Ryan (2000).

Below we examine how the myopic policy changes as a function of the initial inventory levels. We consider two scenarios: (a) The initial inventory of product 2 is fixed at  $x_2 = 30$ , but that of product 1 ( $x_1$ ) varies from 30 to 69, and (b) the initial inventory of product 1 is fixed at  $x_1 = 30$ ,

but  $x_2$  varies from 30 to 69. For both scenarios, Figures 1-3 illustrate the changes of the myopic base-stock levels  $\mathbf{y}^m$ , market shares  $\mathbf{q}^m$  and prices  $\mathbf{p}^m$ , respectively.

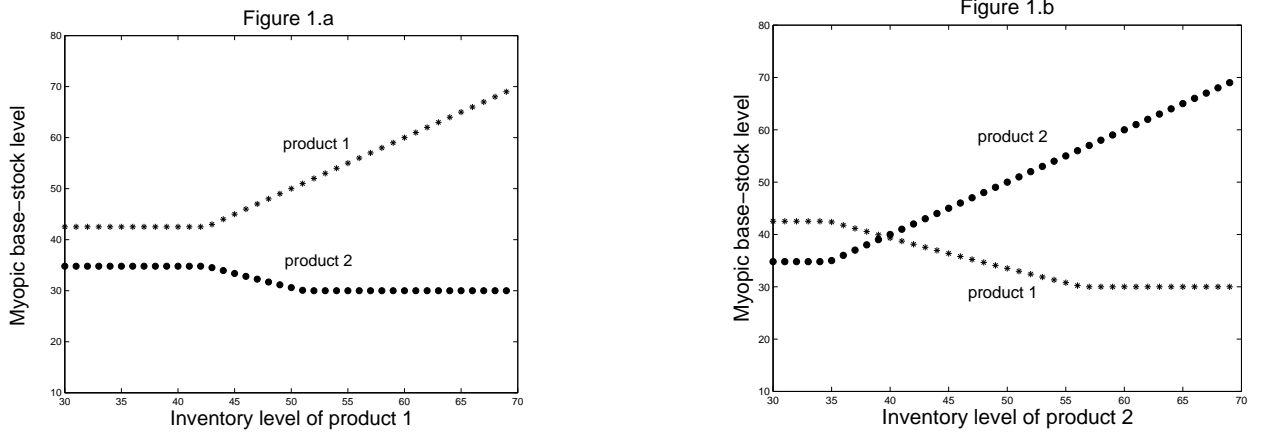


Figure 1: Myopic base-stock level conditioned on the initial inventory of products.

We study scenario (a) first, where  $x_2 = 30 < y_2^m(\emptyset) = 35$ . We observe:

- When  $x_1 \leq y_1^m(\emptyset) = 43$ , from Figure 1.a, for product 1,  $y_1^m = y_1^m(\emptyset)$ , as shown in Theorem 4. In this case,  $(\mathbf{y}^m(\emptyset), \mathbf{q}^m(\emptyset))$  is the optimal policy. Here, the more popular product 1 has a higher base-stock level and market share, although both products have the same price.
- When  $x_1 > y_1^m(\emptyset) = 43$ ,  $y_1^m = x_1$ . In this range, product 1 is overstocked, so no order is placed. For product 2, as shown in Property D, we indeed observe that  $y_2^m(x_1)$  is decreasing in  $x_1$ .

However, when  $43 < x_1 \leq 52$ ,  $y_2^m = y_2^m(x_1) > 30 = x_2$ , so the myopic policy suggests ordering product 2. In other words, in this range, product 1 is overstocked while product 2 is not. Figure 2.a shows the myopic market share of product 1 increases while that of product 2 decreases (in  $x_1$ ). Correspondingly, Figure 3.a shows the price of product 1 decreases while that of product 2 increases.

When  $x_1 > 52$ , we have  $y_2^m = x_2 = 30$ , which means the myopic policy suggests not to order product 2. Thus, in this range, both products are overstocked. Figure 3.a shows that the myopic prices for both products decrease, although the price of product 1 decreases at a higher rate. This is not surprising, given that  $x_1 \gg x_2$ . Interestingly, Figure 2.a shows that while the myopic market share for product 2 decreases, that of product 1 increases. This must be also due to fact that  $x_1 \gg x_2$ .

Now, we examine scenario (b), where  $x_1 = 30 < y_1^m(\emptyset) = 43$ . Here, from Figures 1.b, 2.b and 3.b, we observe that the myopic policy as a function of  $x_2$  demonstrates the same pattern as a

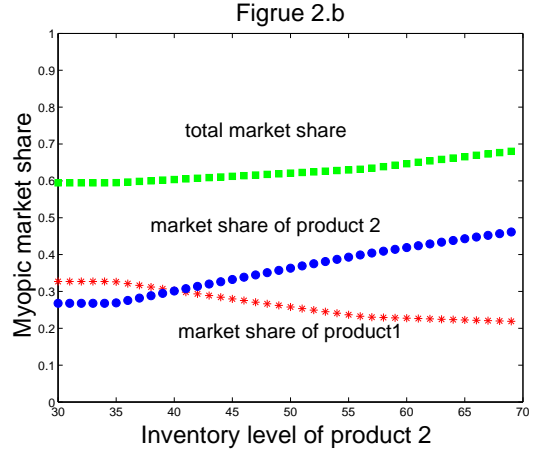
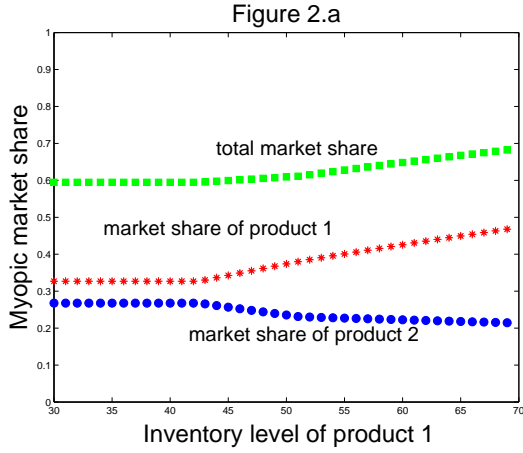


Figure 2: Myopic market share conditioned on the initial inventory of products.

function of  $x_1$  in scenario (a). A new finding in this case is that the myopic base-stock level and market share of the less popular product 2 can be higher than those of the popular product 1, as shown in Figures 1.b and 2.b.

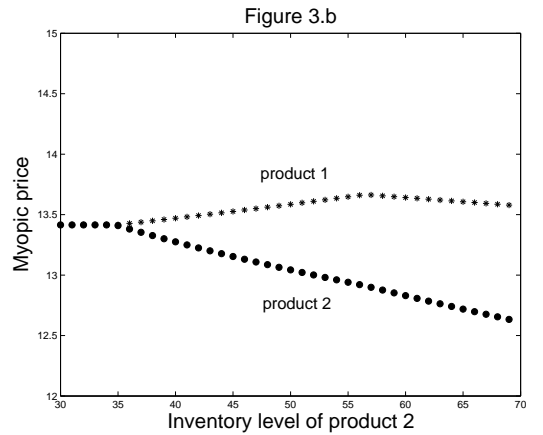
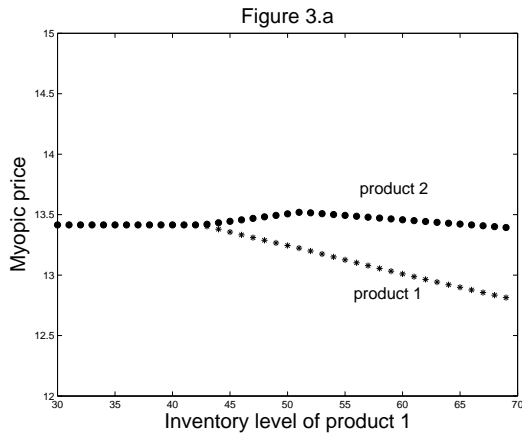


Figure 3: Myopic price conditioned on the initial inventory of products.

## 7 Concluding Remarks

In this paper we studied optimal pricing and replenishment decisions for substitutable products over a finite horizon. We consider price driven substitution, so the random demands are affected by the price vector.

We showed that the optimal policy consists of three components: the not-to-order list, the base-stock levels, and the target market shares. Also, the base-stock levels and the market shares depend only on the initial inventory levels of those products on the not-to-order list (termed overstocking

levels). We developed a branching algorithm to compute the optimal policy.

We also showed that under certain conditions, the optimal base-stock levels possess certain monotonicity property with respect to the overstocking levels. With this property, the optimal policy can be computed much more efficiently.

We further investigated the optimality of the myopic policy and demonstrated the interrelations between inventory and pricing decisions through the myopic policy.

There are several future research directions. One is to develop more efficient algorithms and easy-to-compute approximations for large systems. Another is to consider the lost sales case. The third is to consider fixed ordering costs. Finally, it would be interesting to account for stock-out based substitutions.

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## Appendix

### Proof of Proposition 2

The range of market share,  $\mathcal{Q}$ , comes from the fact that prices should be non-negative. Consider any subset of products,  $\mathcal{J}_s \subseteq \mathcal{J}$ . If we charge zero prices for the products within the subset and set prices to infinity (or the highest level allowed) for the other products, then this subset of products should occupy the highest possible market share,  $[\sum_{l \in \mathcal{J}_s} \exp(\Theta_l)]/[1 + \sum_{l \in \mathcal{J}_s} \exp(\Theta_l)]$ .

From (3), we have  $\exp(\Theta_j - p_j) = q_j/(1 - \sum_{l \in \mathcal{J}} q_l)$ , for  $j \in \mathcal{J}$ . Taking ln on both sides yields the price function. We rewrite

$$r(\mathbf{q}) = \sum_{l \in \mathcal{J}} \{\Theta_l q_l\} - \left(1 - \sum_{l \in \mathcal{J}} q_l\right) \ln \left(1 - \sum_{l \in \mathcal{J}} q_l\right) + \ln \left(1 - \sum_{l \in \mathcal{J}} q_l\right) - \sum_{l \in \mathcal{J}} \{q_l \ln(q_l)\}.$$

The first term  $\sum_{l \in \mathcal{J}} \{\Theta_l q_l\}$  is linear and thus jointly concave. Now, consider the second term. Its first derivative is  $\frac{\partial}{\partial q_j} \left\{ - \left(1 - \sum_{l \in \mathcal{J}} q_l\right) \ln \left(1 - \sum_{l \in \mathcal{J}} q_l\right) \right\} = \ln \left(1 - \sum_{l \in \mathcal{J}} q_l\right) + 1$ , so its Hessian is  $-1/(1 - \sum_{l \in \mathcal{J}} q_l)[\mathbf{1}]$ , where  $[\mathbf{1}]$  is the matrix with all entries equal to 1. The Hessian is negative semi-definite, so the second term is jointly concave. The third term is also concave, with the negative semi-definite Hessian matrix,  $-1/(1 - \sum_{l \in \mathcal{J}} q_l)^2[\mathbf{1}]$ . The last term is strictly concave, with the negative definite diagonal Hessian matrix,  $\mathbf{Diag}(-1/q_j)$ . Therefore,  $r(\mathbf{q})$  is strictly concave.

### Proof of Lemma 1

To show the convexity, fix the random part  $\mathbf{d}_t$  to its realization  $\hat{\mathbf{d}}_t$ . Consider any pair  $(\mathbf{y}, \mathbf{q})$  and  $(\mathbf{y}', \mathbf{q}')$ . Recall that  $\hat{\mathbf{d}}_t(\mathbf{q}) = \lambda_t \mathbf{q} + \mathbf{L}(\mathbf{q}) \epsilon_t$ , where  $\mathbf{L}$  is linear. So,  $\hat{\mathbf{d}}_t(\alpha \mathbf{q} + (1 - \alpha) \mathbf{q}') = \alpha \hat{\mathbf{d}}_t(\mathbf{q}) + (1 - \alpha) \hat{\mathbf{d}}_t(\mathbf{q}')$ . From the convexity of  $\hat{H}_t$ , we have

$$\begin{aligned} \hat{H}_t(\alpha \mathbf{y} + (1 - \alpha) \mathbf{y}' - \hat{\mathbf{d}}_t(\alpha \mathbf{q} + (1 - \alpha) \mathbf{q}')) &= \hat{H}_t(\alpha \mathbf{y} + (1 - \alpha) \mathbf{y}' - \alpha \hat{\mathbf{d}}_t(\mathbf{q}) - (1 - \alpha) \hat{\mathbf{d}}_t(\mathbf{q}')) \\ &= \hat{H}_t(\alpha(\mathbf{y} - \hat{\mathbf{d}}_t(\mathbf{q})) + (1 - \alpha)(\mathbf{y}' - \hat{\mathbf{d}}_t(\mathbf{q}'))) \leq \alpha \hat{H}_t(\mathbf{y} - \hat{\mathbf{d}}_t(\mathbf{q})) + (1 - \alpha) \hat{H}_t(\mathbf{y}' - \hat{\mathbf{d}}_t(\mathbf{q}')). \end{aligned}$$

Therefore,  $H_t(\mathbf{y}, \mathbf{q}) = E \hat{H}_t(\mathbf{y} - \hat{\mathbf{d}}_t(\mathbf{q}))$  is jointly convex in  $(\mathbf{y}, \mathbf{q})$ .

### Proof of Theorem 1

We first observe that, for any  $t$ , the first four terms of  $G_t(\mathbf{y}, \mathbf{p})$  in (10) are jointly concave. The first term is concave in  $\mathbf{q}$ , according to Assumption 1. The second term is linear in  $\mathbf{q}$ , and the third term is linear in  $\mathbf{y}$ . The fourth term is jointly concave in  $(\mathbf{y}, \mathbf{q})$  (see Lemma 1). Given this fact, the rest of the proof follows from a standard induction argument, such as one given by Federgruen and Heching (1999).

### Proof of Proposition 4

(a) Denote  $(\mathbf{y}_t^*, \mathbf{q}_t^*) \in \arg \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q})$  is any optimal solution under the full constraints. When  $x_1 > y_{1t}^*(\emptyset)$  and  $x_2 \leq y_{2t}^*(\emptyset)$ , we want to prove that it is optimal not to order product 1. We assume conversely that any optimal solution must have  $y_{1t}^* > x_1$ , that is, it is optimal only if the retailer orders some positive amount of product 1. Under the converse assumption, there is no optimal solution such that  $y_{1t}^* = x_1$ . Now we have  $y_{1t}^* > x_1 > y_{1t}^*(\emptyset)$ . Set  $\alpha = \frac{x_1 - y_{1t}^*(\emptyset)}{y_{1t}^* - y_{1t}^*(\emptyset)}$ , then  $0 < \alpha < 1$ . Let  $(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ .

We now prove  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is feasible but suboptimal. First, we have  $\mathbf{y}'_t \geq \mathbf{x}$ , since  $y'_{1t} = \alpha y_{1t}^* + (1 - \alpha)y_{1t}^*(\emptyset) = x_1$ , and  $y'_{2t} = \alpha y_{2t}^* + (1 - \alpha)y_{2t}^*(\emptyset) \geq \alpha x_2 + (1 - \alpha)x_2 = x_2$ . Second, we have  $\mathbf{q}'_t \in \mathcal{Q}_t$ , since  $\mathbf{q}'_t = \alpha \mathbf{q}_t^* + (1 - \alpha)\mathbf{q}_t^*(\emptyset)$ , and the control space  $\mathcal{Q}_t$  is convex. Last, it is suboptimal, since  $y'_{1t} = x_1$ .

Because  $(\mathbf{y}_t^*, \mathbf{q}_t^*)$  is an optimal solution, we have  $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Because  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$  is the maximum solution of a relaxed maximization problem, we have  $G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \geq G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)$ . Thus, we obtain  $\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . This contradicts the concavity of  $G_t$ , hence we must have some optimal solution such that  $y_{1t}^* = x_1$ , i.e., it is optimal for the retailer not to order product 1.

(b) Now fix  $y_{1t} = x_1$ , obtain  $(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1)$ , where  $y_{1t}^*(x_1) = x_1$ . Clearly, if  $y_{2t}^*(x_1) \geq x_2$ , then it is optimal to order product 2 up to  $y_{2t}^*(x_1)$ , and set market share at  $(q_{1t}^*(x_1), q_{2t}^*(x_1))$ . Otherwise, by a similar argument of part (a), it is optimal for retailer not to order product 2. In this case, set the optimal market share at  $\mathbf{q}_t^*(\mathbf{x}) \in \arg \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q})$ .

### Proof of Proposition 5

(a) When  $x_1 > y_{1t}^*(\emptyset)$  and  $x_2 > y_{2t}^*(\emptyset)$ , we want to prove it is optimal not to order at least one product. We assume conversely that any optimal solution must have  $\mathbf{y}_t^* > \mathbf{x}$ , that is, it is optimal only if some positive amount of both products is ordered. Then  $\mathbf{y}_t^* > \mathbf{x} > \mathbf{y}^*(\emptyset)$ . Let  $l \in \arg \max_{j \in \{1,2\}} \left\{ \frac{x_j - y_{jt}^*(\emptyset)}{y_{jt}^* - y_{jt}^*(\emptyset)} \right\}$ , and set  $\alpha = \frac{x_l - y_{lt}^*(\emptyset)}{y_{lt}^* - y_{lt}^*(\emptyset)}$ . Then we have  $x_l = \alpha y_{lt}^* + (1 - \alpha)y_{lt}^*(\emptyset)$ . Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)).$$

As in the proof of Proposition 4, we can prove  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is feasible but suboptimal. Because  $(\mathbf{y}_t^*, \mathbf{q}_t^*)$  is an optimal solution, and  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is feasible but suboptimal, we have  $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Because  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$  is a maximizer of the relaxation, we have  $G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \geq G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)$ . Thus, we obtain  $\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ , which contradicts to the concavity of  $G_t$ . Therefore, it is optimal not to order at least one product.

(b) Denote  $k^*$  to be the product that needs not to be ordered, and consider the subproblem indexed by  $k^*$ . Relax  $\mathbf{y} \geq \mathbf{x}$ , and obtain  $(\mathbf{y}_t^*(x_{k^*}), \mathbf{q}_t^*(x_{k^*})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_{k^*} = x_{k^*})$ . Note that  $y_{k^*}^*(x_{k^*}) = x_{k^*}$ . Let  $j = 3 - k^*$ . If  $x_j \leq y_{jt}^*(x_{k^*})$ , then  $\mathcal{J}^+ = \{k^*\}$ , having  $V_t(\mathbf{x}) = V_t^{\{k^*\}}(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{R}^2; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_{k^*} = x_{k^*})$ . Otherwise,  $\mathcal{J}^+ = \mathcal{J}$ , having  $V_t(\mathbf{x}) = V_t^{\{k^*\}}(\mathbf{x}) = V_t^{\mathcal{J}}(\mathbf{x}) = \max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{x}, \mathbf{q})$ . Similar to the proof of Proposition 4, we can show that it is optimal not to order  $j$  if  $x_j > y_{jt}^*(x_{k^*})$ .

### Proof of Proposition 6

Proposition 5 shows it is optimal not to order at least one product. Without loss of generality, let this product be product 1. We would like to prove that it is optimal not to order product 2. Note that  $y_{1t}^* = x_1 > y_{1t}^*(\emptyset)$ , and  $x_2 > y_{2t}^*(\emptyset)$ . From the decreasing property, we have  $y_{2t}^*(\emptyset) \geq y_{2t}^*(x_1)$ . Therefore, we have  $x_2 > y_{2t}^*(x_1)$ . Next we show  $y_{2t}^* = x_2$ . Suppose this is not the case, that is, any optimal solution must have  $y_{1t}^* = x_1$ , and  $y_{2t}^* > x_2$ . Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)),$$

where  $\alpha = \frac{x_2 - y_{2t}^*(x_1)}{y_{2t}^* - y_{2t}^*(x_1)}$ . We have  $x_2 = \alpha y_{2t}^* + (1 - \alpha)y_{2t}^*(x_1)$ .

It can be verified  $\mathbf{y}'_t = \mathbf{x}$ , so  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is feasible and suboptimal. Hence,  $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Because  $\mathbf{y}'_t = \mathbf{x}$ , we have  $G_t(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) \geq G_t(\mathbf{y}_t^*(\mathbf{x}), \mathbf{q}_t^*(\mathbf{x})) \geq G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Then, we obtain

$$\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)G_t(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t),$$

which contradicts the concavity of  $G_t$ . Hence it is also optimal not to order product 2.

The result on Additive- $I$  demand will be shown in Proposition 9.

### Proof of Theorem 2 & Algorithm 2

**Proof :** It is sufficient to show the optimal not-to-order list can be finally reached along one path. On this ‘‘good’’ path, in each step  $m$ , the key is to prove whenever  $\mathcal{A}_m = \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{F}_m^+}) < x_j\}$  is not empty, there exists a node  $k_m^* \in \mathcal{A}_m$  so that it is optimal not to order  $k_m^*$ . With this new node, the path continues to extend until the sequence  $\mathcal{F}_m^+$  along this path becomes a partial list  $\mathcal{J}_m^+$ .

In step 1, we obtain  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ . If  $\mathbf{y}_t^*(\emptyset) \geq \mathbf{x}$ , then the optimal policy is given by  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ . Otherwise, there is some  $j \in \mathcal{F}_0^- = \mathcal{J}$ , such that  $y_{jt}^* < x_j$ , i.e.,  $\mathcal{A}_1 = \{j \mid y_{jt}^*(\emptyset) < x_j\}$  is not empty.

We argue that there always exists a  $k_1^* \in \mathcal{A}_1$ , so that it is optimal not to order it. Assume conversely that it is optimal only if the retailer order some positive amount for all the products in  $\mathcal{A}_1$ , i.e.  $y_{jt}^* > x_j$ , for any  $j \in \mathcal{A}_1$ . Then choose  $l \in \arg \max_{\{j \in \mathcal{A}_1\}} \left\{ \frac{x_j - y_{jt}^*(\emptyset)}{y_{jt}^* - y_{jt}^*(\emptyset)} \right\}$ . Set  $\alpha = \frac{x_l - y_{lt}^*(\emptyset)}{y_{lt}^* - y_{lt}^*(\emptyset)}$ ,

$\alpha \in (0, 1)$ . Note that  $x_j \leq \alpha y_{jt}^* + (1 - \alpha)y_{jt}^*(\emptyset)$ , for all  $j \in \mathcal{A}_1$ , and  $x_l = \alpha y_{lt}^* + (1 - \alpha)y_{lt}^*(\emptyset)$ . Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)).$$

We can show  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is feasible and suboptimal, using a similar argument in the proof of Proposition 5. Thus  $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Because  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$  is a maximum solution of a less constrained problem, we have  $G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \geq G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . So

$$\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t),$$

contradicting to the concavity of  $G_t$ . Hence, there exists at least one  $k_1^* \in \mathcal{A}_1$ , such that it is optimal not to order product  $k_1^*$ . Then we update  $\mathcal{F}_1^+ = \mathcal{F}_0^+ \cup \{k_1^*\}$ , which is a partial list. We now solve a maximization problem with one more equality constraint.

$$\max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}) = \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{F}_1^+} = \mathbf{x}_{\mathcal{F}_1^+}).$$

If  $\mathcal{F}_1^+ \subset \mathcal{J}$ , we go to the next step.

In step  $m$ , assume we already has a partial list  $\mathcal{F}_{m-1}^+$ . We know the optimal solution satisfies  $(\mathbf{y}_t^*, \mathbf{q}_t^*) \in \arg \max_{\mathbf{y} \geq \mathbf{x}; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{F}_{m-1}^+} = \mathbf{x}_{\mathcal{F}_{m-1}^+})$ . We continue to establish a new set of thresholds,  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}))$ , by solving a new relaxed optimization problem,

$$\max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{F}_{m-1}^+} = \mathbf{x}_{\mathcal{F}_{m-1}^+}).$$

If  $\mathcal{A}_m = \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}) < x_j\}$  is a an empty set, then the optimal not-to-order list is made by  $\mathcal{F}_M^+ = \mathcal{F}_{m-1}^+$ , and the algorithm terminates at step  $m$ . Otherwise, we argue that there always exist a  $k_m^* \in \mathcal{A}_m$ , so that it is optimal not to order it. Assume conversely that it is optimal only if the retailer order some positive amount for all the products in  $\mathcal{A}_m$ , i.e.  $y_{jt}^* > x_j$ , for any  $j \in \mathcal{A}_m$ . Then choose  $l \in \arg \max_{\{j \in \mathcal{A}_m\}} \left\{ \frac{x_j - y_{jt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})}{y_{jt}^* - y_{jt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})} \right\}$ . Set  $\alpha = \frac{x_l - y_{lt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})}{y_{lt}^* - y_{lt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})}$ ,  $\alpha \in (0, 1)$ . Note that  $x_j \leq \alpha y_{jt}^* + (1 - \alpha)y_{jt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})$ , for all  $j \in \mathcal{A}_1$ , and  $x_l = \alpha y_{lt}^* + (1 - \alpha)y_{lt}^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})$ . Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})).$$

We can show  $(\mathbf{y}'_t, \mathbf{q}'_t)$  is a feasible and suboptimal solution, by a similar argument in the proof of Proposition 5. Thus  $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . Because  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}))$  is a maximum solution of a relaxed problem, we have  $G_t(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})) \geq G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$ . So

$$\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1 - \alpha)G_t(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{F}_{m-1}^+})) > G_t(\mathbf{y}'_t, \mathbf{q}'_t),$$

contradicting to the concavity of  $G_t$ . Hence, there exists at least one  $k_m^* \in \mathcal{A}_m$ , such that it is optimal not to order product  $k_m^*$ . Then we update the partial list,  $\mathcal{F}_m^+ = \mathcal{F}_{m-1}^+ \cup \{k_m^*\}$ , and continue searching the optimal solution in a maximization problem with more equality constraints.

Continue in the same fashion, the optimal not-to-order list can be obtained along this path  $(k_1^*, \dots, k_M^*)$  with the length equal to  $M - 1$ .  $\mathcal{F}_M^+$  can be  $\emptyset$ , for  $M = 1$ , and  $\mathcal{F}_0^+ \subset \dots \subset \mathcal{F}_{M-1}^+ \subset \mathcal{F}_M^+$ , for  $1 < M \leq J$ . It is obvious that this path must be one of the pathes generated by Algorithm 2.

■

### Proof of Algorithm 3

We prove the result by induction. In each step, the key is to prove that if  $\mathcal{A}_m = \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}) < x_j\}$  is a non-empty set, it is optimal not to order any product in  $\mathcal{A}_m$ .

In step 1, we obtain  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ . If  $\mathbf{y}_t^*(\emptyset) \geq \mathbf{x}$ , then the optimal policy is given by  $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ . Otherwise, there is some  $j \in \mathcal{J}_0^- = \mathcal{J}$ , such that  $y_{jt}^* < x_j$ , i.e.,  $\mathcal{A}_1 = \{j \mid y_{jt}^*(\emptyset) < x_j\}$  is a non-empty set.

From Algorithm 2, there always exists at least one  $k_1 \in \mathcal{A}_1$ , for which it is optimal not to order. Here we would like to show that under Assumption 2, if  $\mathcal{A}_1$  contains more than one product, then it is optimal not to order any of them. First, we make the converse assumption that it is optimal only if to order some product in  $\mathcal{A}_1$ . Hence,  $\mathcal{A}_1$  can be divided into two mutually exclusive non-empty subsets,  $\mathcal{A}_1^+$  and  $\mathcal{A}_1^-$ . The former contains products not to order, and the latter contains products to order. Note that  $k_1 \in \mathcal{A}_1^+$ .

We now show  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}))_{\mathcal{A}_1^-} \leq (\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1^-} < \mathbf{x}_{\mathcal{A}_1^-}$ ,

$$\text{where } \begin{cases} (\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+} = \mathbf{x}_{\mathcal{A}_1^+}); \\ (\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+} = (\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1^+}). \end{cases}$$

The first inequality results from the decreasing property, because  $(\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1^+} < \mathbf{x}_{\mathcal{A}_1^+}$ . The second inequality holds, because  $(\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1} < \mathbf{x}_{\mathcal{A}_1}$  and  $\mathcal{A}_1^- \subset \mathcal{A}_1$ . Let  $\mathcal{J}_1^+ = \mathcal{J}_0^+ \cup \mathcal{A}_1^+ = \mathcal{A}_1^+$ , updating the partial list, and go to the next step.

In step 2, there exists  $\mathcal{A}_2 = \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_1^+}) < x_j\}$ . It is non-empty, because  $\mathcal{A}_1^- \subseteq \mathcal{A}_2$ . According to Algorithm 2, there exists  $k_2 \in \mathcal{A}_2$ , which is overstocked. If  $k_2$  comes from  $\mathcal{A}_1^-$ , then it contradicts the assumption that any element in  $\mathcal{A}_1^-$  is understocked. Therefore,  $k_2 \in \mathcal{A}_2 \setminus \mathcal{A}_1^-$ . Then we can divide  $\mathcal{A}_2$  into two mutually exclusive non-empty subsets,  $\mathcal{A}_2^+$  and  $\mathcal{A}_2^-$ , where we have  $\mathcal{A}_1^- \subseteq \mathcal{A}_2^- \subset \mathcal{A}_2$ ,  $\mathcal{A}_1^- \cap \mathcal{A}_2^+ = \emptyset$ ,  $\mathcal{A}_1^+ \cap \mathcal{A}_2^+ = \emptyset$ , and  $\mathcal{A}_2^+ \subseteq \mathcal{J}_1^+ \setminus \mathcal{A}_1^+$ .

We next show  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}))_{\mathcal{A}_1^-} \leq (\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}))_{\mathcal{A}_1^-} < \mathbf{x}_{\mathcal{A}_1^-}$ ,

$$\text{where } \begin{cases} (\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+} = \mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}); \\ (\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+} = (\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}))_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}). \end{cases}$$

The first inequality comes from the decreasing property, because  $\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_1^+} = \mathbf{x}_{\mathcal{A}_1^+}$ , and  $\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_2^+} < \mathbf{x}_{\mathcal{A}_2^+}$ , where  $\mathcal{A}_2^+ \subset \mathcal{A}_2 = \{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_1^+}) < x_j\}$ , and  $\mathcal{J}_1^+ = \mathcal{A}_1^+$ . The second inequality directly comes from the results in step 1.

Let  $\mathcal{J}_2^+ = \mathcal{J}_1^+ \cup \mathcal{A}_2^+ = \mathcal{A}_1^+ \cup \mathcal{A}_2^+$ , updating the partial list.

By induction, if we insist that any product  $l$  in  $\mathcal{A}_1^-$  is understocked, we have to assume  $\mathcal{A}_1^- \cap \mathcal{A}_m^+ = \emptyset$ , for  $m = 1, 2, \dots$ . However, under Assumption 2, we always have

$$\mathbf{x}_{\mathcal{A}_1^-} > \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_0^+})\right)_{\mathcal{A}_1^-} \geq \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_1^+})\right)_{\mathcal{A}_1^-} \geq \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_2^+})\right)_{\mathcal{A}_1^-} \geq \dots, \quad (18)$$

where  $\mathcal{J}_m^+ = \mathcal{A}_1^+ \cup \dots \cup \mathcal{A}_m^+$ . Then Algorithm 2 continues forever. However,  $\mathcal{J}$  is a finite set, the algorithm cannot infinitely continue adding  $\mathcal{A}_m^+$  on  $\mathcal{J}_{m-1}^+$ . Therefore, sooner or later, we have to put  $l$  on the partial list.

### Proof of Theorem 3

**Proof :** Obviously,  $\hat{H}_t(\mathbf{z}) = \mathbf{h}_t[\mathbf{z}]^+ + \mathbf{b}_t[\mathbf{z}]^-$  is strictly convex in  $\mathbf{z}$ . We would like to show  $G_t(\mathbf{y}, \mathbf{q})$  is strictly jointly concave in  $(\mathbf{y}, \mathbf{q})$ . It is sufficient to show that the composite function,  $E \left\{ \beta \hat{R}_t(\mathbf{q}) - \hat{H}_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) \right\}$ , is strictly joint convex in  $(\mathbf{y}, \mathbf{q})$ . We consider two cases. In each case, either  $R_t(\mathbf{q})$  or  $-H_t(\mathbf{y}, \mathbf{q})$  is strictly concave in  $(\mathbf{y}, \mathbf{q})$ . Let  $(\mathbf{y}, \mathbf{q})$  and  $(\mathbf{y}', \mathbf{q}')$  be two sets of decision variables, and  $(\mathbf{y}, \mathbf{q}) \neq (\mathbf{y}', \mathbf{q}')$ .

Case 1:  $\mathbf{q} \neq \mathbf{q}'$ . We have  $R_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}') > \alpha R_t(\mathbf{q}) + (1-\alpha)R_t(\mathbf{q}')$ , because we assume  $r(\mathbf{q})$  is strictly concave.

Case 2:  $\mathbf{q} = \mathbf{q}'$ . We must have  $\mathbf{y} \neq \mathbf{y}'$ , because we assume  $(\mathbf{y}, \mathbf{q}) \neq (\mathbf{y}', \mathbf{q}')$ . Let  $\hat{\mathbf{d}}_t(\mathbf{q})$  be any realization of  $\mathbf{d}_t(\mathbf{q})$ . We have  $\mathbf{y} - \hat{\mathbf{d}}_t(\mathbf{q}) \neq \mathbf{y}' - \hat{\mathbf{d}}_t(\mathbf{q}')$ , then we have

$$\hat{H}_t\left(\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' - \hat{\mathbf{d}}_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}')\right) < \alpha\hat{H}_t\left(\mathbf{y} - \hat{\mathbf{d}}_t(\mathbf{q})\right) + (1-\alpha)\hat{H}_t\left(\mathbf{y}' - \hat{\mathbf{d}}_t(\mathbf{q}')\right).$$

Therefore,  $\beta R_t(\mathbf{q}) - H_t(\mathbf{y}, \mathbf{q})$  is strictly jointly concave in  $(\mathbf{y}, \mathbf{q})$ . Then  $G_t(\mathbf{y}, \mathbf{q})$  is strictly concave in  $(\mathbf{y}, \mathbf{q})$ , and the algorithm can guarantee a unique optimal solution. ■

**Lemma 3** *Let  $\mathbf{A}$  be an  $N \times N$  M-matrix. Then*

(a) *It is an M-matrix, if and only if each principal submatrix of  $\mathbf{A}$  is a nonsingular M-matrix.*

(b) *For any nonempty proper subset  $\mathcal{N}^+$  of  $\mathcal{N}$ , we have  $\mathbf{A}(\mathcal{N}^+)^{-1} \mathbf{A}(\mathcal{N}^+, \mathcal{N}^-) \leq 0$ , and*

$$\mathbf{A}(\mathcal{N}^-, \mathcal{N}^+) \mathbf{A}(\mathcal{N}^+)^{-1} \leq 0.$$

Part (a) (Bru et al. 2005, p164) implies that  $\mathbf{A}(\mathcal{N}^+)$  is also an M-matrix, thereby  $\mathbf{A}(\mathcal{N}^+)^{-1}$  is an inverse M-matrix, which is nonnegative. On the other hand, we have  $\mathbf{A}(\mathcal{N}^+, \mathcal{N}^-)$  non-positive, since it is an off-diagonal submatrix of an M-matrix.

**Lemma 4** *Let  $\mathbf{A}$  be a  $N \times N$  inverse M-matrix. Then*

- (a) *All principal minors of  $\mathbf{A}$  are positive.*
- (b) *Each principal submatrix of  $\mathbf{A}$  is an inverse M-matrix.*
- (c) *For any permutation matrix  $\mathbf{P}$  of order  $N$ ,  $\mathbf{PAP}^T$  is an inverse M-matrix.*
- (d) *For any nonempty proper subset  $\mathcal{N}^+$  of  $\mathcal{N}$ , we have  $\mathbf{A}(\mathcal{N}^+)^{-1} \mathbf{A}(\mathcal{N}^+, \mathcal{N}^-) \geq 0$ , and  $\mathbf{A}(\mathcal{N}^-, \mathcal{N}^+) \mathbf{A}(\mathcal{N}^+)^{-1} \geq 0$ .*
- (e) *The Schur complement  $\mathbf{A}/\mathcal{N}^+$  is an inverse M-matrix.*

The proof of (a)  $\sim$  (d) can be found in Lemma 2.1 (Chen 2004, p54). The proof of (e) can be found in Imam (1984).

### Proof of Corollary 3

To ease the notation, we suppress the subscript  $t$  in the proof. Define  $\mathbf{A} = [-\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q})]^{-1}$ , which is an inverse M-matrix. We construct a Lagrangian function,

$$\tilde{L}(\mathbf{q}, \gamma_{\mathcal{L}^+}) = \beta r(\mathbf{q}) - \mathbf{c}\mathbf{q} + \gamma_{\mathcal{L}^+}(\mathbf{q}_{\mathcal{L}^+} - \hat{\mathbf{q}}_{\mathcal{L}^+}).$$

Let  $\gamma_j = 0$ , for  $j \in \mathcal{L}^-$ . We have the first derivative of the Lagrangian function. That is,  $\frac{\partial}{\partial q_j} \tilde{L}(\mathbf{q}, \gamma_{\mathcal{L}^+}) = \beta \frac{\partial}{\partial q_j} r(\mathbf{q}) - c_j + \gamma_j$ , for  $j \in \mathcal{J}$ . At optimality, we have

$$\beta \frac{\partial}{\partial q_j} r(\mathbf{q}^*) - c_j + \gamma_j^* = 0, \text{ for } j \in \mathcal{J}.$$

Note that  $\gamma_j^* = 0$ , for  $j \in \mathcal{L}^-$ . These  $J$  equations admit the implicit function,  $\mathbf{q}^*(\gamma_{\mathcal{L}^+}^*)$ . We use the theorem of implicit function, having

$$[\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{q}^*(\gamma_{\mathcal{L}^+}^*)] = -[\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q}^*)]^{-1} [\nabla_{\gamma_{\mathcal{L}^+}} \gamma^*] = \mathbf{A}(\mathcal{J}, \mathcal{L}^+).$$

Note that  $[\nabla_{\gamma_{\mathcal{L}^+}} \gamma^*] = \mathbf{I}(\mathcal{J}, \mathcal{L}^+)$ .

Now we consider the constraint such that  $\mathbf{q}_{\mathcal{L}^+}^*(\gamma_{\mathcal{L}^+}^*) = \hat{\mathbf{q}}_{\mathcal{L}^+}$ , which admits an implicit function  $\gamma_{\mathcal{L}^+}^*(\hat{\mathbf{q}}_{\mathcal{L}^+})$ . According to the theorem of implicit function, we have

$$\nabla_{\hat{\mathbf{q}}_{\mathcal{L}^+}} \gamma_{\mathcal{L}^+}^*(\hat{\mathbf{q}}_{\mathcal{L}^+}) = [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{q}_{\mathcal{L}^+}^*(\gamma_{\mathcal{L}^+}^*)]^{-1} = [\mathbf{A}(\mathcal{L}^+)]^{-1}.$$

We consider any  $j \in \mathcal{L}^+$ , by the chain rule, we have

$$[\nabla_{\hat{\mathbf{q}}_{\mathcal{L}^+}} \mathbf{q}_{\mathcal{L}^-}^*(\hat{\mathbf{q}}_{\mathcal{L}^+})] = [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{q}_{\mathcal{L}^-}^*(\gamma_{\mathcal{L}^+}^*)][\nabla_{\hat{\mathbf{q}}_{\mathcal{L}^+}} \gamma_{\mathcal{L}^+}^*(\hat{\mathbf{q}}_{\mathcal{L}^+})] = \mathbf{A}(\mathcal{L}^-, \mathcal{L}^+)[\mathbf{A}(\mathcal{L}^+)]^{-1} \leq 0,$$

which results from Lemma 3 (b).

## Proof of Proposition 7

In each period  $t \in \{1, \dots, T\}$ , we have

$$\tilde{H}_t(\mathbf{s}) = E\hat{H}_t(\mathbf{s} - \varepsilon_t) = \sum_{j=1}^2 \left\{ h_{jt} \int_{-\infty}^{s_j} (s_j - \epsilon_j) f_{\varepsilon_{jt}}(\epsilon_j) d\epsilon_j + b_{jt} \int_{s_j}^{\infty} (\epsilon_j - s_j) f_{\varepsilon_{jt}}(\epsilon_j) d\epsilon_j \right\}.$$

Then  $\tilde{H}_t(\mathbf{s})$  is twice differentiable, and its Hessian matrix,  $[\nabla_{\mathbf{s}} \tilde{H}_t(\mathbf{s})]$ , is a positive diagonal matrix with the entry  $(j, j)$  equal to  $(h_{jt} + b_{jt}) f_{\varepsilon_{jt}}(s_j)$ .

We prove the property by induction. Recall we have

$$\tilde{G}_t(\mathbf{s}, \mathbf{q}) = \beta \mu_{\Lambda_t} r(\mathbf{q}) - \mu_{\Lambda_t} \mathbf{c}_t \mathbf{q} + (\beta \mathbf{c}_{t+1} - \mathbf{c}_t) \mathbf{s} - \tilde{H}_t(\mathbf{s}) + \beta EV_{t+1}(\mathbf{s} - \varepsilon_t).$$

In period  $T + 1$ , we assume that  $V_{T+1}(\mathbf{x}) = 0$ . It is obvious that  $\tilde{G}_T(\mathbf{s}, \mathbf{q})$  is twice differentiable in  $(\mathbf{s}, \mathbf{q})$ , and  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_T(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix, since it is a positive diagonal matrix.

Now we assume the proposition holds in period  $t + 1$ , for  $1 \leq t \leq T$ , we proceed to period  $t$ . By the assumption,  $\tilde{G}_t(\mathbf{s}, \mathbf{q})$  is twice differentiable in  $\mathbf{s}$  almost everywhere, and  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix.  $\tilde{G}_t(\mathbf{s}, \mathbf{q})$  is also twice differentiable in  $\mathbf{q}$ , since we assume  $r(\mathbf{q})$  is twice differentiable. We would like to show (a)  $V_t(\mathbf{x})$  is twice differentiable everywhere in  $\mathbb{R}^2$ , (b)  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is either a positive diagonal matrix or an inverse M-matrix, and (c)  $[\nabla_{\mathbf{s}}^2 (\tilde{H}_{t-1}(\mathbf{s}) - \beta EV_t(\mathbf{s} - \varepsilon_{t-1}))]$  is an inverse M-matrix, therefore  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_{t-1}(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix.

We define a differentiable Lagrangian function,  $\tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma) = \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma(\mathbf{s} + \mu_{\Lambda_t} \mathbf{q} - \mathbf{x})$ . At optimality,  $\mathbf{y} = \mathbf{y}_t^*$ ,  $\mathbf{q} = \mathbf{q}_t^*$ , and  $\gamma = \gamma_t^*$ , which are the corresponding Lagrangian multipliers. We have  $V_t(\mathbf{x}) = \tilde{L}_t(\mathbf{s}_t^*, \mathbf{q}_t^*, \gamma_t^*) = \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*) + \gamma_t^*(\mathbf{s}_t^* + \mu_{\Lambda_t} \mathbf{q}_t^* - \mathbf{x})$ , where  $\gamma_t^*$  can be divided to  $\mathbf{0}_{\mathcal{J}^-}$  and  $\gamma_{t\mathcal{J}^+}^*$ .

First, we prove (a), the twice differentiability, under the condition of strictly complementary slackness. Fiacco (1976) (Theorem 2.1 (b)) states that, when the strictly complementary slackness holds, the optimal values,  $\mathbf{y}_t^*$ ,  $\mathbf{q}_t^*$ , and  $\gamma_t^*$  are all differentiable in  $\mathbf{x}$ . We will show the strictly complementary slackness at the end of proof.

By the Envelope Theorem, we have

$$\frac{\partial V_t(\mathbf{x})}{\partial x_j} = \frac{\partial \tilde{L}_t}{\partial x_j} = -\gamma_{jt}^*, \text{ for } j \in \mathcal{J}.$$

Because  $\gamma_{jt}^*$  is differentiable in  $\mathbf{x}$ ,  $\frac{\partial}{\partial x_j} V_t(\mathbf{x})$  is differentiable in  $\mathbf{x}$ , thus  $V_t(\mathbf{x})$  is twice differentiable.

Second, we prove (b), under the condition of strictly complementary slackness. We need to consider two cases.

In case 1,  $\gamma_{1t}^* = 0$  or  $\gamma_{2t}^* = 0$ .

Fiacco (1976) (Theorem 2.1 (c)) states that if  $\gamma_{jt}^* = 0$  holds for some  $(x_1, x_2)$ , it holds for a neighborhood of  $(x_1, x_2)$ . So,  $\frac{\partial^2}{\partial x_1 \partial x_2} V_t(\mathbf{x}) = 0$ , and  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is a nonnegative diagonal matrix,

with three possible formats,

$$\begin{bmatrix} 0 & 0 \\ 0 & -\frac{\partial^2}{\partial x_2^2} V_t(\mathbf{x}) \end{bmatrix}, \begin{bmatrix} -\frac{\partial^2}{\partial x_1^2} V_t(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In case 2,  $\gamma_{1t}^* > 0$  and  $\gamma_{2t}^* > 0$ .

We have  $\frac{\partial}{\partial s_j} \tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma) = \frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma_j$ , for  $j \in \mathcal{J}$ . At optimality, we must have

$$-\gamma_{jt}^* = \frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*), \text{ for } j \in \mathcal{J}. \quad (19)$$

Note that  $\frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*) = (\beta c_{1-t+1} - c_{1-t}) - \frac{\partial}{\partial s_1} \tilde{H}_t(\mathbf{s}_t^*) + \beta \frac{\partial}{\partial s_1} E\{V_t(\mathbf{s}_t^* - \varepsilon_t)\}$  is the function of  $\mathbf{s}_t^*$  only, and not the function of  $\mathbf{q}_t^*$ .

We have  $\frac{\partial}{\partial q_j} \tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma) = \beta \mu_{\Lambda_t} \frac{\partial r(\mathbf{q})}{\partial q_j} - \mu_{\Lambda_t} c_{jt} + \mu_{\Lambda_t} \gamma_j$ , for  $j \in \mathcal{J}$ . At optimality, we have

$$\beta \frac{\partial r(\mathbf{q}_t^*)}{\partial q_j} = c_{jt} - \gamma_{jt}^*, \text{ for } j \in \mathcal{J}. \quad (20)$$

According to (19) and (20), we have

$$\beta \frac{\partial r(\mathbf{q}_t^*)}{\partial q_j} = c_{jt} + \frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*), \text{ for } j \in \mathcal{J},$$

which admit the implicit function  $\mathbf{q}_t^*(\mathbf{s}_t^*)$ . So we have

$$\nabla_{\mathbf{s}} \mathbf{q}_t^*(\mathbf{s}_t^*) = [\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1} [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]$$

From the proof of (a), we have  $\frac{\partial V_t(\mathbf{x})}{\partial x_j} = -\gamma_{jt}^*$ , for  $j \in \mathcal{J}$ . Therefore, the Hessian of  $V_t(\mathbf{x})$  can be expressed as  $\nabla_{\mathbf{x}}^2 V_t(\mathbf{x}) = -\nabla_{\mathbf{x}} \gamma_t^*$ . According to (19), we use the chain rule to derive

$$\nabla_{\mathbf{x}}^2 V_t(\mathbf{x}) = -\nabla_{\mathbf{s}} \gamma_t^* \nabla_{\mathbf{x}} \mathbf{s}_t^* = [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)] [\nabla_{\mathbf{x}} \mathbf{s}_t^*].$$

The relationship of  $\mathbf{s}_t^*$  and  $\mathbf{x}$  is established in the constraints such that

$$s_{jt}^* + \mu_{\Lambda_t} q_{jt}^*(\mathbf{s}_t^*) - x_{jt} = 0, \text{ for } j \in \mathcal{J}.$$

The inequality constraints must be bind according to the complementary slackness, because we assume  $\gamma_{jt}^* > 0$ , for all  $j \in \mathcal{J}$ . Thus,  $\nabla_{\mathbf{x}} \mathbf{s}_t^*$  can be obtained by the implicit function theorem.

$$\nabla_{\mathbf{x}} \mathbf{s}_t^* = [\mathbf{I} + \mu_{\Lambda_t} \nabla_{\mathbf{s}} \mathbf{q}_t^*(\mathbf{s}_t^*)]^{-1} = \left[ \mathbf{I} + \frac{\mu_{\Lambda_t}}{\beta} [\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1} [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)] \right]^{-1}.$$

Therefore,  $\nabla_{\mathbf{x}}^2 V_t(\mathbf{x}) = [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)] \left[ \mathbf{I} + \frac{\mu_{\Lambda_t}}{\beta} [\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1} [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)] \right]^{-1}$ , or equivalently,

$$[\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]^{-1} = [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]^{-1} + \frac{\mu_{\Lambda_t}}{\beta} [\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1}. \quad (21)$$

From Assumption 3,  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]$  is an inverse M-matrix. So  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1}$  is an M-matrix. Furthermore,  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]^{-1}$  is an M-matrix, due to the assumption of induction. The summation of  $\frac{\mu_{\Lambda_t}}{\beta} [-\nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1}$  and  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]^{-1}$  is still an M-matrix, in that the class of symmetric M-matrices is closed under addition (Fallat and Johnson 1999, p152). Therefore  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is an inverse M-matrix, if  $\gamma_{jt}^* > 0$ , for all  $j \in \mathcal{J}$ .

Third, we prove part (c). Clearly,  $[\nabla_{\mathbf{s}}^2 \tilde{H}_{t-1}(\mathbf{s})]$  is a positive diagonal matrix, as we have proved at the beginning. According to the Leibniz's Rule, we have  $[\nabla_{\mathbf{s}}^2 EV_t(\mathbf{s} - \varepsilon_{t-1})] = E[\nabla_{\mathbf{s}}^2 V_t(\mathbf{s} - \varepsilon_{t-1})]$ . Due to the results in the two cases of (b), we have  $[\nabla_{\mathbf{s}}^2 (\tilde{H}_{t-1}(\mathbf{s}) - \beta EV_t(\mathbf{s} - \varepsilon_{t-1}))]$  is a positive definite and nonnegative matrix, therefore  $[-\nabla_{\mathbf{s}}^2 \tilde{G}_{t-1}(\mathbf{s}, \mathbf{q})]$  is an inverse M-matrix (see Corollary 1).

Last, we show the strictly complementary slackness holds for  $(x_1, x_2)$  almost everywhere. To complete the proof, we discuss the situations in which the strictly complementary slackness might collapse, which means both  $\gamma_{jt}^* = 0$  and  $s_{jt}^* + \mu_{\Lambda_t} q_{jt}^* - x_j = 0$ .

Consider  $\gamma_{1t}^* = 0$  and  $\gamma_{2t}^* = 0$ . Equations (19) give a unique solution of  $\mathbf{s}_t^*$ , and equations (20) give a unique solution of  $\mathbf{q}_t^*$ . So, the collapse of the strictly complementary slackness requires either  $s_{1t}^* + \mu_{\Lambda_t} q_{1t}^* - x_1 = 0$  or  $s_{2t}^* + \mu_{\Lambda_t} q_{2t}^* - x_2 = 0$ .

Consider  $\gamma_{1t}^* = 0$  and  $\gamma_{2t}^* > 0$ . The complementary slackness collapses if  $s_{1t}^* + \mu_{\Lambda_t} q_{1t}^* - x_1 = 0$ . We will show, for any  $x_2$ , there is a unique  $x_1$ , which satisfies  $x_1 = s_{1t}^* + \mu_{\Lambda_t} q_{1t}^*$ . Equations (19) admit a unique solution  $(s_{1t}^*(\gamma_{2t}^*), s_{2t}^*(\gamma_{2t}^*))$ , and equations (20) admit a unique solution  $(q_{1t}^*(\gamma_{2t}^*), q_{2t}^*(\gamma_{2t}^*))$ . Because  $\gamma_{2t}^* > 0$ , we must have  $s_{2t}^*(\gamma_{2t}^*) + \mu_{\Lambda_t} q_{2t}^*(\gamma_{2t}^*) - x_2 = 0$ . Thus for any given  $x_2$ , there is a unique  $\gamma_{2t}^*$  makes valid the complementary slackness. So there is at most one  $x_1$  which satisfies  $x_1 = s_{1t}^*(\gamma_{2t}^*) + \mu_{\Lambda_t} q_{1t}^*(\gamma_{2t}^*)$ .

### Proof of Proposition 8

Consider any given  $\mathcal{L}^+ \subset \mathcal{J}$ , and  $\mathcal{L}^- = \mathcal{J} \setminus \mathcal{L}^+$ , we have

$$(\mathbf{s}_t^*(\mathbf{x}_{\mathcal{L}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{L}^+})) \in \arg \max_{\mathbf{s} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} \tilde{G}_t(\mathbf{s}, \mathbf{q} \mid \mathbf{s}_{\mathcal{L}^+} + \mu_{\Lambda_t} \mathbf{q}_{\mathcal{L}^+} = \mathbf{x}_{\mathcal{L}^+}).$$

Due to the property (b) and (c) of Lemma 4, W.L.O.G., we assume that the first  $J - m$  products are in  $\mathcal{L}^-$ , and the last  $m$  products are in  $\mathcal{L}^+$ .

We define the Lagrangian function,  $\tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma_{\mathcal{L}^+}) = \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma_{\mathcal{L}^+}(\mathbf{s}_{\mathcal{L}^+} + \mu_{\Lambda_t} \mathbf{q}_{\mathcal{L}^+} - \mathbf{x}_{\mathcal{L}^+})$ . If define  $\gamma$  to consist of  $\mathbf{0}_{\mathcal{L}^-}$  and  $\gamma_{\mathcal{L}^+}$ , we can simply rewrite the Lagrangian function,

$$\tilde{L}_t(\mathbf{s}, \mathbf{q}, \gamma) = \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma(\mathbf{s} + \mu_{\Lambda_t} \mathbf{q} - \mathbf{x}).$$

We have the first derivative of the Lagrangian function:

$$\begin{cases} \frac{\partial \tilde{L}_t}{\partial s_j} = \frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}, \mathbf{q}) + \gamma_j = (\beta c_{j \ t+1} - c_{jt}) - \frac{\partial}{\partial s_j} \tilde{E} H_t(\mathbf{s}) + \beta \frac{\partial}{\partial s_j} E\{V_t(\mathbf{s} - \varepsilon_t)\} + \gamma_j; \\ \frac{\partial \tilde{L}_t}{\partial q_j} = \beta \mu_{\Lambda_t} \frac{\partial r(\mathbf{q})}{\partial q_j} - \mu_{\Lambda_t} c_{jt} + \mu_{\Lambda_t} \gamma_j. \end{cases}$$

The complementary slackness can be expressed as  $\gamma_j(s_j + \mu_{\Lambda_t} q_j - x_j) = 0$ , where  $\gamma_j = 0$ , for  $j \in \mathcal{L}^-$  and  $s_k + \mu_{\Lambda_t} q_k - x_k = 0$ , for  $k \in \mathcal{L}^+$ . At optimality, we have

$$\begin{cases} \frac{\partial}{\partial s_j} \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*) = -\gamma_{jt}^*; \\ \beta \frac{\partial r(\mathbf{q}_t^*)}{\partial q_j} = c_{jt} - \gamma_{jt}^*. \end{cases} \quad (22)$$

The equations define the implicit functions  $\mathbf{s}_t^*(\gamma_{t\mathcal{L}^+}^*)$  and  $\mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*)$ . By Assumption 3,  $|\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)| < 0$ . According the theorem of implicit function, we have

$$\begin{cases} [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*)]_{J \times M} = -[\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]_{J \times J}^{-1} [\nabla_{\gamma_{\mathcal{L}^+}}(\gamma_t^*)]_{J \times M}, \text{ where } [\nabla_{\gamma_{\mathcal{L}^+}}(\gamma_t^*)] = \mathbf{I}(\mathcal{J}, \mathcal{L}^+), \\ [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{s}_t^*(\gamma_{t\mathcal{L}^+}^*)]_{J \times M} = -[\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]_{J \times J}^{-1} [\nabla_{\gamma_{\mathcal{L}^+}}(\gamma_t^*)]_{J \times M}. \end{cases}$$

Note  $\mathbf{I}$  is a  $J$ -dimension identity matrix. The base-stock level is equal to the expected demand plus the safety stock,  $\mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*) = \mathbf{s}_t^*(\gamma_{t\mathcal{L}^+}^*) + \mu_{\Lambda_t} \mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*)$ , then we have

$$[\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*)] = - \left[ [\nabla_{\mathbf{s}}^2 \tilde{G}_t(\mathbf{s}_t^*, \mathbf{q}_t^*)]^{-1} + \mu_{\Lambda_t} [\beta \nabla_{\mathbf{q}}^2 r(\mathbf{q}_t^*)]^{-1} \right] [\nabla_{\gamma_{\mathcal{L}^+}}(\gamma_t^*)] = \mathbf{A}_t(\mathcal{J}, \mathcal{L}^+).$$

Thereby we have  $[\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{y}_{t\mathcal{L}^-}^*(\gamma_{t\mathcal{L}^+}^*)] = \mathbf{A}_t(\mathcal{L}^-, \mathcal{L}^+)$ , and  $[\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)] = \mathbf{A}_t(\mathcal{L}^+)$ . Now we consider the constraints,

$$\mathbf{x}_{\mathcal{L}^+} = \mathbf{s}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*) + \mu_{\Lambda_t} \mathbf{q}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*) = \mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*),$$

which admits the implicit function  $\gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})$ . Using the implicit function theorem, we have

$$[\nabla_{\mathbf{x}_{\mathcal{L}^+}} \gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})] = [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)]^{-1} = \mathbf{A}_t(\mathcal{L}^+)^{-1}.$$

Now we show Property D. Consider  $j \in \mathcal{L}^-$ . By the chain rule, we have

$$[\nabla_{\mathbf{x}_{\mathcal{L}^+}} y_{jt}^*]_{1 \times M} = [\nabla_{\gamma_{\mathcal{L}^+}} y_{jt}^*(\gamma_{t\mathcal{L}^+}^*)]_{1 \times M} [\nabla_{\mathbf{x}_{\mathcal{L}^+}} \gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})]_{M \times M} = \mathbf{A}_t(j, \mathcal{L}^+) \mathbf{A}_t(\mathcal{L}^+)^{-1}.$$

Conclusion (a) immediately results from the above formula.

Now prove (b). Select an index  $k \in \mathcal{J}$ , and define  $x_k = y_k^*(\emptyset)$ . Add the constraint such that  $y_k = x_k$ , and consider the situation that  $\mathcal{L}_1^+$  has only one index  $k$ , then  $\mathcal{L}_1^- = \mathcal{J} \setminus \{k\}$ . Property D requires  $[\nabla_{x_k} y_{jt}^*] \leq 0$ , for any  $j \in \mathcal{L}_1^-$ . Thus  $\mathbf{A}_t^*(j, k) \mathbf{A}_t^*(k)^{-1} \leq 0$ . We have already known that  $r(\mathbf{q})$  is strictly concave in  $r(\mathbf{q})$ , then  $\tilde{G}_t(\mathbf{s}, \mathbf{q})$  is strictly concave also (the proof is similar to that of Theorem 3). Hence,  $\mathbf{A}_t^*$  is positively definite, having the positive diagonal entry  $\mathbf{A}_t^*(k)$ , for any  $k \in \mathcal{J}$ . The condition  $\mathbf{A}_t^*(j, k) \mathbf{A}_t^*(k)^{-1} \leq 0$  implies that  $\mathbf{A}_t^*(j, k) \leq 0$ , for any  $k \in \mathcal{J}$  and  $j \in \mathcal{J} \setminus \{k\}$ . It leads to that  $\mathbf{A}_t^*$  is a positive definite Z-matrix. So we have  $\mathbf{A}_t^*$  an M-matrix (Fallat and Johnson 1999, p152).

Consider part (c). If  $\mathbf{A}_t$  is an M-matrix, then  $[\nabla_{\mathbf{x}_{\mathcal{L}^+}} y_{jt}^*] = \mathbf{A}_t(j, \mathcal{L}^+) \mathbf{A}_t(\mathcal{L}^+)^{-1} \leq 0$ . Therefore Property D in Assumption 2 holds.

Part (d) is obvious, in that the summation of symmetric M-matrices is an M-matrix (Fallat and Johnson 1999, p152).

## Proof of Proposition 9

We define the Lagrangian vector,  $\gamma = (\gamma_j)_{j \in \mathcal{J}}$ , and the Lagrangian function,  $L_t(\mathbf{y}, \mathbf{q}, \gamma) = G_t(\mathbf{y}, \mathbf{q}) + \gamma(\mathbf{y} - \mathbf{x})$ . At optimality,  $\mathbf{y} = \mathbf{y}_t^*$ ,  $\mathbf{q} = \mathbf{q}_t^*$ , and  $\gamma = \gamma_t^*$ , which are corresponding Lagrangian multipliers. We have  $V_t(\mathbf{x}) = L_t(\mathbf{y}, \mathbf{q}, \gamma) = G_t(\mathbf{y}, \mathbf{q}) + \gamma(\mathbf{y} - \mathbf{x})$ .

(a) First, we prove the twice differentiability under the condition of strictly complementary slackness. Fiacco (1976) (Theorem 2.1 (b)) states that, when the strictly complementary slackness holds, the optimal values,  $\mathbf{y}_t^*$ ,  $\mathbf{q}_t^*$ , and  $\gamma_t^*$  are all differentiable in  $\mathbf{x}$ .

By the Envelope Theorem, we have

$$\frac{\partial V_t(\mathbf{x})}{\partial x_j} = \frac{\partial L_t}{\partial x_j} = -\gamma_{jt}^*, \text{ for } j = 1, \dots, J.$$

Because  $\gamma_{jt}^*$  is differentiable in  $\mathbf{x}$ ,  $\frac{\partial V_t(\mathbf{x})}{\partial x_j}$  is differentiable in  $\mathbf{x}$ , thus  $V_t(\mathbf{x})$  is twice differentiable.

Second, we prove the submodularity. We use chain rule to derive the second derivatives of  $V_t(\mathbf{x})$ . To avoid overlapping, in the proof of part (a), we only look at the case in which  $\gamma_{jt}^* > 0$ , for all  $j \in \mathcal{J}$ . The case that  $\gamma_t^*$  is divided to  $\mathbf{0}_{\mathcal{J}^-}$  and  $\gamma_{t\mathcal{J}^+}^*$  will be discussed in the proof of (b).

We have  $\frac{\partial L_t}{\partial y_j} = (\beta c_{j,t+1} - c_{j,t}) - \frac{\partial}{\partial y_j} E\{\hat{H}_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) - \beta V_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q}))\} + \gamma_j$ . At optimality,

$$-\gamma_{jt}^* = (\beta c_{j,t+1} - c_{j,t}) + \frac{\partial}{\partial y_j} E\left\{-\hat{H}_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*)) + \beta V_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))\right\}. \quad (23)$$

Similarly,  $\frac{\partial L_t}{\partial q_j} = \beta \mu_{\Lambda_t} \frac{\partial r(\mathbf{q})}{\partial q_j} - \beta \mu_{\Lambda_t} c_{j,t} - \frac{\partial}{\partial q_j} E\{\hat{H}_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) + \beta V_t(\mathbf{y} - \mathbf{d}_t(\mathbf{q}))\}$ . At optimality,

$$\beta \mu_{\Lambda_t} \frac{\partial r(\mathbf{q}_t^*)}{\partial q_1} - \beta \mu_{\Lambda_t} c_{1,t} + \frac{\partial}{\partial q_1} E\left\{-\hat{H}_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*)) + \beta V_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))\right\} = 0. \quad (24)$$

Define  $EW_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*)) = E\{-\hat{H}_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*)) + \beta V_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))\}$ . Recall we have  $\frac{\partial V_t(\mathbf{x})}{\partial x_j} = -\gamma_{jt}^*$ , for  $j = 1, \dots, J$ . Therefore, the second derivative of  $V_t(\mathbf{x})$  can be expressed as  $\nabla_{\mathbf{x}}^2 V_t(\mathbf{x}) = -\nabla_{\mathbf{x}} \gamma_t^*$ . Apply the chain rule to (23):

$$\nabla_{\mathbf{x}}^2 V_t(\mathbf{x}) = [\nabla_{\mathbf{y}}^2 EW_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))] \nabla_{\mathbf{x}} \mathbf{y}_t^* + [\nabla_{\mathbf{q}} [\nabla_{\mathbf{y}} EW_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))]^T] \nabla_{\mathbf{y}} \mathbf{q}_t^* \nabla_{\mathbf{x}} \mathbf{y}_t^*,$$

where  $\nabla_{\mathbf{x}} \mathbf{y}_t^* = \mathbf{I}$ . Equations (24) admit the implicit function  $\mathbf{q}_t^*(\mathbf{y}_t^*)$ . We have

$$\nabla_{\mathbf{y}} \mathbf{q}_t^* = - \left[ \nabla_{\mathbf{q}} E\left\{\beta \hat{R}_t(\mathbf{q}_t^*) + W_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))\right\} \right]^{-1} \left[ \nabla_{\mathbf{y}} [\nabla_{\mathbf{q}} EW_t(\mathbf{y}_t^* - \mathbf{d}_t(\mathbf{q}_t^*))]^T \right].$$

Recall  $G_t(\mathbf{y}, \mathbf{q})$  in (10), we can organize the above results as  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})] = \mathbf{Schur}_t$ , where

$$-\mathbf{Schur}_t = \left[ \nabla_{\mathbf{y}}^2 G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) \right] - \left[ \nabla_{\mathbf{q}} [\nabla_{\mathbf{y}} G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)]^T \right] \left[ \nabla_{\mathbf{q}}^2 G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) \right]^{-1} \left[ \nabla_{\mathbf{y}} [\nabla_{\mathbf{q}} G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)]^T \right]. \quad (25)$$

Let  $[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)]$  be partitioned as in (12), and its Schur complement,

$$\mathbf{Schur}_t = \left[ \nabla_{\mathbf{y}, \mathbf{q}}^2 [-G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)] \right] / \left[ \nabla_{\mathbf{q}}^2 [-G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)] \right],$$

is defined as in (13). If the Schur complement is an inverse M-matrix, then  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is an inverse M-matrix.

If  $\gamma_{t\mathcal{J}^-}^* = \mathbf{0}$ , and  $\gamma_{t\mathcal{J}^+}^* > \mathbf{0}$ , then  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})]$  is a matrix such that  $[-\nabla_{\mathbf{x}}^2 V_t(\mathbf{x})](\mathcal{J}^+) = \mathbf{Schur}_t(\mathcal{J}^+)$ , and the other blocks are entry-wise zero matrices.

(b) We want to show if  $\mathbf{x}_{\mathcal{L}^+}$  increases, then  $y_{jt}^*(\mathbf{x}_{\mathcal{L}^+})$  decreases, for any  $j \in \mathcal{L}^- = \mathcal{J} \subset \mathcal{L}^+$ , where  $(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{L}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{L}^+})) \in \arg \max_{\mathbf{y} \in \mathbb{R}^J; \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{L}^+} = \mathbf{x}_{\mathcal{L}^+})$ . To ease the notation, W.L.O.G., we assume  $\mathcal{L}^- = \{1, \dots, J-M\}$ , and  $\mathcal{L}^+ = \{J-M+1, \dots, J\}$ . We define the Lagrangian function,  $L_t(\mathbf{y}, \mathbf{q}, \gamma) = G_t(\mathbf{y}, \mathbf{q}) + \gamma_{\mathcal{L}^+}(\mathbf{y}_{\mathcal{L}^+} - \mathbf{x}_{\mathcal{L}^+})$ . For brevity, we can write  $L_t(\mathbf{y}, \mathbf{q}, \gamma) = G_t(\mathbf{y}, \mathbf{q}) + \gamma(\mathbf{y} - \mathbf{x})$ , where we define  $\gamma_{\mathcal{L}^-} = \mathbf{0}_{\mathcal{L}^-}$ .

At optimality,  $\gamma_t^*$  consists of  $\mathbf{0}_{\mathcal{L}^-}$ , and  $\gamma_{t\mathcal{L}^+}^*$ . We have the following equations at optimality.

$$\nabla_{\mathbf{y}} L_t = \nabla_{\mathbf{y}} G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + \gamma_t^* = \mathbf{0}; \quad (26)$$

$$\nabla_{\mathbf{q}} L_t = \nabla_{\mathbf{q}} G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) = \mathbf{0}; \quad (27)$$

$$\mathbf{y}_{t\mathcal{L}^+}^* = \mathbf{x}_{\mathcal{L}^+}. \quad (28)$$

Hence, equations of (26) and (27) together admit the implicit function  $\mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*)$  and  $\mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*)$ . Then (28) can be equivalently expressed as  $\mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*) = \mathbf{x}_{\mathcal{L}^+}$ , which also implies an implicit function  $\gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})$ . By the implicit function theorem, we have  $\nabla_{\mathbf{x}_{\mathcal{L}^+}} \gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+}) = [\nabla_{\gamma_{\mathcal{L}^+}} [\mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)]]^{-1}$ .

By the chain rule, we have

$$\nabla_{\mathbf{x}_{\mathcal{L}^+}} \begin{bmatrix} \mathbf{y}_t^* \\ \mathbf{q}_t^* \end{bmatrix} = \nabla_{\gamma_{\mathcal{L}^+}} \begin{bmatrix} \mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*) \\ \mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*) \end{bmatrix} [\nabla_{\mathbf{x}_{\mathcal{L}^+}} [\gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})]] = \nabla_{\gamma_{\mathcal{L}^+}} \begin{bmatrix} \mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*) \\ \mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*) \end{bmatrix} [\nabla_{\gamma_{\mathcal{L}^+}} [\mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)]]^{-1},$$

where  $\mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*)$  and  $\mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*)$  are defined to be implicit functions by (26) and (27).

$$\nabla_{\gamma_{\mathcal{L}^+}} \begin{bmatrix} \mathbf{y}_t^*(\gamma_{t\mathcal{L}^+}^*) \\ \mathbf{q}_t^*(\gamma_{t\mathcal{L}^+}^*) \end{bmatrix} = - \left[ \nabla_{\mathbf{y}, \mathbf{q}}^2 G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) \right]_{2J \times 2J}^{-1} \begin{bmatrix} \mathbf{0}_{(J-M) \times M} \\ \mathbf{I}_{M \times M} \\ \mathbf{0}_{J \times M} \end{bmatrix}_{2J \times K}$$

Let  $[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G_t(\mathbf{y}_t^*, \mathbf{q}_t^*)]$  be partitioned as in (12), and its Schur complement is defined as in (13). If the Schur complement is an inverse M-matrix, then  $\mathbf{C}_{11} = (\mathbf{B}/\mathbf{B}_{22})^{-1}$  is an M-matrix, where the diagonal entries are positive, and the off-diagonal entries are non-negative. We have  $\nabla_{\gamma_{\mathcal{L}^+}} [\mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)] = \mathbf{C}_{11}(\mathcal{J}, \mathcal{L}^+)$ . Moreover, we have  $\mathbf{C}_{11}(\mathcal{L}^+)$  is an M-matrix (Lemma 3), then  $[\nabla_{\gamma_{\mathcal{L}^+}} [\mathbf{y}_{t\mathcal{L}^+}^*(\gamma_{t\mathcal{L}^+}^*)]]^{-1} = [\mathbf{C}_{11}(\mathcal{L}^+)]^{-1}$  is an inverse M-matrix, having all the entries non-negative. By the chain rule and Lemma 3, we have

$$[\nabla_{\mathbf{x}_{\mathcal{L}^+}} [\mathbf{y}_{t\mathcal{L}^-}^*(\mathbf{x}_{\mathcal{L}^+})]] = [\nabla_{\gamma_{\mathcal{L}^+}} \mathbf{y}_{t\mathcal{L}^-}^*(\gamma_{t\mathcal{L}^+}^*)] [\nabla_{\mathbf{x}_{\mathcal{L}^+}} \gamma_{t\mathcal{L}^+}^*(\mathbf{x}_{\mathcal{L}^+})] = \mathbf{C}_{11}(\mathcal{L}^-, \mathcal{L}^+) \mathbf{C}_{11}(\mathcal{L}^+)^{-1} \leq \mathbf{0}.$$

### Proof of Proposition 10

The solution results from the first order condition, by solving  $\nabla_{(\mathbf{y}, \mathbf{q})} G(\mathbf{y}, \mathbf{q}) = 0$ . For Additive-I demand, the inventory related cost  $H(\mathbf{y}, \mathbf{q})$  can be specified as

$$H = \sum_{j=1}^J \left\{ h_j \int_{-\infty}^{y_j - \mu_\Lambda q_j} (y_j - \mu_\Lambda q_j - \epsilon_j) f_{\epsilon_j}(\epsilon_j) d\epsilon_j + b_j \int_{y_j - \mu_\Lambda q_j}^{\infty} (-y_j + \mu_\Lambda q_j + \epsilon_j) f_{\epsilon_j}(\epsilon_j) d\epsilon_j \right\}.$$

$$\text{So } \begin{cases} F_{\epsilon_j}(y_j - \mu_\Lambda q_j) = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ F_{\epsilon_j}(y_j - \mu_\Lambda q_j) = \frac{b_j - \beta \frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j}. \end{cases}$$

For Additive-**Diag**( $\mathbf{q}$ ) demand,  $H(\mathbf{y}, \mathbf{q})$  can be expressed as

$$H = \sum_{j=1}^J \left\{ h_j \int_{-\infty}^{(y_j - \mu_\Lambda q_j)/q_j} (y_j - \mu_\Lambda q_j - \epsilon_j q_j) f_{\epsilon_j}(\epsilon_j) d\epsilon_j + b_j \int_{(y_j - \mu_\Lambda q_j)/q_j}^{\infty} (-y_j + \mu_\Lambda q_j + \epsilon_j q_j) f_{\epsilon_j}(\epsilon_j) d\epsilon_j \right\}.$$

$$\text{So } \begin{cases} \int_{-\infty}^{\frac{y_j - \mu_\Lambda q_j}{q_j}} f_{\epsilon_j}(\epsilon_j) d\epsilon_j = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ \int_{-\infty}^{\frac{y_j - \mu_\Lambda q_j}{q_j}} \epsilon_j f_{\epsilon_j}(\epsilon_j) d\epsilon_j = \mu_\Lambda \left\{ \frac{b_j - \beta \frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j} - F_{\epsilon_j} \left( \frac{y_j - \mu_\Lambda q_j}{q_j} \right) \right\}. \end{cases}$$

For the multiplicative demand, the inventory related cost is

$$H(\mathbf{y}, \mathbf{q}) = \sum_{j=1}^J \left\{ h_j \int_0^{y_j/q_j} (y_j - \lambda q_j) f_\Lambda(\lambda) d\lambda + b_j \int_{y_j/q_j}^{\infty} (-y_j + \lambda q_j) f_\Lambda(\lambda) d\lambda \right\}.$$

$$\text{So } \begin{cases} \int_0^{y_j/q_j} f_\Lambda(\lambda) d\lambda = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ \int_0^{y_j/q_j} \lambda f_\Lambda(\lambda) d\lambda = \mu_\Lambda \frac{b_j - \beta \frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j}. \end{cases}$$

The results follow by further simplification of the above expressions.

### Proof of Corollary 4

We only prove the multiplicative case; others are similar. By Proposition 8, let  $[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G(\mathbf{y}, \mathbf{q})]$  be partitioned as in (12), and its Schur complement is defined as in (13). If the Schur complement is an inverse M-matrix, then Assumption 2 holds. Under the multiplicative demand, we have

$$H(\mathbf{y}, \mathbf{q}) = \sum_{j=1}^J \left\{ h_j \int_0^{y_j/q_j} (y_j - \lambda q_j) f_\Lambda(\lambda) d\lambda + b_j \int_{y_j/q_j}^{\infty} (-y_j + \lambda q_j) f_\Lambda(\lambda) d\lambda \right\}.$$

Clearly we have  $\frac{\partial^2}{\partial y_j \partial y_k} EH = 0$ ,  $\frac{\partial^2}{\partial y_j \partial q_k} EH = 0$ , and  $\frac{\partial^2}{\partial q_j \partial q_k} EH = 0$ , for any  $j \neq k$ . We also have

$$\frac{\partial^2}{\partial y_j^2} H(\mathbf{y}, \mathbf{q}) = \frac{h_j + b_j}{q_j} f_\Lambda\left(\frac{y_j}{q_j}\right),$$

$$\begin{aligned}\frac{\partial^2}{\partial y_j \partial q_j} H(\mathbf{y}, \mathbf{q}) &= -\frac{y_j}{q_j} \left( \frac{h_j + b_j}{q_j} f_\Lambda\left(\frac{y_j}{q_j}\right) \right), \\ \frac{\partial^2}{\partial q_j^2} H(\mathbf{y}, \mathbf{q}) &= \left( \frac{y_j}{q_j} \right)^2 \left( \frac{h_j + b_j}{q_j} f_\Lambda\left(\frac{y_j}{q_j}\right) \right), \text{ for any } j \in \mathcal{J}.\end{aligned}$$

Define  $\chi_j = \frac{h_j + b_j}{q_j} f_\Lambda\left(\frac{y_j}{q_j}\right)$ . Then we have

$$[-\nabla_{\mathbf{y}, \mathbf{q}}^2 G(\mathbf{y}, \mathbf{q})] = \begin{bmatrix} \mathbf{Diag}(\chi_j) & \mathbf{Diag}\left(-\frac{y_j}{q_j} \chi_j\right) \\ \mathbf{Diag}\left(-\frac{y_j}{q_j} \chi_j\right) & \mathbf{Diag}\left(\left(\frac{y_j}{q_j}\right)^2 \chi_j\right) + (\beta \mu_\Lambda) [-\nabla_{\mathbf{q}}^2 r(\mathbf{q})] \end{bmatrix}.$$

Then the Schur Complement can be expressed as

$$\mathbf{Schur} = \mathbf{Diag}(\chi_j) - \mathbf{Diag}\left(\frac{y_j}{q_j} \chi_j\right) \left[ \mathbf{Diag}\left(\left(\frac{y_j}{q_j}\right)^2 \chi_j\right) + (\beta \mu_\Lambda) [-\nabla_{\mathbf{q}}^2 r(\mathbf{q})] \right]^{-1} \mathbf{Diag}\left(\frac{y_j}{q_j} \chi_j\right).$$

Zhang (2005, p 152) demonstrates the parallel sum of two positive definite matrices,  $\mathbf{A}$  and  $\mathbf{B}$ , in which we have

$$[\mathbf{A}^{-1} + \mathbf{B}^{-1}]^{-1} = \mathbf{A} [\mathbf{A} + \mathbf{B}]^{-1} \mathbf{B} = \mathbf{B} [\mathbf{A} + \mathbf{B}]^{-1} \mathbf{A} = \mathbf{A} - \mathbf{A} [\mathbf{A} + \mathbf{B}]^{-1} \mathbf{A}.$$

By using the above property, we obtain

$$\begin{aligned}\mathbf{Schur} &= \mathbf{Diag}\left(\frac{y_j}{q_j}\right)^{-1} \left[ \mathbf{Diag}\left(\left(\frac{y_j}{q_j}\right)^2 \chi_j\right)^{-1} + \frac{1}{\beta \mu_\Lambda} [-\nabla_{\mathbf{q}}^2 r(\mathbf{q})]^{-1} \right]^{-1} \mathbf{Diag}\left(\frac{y_j}{q_j}\right)^{-1} \\ &= \left[ \mathbf{Diag}(\chi_j)^{-1} + \frac{1}{\beta \mu_\Lambda} \left[ \mathbf{Diag}\left(\frac{y_j}{q_j}\right) [-\nabla_{\mathbf{q}}^2 r(\mathbf{q})] \mathbf{Diag}\left(\frac{y_j}{q_j}\right) \right]^{-1} \right]^{-1}\end{aligned}$$

Because  $[-\nabla_{\mathbf{q}}^2 r(\mathbf{q})]$  is an inverse M-matrix,  $\left[ \mathbf{Diag}\left(\frac{y_j}{q_j}\right) [-\nabla_{\mathbf{q}}^2 r(\mathbf{q})] \mathbf{Diag}\left(\frac{y_j}{q_j}\right) \right]$  is an inverse M-matrix (see Lemma 1, Wang et al. 2000, p 24). Therefore  $\mathbf{Schur}^{-1}$  is an M-matrix, since the summation of two symmetric M-matrices is an M-matrix (see Fallat and Johnson 1999, p152).