

Optimizing Replenishment Intervals for Two-echelon Distribution Systems with Stochastic Demand

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We consider a periodic-review inventory system with one warehouse and N non-identical retailers. Each location replenishes inventory according to a base-stock policy in a fixed time interval. A fixed cost is incurred for each inventory reorder. The warehouse fills the retailers' orders in the same sequence as the original demand. The objective is to minimize the long-run average system cost per period. This paper provides an approach to obtain the optimal base-stock levels and reorder intervals. Specifically, for fixed reorder intervals, we show that the optimal base-stock levels can be obtained by generalizing a result in the literature. To find the optimal reorder intervals, we derive solution bounds by constructing bounds for the cost of each stage. The optimal reorder intervals can therefore be found by a complete enumeration. To reduce the computation, we prove a useful property: If the optimal policy is an integer-ratio policy, the optimal reorder interval of the warehouse is no shorter than the minimum of those of the retailers. A numerical study suggests that the optimal policy tends to be an integer-ratio policy and that the power-of-two solution generated from the corresponding deterministic model may perform poorly under some conditions.

1. Introduction

We consider a one-warehouse-multi-retailer system (OWMR) in which each stage replenishes inventory periodically. There are N non-identical retailers, each facing independent Poisson demand with complete backlogging. The retailers replenish their inventories from the warehouse, which in turn orders from an outside supplier with unlimited stock. Let j be the stage index, where $j = 0$ represents the warehouse and $j = 1, \dots, N$ represents retailer j . Each location implements a stationary echelon (S, T) policy. Specifically, retailer j reviews its inventory order position (inventory on order + inventory on hand - backorders) in every T_j periods and orders up to a base-stock level S_j ; the warehouse reviews its echelon inventory order position (inventory on order + inventory on hand + inventory at or in transit to the retailers - backorders at the retailers) in every T_0 periods and orders up to an echelon base-stock level S_0 . We call T_j reorder interval. We assume that

the order schedules are *synchronized*, i.e., a retailer, whenever possible, places an order when the warehouse receives a shipment (see §2 for an example). We also assume that the warehouse applies the virtual allocation rule (e.g., Axsäter 1993, Graves 1996), in which the retailers' orders are filled in the same sequence as the occurrence of the demand at the retailers¹. A linear echelon holding cost h_j is incurred per unit per period for stage $j(\geq 0)$ and a backorder cost b_j per unit per period for retailer $j(\geq 1)$. In addition, a fixed order cost K_j is incurred at stage $j(\geq 0)$ for each inventory reorder. The objective is to minimize the long-run average total cost per period.

This type of periodic replenishment scheme is commonly implemented in practice due to simplicity – it is just simpler for firms to plan labor, coordinate production, and deliver materials according to fixed schedules. Despite its common usage, it is not clear how to find the optimal reorder intervals. In the deterministic demand regime, although the optimal policy is not known, there exists a very simple integer-ratio policy that guarantees 94 %-effectiveness. (Let \mathbb{N} denote the set of positive integers, and T_b be some base period. An integer-ratio policy satisfies the following conditions: $T_j = n_j T_b$, $n_j \in \mathbb{N}$ for all j and T_j/T_0 or $T_0/T_j \in \mathbb{N}$ for $j = 1, \dots, N$.) If the base period T_b can be chosen as a non-integer value, there exists a power-of-two (PO2) policy that guarantees 98% effectiveness (Roundy 1985). (A PO2 policy is an integer-ratio policy where $T_j = 2^{k_j} T_b$, $k_j \in \mathbb{N}$ for all j .) It is commonly viewed that the PO2 reorder intervals obtained by solving the corresponding deterministic model can be a good approximation for the stochastic model. In fact, this approach has been used in the literature (e.g., Chen and Zheng 1997, Rao 2003). However, it is not clear whether the PO2 policy or, in general, the integer-ratio policy is effective for the stochastic OWMR model.

This study aims to answer three very fundamental questions. First, how to obtain the optimal (S, T) policy for the stochastic OWMR problem? A recent study by Liu and Song (2010) pointed out a mistake in Rao (2003) and showed that the total cost function of a single-stage system is not jointly convex in S and T . Moreover, when the demand is discrete, the cost function with optimized S for a fixed T is not even unimodal in T . They developed an enumeration algorithm to search for the optimal policy. With this recent finding, it is conceivable that finding the system-wide optimal policy for the OWMR model would be extremely difficult. Second, what properties do the optimal reorder intervals possess? Is the integer-ratio policy a good candidate for designing heuristics? Third, how effective is the PO2 solution obtained from the corresponding deterministic

¹The virtual allocation rule is commonly seen in practice. For example, Wal-Mart's distribution center assigns replenishment stocks to the demands as they occurred. The assigned stocks will be loaded onto a truck and shipped to the retail stores according to some fixed schedule (Chandran, 2003).

model? Under what conditions does the deterministic PO2 solution perform poorly?

We develop an approach to find the optimal (S, T) policy. Since we do not know whether the integer-ratio policy is optimal, we do not impose any restrictions on the reorder intervals except that they are integer multiples of some integer base period (i.e., $T_j = n_j T_b$ and $n_j, T_b \in \mathbb{N}$ for $j = 0, \dots, N$). In other words, the reorder interval of the warehouse may be shorter than that of the retailer and the considered (S, T) policy may not be an integer-ratio policy. We first characterize inventory variables and provide a bottom-up recursion to evaluate the average system cost. With fixed reorder intervals, we show that the optimal echelon base-stock levels can be obtained by generalizing the result of Axsäter (1990) (see Appendix A). Our focus is to provide a method for finding the optimal reorder intervals. This is achieved by three results we establish: (1) We allocate the system cost into each stage by decomposing the retailer’s cost. We then construct a *stage-independent* lower bound to each of the allocated stage cost functions. (2) We construct a lower bound to the system cost by converting the distribution system into a single-stage system with the original problem data. (3) We prove that these lower-bound cost functions are unimodal in the reorder interval. With these results, we generate bounds for the optimal reorder interval for each stage. The optimal reorder intervals can therefore be found by enumerating all feasible solutions.

To the best of our knowledge, this is the first paper that solves the reorder intervals for the stochastic OWMR model. We also prove an interesting property that helps computation by reducing the number of the feasible solutions: If the optimal policy is an integer-ratio policy, the reorder interval of the warehouse must be no shorter than the minimum of those of the retailers. This result corresponds to the “last-minute property” (the warehouse orders only when at least one retailer orders) established by Schwarz (1973) for the deterministic OWMR model. In a numerical study, we find that the optimal solution is always an integer-ratio policy. This provides an evidence that the inter-ratio policy is a good candidate for designing a heuristic policy. With this finding, it is natural to test whether the PO2 solution obtained from the corresponding deterministic model is effective. To our surprise, the deterministic PO2 solution can perform poorly when $K_0/(h_0\lambda_0)$ is significantly smaller than those of the retailers, where λ_j is the demand rate for the retailer j , and $\lambda_0 = \sum_{i=1}^N \lambda_i$. (See §5 for an explanation).

We provide a brief literature review by focusing on the stochastic distribution system with reorder intervals. Graves (1996) considered a general distribution system in which the inventory is controlled by a local (S, T) policy. Graves found that most of the safety stock should be held

at the retailer sites and that the virtual allocation rule is near optimal. Axsäter (1993) studied a special case of Graves’s model in that the retailers are identical and the order schedule is nested, i.e., $T_0/T_j \in \mathbb{N}$ for $j = 1, \dots, N$. He demonstrated that the local (S, T) policy under the virtual allocation rule in Graves (1996) is essentially the same as the echelon (S, T) policy.

There are papers aiming to find optimal or near-optimal reorder intervals in a context of distribution systems. Çetinkaya and Lee (2000) considered a warehouse that replenishes its stock according to an (s, S) policy and ships stocks to retailers in every fixed interval. Gürbüz et al. (2007) considered a replenishment policy in which retailers are replenished simultaneously in every fixed interval. Chu and Shen (2010) considered the system with a pre-specified service level at each location. By assuming normal approximation on the demand and stationary safety stock policy at the warehouse, they developed a worst-case bound for the power-of-two solution. Marklund (2011) considered a continuous-review system in which the warehouse implements a local (r, Q) policy and allocates the stock according to the virtual allocation rule. The retailers implement an (S, T) policy. Marklund assumed batch size at the warehouse is fixed (i.e., no fixed cost consideration at the warehouse). He provided a method to evaluate the average system cost and proposed heuristics to generate the policy parameters.

There is a stream of research on the distribution model aiming to study the impact of order schedules on bullwhip effect. Noteworthy examples include Lee et al. (1997), Cachon (1999), Chen and Samroengraja (2004), Cheung and Zhang (2008). Finally, our model is a generalization of stochastic joint replenishment problem (JRP) studied by Atkins and Iyogun (1988), who showed that the periodic review (S, T) policy outperforms the can-order policy (Silver 1981, Federgruen et al. 1984) when the major fixed cost is large. The optimization method developed in the paper can be used for solving the JRP in Atkins and Iyogun (1988).

The rest of the paper is organized as follows. §2 introduces the model and the notation. §3 characterizes the inventory variables and evaluates the total cost per period. §4 provides an approach for finding the optimal reorder intervals. §5 conducts a numerical study to examine the optimal solution and the effectiveness of the deterministic PO2 solution. §6 concludes. Appendix A shows how to optimize the base-stock levels when the reorder intervals are fixed. Appendix B presents an evaluation scheme for serial systems with non-nested (S, T) policies. Appendix C provides proofs.

2. The Model

We consider a periodic-review, two-echelon distribution system in which a single warehouse supplies N non-identical retailers. Time periods are indexed $0, 1, 2, \dots$. Let $[t, t + \tau)$ and $[t, t + \tau]$ denote the time interval over periods $t, t + 1, \dots, t + \tau - 1$ and periods $t, t + 1, \dots, t + \tau$, respectively. Retailer j faces a Poisson demand with stationary rate λ_j . The retailers' demands are independent of each other. Let $D_j[t, t + \tau)$ and $D_j[t, t + \tau]$ denote the cumulative demand over time in $[t, t + \tau)$ and $[t, t + \tau]$. There is a constant lead time L_j for stage j , $j = 0, \dots, N$. Let $L_{[0,j]} = L_0 + L_j$. Each stage implements a stationary echelon (S, T) policy. More specifically, at the beginning of every T_j periods, retailer j orders up to a base-stock level S_j if its inventory order position (inventory on order + inventory on hand - backorders) is less than S_j . Similarly, at the beginning of every T_0 periods, the warehouse orders up to an echelon base-stock level S_0 if its echelon inventory order position (inventory on order + inventory on hand + inventory at or in transit to the retailers - backorders at the retailers) is less than S_0 . We call these T_j -th periods *order period* and the moment of placing an order *order epoch*. Let h_j be the echelon holding cost rate for stage j , $j = 0, \dots, N$, and the local holding cost rate $H_j = h_0 + h_j$ for $j = 1, \dots, N$. Unmet demand is fully backlogged at each stage. Let b_j be the backorder cost rate for stage j , $j = 1, \dots, N$. Finally, there is a fixed cost K_j associated with each inventory reorder at stage j . The objective is to obtain the policy such that the total system cost per period is minimized.

We assume that the reorder intervals are an integer multiple of some integer base period. Without loss of generality, let the base period be one. The ordering activities between the warehouse and the retailers are coordinated in a synchronized manner: Whenever possible, retailer j places an order when the warehouse receives a shipment. For example, consider a one-warehouse-two-retailer system with $L_0 = 3$, $T_0 = 2$, $T_1 = 1$, and $T_2 = 3$. Suppose that the warehouse places an order at the beginning of period $t, t + 2, t + 4, \dots$. Consider the order epoch t . This order placed at t will arrive at the beginning of period $t + L_0 = t + 3$. This is the moment that both retailers place an order. Thus, the order periods for retailer 1 are $t + 3, t + 4, t + 5, \dots$, and for retailer 2 are $t + 3, t + 6, t + 9, \dots$. We term such an order coordination as *synchronized ordering rule*. Let

$$T = lcm\{T_0, T_1, \dots, T_N\},$$

where *lcm* is an operator that generates the least common multiplier of T_j 's. If we define t and $t + L_0$ as the starting time of a *cycle* for the warehouse and the retailers, respectively, the next cycle will start in T periods later under the synchronized ordering rule. Note that we do not impose any

restrictions on the reorder intervals other than integers.

One concern for the synchronized ordering rule is that it may lead to a bigger bullwhip effect when the system is controlled by local policies (Lee et al. 1997). However, Cheung and Zhang (2008) pointed out that a high bullwhip effect may not necessarily lead to a high system cost. Moreover, the bullwhip effect has less impact in our model because demand is learned by the warehouse when it occurs (i.e., echelon control).

Under the echelon (S, T) policy, the retailer fills the incoming demand as if it were a single-stage system. That is, when a unit of demand arrives, the retailer fills the demand and the retailer's net inventory level (= inventory on hand minus backorders) is reduced by one unit. The retailer does not place an order until its next order epoch. On the other hand, the warehouse immediately learns this arrived demand and assigns one unit (if available) to this demand. These assigned inventory units will be shipped at the retailer's order epoch. Thus, one may view the retailers' order epochs as "shipping times" planned by the warehouse.

Because the centralized information is available, we assume that the warehouse commits its inventory, if available, to fill the incoming orders according to the sequence of the demands occurring at the retailer site. If the warehouse does not have an uncommitted unit to fill an arrived demand, the warehouse creates a backorder and adds this to the current list of outstanding orders. When inventory becomes available, the outstanding orders are filled in the sequence in which they are created. This is the so-called *virtual allocation* rule (Graves 1996).

We end this section by listing the sequence of events in a period. For the warehouse, (1) an order, if any, is received from retailer j ; (2) an order is placed with the outside supplier if the period is an order period; (3) a shipment, if any, is received; (4) a shipment is sent to retailer j if the period is retailer j 's order period. For retailer j , order placement occurs at the beginning of retailer j 's order period, while customer demand arrives during the period. Costs are evaluated at the end of the period for all stages.

3. Evaluation

Under the echelon (S, T) policy, the system forms a regenerative process with a cycle of T periods. Specifically, consider a warehouse order epoch t at which the warehouse orders up to S_0 . This order will arrive at the beginning of period $t + L_0$. We assume that this is the moment that each retailer j will simultaneously place an order up to S_j . Thus, the regenerative cycle that we consider in the subsequent analysis is $[t, t + T - 1]$ for the warehouse and $[t + L_0, t + L_0 + T - 1]$ for each retailer

j . We call t and $t + L_0$ *regenerative epoch* for the warehouse and the retailers, respectively. To evaluate the total cost per period, we only need to characterize the distribution of the inventory variables in the above regenerative cycle. The long-run average total cost per period is equal to the expected total cost incurred in the cycle divided by the cycle length T .

We first define inventory variables below: For stage $j, j = 0, \dots, N$,

$IOP_j(n)$ = echelon inventory order position after ordering at stage j at the beginning of period n ,

$IL_j^-(n)$ = echelon inventory level at stage j at the beginning of period n ,

$IL_j(n)$ = echelon inventory level at stage j at the end of period n .

Note that for retailer j , the above echelon inventory variables are equivalent to the local inventory variables at they are the most downstream stages.

We describe the inventory dynamics in the regenerative cycle. Let r be the period index in the considered regenerative cycle, i.e.,

$$r = 0, 1, 2, \dots, T - 1.$$

Also, define $[a]$ as the roundoff operator, which returns the greatest integer less than or equal to a , a real number. Define

$$r_j(r) = \left\lfloor \frac{r}{T_j} \right\rfloor T_j, \quad j = 0, 1, 2, \dots, N.$$

Thus, for $r = 0, 1, \dots, T - 1$,

$$r_j(r) = 0, T_j, 2T_j, \dots, \left\lfloor \frac{T-1}{T_j} \right\rfloor T_j.$$

Since t and $t + L_0$ are regenerative epochs for the warehouse and the retailers, respectively, the warehouse's order periods are $t + r_0(r)$, and the retailer j 's order periods are $t + L_0 + r_j(r)$. As we shall see, the order decision for the warehouse at the beginning of period $t + r_0(r)$ will directly or indirectly determine the retailer j 's inventory levels $IL_j(t + L_{[0,j]} + r)$.

Let $D_0[s, t] = \sum_{j=1}^N D_j[s, t]$ and $D_0[s, t] = \sum_{j=1}^N D_j[s, t]$. The inventory dynamics for the warehouse are

$$IOP_0(t + r_0(r)) = S_0, \tag{1}$$

$$\begin{aligned} IOP_0(t + r) &= IOP_0(t + r_0(r)) - D_0[t + r_0(r), t + r] \\ &= S_0 - D_0[t + r_0(r), t + r], \end{aligned} \tag{2}$$

$$IL_0^-(t + L_0 + r) = IOP_0(t + r) - D_0[t + r, t + L_0 + r], \quad (3)$$

$$IL_0(t + L_0 + r) = IOP_0(t + r) - D_0[t + r, t + L_0 + r]. \quad (4)$$

Equation (1) means that the warehouse's echelon inventory order position after ordering is equal to S_0 . Equation (2) shows the warehouse's echelon inventory order position at any given time $t + r$ in the regenerative cycle. Equations (3) and (4) specify the echelon inventory level of the warehouse at the beginning and the end of period $t + L_0 + r$, respectively.

Now, we consider retailer j . Since retailer j 's order may not be fully filled by the warehouse when it orders and the unfilled order is backlogged at the warehouse, we define IP_j , B_0 and B_{0j} to reflect this situation:

$$IP_j(n) = \text{inventory in-transit position at the beginning of period } n \text{ at retailer } j,$$

$$B_0(n) = \text{total number of warehouse backorders at the beginning of period } n,$$

$$B_{0j}(n) = \text{number of warehouse backorders that belong to retailer } j.$$

Notice that the difference between $IOP_j(n)$ and $IP_j(n)$ is the inventory ordered by retailer j but not filled by the warehouse. Thus, $B_{0j}(n) = IOP_j(n) - IP_j(n)$ and $B_0(n) = \sum_{j=1}^N B_{0j}(n)$.

The corresponding regenerative cycle for stage j includes periods $t + L_0 + r$, $r = 0, 1, \dots, T - 1$. Suppose that $IP_j(t + L_0 + r)$ is known, it will determine $IL_j(t + L_{[0,j]} + r)$ as follows:

$$IL_j(t + L_{[0,j]} + r) = IP_j(t + L_0 + r) - D_j[t + L_0 + r, t + L_{[0,j]} + r]. \quad (5)$$

After all IL_j in each period of the regenerative cycle is obtained, the total cost of the system per period is

$$\begin{aligned} C(\mathbf{S}, \mathbf{T}) = & \sum_{j=0}^N \frac{K_j}{T_j} + \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0 IL_0(t + L_0 + r) + \sum_{j=1}^N h_j IL_j(t + L_{[0,j]} + r) \right. \\ & \left. + (b_j + H_j)[IL_j(t + L_{[0,j]} + r)]^- \right], \end{aligned} \quad (6)$$

where $[x]^- = \min\{0, -x\}$. The first term in $C(\mathbf{S}, \mathbf{T})$ represents the average total fixed cost per period² and the rest represents the average inventory holding and backorder cost per period.

Our remaining task is to characterize $IP_j(t + L_0 + r)$. First, for retailer j , notice that $IP_j(t + L_0 + r)$ depends on $IP_j(t + L_0 + r_j(r))$, i.e.,

$$IP_j(t + L_0 + r) = IP_j(t + L_0 + r_j(r)) - D_j[t + L_0 + r_j(r), t + L_0 + r]. \quad (7)$$

²Technically, the fixed cost term should be $\sum_{j=0}^N K_j \Pr(D[T_j] > 0)/T_j$. Following Section 5 of Shang and Zhou (2010), it is easy to show that all of our results can be carried over to the exact cost expression. We ignore $\Pr(D[T_j] > 0)$ in the cost expression here because this probability is close to one.

Since retailer j may not be able to receive the full quantity it ordered in period $t + L_0 + r_j(r)$, $IP_j(t + L_0 + r_j(r))$ is not necessary equal to S_j . More specifically,

$$\begin{aligned} IP_j(t + L_0 + r_j(r)) &= IOP_j(t + L_0 + r_j(r)) - B_{0j}(t + L_0 + r_j(r)) \\ &= S_j - B_{0j}(t + L_0 + r_j(r)). \end{aligned} \quad (8)$$

To characterize $B_{0j}(t + L_0 + r_j(r))$, we first have to characterize $B_0(t + L_0 + r_j(r))$. Define $IL'_0(n)$ and $IOP'_0(n)$ the *local* inventory level and inventory order position for the warehouse at the beginning of period n , respectively. By definition,

$$\begin{aligned} B_0(t + L_0 + r_j(r)) &= [IL'_0(t + L_0 + r_j(r))]^- \\ &= [IOP'_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)]]^- \\ &= \left[IOP_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{i=1}^N IOP_i(t + r_j(r)) \right]^- \\ &= \left[IOP_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{i=1}^N S_i \right]^- . \end{aligned} \quad (9)$$

The last term in (9) is due to the fact that $IOP_i(t + r_j(r)) = S_i$ under the virtual allocation rule.

Furthermore, from (2), we can re-write (9) as follows:

$$\begin{aligned} &B_0(t + L_0 + r_j(r)) \\ &= \left[S_0 - D_0[t + r_0(r_j(r)), t + r_j(r)] - D_0[t + r_j(r), t + L_0 + r_j(r)] - \sum_{j=1}^N S_j \right]^- \\ &= \left[s_0 - D_0[t + r_0(r_j(r)), t + r_j(r)] - D_0[t + r_j(r), t + L_0 + r_j(r)] \right]^- \\ &= \left[s_0 - D_0[t + r_0(r_j(r)), t + L_0 + r_j(r)] \right]^- \end{aligned} \quad (10)$$

where $s_0 = S_0 - \sum_{j=1}^N S_j$, the local base-stock level for the warehouse. Note that $r_0(r_j(r)) \leq r_j(r)$, which implies that the time interval in D_0 in (10) must be at least L_0 periods. Specifically, $0 \leq r_j(r) - r_0(r_j(r)) < T_0$ where $r_j(r) - r_0(r_j(r)) = 0$ when T_j is an integer multiple of T_0 .

Under the virtual allocation rule, we can apply binomial disaggregation on B_0 to obtain the distribution of B_{0j} because demand arrival at each retailer follows an independent Poisson process.

That is, for any period n , we have

$$P(B_{0j}(n) = k | B_0(n) = m) = \binom{m}{k} \left(\frac{\lambda_j}{\lambda_0} \right)^k \left(\frac{\lambda_0 - \lambda_j}{\lambda_0} \right)^{m-k}, \quad k = 0, 1, \dots, m, \quad (11)$$

where $\lambda_0 = \sum_{j=1}^N \lambda_j$. Since this conditional probability is independent of time, we shall omit the period index n in the further analysis. Consequently, the distribution of B_{0j} is

$$\begin{aligned}
& \mathbb{P}(B_{0j}(t + L_0 + r_j(r)) = k) \\
&= \sum_{m=k}^{\infty} \mathbb{P}(B_0(t + L_0 + r_j(r)) = m) \mathbb{P}(B_{0j} = k | B_0 = m) \\
&= \sum_{m=k}^{\infty} \mathbb{P}(D_0[L_0 + r_j(r) - r_0(r_j(r))] = s_0 + m) \binom{m}{k} \left(\frac{\lambda_j}{\lambda_0}\right)^k \left(\frac{\lambda_0 - \lambda_j}{\lambda_0}\right)^{m-k} \quad (12)
\end{aligned}$$

Note that the distribution of B_{0j} is independent of time period t and depends on s_0 , T_0 and T_j . After B_{0j} is characterized, we can use (7) and (8) to further obtain the distribution of IP_j . Finally, we apply (5) to obtain the distribution of IL_j for each period in the considered regenerative cycle. The average total cost per period is shown in (6).

A Bottom-up Evaluation Scheme

We provide a bottom-up recursion to simplify the evaluation of $C(\mathbf{S}, \mathbf{T})$ and to facilitate the subsequent analysis. This procedure is to first evaluate the retailer j 's cost by assuming that the retailer has ample supply. Then, we evaluate the echelon cost of the warehouse, which is equivalent to the total system cost. More specifically, let $\mathbb{M}_x(y)$ be an operator that returns the remainder of y divided by x , where x is a positive integer and y is a nonnegative integer.

Let $D_j[\tau]$ and $D_j[\tau]$ denote the total demand in τ and $\tau + 1$ periods, respectively, at retailer j . For $j = 1, \dots, N$, define

$$G_j(y, T_j, r) = \mathbb{E}[h_j(y - D_j[L_j + \mathbb{M}_{T_j}(r)]) + (b_j + H_j)(y - D_j[L_j + \mathbb{M}_{T_j}(r)])^-], \quad (13)$$

and

$$G_0(\mathbf{S}, \mathbf{T}, r) = \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N G_j(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right], \quad (14)$$

where $B_{0j}(L_0 + r_j(r))$ is the steady state of $B_{0j}(t + L_0 + r_j(r))$, which can be found by removing the time index t in (10)-(12). Let

$$G(\mathbf{S}, \mathbf{T}) \stackrel{def}{=} \frac{1}{T} \sum_{r=0}^{T-1} G_0(\mathbf{S}, \mathbf{T}, r).$$

Proposition 1 *For given echelon base-stock policies with parameters (\mathbf{S}, \mathbf{T}) , the average total cost per period is*

$$C(\mathbf{S}, \mathbf{T}) = \sum_{j=0}^N \frac{K_j}{T_j} + G(\mathbf{S}, \mathbf{T}),$$

where $\sum_{j=0}^N (K_j/T_j)$ is the average total fixed order cost per period and $G(\mathbf{S}, \mathbf{T})$ is the average inventory holding and backorder cost per period.

Proposition 1 can be proven by comparing the total cost obtained by using the above recursion with the total cost obtained in (6). We omit the details for brevity.

4. Optimization

This section discusses how to obtain the optimal (S, T) policies. When the reorder intervals are fixed, we show that Axsäter's (1990) algorithm can be revised to obtain the optimal echelon base-stock levels (see Appendix A). As a result, our focus is to show how to optimize the reorder intervals.

We suggest a complete enumeration to find the optimal reorder intervals. To facilitate the search, we construct bounds for the optimal reorder interval T_j^* , $j = 0, \dots, N$. These solution bounds can be obtained by constructing a lower-bound function for each stage's cost and for the entire system cost. More specifically, in §4.1, we allocate the total system cost into each stage by decomposing the retailer's cost. In §4.2, we construct a lower bound to the allocated warehouse's cost and retailer's cost. These lower bounds are a function of each stage's reorder interval. In §4.3, we show that the total system cost is bounded below by the total cost obtained from a single-stage system with a reorder interval T_0 . Finally, in §4.4, we show how to construct upper and lower bounds for the optimal reorder intervals by using these stage and system cost bound functions. We also prove a property that helps the computation.

4.1 Decomposition of the System Cost

In serial inventory systems, it is a common technique to use the so-called induced-penalty cost function (the cost charged to an upstream stage if it cannot satisfy its downstream stage's order) to decouple the system cost (e.g., Chen and Zheng 1998, Shang and Zhou 2010). The reason why such a technique works for the serial system is that the optimal base-stock level of a downstream stage is independent of those of its upstream stages when the reorder intervals are fixed. However, this is not true for the distribution system, in which the retailer's optimal base-stock level depends on that of the warehouse (see Appendix A).

Here, we provide a different approach to allocate $G(\mathbf{S}, \mathbf{T})$ into each stage. This approach is to decompose the retailer's cost function in *each* period r in the considered regenerative cycle. More

specifically, for $j = 1, 2, \dots, N$ and $r = 1, 2, \dots, T - 1$, define $g_j(y, T_j, r) = G_j(y, T_j, r)$. Let

$$S_j(T_j, r) = \operatorname{argmin}_y g_j(y, T_j, r),$$

and

$$g_{jj}(y, T_j, r) = \begin{cases} g_j(S_j(T_j, r), T_j, r), & \text{if } y \leq S_j(T_j, r), \\ g_j(y, T_j, r), & \text{otherwise.} \end{cases}$$

Let

$$g_{j0}(y, T_j, r) = g_j(y, T_j, r) - g_{jj}(y, T_j, r).$$

Notice that, for fixed T_j and r , $g_{jj}(y, T_j, r)$ is a constant before $S_j(T_j, r)$ and convex and increasing in y after $S_j(T_j, r)$. The minimum value is $g_j(S_j(T_j, r), T_j, r)$. On the other hand, $g_{j0}(y, T_j, r)$ is convex and decreasing in y with minimum value zero after $S_j(T_j, r)$.

We provide an explanation for this decomposition. The $g_j(y, T_j, r)$ function is the expected inventory holding and backorder cost for retailer j in period $t + L_{[0,j]} + r_j(r) + \mathbb{M}_{T_j}(r)$, referred to as period r below, when retailer j 's $IP_j(t + L_0 + r_j(r)) = y$. $S_j(T_j, r)$ represents the IP_j level that leads to retailer j 's minimum cost in period r . Notice that $S_j(T_j, r)$ is different in each period r . From retailer j 's perspective, it would be the best if the inventory position $IP_j(t + L_0 + r_j(r))$ can achieve $S_j(T_j, r)$ for period r . Unfortunately, according to the considered (S, T) policy, achieving different $S_j(T_j, r)$ for different periods r is impossible because retailer j is only allowed to place an order in period $t + L_0 + r_j(r)$. We term $g_{jj}(y, T_j, r)$ *minimum period cost* for retailer j , and $g_{j0}(y, T_j, r)$ *period penalty cost* generated from retailer j .

Note that $g_{j0}(y, T_j, r)$ is different from the induced-penalty cost function discussed in the serial multi-echelon system in that the penalty cost function charged to the warehouse here is a function of $S_j(T_j, r)$, not the retailer j 's target base-stock level S_j at its ordering epoch.

Since $g_j(y, T_j, r) = g_{j0}(y, T_j, r) + g_{jj}(y, T_j, r)$ for any y with fixed T_j and r , the total inventory and backorder cost can be rewritten as

$$\begin{aligned} G(\mathbf{S}, \mathbf{T}) &= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N g_{j0}(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right. \\ &\quad \left. + \sum_{j=1}^N g_{jj}(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right]. \end{aligned}$$

We allocate the penalty cost g_{j0} to the warehouse in each period r , and define the allocated warehouse cost as

$$\hat{g}_0(\mathbf{S}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N g_{j0}(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right]. \quad (15)$$

Furthermore, we define the allocated retailer cost as

$$\widehat{g}_j(\mathbf{S}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[g_{jj}(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right]. \quad (16)$$

Let $c_j(\mathbf{S}, \mathbf{T}) = K_j/T_j + \widehat{g}_j(\mathbf{S}, \mathbf{T})$ for $j = 0, \dots, N$. Then, we have $G(\mathbf{S}, \mathbf{T}) = \sum_{j=0}^N \widehat{g}_j(\mathbf{S}, \mathbf{T})$ and $C(\mathbf{S}, \mathbf{T}) = \sum_{j=0}^N c_j(\mathbf{S}, \mathbf{T})$. This completes the decomposition of the total cost per period.

4.2 A Lower Bound on the Stage Cost

We proceed to construct a lower bound function for $\widehat{g}_j(\mathbf{S}, \mathbf{T})$ for $j = 0, \dots, N$. As we shall see, the lower-bound cost functions are a function of each stage's reorder interval and therefore they are independent. The independence feature is important for us to construct the solution bounds.

We first construct a lower bound for the retailer j 's cost. We have, from (16),

$$\begin{aligned} \widehat{g}_j(\mathbf{S}, \mathbf{T}) &\geq \frac{1}{T} \sum_{r=0}^{T-1} g_{jj}(S_j(T_j, r), T_j, r) = \frac{1}{T_j} \sum_{r=0}^{T_j-1} g_{jj}(S_j(T_j, r), T_j, r) \\ &= \frac{1}{T_j} \sum_{r=0}^{T_j-1} g_j(S_j(T_j, r), T_j, r) \stackrel{def}{=} \underline{g}_j(T_j). \end{aligned}$$

The inequality holds because $g_{jj}(S_j(T_j, r), T_j, r)$ is the minimum cost for retailer j in period $t + L_0 + r_j(r) + M_{T_j}(r)$. Proposition 2 summarizes this result.

Proposition 2 $\widehat{g}_j(\mathbf{S}, \mathbf{T}) \geq \underline{g}_j(T_j)$.

Now, define

$$\underline{c}_j(T_j) \stackrel{def}{=} \frac{K_j}{T_j} + \underline{g}_j(T_j).$$

Proposition 3 $\underline{c}_j(T_j)$ is unimodal in T_j , $j = 1, \dots, N$.

Recall that $\underline{c}_j(T_j)$ is a lower bound to the retailer j 's cost $c_j(\mathbf{S}, \mathbf{T})$. For fixed (S_i, T_i) , $i \neq j$, $c_j(\mathbf{S}, \mathbf{T})$ has the same structure as the average total cost per period for a single-stage (S, T) system. A recent study by Liu and Song (2010) showed that the total cost function of a single-stage system with the base-stock level optimized for a fixed T is not unimodal in T . Proposition 3 indicates that, although the exact cost function is not unimodal, the lower bound cost function we derived is. Indeed, this unimodality result can be used to construct an alternative algorithm which is more efficient than Liu and Song's for the single-stage system. Here, it helps construct the solution bounds for our distribution model.

We next construct a lower bound to $\widehat{g}_0(\mathbf{S}, \mathbf{T})$ in (15). The first term in (15) is a function of (S_0, T_0) . Our goal is to construct a lower bound that is a function of (S_0, T_0) to the second term. If we can achieve this goal, we can reduce $1/T \sum_{r=0}^{T-1}$ to $1/T_0 \sum_{r=0}^{T_0-1}$ and the entire function will depend only on (S_0, T_0) .

First, for any retailer j , notice that $S_j - B_{0j}(t + L_0 + r_j(r)) = IP_j(t + L_0 + r_j(r))$. By definition,

$$\begin{aligned} IP_j(t + L_0 + r_j(r)) &\leq IL_0^-(t + L_0 + r_j(r)) \\ &= IOP_0(t + r_j(r)) - D_0[t + r_j(r), t + L_0 + r_j(r)] \quad [\text{from (3)}] \\ &= S_0 - D_0[t + r_0(r_j(r)), t + L_0 + r_j(r)]. \quad [\text{from (2)}] \end{aligned}$$

Since $g_{j0}(y, T_j, r)$ is convex and decreasing in y , we have

Proposition 4 For fixed T_j and r ,

$$\mathbb{E}[g_{j0}(S_j - B_{0j}(L_0 + r_j(r)), T_j, r)] \geq \mathbb{E}[g_{j0}(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r_j(r))], T_j, r)].$$

The right-hand side of the inequality in Proposition 4 is a lower bound to the period penalty cost incurred from retailer j . This lower bound still depends on the retailer j 's reorder interval T_j . The next result develops a lower bound to the above right-hand side term by removing this dependency.

Proposition 5

$$\mathbb{E}\left[g_{j0}(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r_j(r))], T_j, r)\right] \geq \mathbb{E}\left[g_{j0}(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r_0(r))], T_0, r)\right].$$

Proposition 5 states that by regulating the retailer j 's reorder interval T_j to T_0 , the resulting period penalty cost function is smallest among all other possibilities. Since $\mathbb{M}_{T_0}(r_0(r)) = 0$,

$$\mathbb{E}\left[g_{j0}(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r_0(r))], T_0, r)\right] = \mathbb{E}\left[g_{j0}(S_0 - D_0[L_0], T_0, r)\right]. \quad (17)$$

It is interesting to compare this result with the one established in Shang and Zhou (2010), who studied a serial system with nested reorder interval policies (i.e., the reorder interval at an upstream stage is an integer multiple of that of its immediate downstream stage) under the synchronized ordering rule. They constructed a lower bound to the induced penalty cost function by regulating the reorder intervals of the downstream stages. Although the idea of constructing bounds is similar, our result here is different from Shang and Zhou's in the following two aspects. First, we construct a lower bound to the *period* penalty cost by regulating the retailer's reorder interval, not to the

induced penalty cost. Second, the reorder interval of the warehouse may be smaller than the reorder interval of the retailer in our model.

We substitute the right-hand-side term in (17) for the exact period penalty cost function in $\widehat{g}_0(\mathbf{S}, \mathbf{T})$, and define the resulting function as $\underline{g}_0(S_0, T_0)$, i.e.,

$$\underline{g}_0(S_0, T_0) = \frac{1}{T_0} \sum_{r=0}^{T_0-1} \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N g_{j0}(S_0 - D_0[L_0], T_0, r) \right].$$

The function $\underline{g}_0(S_0, T_0)$ is a lower bound to the warehouse cost $\widehat{g}_0(\mathbf{S}, \mathbf{T})$.

Proposition 6 *For fixed T_0 , $\underline{g}_0(S_0, T_0)$ is convex in S_0 .*

Let $S_0(T_0) = \operatorname{argmin}_{S_0} \underline{g}_0(S_0, T_0)$, and $\underline{g}_0(T_0) = \underline{g}_0(S_0(T_0), T_0)$. We have

$$G(\mathbf{S}, \mathbf{T}) = \sum_{j=0}^N \widehat{g}_j(\mathbf{S}, \mathbf{T}) \geq \sum_{j=0}^N \underline{g}_j(T_j). \quad (18)$$

Equation (18) shows a lower bound to the inventory holding and backorder cost per period for any given (S, T) policies. This lower bound is the sum of N separable cost functions, each with stage j 's control parameter T_j . In order to construct solution bounds, we need one more result discussed in the next section.

4.3 A Lower Bound to the System Cost

This section constructs a lower bound to the average system cost per period. As we shall see, this lower bound is a function of warehouse's control parameters (S_0, T_0) .

Define

$$h = \min\{h_1, \dots, h_N\}, \quad H = h_0 + h, \quad b = \min\{b_1, \dots, b_N\},$$

$$IP_r(n) = \sum_{j=1}^N IP_j(n) = \text{the sum of inventory in-transit positions of the retailers in period } n.$$

Recall the average inventory holding and backorder cost $G(\mathbf{S}, \mathbf{T})$ in (6).

$$\begin{aligned} G(\mathbf{S}, \mathbf{T}) &= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0 IL_0(t + L_0 + r) + \sum_{j=1}^N \left(h_j IL_j(t + L_{[0,j]} + r) + (b_j + H_j) [IL_j(t + L_{[0,j]} + r)]^- \right) \right] \\ &\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0 IL_0(t + L_0 + r) + \sum_{j=1}^N \left(h IL_j(t + L_{[0,j]} + r) + (b + H) [IL_j(t + L_{[0,j]} + r)]^- \right) \right] \\ &\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0 IL_0(t + L_0 + r) + h \left(IP_r(t + L_0 + r) - \sum_{j=1}^N D_j[L_j] \right) \right] \end{aligned}$$

$$+(b + H) \left(IP_r(t + L_0 + r) - \sum_{j=1}^N D_j[L_j] \right)^- \Big] \stackrel{def}{=} G^S(\mathbf{S}, \mathbf{T}). \quad (19)$$

The cost function $G^S(\mathbf{S}, \mathbf{T})$ in (19) can be viewed as the inventory holding and backorder cost generated from a two-stage serial inventory system, in which the upstream stage has the parameters as those in the warehouse and the downstream stage has holding cost rate h , backorder cost rate b , and inventory position $IP_r(t + L_0 + r)$. In Appendix B, we provide a simple recursion to evaluate $G^S(\mathbf{S}, \mathbf{T})$.

We now develop a lower bound to $G^S(\mathbf{S}, \mathbf{T})$. This lower bound cost is obtained from a single-stage (S, T) system with policy control parameters (S_0, T_0) .

We first define this single-stage system. Consider a single-stage system with holding cost rate h_0 , backorder cost rate b , and the demand during lead time $D_0[L_0 + \mathbb{M}_{T_0}(r)] + \sum_{j=1}^N D_j[L_j]$. The cost function for this single-stage system for a given period r , $r = 0, 1, \dots, T_0 - 1$, is

$$\mathbb{G}(y, r) = \mathbb{E} \left[h_0 \left(y - D_0[L_0 + \mathbb{M}_{T_0}(r)] - \sum_{j=1}^N D_j[L_j] \right) + (b + h_0) \left(y - D_0[L_0 + \mathbb{M}_{T_0}(r)] - \sum_{j=1}^N D_j[L_j] \right)^- \right].$$

Let $S_0^\ell(T_0, r) = \arg \min_y \mathbb{G}(y, r)$, and

$$G^\ell(T_0) = \frac{1}{T_0} \sum_{r=0}^{T_0-1} \mathbb{G}(S_0^\ell(T_0, r), r).$$

Let $C^\ell(T_0) = K_0/T_0 + G^\ell(T_0)$ and $\pi = h_0 \sum_{j=1}^N \lambda_j L_j$, the total inventory pipeline inventory holding cost. We have

Proposition 7 (1) $C^\ell(T_0)$ is unimodal in T_0 . (2) $G^S(\mathbf{S}, \mathbf{T}) \geq \mathbb{G}(S_0, T_0) + \pi \geq G^\ell(T_0) + \pi$.

4.4 Bounds on the Optimal Reorder Intervals

We shall use the system cost bound $C^\ell(T_0)$ to find the bounds for T_0^* , and use the retailer cost bound $\underline{c}_j(T_j)$ to find the bounds for T_j^* , $j = 1, \dots, N$.

We start with the construction of bounds for T_0^* . Let C^h be the cost obtained from any heuristic policy (say, the deterministic power-of-two reorder intervals and the corresponding optimal base-stock levels obtained from the algorithm in Appendix A). Then,

$$C^h \geq C(\mathbf{S}^*, \mathbf{T}^*) \geq C^\ell(T_0^*) + \pi.$$

Since $C^\ell(T_0)$ is unimodal in T_0 and C^ℓ goes to infinity when $T_0 \rightarrow \infty$, a lower bound \underline{T}_0 and an upper \bar{T}_0 can be obtained as follows:

$$\bar{T}_0 = \max \left\{ T_0 \mid C^\ell(T_0) \leq C^h - \pi \right\}, \quad \underline{T}_0 = \min \left\{ T_0 \mid C^\ell(T_0) \leq C^h - \pi \right\}.$$

We next construct the bounds for T_j^* , $j = 1, \dots, N$. First, we can obtain the minimum value of $\underline{c}_0(T_0) \stackrel{\text{def}}{=} K_0/T_0 + \underline{g}_0(T_0)$ by searching $T_0 \in [\underline{T}_0, \bar{T}_0]$. Denote the resulting minimum value \underline{c}_0 . Then,

$$C^h \geq C(\mathbf{S}^*, \mathbf{T}^*) \geq \underline{c}_0(T_0^*) + \sum_{j=1}^N \underline{c}_j(T_j^*) \geq \underline{c}_0 + \sum_{j=1}^N \underline{c}_j(T_j^*).$$

From Proposition 3, we can find the minimal value of $\underline{c}_j(T_j)$, referred to as \underline{c}_j , $j = 1, \dots, N$. Thus, an upper bound \bar{T}_j and a lower bound \underline{T}_j to T_j^* can be obtained from the following inequalities:

$$\bar{T}_j = \max \left\{ T_j \mid \underline{c}_j(T_j) \leq C^h - \underline{c}_0 - \sum_{i \neq j} \underline{c}_i \right\}, \quad \underline{T}_j = \min \left\{ T_j \mid \underline{c}_j(T_j) \leq C^h - \underline{c}_0 - \sum_{i \neq j} \underline{c}_i \right\}.$$

To find the optimal reorder intervals, we need to enumerate the policies such that $T_j \in [\underline{T}_j, \bar{T}_j]$, and use their corresponding optimal base-stock levels. Below we prove a property that can speed up the search.

Proposition 8 *If T_j^*/T_0^* or $T_0^*/T_j^* \in \mathbb{N}$ for $j = 1, \dots, N$, then $T_0^* \geq \min\{T_1^*, \dots, T_N^*\}$.*

Proposition 8 states that if the optimal policy is an integer-ratio policy, the reorder interval of the warehouse must be no less than the minimum of those for the retailers. This property can help us avoid evaluating non-optimal policies. For example, we do not need to evaluate the policy with $(T_0, T_1, T_2) = (2, 4, 8)$ during the search when it is a feasible policy within the bounds we derived. We notice that this property corresponds to the “last-minute” property, i.e., the warehouse orders only when at least one retailer orders, established by Schwarz (1973) for the deterministic model. We have the same property here if the optimal policy belongs to an integer-ratio policy. Unfortunately, unlike the deterministic demand model, we are not able to analytically prove that the optimal policy must be an integer-ratio policy.

5. Numerical Study

The objective of the numerical study is two-fold. First, we provide several observations on the optimal reorder intervals. Second, we examine the effectiveness of the PO2 solution generated from the corresponding deterministic model. We identify conditions under which the deterministic PO2 solution performs poorly.

5.1 Properties of the Optimal Policy

The test bed includes 128 one-warehouse-two-retailer systems, which are generated from the following procedure. We fix the retailer 1's parameters and vary the parameters for the warehouse and retailer 2. More specifically, the parameters for retailer 1 are $K_1 = h_1 = L_1 = \lambda_1 = 1$ and $b_1 = 3$. For retailer 2, we set $h_2 = 1$. The rest of the parameters for the warehouse and the retailer 2 are chosen from the following sets:

$$h_0 \in \{1, 2\}, \quad K_0, K_2 \in \{0.25, 16\}, \quad L_0, L_2 \in \{1, 3\}, \quad b_2 \in \{3, 18\}, \quad \lambda_2 \in \{0.5, 1\}.$$

In these examples, we search for the optimal reorder intervals, and then find their corresponding optimal base-stock levels by using the algorithm shown in Appendix A.

Below we provide several observations on the optimal reorder intervals.

- (1) As stated, we do not restrict the inventory policy to be an integer-ratio policy, so one would expect to obtain a non-integer-ratio optimal policy. Interestingly, we do not find such a case in our test bed. This suggests that an integer-ratio policy is a good candidate when designing a heuristic for the reorder intervals.
- (2) From the perspective of coordinating shipping schedules, it is important to know which retailers would have the same reorder interval as the warehouse. For the identical retailer system, we find that when $K_j/(h_j\lambda_j) > K_0/(h_0\lambda_0)$, the optimal reorder intervals tend to be the same among all locations. This can be explained as follows: For the EOQ model, it is known that the optimal reorder interval is proportional to $K/(h\lambda)$. Thus, when $K_j/(h_j\lambda_j)$ is greater than $K_0/(h_0\lambda_0)$, the retailers should use a bigger reorder interval than the warehouse. However, from Proposition 8, T_0^* should be no less than T_j^* (for integer-ratio policies as observed). These two facts together explain why $T_0^* = T_j^*$. This observation seems to apply to the non-identical retailer system as well. Suppose that $K_0/(h_0\lambda_0)$ is smaller than $K_j/(h_j\lambda_j)$ where j is the retailer who has the smallest $K_j/(h_j\lambda_j)$, then T_0^* is always equal to T_j^* . In fact, when the cost ratios $K_j/(h_j\lambda_j)$ are not significantly different between the retailers, the optimal reorder intervals for all stages tend to be the same.
- (3) It is conceivable that when the backorder cost b_j increases, the retailer j 's optimal reorder interval T_j^* will decrease. This is because a shorter reorder interval makes the retailer more responsive to the demand, which will reduce the backorders. Interestingly, we find that a larger backorder cost at one retailer may also shorten the optimal reorder intervals of the other

retailers. For example, in the case with $(K_0, K_1, K_2) = (16, 1, 16)$, $(L_0, L_1, L_2) = (1, 1, 3)$, $(h_0, h_1, h_2) = (2, 1, 1)$, when b_2 increases from 3 to 18, the optimal reorder intervals are changed from $(T_0^*, T_1^*, T_2^*) = (6, 6, 6)$ to $(4, 4, 4)$. This phenomenon is due to the benefit of order coordination. From the system perspective, although reducing T_1^* from 6 to 4 may increase the order cost per period, this additional cost is offset by the benefit received from the order coordination between these two retailers (due to demand consolidation). We find that the demand rate has a similar effect: When the demand rate at a retailer increases, the optimal reorder intervals of all retailers become smaller.

- (4) We observe that when h_0 decreases, depending on K_0 , the retailers' optimal reorder intervals may increase or decrease. This observation can be explained as follows. When h_0 decreases, the warehouse tends to stock more. The warehouse can increase the stock level by either increasing T_0 or S_0 . If K_0 is large, it is better off for the system to increase T_0 because the benefit from reducing the order frequency would be significant. In this situation, the retailer's reorder interval may increase due to the benefit of order coordination. On the other hand, if K_0 is small, increasing T_0 may not gain a significant benefit. Instead, the system may benefit more by increasing S_0 because the retailer can be more responsive to the demand by ordering more frequently or, equivalently, reducing the reorder interval. (That is, the warehouse should stock more if the retailer orders in every week rather than every other week due to the demand disaggregation effect.)
- (5) The impact of lead time on the optimal reorder interval is not clear. To offset a longer lead time, one might expect the retailer will set a shorter reorder interval to retain responsiveness. However, this intuition holds in some cases, but not all. For example, for the case with $(K_0, K_1, K_2) = (16, 1, 16)$, $(h_0, h_1, h_2) = (2, 1, 1)$, $(L_0, L_1, L_2) = (1, 1, 1)$, $(b_1, b_2) = (3, 3)$, and $(\lambda_1, \lambda_2) = (1, 1)$, when L_0 and L_2 increase from 1 to 3, (T_0^*, T_1^*, T_2^*) is increased from $(4, 4, 4)$ to $(6, 6, 6)$. But this is the biggest difference we observe. In most cases, we do not see a significant change on the optimal reorder intervals when the lead time changes.
- (6) Overall, compared with the optimal base-stock levels, the optimal reorder intervals are relatively insensitive to the change of the system parameters. In other words, if a supply chain chooses right reorder intervals, the system can achieve a high efficiency by adjusting the base-stock levels.

The last point above raises a question of how to find a simple yet effective heuristic reorder

intervals. From the first observation, one should consider integer-ratio policies as an effective heuristic. Since the power-of-two policy generated from the corresponding deterministic model is an integer-ratio policy and it is commonly viewed that such a policy should perform well for the stochastic model, we examine its effectiveness in the next section.

5.2 Effectiveness of the Power-of-Two Solution

This section examines the effectiveness of the deterministic PO2 solution. We use the Poisson demand rate as the demand rate for the deterministic model, and apply Roundy's (1985) algorithm to generate power-of-two reorder intervals, denoted as $\mathbf{T}^d = (T_0^d, T_1^d, \dots, T_N^d)$. We then use \mathbf{T}^d to find the corresponding optimal base-stock levels. Let the resulting cost be $C(\mathbf{T}^d)$. Define the percentage error as

$$\frac{C(\mathbf{T}^d) - C(\mathbf{S}^*, \mathbf{T}^*)}{C(\mathbf{S}^*, \mathbf{T}^*)} \times 100\%.$$

We test the same 128 instances. The average percentage error is 2.94% with maximum error of 11.91%. Although the overall performance is reasonably effective, we notice that the performance of the heuristic is quite variable. For example, there are 34 instances (26.5% of the 128 instances) whose percentage errors are more than 5%.

We find that the cost ratio $K_j/(h_j\lambda_j)$ is the key driver that affects the effectiveness of the PO2 solution. More specifically, when $K_0/(h_0\lambda_0)$ is significantly larger (smaller) than $K_j/(h_j\lambda_j)$, $j = 1, 2$, the PO2 solution performs the best (worst). For example, for the 16 instances with $(K_0, K_1, K_2) = (16, 1, 0.25)$ and $h_0 \in \{1, 2\}$, $(h_1, h_2) = (1, 1)$, and $(\lambda_1, \lambda_2) = (1, 1)$, the average percentage error is 0.96% with a maximum error of 1.65%. On the other hand, for the 16 instances with the same parameters except that $(K_0, K_1, K_2) = (0.25, 1, 16)$, the average percentage error is 7.19% with a maximum error of 11.91%.

The above observation can be explained as follows. The algorithm that generates the PO2 solution includes two steps, clustering and minimization (Roundy 1985). Based on the cost ratios $K_j/(h_j\lambda_j)$, the clustering step assigns each retailer into one of three clusters G , E , and L where G (E , L , respectively) includes the retailers whose reorder intervals are larger than (equal to, smaller than, respectively) the warehouse's. Under the condition of $K_0/(h_0\lambda_0) < K_j/(h_j\lambda_j)$, the algorithm will allocate the warehouse and the retailer with the smallest $K_j/(h_j\lambda_j)$, say, retailer 1, into the E cluster, leaving the other retailer, say, retailer 2, in the G cluster. Consequently, $T_0^d = T_1^d < T_2^d$. However, the optimal solution T_0^* , T_1^* , and T_2^* tend to be the same. One reason that leads to this difference is the order (demand) pooling effect. In the deterministic model, since

there is no demand variability, the retailers are clustered only based on the cost ratios. However, with stochastic demand, determining an effective clustering scheme is more complex. For example, when the retailers use the same reorder interval, the warehouse will face a less variable order stream from the retailers, which, in turn, helps to reduce the inventory holding cost. This may explain why the optimal reorder intervals tend to be the same in the stochastic demand model when $K_j/(h_j\lambda_j)$ are not significantly different between the retailers.

6. Concluding Remarks

This paper studies a one-warehouse, multi-retailer system in which (S, T) policies are implemented. When the demand is deterministic, it is known that there exists a very simple PO2 solution that yields a near-optimal cost. However, fewer results are known for the corresponding stochastic model. In this paper, we first derive a simple bottom-up recursion to evaluate the system cost for any given (S, T) policies. We then provide a method to obtain the optimal policy parameters. In a numerical study, we observe the behaviors of the optimal reorder intervals. Among others, we find that the optimal reorder intervals tend to follow integer-ratio relations. Thus, an integer-ratio policy is a good candidate for designing a heuristic policy. Nevertheless, we find that the deterministic PO2 solution can be very ineffective when the cost ratio of fixed order cost to the holding cost times the demand rate of the warehouse is significantly smaller than those of the retailers. This conclusion is in contrast to a common belief that a deterministic PO2 solution in general is an effective approximation for the stochastic model. Finally, we find that the inefficiency of the PO2 solution is due to that the clustering scheme ignores the benefit of the retailers' order pooling. Thus, it is an interesting open question whether one could propose a more effective heuristic that clusters the retailers based on not only the above cost ratios but also the benefit of order pooling between the retailers.

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Appendix A: Optimization of Base-Stock Levels with Fixed \mathbf{T}

This appendix shows how to optimize the optimal echelon base-stock levels when the reorder intervals \mathbf{T} are fixed. It is worth mentioning that Axsäter (1993) provides an approach (which is a generalization of Axsäter (1990) on base-stock systems) for finding the optimal base-stock levels with fixed reorder intervals. Axsäter's model is a special case of ours because he assumes identical retailers and that the reorder interval of the warehouse is an integer multiple of that of the retailer (i.e., nested integer-ratio policies). Below we scratch the idea of how his approach can be generalized to our model. The proofs for Propositions 9-11 are shown in Appendix C.

We present the analysis from the local policy perspective, because B_0 is a function of s_0 , and S_j is the same as the local base-stock levels s_j .

Define the local inventory holding and backorder cost for the retailer j as follows:

$$f_j(y, r) = \mathbb{E}[H_j(y - D_j[L_j + \mathbb{M}_{T_j}(r)]) + (b_j + H_j)(y - D_j[L_j + \mathbb{M}_{T_j}(r)])^-], \quad j = 1, \dots, N,$$

and

$$f_0(y, r) = h_0 \mathbb{E} \left[y - D_0[L_0 + \mathbb{M}_{T_0}(r)] + \sum_{j=1}^N D_j[L_j + \mathbb{M}_{T_j}(r)] + \sum_{j=1}^N \frac{\lambda_j}{\lambda_0} (y - D_0[L_0 + M_{T_0}(r_j(r))))^- \right].$$

Proposition 9

$$C(\mathbf{S}, \mathbf{T}) = \sum_{j=0}^N \frac{K_j}{T_j} + \frac{1}{T} \sum_{r=0}^{T-1} \left(f_0(s_0, r) + \sum_{j=1}^N \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)] \right) + h_0 \sum_{j=1}^N (\lambda_j L_j).$$

Note that $f_0(s_0, r)$ is the warehouse inventory holding cost; $f_j(s_j - B_{0j}(L_0 + r_j(r)), r)$ is the retailer j 's inventory holding and backorder cost; the last term is the average holding cost of pipeline inventory per period, which is constant.

For convenience, let us define

$$\widehat{f}_0(s_0) = \frac{1}{T} \sum_{r=0}^{T-1} f_0(s_0, r), \quad \widehat{f}_j(s_0, s_j) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)].$$

Here, $\widehat{f}_j(\cdot, \cdot)$ is a function of s_0 because B_{0j} is a function of s_0 .

Proposition 10 For fixed \mathbf{T} and s_0 , $\widehat{f}_j(s_0, s_j)$ is convex in s_j .

With Proposition 10, we can find the best local base-stock level $s_j(s_0)$ for each retailer j . That is,

$$s_j(s_0) = \arg \min_{s_j} \widehat{f}_j(s_0, s_j).$$

Substituting $s_j(s_0)$ for s_j in $C(\mathbf{S}, \mathbf{T})$, the objective function becomes a function of s_0 , i.e., $C(s_0) = \widehat{f}_0(s_0) + \sum_{j=1}^N \widehat{f}_j(s_0, s_j(s_0))$. Unfortunately, $C(s_0)$ is not convex in s_0 , so we have to construct bounds for the optimal s_0 , denoted as s_0^* , and conduct a search over the feasible interval.

Following Axsäter's (1990) approach, we next provide bounds for s_0^* . Let $s_j^\ell = s_j(\infty)$ and $s_j^u = s_j(0)$. Define

$$s_0^u = \arg \min_{s_0} \left\{ \widehat{f}_0(s_0) + \sum_{j=1}^N \widehat{f}_j(s_0, s_j^\ell) \right\}$$

and

$$s_0^\ell = \arg \min_{s_0} \left\{ \widehat{f}_0(s_0) + \sum_{j=1}^N \widehat{f}_j(s_0, s_j^u) \right\}.$$

Then,

Proposition 11 (1) $s_j^\ell \leq s_j^* \leq s_j^u$; (2) $s_0^\ell \leq s_0^* \leq s_0^u$.

With Proposition 11, we can search over all possible s_0 between s_0^ℓ and s_0^u . After s_0^* is found, the optimal base-stock level for retailer j is $s_j^* = s_j(s_0^*)$.

Appendix B: A Recursion to Evaluate $G^S(\mathbf{S}, \mathbf{T})$

We provide a simple recursion to evaluate $G^S(\mathbf{S}, \mathbf{T})$. Let $S_R = \sum_{j=1}^N S_j$. Then,

$$G^S(\mathbf{S}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} F_0(\mathbf{S}, \mathbf{T}, r),$$

where

$$\begin{aligned} F_0(\mathbf{S}, \mathbf{T}, r) &= \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) \right. \\ &\quad \left. + F_1 \left(\min \left\{ S_R, S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)] + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] \right\}, \mathbf{T}, r \right) \right], \\ F_1(y, \mathbf{T}, r) &= \mathbb{E} \left[h \left(y - \sum_{j=1}^N D_j[L_j + \mathbb{M}_{T_j}(r)] \right) + (b + H) \left(y - \sum_{j=1}^N D_j[L_j + \mathbb{M}_{T_j}(r)] \right)^- \right]. \end{aligned}$$

We omit the proof for brevity. A complete proof is available from the authors.

Notice that the recursion is similar to the one presented in Shang and Zhou (2010) to evaluate the inventory holding and backorder cost for a serial system with nested echelon (S, T) policies. In fact, the above recursion can be used to evaluate a two-stage serial system with *nonnested* echelon (S, T) policies by setting $N = 1$ (one retailer) in the recursion.

Appendix C: Proofs

Proposition 3

To show $\underline{c}_j(T_j)$ is unimodal, we need to show $\underline{c}_j(T_j + 1) - \underline{c}_j(T_j)$ is always nonnegative after some T_j . It can be shown that $g_j(S_j(T_j + 1, T_j), T_j + 1, T_j)$ is increasing in T_j . Thus,

$$\begin{aligned} & \underline{c}_j(T_j + 1) - \underline{c}_j(T_j) \\ = & \frac{1}{T_j(T_j + 1)} \left(T_j g_j(S_j(T_j + 1, T_j), T_j + 1, T_j) - \sum_{r=0}^{T_j-1} g_j(S_j(T_j, r), T_j, r) - K_j \right) \end{aligned} \quad (20)$$

Let

$$\Delta(T_j) \stackrel{def}{=} T_j g_j(S_j(T_j + 1, T_j), T_j + 1, T_j) - \sum_{r=0}^{T_j-1} g_j(S_j(T_j, r), T_j, r).$$

Notice that

$$\begin{aligned} & \Delta(T_j + 1) \\ = & (T_j + 1) g_j(S_j(T_j + 2, T_j + 1), T_j + 2, T_j + 1) - \sum_{r=0}^{T_j} g_j(S_j(T_j + 1, r), T_j + 1, r) \\ = & (T_j + 1) g_j(S_j(T_j + 2, T_j + 1), T_j + 2, T_j + 1) - (T_j + 1) g_j(S_j(T_j + 1, T_j), T_j + 1, T_j) + \Delta(T_j) \\ = & (T_j + 1) \left(g_j(S_j(T_j + 2, T_j + 1), T_j + 2, T_j + 1) - g_j(S_j(T_j + 1, T_j), T_j + 1, T_j) \right) + \Delta(T_j) \\ \geq & \Delta(T_j), \end{aligned}$$

where the inequality follows from that $g_j(S_j(T_j + 1, T_j), T_j + 1, T_j)$ is increasing in T_j . So the bracketed terms in (20) are increasing in T_j , which implies the unimodality of $\underline{c}_j(T_j)$.

Proposition 4

Note that, for any order period t of stage j ,

$$S_j - B_{0j}(L_0 + r_j(r)) = IP_j(t + L_0 + r_j(r)),$$

$$S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r_j(r))] = IL_0^-(t + L_0 + r_j(r)),$$

and $IL_0^-(t + L_0 + r_j(r)) \geq IP_j(t + L_0 + r_j(r))$. For simplicity, we suppress $(t + L_0 + r_j(r))$ from IP_j and IL_0^- since we also consider the same time period. Thus,

$$\mathbb{E} \left[\sum_{j=1}^N g_{j0}(IP_j, T_j, r) \right]$$

$$\begin{aligned}
&= \sum_{j=1}^N \left[\sum_{n=-\infty}^{S_j} g_{j0}(n, T_j, r) \mathbb{P}(IP_j = n) \right] \\
&\geq \sum_{j=1}^N \sum_{n_0=-\infty}^{S_0} \left[\sum_{n=-\infty}^{n_0} g_{j0}(n_0, T_j, r) \left[\mathbb{P}(IP_j = n | IL_0^- = n_0) \mathbb{P}(IL_0^- = n_0) \right] \right] \\
&= \sum_{j=1}^N \sum_{n_0=-\infty}^{S_0} g_{j0}(n_0, T_j, r) \mathbb{P}(IL_0^- = n_0) \left[\sum_{n=-\infty}^{n_0} \mathbb{P}(IP_j = n | IL_0 = n_0) \right] \\
&= \mathbb{E} \left[\sum_{j=1}^N g_{j0}(IL_0^-, T_j, r) \right],
\end{aligned}$$

where the inequality follows from that g_{j0} is nonincreasing.

Proposition 5

By definition

$$\mathbb{E}[g_{j0}(y - D_0[L_0 + \mathbb{M}_{T_0}(r_0(r))], T_0, r)] = \mathbb{E}[g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r) - g_j(S_j(T_0, r), T_0, r)]$$

and

$$\begin{aligned}
&\mathbb{E} \left[g_{j0} \left(y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], T_j, r \right) \right] \\
&= \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right].
\end{aligned}$$

So to prove the result, we need to show, for $r = 0, 1, \dots, T - 1$,

$$\begin{aligned}
&\mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\
&\geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r) - g_j(S_j(T_0, r), T_0, r)]. \tag{21}
\end{aligned}$$

It is clear that we just need to verify $r = 0, 1, \dots, lcm(T_0, T_j) - 1$. Let $lcm(T_0, T_j) = nT_0 = mT_j$.

Consider the following two cases differentiated by $n \geq m$ or $n < m$.

Case 1: $n < m$ and so $T_0 > T_j$. Thus we can write

$$T_0 = \left\lfloor \frac{m}{n} \right\rfloor T_j + \left(\frac{m}{n} - \left\lfloor \frac{m}{n} \right\rfloor \right) T_j,$$

where T_j must be divisible by n as m and n are relatively prime. Note that $(\frac{m}{n} - \lfloor \frac{m}{n} \rfloor)$ could take value $1/n, \dots, (n-1)/n$. For ease of presentation, we assume it is $1/n$ or $T_0 = \lfloor \frac{m}{n} \rfloor T_j + \frac{1}{n} T_j$. Other cases can be similarly proved using the following procedure.

To prove the inequality (21) for all $r = 0, 1, \dots, \text{lcm}(T_0, T_j) - 1$, we consider different regions of r .

Subcase 1. $r = 0, \dots, T_j - 1$.

In this case, $\lfloor \frac{r}{T_j} \rfloor T_j = 0$ and from the definition of $g_j(y, T, r)$ and $S_j(r, T)$,

$$\begin{aligned} & \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r) - g_j(S_j(T_0, r), T_0, r)] \\ &= \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right], \end{aligned}$$

because $S_j(T_j, r) = S_j(T_0, r)$ and $g_j(y, T_j, r) = g_j(y, T_0, r)$ in this case.

Subcase 2. $r = T_j, \dots, T_0 - 1$.

Note that, for $r < T_0$, $S_j(T_0, r)$ is increasing in r because

$$g_j(y, T_0, r) = \mathbb{E}[h_j(y - D_j[L_j + r]) + (b_j + H_j)(D_j[L_j + r] - y)^+]$$

is submodular in y and r as $\partial g_j(y, T_0, r)/\partial y = h_j - (b_j + H_j)\mathbb{P}(D_j[L_j + r] > y)$ is decreasing in r .

Moreover, in this case,

$$\mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) = \left\lfloor \frac{r}{T_j} \right\rfloor T_j,$$

from the definition of $g_j(y, T_0, r)$ and $\mathbb{M}_{T_0}(r) = \mathbb{M}_{T_j}(r) + \lfloor \frac{r}{T_j} \rfloor T_j$,

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \left\lfloor \frac{r}{T_j} \right\rfloor T_j \right], S_j(T_j, r) \right\}, T_j, r \right) \right] \\ & \geq \mathbb{E} \left[g_j \left(\min \left\{ y - D_0[L_0] - D_j \left[\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right], S_j(T_j, r) \right\}, T_j, r \right) \right] \\ & = \mathbb{E} \left[g_j \left(\min \left\{ y - D_0[L_0], S_j(T_j, r) + D_j \left[\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right] \right\}, T_0, r \right) \right] \\ & \geq \mathbb{E}[g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r)], \end{aligned} \tag{22}$$

where the first inequality follows from that $g_j(\min\{y, S_j(T_j, r)\}, T_j, r)$ is decreasing in y and the second inequality from the convexity of $g_j(y, T_0, r)$ and the optimality of $S_j(T_0, r)$ for $g_j(y, T_0, r)$. Meanwhile, as $g_j(S_j(T_0, r), T_0, r) \geq g_j(S_j(T_j, r), T_j, r)$ for $T_j \leq r < T_0$ because $\mathbb{M}_{T_0}(r) > \mathbb{M}_{T_j}(r)$, the following inequality is valid,

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \left\lfloor \frac{r}{T_j} \right\rfloor T_j \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r) - g_j(S_j(T_0, r), T_0, r)], \end{aligned}$$

and so is inequality (21).

Subcase 3. $r = \ell T_0, \ell T_0 + 1, \dots, (\ell + 1)T_0 - 1$ for $1 \leq \ell \leq n - 1$.

- (a) First consider $r = \ell T_0 + T_j, \ell T_0 + T_j + 1, \dots, (\ell + 1)T_0 - 1$. Note that for $r = \ell T_0 + (k - 1)T_j, \dots, \ell T_0 + kT_j - 1$ for $1 < k \leq \lfloor \frac{m}{n} \rfloor T_j$,

$$\mathbb{M}_{T_0}(r) = (k - 1)T_j, (k - 1)T_j + 1, \dots, kT_j - 1, \quad (23)$$

$$\begin{aligned} & \mathbb{M}_{T_j}(r) \quad (24) \\ = & \begin{cases} \frac{\ell}{n}T_j, \frac{\ell}{n}T_j + 1, \dots, T_j - 1, & r = \ell T_0 + (k - 1)T_j, \dots, \ell T_0 + (k - 1)T_j + \frac{n - \ell}{n}T_j - 1, \\ 0, 1, \dots, \frac{\ell}{n}T_j - 1, & r = \ell T_0 + (k - 1)T_j + \frac{n - \ell}{n}T_j, \dots, \ell T_0 + kT_j - 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \quad (25) \\ = & \begin{cases} ((k - 1) - \frac{\ell}{n})T_j, \dots, ((k - 1) - \frac{\ell}{n})T_j, & r = \ell T_0 + (k - 1)T_j, \dots, \ell T_0 + (k - 1)T_j + \frac{n - \ell}{n}T_j - 1, \\ (k - \frac{\ell}{n})T_j, \dots, (k - \frac{\ell}{n})T_j, & r = \ell T_0 + (k - 1)T_j + \frac{n - \ell}{n}T_j, \dots, \ell T_0 + kT_j - 1. \end{cases} \end{aligned}$$

With these, we can show that $\mathbb{M}_{T_0}(r) = \mathbb{M}_{T_j}(r) + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right)$. Following the arguments as those in deriving (22),

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) \right] \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r)]. \end{aligned}$$

Moreover, as in this range, $\mathbb{M}_{T_0}(r) \geq \mathbb{M}_{T_j}(r)$ and so $g_j(S_j(T_0, r), T_0, r) \geq g_j(S_j(T_j, r), T_j, r)$.

Thus, (21) is true.

- (b) For $r = \ell T_0 + \lfloor \frac{m}{n} \rfloor T_j, \ell T_0 + \lfloor \frac{m}{n} \rfloor T_j + 1, \dots, (\ell + 1)T_0 - 1$,

$$\begin{aligned} \mathbb{M}_{T_0}(r) &= \left\lfloor \frac{m}{n} \right\rfloor T_j, \left\lfloor \frac{m}{n} \right\rfloor T_j + 1, \left\lfloor \frac{m}{n} \right\rfloor T_j + \frac{1}{n}T_j - 1, \\ \mathbb{M}_{T_j}(r) &= \frac{\ell}{n}T_j, \frac{\ell}{n}T_j + 1, \dots, \frac{\ell + 1}{n}T_j - 1, \\ \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) &= \left\lfloor \frac{m}{n} \right\rfloor T_j - \frac{\ell}{n}T_j. \end{aligned}$$

As $\mathbb{M}_{T_0}(r) = \mathbb{M}_{T_j}(r) + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right)$, the inequality (21) is also valid following the preceding argument.

- (c) For $r = \ell T_0, \ell T_0 + 1, \dots, \ell T_0 + T_j - 1$. Note that in this case, $\mathbb{M}_{T_0}(r)$ and $\mathbb{M}_{T_j}(r)$ are given in (23) and (24) respectively with $k = 1$. Therefore, it can be seen that when $r = \ell T_0, \ell T_0 +$

$1, \dots, \ell T_0 + \frac{n-\ell}{n}T_j - 1$, $\mathbb{M}_{T_0}(r) \leq \mathbb{M}_{T_j}(r)$ and so $S_j(T_j, r) \geq S_j(T_0, r)$ but $g_j(S_j(T_0, r), T_0, r) \leq g_j(S_j(T_j, r), T_j, r)$. However, we can show that

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_j, r)\}, T_j, r)] - g_j(S_j(T_j, r), T_j, r) \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_j, r)] - g_j(S_j(T_0, r), T_j, r) \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r)] - g_j(S_j(T_0, r), T_0, r), \end{aligned}$$

where the first inequality follows from that $g_{j0}(y, T_j, r)$ is decreasing in y ; the second inequality follows from the convexity of $g_j(y, T_j, r)$ and the optimality of $S_j(T_j, r)$ for $g_j(y, T_j, r)$; the third inequality follows from that $\mathbb{E}[h_j(y - D_j[L_j + r]) + (b_j + H_j)(D_j[L_j + r] - y)^+]$ is submodular in (y, r) and $\mathbb{M}_{T_0}(r) \leq \mathbb{M}_{T_j}(r)$.

- (d) For $r = \ell T_0 + \frac{n-\ell}{n}T_j, \dots, \ell T_0 + T_j - 1$, from (23) and (24), $\mathbb{M}_{T_0}(r) \geq \mathbb{M}_{T_j}(r)$. Moreover, as $\mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) = T_j - \ell/nT_j$, $\mathbb{M}_{T_0}(r) = \mathbb{M}_{T_j}(r) + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right)$ and so the inequality (21) follows the same argument as we prove the case $r = \ell T_0 + T_j, \dots, (\ell + 1)T_0 - 1$.

Case 2: $n \geq m$ and so $T_j \geq T_0$ and

$$T_j = \left\lfloor \frac{n}{m} \right\rfloor T_0 + \left(\frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor \right) T_0.$$

Similar to the analysis of Case 1, we discuss different ranges of r . And although $\frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor$ can take values $\frac{1}{m}, \dots, \frac{m-1}{m}$, we again only consider the case where it is $1/m$, or $T_j = \left\lfloor \frac{n}{m} \right\rfloor T_0 + \frac{1}{m}T_0$.

Subcase 1. $r = 0, 1, \dots, T_0 - 1$.

The inequality (21) still holds as equality.

Subcase 2. $r = T_0, T_0 + 1, \dots, T_j - 1$.

$\mathbb{M}_{T_0}(r) \leq \mathbb{M}_{T_j}(r)$ and so

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\ & = \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_j, r)\}, T_j, r)] - g_j(S_j(T_j, r), T_j, r) \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_j, r)] - g_j(S_j(T_0, r), T_j, r) \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r)] - g_j(S_j(T_0, r), T_0, r), \end{aligned}$$

where the first inequality follows from the convexity of $g_j(y, T_j, r)$ and the optimality of $S_j(T_j, r)$ while the second inequality follows from the submodularity of $g_j(y, T, r)$ on (y, r) and $\mathbb{M}_{T_0}(r) \leq \mathbb{M}_{T_j}(r)$.

Subcase 3. $r = \ell T_j, \ell T_j + 1, \dots, (\ell + 1)T_j - 1$, for $1 \leq \ell \leq m - 1$.

Note that for $r = \ell T_j + (k - 1)T_0, \dots, \ell T_j + kT_0 - 1$ with $1 \leq \ell \leq m - 1$ and $1 \leq k \leq \lfloor \frac{n}{m} \rfloor$,

$$\mathbb{M}_{T_0}(r) = \begin{cases} \frac{\ell}{m}T_0, \frac{\ell}{m}T_0 + 1, \dots, T_0 - 1, & r = \ell T_j + (k - 1)T_0, \dots, \ell T_j + (k - 1)T_0 + \frac{m-\ell}{m}T_0 - 1 \\ 0, 1, \dots, \frac{\ell}{m}T_0 - 1, & r = \ell T_j + (k - 1)T_0 + \frac{m-\ell}{m}T_0, \dots, \ell T_j + kT_0 - 1, \end{cases} \quad (26)$$

$$\mathbb{M}_{T_j}(r) = (k - 1)T_0, (k - 1)T_0 + 1, \dots, kT_0 - 1, \quad (27)$$

$$\mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) = \frac{\ell}{m}T_0. \quad (28)$$

(a) If $k > 1$, as $\mathbb{M}_{T_0}(r) \leq \mathbb{M}_{T_j}(r)$, we can use the same arguments as in the subcase 2 of this case to verify (21).

(b) For $k = 1$ and $r = \ell T_j, \dots, \ell T_j + \frac{m-\ell}{m}T_0 - 1$, $\mathbb{M}_{T_0}(r) \geq \mathbb{M}_{T_j}(r)$ and $\mathbb{M}_{T_0}(r) = \mathbb{M}_{T_j}(r) + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right)$. So $g_j(S_j(T_0, r), T_0, r) \geq g_j(S_j(T_j, r), T_j, r)$ and

$$\begin{aligned} & \mathbb{E} \left[g_j \left(\min \left\{ y - D_0 \left[L_0 + \mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right], S_j(T_j, r) \right\}, T_j, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\ & \geq \mathbb{E} \left[g_j \left(\min \left\{ y - D_0[L_0], S_j(T_j, r) + D_j \left[\mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right) \right] \right\}, T_0, r \right) - g_j(S_j(T_j, r), T_j, r) \right] \\ & \geq \mathbb{E} [g_j(\min\{y - D_0[L_0], S_j(T_0, r)\}, T_0, r) - g_j(S_j(T_0, r), T_0, r)], \end{aligned}$$

where the last inequality follows also from the optimality of $S_j(T_0, r)$.

(c) If $k = 1$ and $r = \ell T_j + \frac{m-\ell}{m}T_0, \dots, \ell T_j + T_0 - 1$, then $\mathbb{M}_{T_0}(r) < \mathbb{M}_{T_j}(r)$ and the inequality (21) can be proved similarly as subcase 2.

(d) Finally, for $r = \ell T_j + \lfloor \frac{n}{m} \rfloor T_0, \ell T_j + \lfloor \frac{n}{m} \rfloor T_0 + 1, \dots, \ell T_j + \lfloor \frac{n}{m} \rfloor T_0 + \frac{1}{m}T_0 - 1$, $\mathbb{M}_{T_0} \left(\left\lfloor \frac{r}{T_j} \right\rfloor T_j \right)$ is the same as (28) while

$$\begin{aligned} \mathbb{M}_{T_0}(r) &= \frac{\ell}{m}T_0, \frac{\ell}{m}T_0 + 1, \dots, \frac{\ell + 1}{m}T_0 - 1 \\ \mathbb{M}_{T_j}(r) &= \lfloor \frac{n}{m} \rfloor T_0, \lfloor \frac{n}{m} \rfloor T_0 + 1, \dots, \lfloor \frac{n}{m} \rfloor T_0 + \frac{1}{m}T_0 - 1, \end{aligned}$$

which shows that $\mathbb{M}_{T_0}(r) < \mathbb{M}_{T_j}(r)$ and so again the inequality (21) can be proved similarly as subcase 2.

Therefore, the proof is complete.

Proposition 6

It is clear that, for each T_j and r , $g_{j0}(y, T_j, r)$ is convex in y from its definition. Thus, from its definition, the convexity of $g_0(S_0, T_0)$ in S_0 follows.

Proposition 7

(1). It follows an analogous proof as that of Proposition 3.

(2). For simplicity, let $D'_j(r) = D_j[L_j + \mathbb{M}_{T_j}(r)]$ and $D'_j[r] = D_j[L_j + \mathbb{M}_{T_j}(r)]$ for $j = 0, \dots, N$.

From Appendix B,

$$\begin{aligned}
G^S(\mathbf{S}, \mathbf{T}) &= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D'_0[r]) + F_1 \left(\min \left\{ S_R, S_0 - D'_0(r) + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] \right\}, r \right) \right] \\
&= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D'_0[r]) + h \left(\min \left\{ S_R, S_0 - D'_0(r) + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] \right\} - \sum_{j=1}^N D'_j[r] \right) \right. \\
&\quad \left. + (b + H) \left(\min \left\{ S_R, S_0 - D'_0(r) + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] \right\} - \sum_{j=1}^N D'_j[r] \right)^- \right] \\
&\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} [h_0(S_0 - D'_0[r]) \\
&\quad + (b + h_0) \left(\min \left\{ S_R, S_0 - D'_0(r) + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] \right\} - \sum_{j=1}^N D'_j[r] \right)^-] \\
&\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D'_0[r]) + (b + h_0) \left(S_0 - D'_0(r) + \sum_{j=1}^N D_j[\mathbb{M}_{T_j}(r)] - \sum_{j=1}^N D'_j[r] \right)^- \right] \\
&= \frac{1}{T_0} \sum_{r=0}^{T_0-1} \mathbb{E} \left[h_0(S_0 - D'_0(r) - \sum_{j=1}^N D_j[L_j]) \right. \\
&\quad \left. + (b + h_0) \left(S_0 - D'_0(r) - \sum_{j=1}^N D_j[L_j] \right)^- \right] + h_0 \sum_{j=1}^N \lambda_j (L_j + 1) - h_0 \lambda_0 \\
&= \mathbb{G}(S_0, T_0) + h_0 \sum_{j=1}^N \lambda_j L_j,
\end{aligned}$$

where the first inequality follows by dropping a positive term $h(\cdot)^+$ and the second inequality from that $(\cdot)^-$ is decreasing. So the first inequality in part (2) follows. The second inequality in part (2) follows from the definition of $S_0^\ell(T_0, r)$.

Proposition 8

Let the optimal reorder intervals be (T_0, T_1, \dots, T_N) . Given T_i satisfies integer ratio relations, we want to show $T_0 \geq \min_{j \geq 1} \{T_j\}$. Consider two retailers with $T_1 \leq T_2$. Due to integer ratio constraint, $\mathbb{M}_{T_1}(T_2) = 0$. Suppose $T_0 < T_1$ and so $\mathbb{M}_{T_0}(T_1) = 0$ and denote $n_1 = T_1/T_0$. And it can

be seen that $T = T_2$ by its definition. We compare the costs of two systems: one with (T_0, T_1, T_2) and the other with (T_1, T_1, T_2) . (S_0, S_1, S_2) are given echelon base-stock levels.

For $r = 0, 1, \dots, T_2 - 1$, we first study how $B_0(t + L_0 + r_j(r))$ differs in these two systems for all $j \geq 1$. Recall that

$$B_0(t + L_0 + r_j(r)) = \left[s_0 - D_0[t + r_0(r_j(r)), t + L_0 + r_j(r)] \right]^-.$$

For $j = 2$, it is clear that with given s_0 , $B_0(t + L_0 + r_j(r))$ is the same in both systems as $r_j(r) = 0$. So we just consider $j = 1$. When the warehouse reorder interval is T_1 , $r_0(r_1(r)) = \left\lfloor \frac{r}{T_1} \right\rfloor T_1$; when its reorder interval is T_0 ,

$$r_0(r_1(r)) = \left\lfloor \frac{\left\lfloor \frac{r}{T_1} \right\rfloor T_1}{T_0} \right\rfloor T_0 = n_1 \left\lfloor \frac{r}{T_1} \right\rfloor T_0 = \left\lfloor \frac{r}{T_1} \right\rfloor T_1$$

since $\mathbb{M}_{T_0}(T_1) = 0$ and $n_1 = T_1/T_0$. So with given s_0 , these two systems share the same $B_0(t + L_0 + r_j(r))$ for each r and so the same $B_{0j}(t + L_0 + r_j(r))$.

The system inventory related cost is evaluated as

$$\frac{1}{T} \sum_{r=0}^{T-1} G_0(\mathbf{S}, \mathbf{T}, r) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N G_j(S_j - B_{0j}(L_0 + r_j(r)), T_j, r) \right],$$

where

$$G_j(y, T_j, r) = \mathbb{E}[h_j(y - D_j[L_j + \mathbb{M}_{T_j}(r)]) + (b_j + H_j)(y - D_j[L_j + \mathbb{M}_{T_j}(r)])^-].$$

It can be seen that if one increases T_0 to T_1 , both T and $\mathbb{E}[\sum_{j=1}^N G_j(S_j - B_{0j}(L_0 + r_j(r)), T_j, r)]$ remain unchanged. However, the average fixed cost at the warehouse decreases and the holding cost term of the warehouse also decreases as $D_0[L_0 + \mathbb{M}_{T_0}(r)]$ increases. Hence, the system with (T_0, T_1, T_2) incurs a higher cost than the system with (T_1, T_1, T_2) that uses the same base-stock levels. Therefore, the result holds for a two-retailer system. It can be seen that the proof can be readily extended to a system with general multiple retailers.

Proposition 9

Recall that

$$G_j(y, T_j, r) = \mathbb{E}[h_j(y - D_j[L_j + \mathbb{M}_{T_j}(r)]) + (b_j + H_j)(y - D_j[L_j + \mathbb{M}_{T_j}(r)])^-].$$

and

$$G_0(\mathbf{S}, \mathbf{T}, r) = \mathbb{E} \left[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)]) + \sum_{j=1}^N G_j(S_j - B_{0j}(L_0 + r_j(r)), r) \right]$$

Because $s_j = S_j$ and $s_0 = S_0 - \sum_{j=1}^N S_j$,

$$\begin{aligned} & G_0(\mathbf{S}, \mathbf{T}, r) \\ &= \mathbb{E}[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)])] + \sum_{j=1}^N \mathbb{E}[h_j(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathbb{M}_{T_j}(r)]) \\ &\quad + (b_j + H_j)(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathbb{M}_{T_j}(r)])^-] \\ &= \mathbb{E}[h_0(S_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)])] - \sum_{j=1}^N \mathbb{E}[h_0(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathbb{M}_{T_j}(r)])] \\ &\quad + \sum_{j=1}^N \mathbb{E} \left[H_j(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathbb{M}_{T_j}(r)]) \right. \\ &\quad \left. + (b_j + H_j)(s_j - B_{0j}(L_0 + r_j(r)) - D_j[L_j + \mathbb{M}_{T_j}(r)])^- \right] \\ &= \mathbb{E}[h_0(s_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)])] + \sum_{j=1}^N \mathbb{E}[h_0(B_{0j}(L_0 + r_j(r)))] - \sum_{j=1}^N \mathbb{E}[h_0(-D_j[L_j + \mathbb{M}_{T_j}(r)])] \\ &\quad + \sum_{j=1}^N \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)] \\ &= h_0 \mathbb{E} \left[(s_0 - D_0[L_0 + \mathbb{M}_{T_0}(r)] + \sum_{j=1}^N (D_j[\mathbb{M}_{T_j}(r)]) + \sum_{j=1}^N \frac{\lambda_j}{\lambda_0} (s_0 - D_0[L_0 + r_j(r) - r_0(r_j(r))])^- \right] \\ &\quad + \sum_{j=1}^N h_0(\lambda_j(L_j)) + \sum_{j=1}^N \mathbb{E}[f_j(s_j - B_{0j}(L_0 + r_j(r)), r)]. \end{aligned}$$

As $h_0 = H_0$, so from Proposition 1, the result follows.

Proposition 10

The convexity on s_j follows directly from the definition of $\widehat{f}_j(s_0, s_j)$. We omit the detailed proof for brevity.

Proposition 11

We need to first show that $\widehat{f}_j(s_0, s_j)$ is supermodular in s_0 and s_j , or

$$\widehat{f}_j(s_0, s_j + 1) - \widehat{f}_j(s_0, s_j) \leq \widehat{f}_j(s_0 + 1, s_j + 1) - \widehat{f}_j(s_0 + 1, s_j). \quad (29)$$

For notational simplicity, we denote $D_j[L_j + \mathbb{M}_{T_j}(r)]$ by D_j . In addition, to emphasize the dependency of B_{0j} on s_0 and for brevity, let $B_{0j}(s_0)$ denote $B_{0j}(L_0 + r_j(r))$. Note that

$$\begin{aligned}
& \widehat{f}_j(s_0, s_j + 1) - \widehat{f}_j(s_0, s_j) \\
= & H_j + \frac{1}{T} \sum_{r=0}^{T-1} (b_j + H_j) \mathbb{E}[(s_j + 1 - B_{0j}(s_0) - D_j)^- - (s_j - B_{0j}(s_0) - D_j)^-] \\
\leq & H_j + \frac{1}{T} \sum_{r=0}^{T-1} (b_j + H_j) \mathbb{E}[(s_j + 1 - B_{0j}(s_0 + 1) - D_j)^- - (s_j - B_{0j}(s_0 + 1) - D_j)^-] \\
= & \widehat{f}_j(s_0 + 1, s_j + 1) - \widehat{f}_j(s_0 + 1, s_j)
\end{aligned}$$

because $()^-$ is convex and $B_{0j}(s_0 + 1)$ is smaller than $B_{0j}(s_0)$. So the inequality (29) is valid. With this result and the convexity of $\widehat{f}_j(s_0, s_j)$ in s_j , part (1) follows.

To show part (2), from (29), we have, for any $s_j \geq s_j^\ell$,

$$\begin{aligned}
0 & \geq \widehat{f}_0(s_0^u) + \sum_{j=1}^N \widehat{f}_j(s_j^\ell, s_0^u) - [\widehat{f}_0(s_0^u + 1) + \sum_{j=1}^N \widehat{f}_j(s_j^\ell, s_0^u + 1)] \\
& \geq \widehat{f}_0(s_0^u) + \sum_{j=1}^N \widehat{f}_j(s_j, s_0^u) - [\widehat{f}_0(s_0^u + 1) + \sum_{j=1}^N \widehat{f}_j(s_j, s_0^u + 1)]
\end{aligned}$$

where the first inequality follows from the definition of s_0^u . As the optimal s_j^* will be greater than s_j^ℓ , s_0^u is an upper bound for s_0^* . Similarly, we can verify the lower bound s_0^ℓ .