Simple Heuristics for Optimal Inventory Policies in Supply Chains

Kevin Shang

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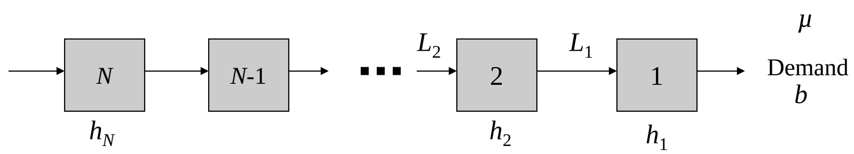
Motivations for Simple Heuristics

- The form of the optimal policy for most supply chains is unknown
- Hard to solve - exact optimization algorithms involve recursive calculations between stages across time
- Little intuition from the algorithm
- Hard to communicate
- Hard to coordinate the supply chain
- Difficult to solve joint problems (inventory/pricing, inventory/cash flows, etc.)

The Basic Model



- i.i.d demand between periods with mean μ
- The objective is to minimize the average total cost per period
- Centralized vs. local information
- Continuous review vs. periodic review

Systems and Inventory Policies

- Continuous-review system
  - Poisson demand
  - Echelon (r,Q) policies
  - Assuming a fixed cost k_j for Q_j
  - Satisfying integer-ratio constraints: $Q_j = n_j Q_{j\text{-}1}$
- Periodic-review system
  - i.i.d demand between periods
  - Echelon (s,T) policies
  - Assuming a fixed cost K_j for T_j
  - Satisfying integer-ratio constraints: $T_j = n_j T_{j\text{-}1}$

Single-stage Systems

- (r,Q) policies

$$C(r,Q) = \frac{k\lambda + \sum_{x=1}^{Q} G(r+x)}{Q}$$

where $G(y) = E[h(y - D(L))^{+} + b(y - D(L))^{-}]$

- (s,T) policies

$$C(s,T) = \frac{K}{T} + G(s,T)$$

where

$$G(y,T) = \frac{1}{T}\left(\sum_{\tau=0}^{T-1} \mathrm{E}[h(y - D[L+\tau])^{+} + b(y - D[L+\tau])^{-}]\right)$$

CONTINUOUS-REVIEW SYSTEM

Echelon (r,Q) Policies

- Cost evaluation for any given (r,Q) policies:

$$G_1(y) = \mathsf{E}\big[h_1(y - D[L_1]) + (b + h_{[1,N]})(y - D[L_1])^-\big]$$

For $j = 2, \ldots, N$,

$$G_j(y) = \mathsf{E}\big[h_j(y - D[L_j]) + G_{j-1}(O_{j-1}[y - D[L_j]])\big]$$

$$\text{where } O_j[x] = \begin{cases} x, & \text{if } x \le r_j, \\ x - mQ_j, & \text{otherwise} \end{cases}$$

$$C(\mathbf{r}, \mathbf{Q}) = \textstyle\sum_{i=1}^{N} (k_j \lambda)/Q_j + \sum_{x=1}^{Q_N} G_N(r_N + x)]/Q_N$$

- Optimal reorder point r_j* can be found recursively from stage 1, 2,…, until stage N

$$r_1^* = \arg\min_y \bar{G}_1(y) \triangleq \textstyle\sum_{x=1}^{Q_1} G_1(y + x)$$

$$r_j^* = \arg\min_y \bar{G}_j(y) \triangleq \textstyle\sum_{x=1}^{Q_j} G_j(y + x)$$

Heuristic for optimal reorder points

- For fixed **Q**, r_j* is bounded by the optimal re f two single-stage systems

$$G_j^\ell(y) = \mathsf{E}\big[h_j(y - \tilde{D}_j)^+ + (b + h_{[j+1,N]})(y - \tilde{D}_j)^-\big]$$

$$G_j^u(y) = \mathsf{E}\big[h_{[1,j]}(y - \tilde{D}_j)^+ + (b + h_{[j+1,N]})(y - \tilde{D}_j)^-\big]$$

$$r_j^u = \arg\min_y \left(\textstyle\sum_{x=1}^{Q_j} G_j^\ell(y + x)\right)$$

$$r_j^\ell = \arg\min_y \left(\textstyle\sum_{x=1}^{Q_j} G_j^u(y + x)\right)$$

LB system ➔ UB solution

Holding

Ba

UB system ➔ LB solution

Holding cost: $h_{[1,j]}$
Backorder cost: $b+h_{[j+1,N]}$
Lead time: $L_{[1,j]}$
Batch size: Q_j

Theorem 1 (Shang and Song [34]). *For $j = 1, \ldots, N$, (1) $G_j^u(y) + \tau_j \ge G_j(y) \ge G_j^\ell(y) + \tau_j$ for all y, and (2) $r_j^\ell \le r_j^* \le r_j^u$. When $j = 1$, the above inequalities reduce to equalities.*

Heuristics for Optimal Batch Sizes

- We propose two heuristics for finding the optimal batch sizes
- These two heuristics are inspired by solving the following problem:

$$\min \quad \sum_{j=1}^{N} C_j^{\ell}(Q_j)$$

$$s.t. \quad Q_j \geq Q_{j-1}, \quad j = 2, \ldots, N$$

$$\text{where} \quad C_j^{\ell}(Q_j) = \frac{k_j \lambda}{Q_j} + \frac{1}{Q_j} \sum_{x=1}^{Q_j} G_j^{\ell}(r_j(Q_j) + x)$$

$$\text{and} \quad r_j(Q_j) = \arg\min_y \sum_{x=1}^{Q_j} G_j^{\ell}(y + x) \,,$$

$$G_j^{\ell}(y) = \mathsf{E}\left[h_j(y - \tilde{D}_j)^+ + (b + h_{[j+1,N]})(y - \tilde{D}_j)^-\right]$$

- A clustering algorithm generates a solution

Simplifying the Computation

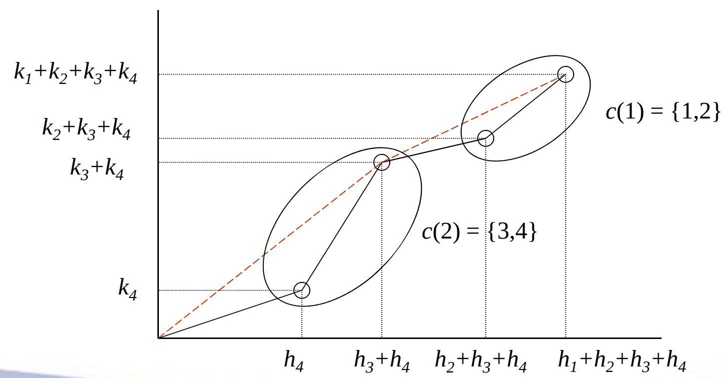
The solution obtained from the cluster algorithm is close to the solution obtained from the following two-step procedure (Shang 2008)

- ❑ Clustering
- ❑ Minimization

Clustering

Let $\{c(1), c(2),\ldots, c(M)\}$ be the clusters

For the cluster $c(m)$, assume there are $n(m)$ stages, $m=1,\ldots,M$



Minimization

Let $Q_{c(m)}$ is the batch size for the stages in $c(m)$, and define

$$h[m] = \sum_{i\in c(m)} h_i, \quad h'[m] = \sum_{i\in c(m)} h_{[i,N]}, \quad \text{and} \quad k[m] = \sum_{i\in c(m)} k_i.$$

Heuristic 1. $Q_{c(m)}$ is the solution of the following problem:

$$\min_Q \left\{ \frac{\lambda \;\; k[m] + \sum_{x=1}^{Q} G^{\ell}_{c(m)}(R^{\ell}(Q)+x)}{Q} \right\},$$

$$\text{s.t.} \quad Q = qQ_{c(m-1)}, \quad q \in \Im^{+}, \; m>1,$$

where

$$R^{\ell}(Q) = \arg\min_y \left\{ \sum_{x=1}^{Q} G^{\ell}_{c(m)}(y+x) \right\}, \quad G^{\ell}_{c(m)}(y) = \sum_{i\in c(m)} G^{\ell}_i(y),$$

and

$$G^{\ell}_i(y) = \mathsf{E}\big[h_i(y-\tilde{D}_i) + (b + h_{[i,N]})(y-\tilde{D}_i])^{-}\big].$$

Minimization

Heuristic 2. Suppose that $c(m)$ contains stage $i \in \{v, v+1, \ldots, v+n(m)-1\}$. $Q_{c(m)}$ is the solution of the following problem:

$$\min_Q \left\{ \frac{\lambda \quad k[m] + \sum_{x=1}^{Q} G_{c(m)}(R(Q)+x)}{Q} \right\},$$
$$\text{s.t. } Q = qQ_{c(m-1)}, \quad q \in \Im^+, \, m > 1,$$

where

$$R(Q) = \arg\min_y \left\{ \sum_{x=1}^{Q} G_{c(m)}(y+x) \right\},$$

and

$$G_{c(m)}(y) = \mathsf{E}\big[h[m](y - \tilde{D}_{v+n(m)-1}) + (n(m)b + h'[m])(y - \tilde{D}_{v+n(m)-1})^-\big].$$

Effectiveness of the Heuristics

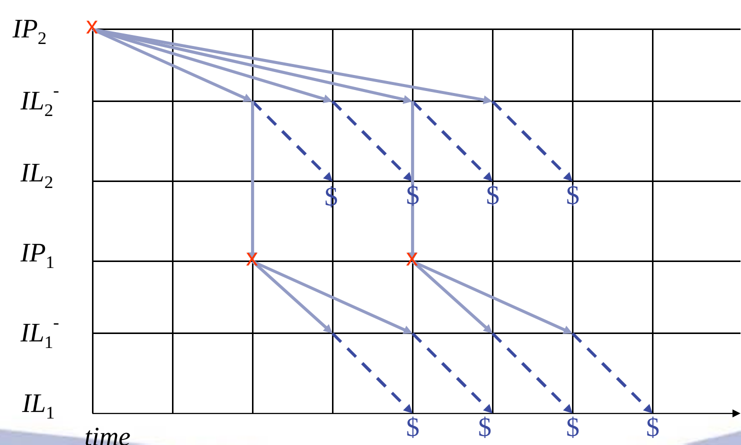
Table 1. Performance summary for Variant 1, Variant 2, and the EOQ heuristic in Group 1.

	Variant 1		Variant 2		The EOQ heuristic	
	$b=10$	$b=50$	$b=10$	$b=50$	$b=10$	$b=50$
0%	193	309	137	307	31	5
(0.0%, 0.5%]	262	157	316	153	378	350
(0.5%, 1.0%]	25	26	24	30	57	128
(1.0%, 1.5%]	12	8	11	10	6	15
(1.5%, 2.0%]	4	4	6	4	6	6
(2.0%, 2.5%]	8	0	10	0	20	0
(2.5%, 3.0%]	0	1	0	1	6	4
3.0% above	8	7	8	7	8	4
Average (%)	0.21	0.13	0.24	0.14	0.41	0.40

PERIODIC-REVIEW SYSTEM

Echelon (s,T) Policies: Inventory Dynamics

- Two-stage system, T_2=4, T_1=2, L_2=2, L_1=1



Optimizing Base-stock Levels with Fixed T: Chao and Zhou (2009)

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- Focus on a regenerative cycle. Characterize IP_j, IL_j for stage $j = N, N\text{-}1, \ldots, 1$
- A bottom-up recursion: Define $\mathbf{T_j} = (T_1, T_2, \ldots, T_j)$

$$G_1(y,\mathbf{T_1}) = \frac{1}{T_1}\left(\sum_{\tau=0}^{T_1-1} \mathrm{E}[h_1(y - D[L_1+\tau]) + (b + h_{[1,N]})(y - D[L_1+\tau])^-]\right)$$

$$s_1(\mathbf{T_1}) = \mathrm{argmin}_y\ G_1(y,\mathbf{T_1})$$

For $j = 2, \ldots, N$,

$$G_j(y,\mathbf{T_j}) = \frac{1}{T_j}\sum_{\tau=0}^{T_j-1} \mathrm{E}\left[\overbrace{h_j\left(y - D\left[L_j+\tau\right]\right)}^{\text{inventory holding cost}} + G_{j-1}\left(\overbrace{\min\left\{\left[y - D\left[L_j + \left\lfloor\frac{\tau}{T_{j-1}}\right\rfloor T_{j-1}\right)\right], s_{j-1}\left(\mathbf{T_{j\text{-}1}}\right)\right\}, \mathbf{T_{j-1}}}^{\text{cost for echelon } j\text{-}1}\right)\right]$$

$$s_j(\mathbf{T_j}) = \mathrm{argmin}_y\ G_j(y,\mathbf{T_j})$$

$$C(\mathbf{T}) = \sum_{j=1}^{N}\left(\frac{K_j}{T_j}\right) + G_N(s_N(\mathbf{T_N}),\mathbf{T_N}) \quad \text{(Average total cost per period)}$$

For simplicity, let $s_j^* = s_j(\mathbf{T_j})$

Heuristic for Optimal Base-stock Levels

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- s_j* is bounded by the solutions obtained from two single-stage systems with (s,T) policies

$$G_j^{\ell}(y) = \frac{1}{T_j}\sum_{\tau=0}^{T_j-1} \mathsf{E}[h_j(y - D[L_{[1,j]}+\tau]) + (b + h_{[j,N]})(y - D[L_{[1,j]}+\tau])^-],$$

$$G_j^{u}(y) = \frac{1}{T_j}\sum_{\tau=0}^{T_j-1} \mathsf{E}[h_{[1,j]}(y - D[L_{[1,j]}+\tau]) + (b + h_{[1,N]})(y - D[L_{[1,j]}+\tau])^-].$$

LB system ➔ UB solution

UB system ➔ LB solution

Holding cost: $h_{[1,j]}$
Backorder cost: $b+h_{[j+1,N]}$
Lead time: $L_{[1,j]}$
Reorder interval: T_j

Let

$$s_j^u = \arg\min_y G_j^{\ell}(y), \qquad s_j^{\ell} = \arg\min_y G_j^{u}(y).$$

Proposition 3 (Shang and Zhou [35]). *For* $j = 1, \ldots, N$,

(1) $G_j^{\ell}(y) + \pi_j \le G_j(y) \le G_j^u(y) + \pi_j$;
(2) $s_j^{\ell} \le s_j^* \le s_j^u$.

Heuristic for Optimal Reorder Intervals

$$\text{Min}_{\mathbf{T}} \ \sum_{j=1}^{N}\left(\frac{K_j}{T_j}\right)+G_N(\mathbf{T_N})$$

subject to

$$T_{j+1}=n_jT_j,$$
$$T_j, n_j \in \text{ positive integer},$$
$$j=1,\ldots,N-1$$

Decomposition

$$\text{Min}_{\mathbf{T}} \ \sum_{j=1}^{N}\left(\frac{K_j}{T_j}+g_j(\mathbf{T_j})\right)$$

subject to

$$T_{j+1}=n_jT_j,$$
$$T_j, n_j \in \text{ positive integer},$$
$$j=1,\ldots,N-1$$

Lower bound

Upper bound

$$\text{Min}_{\mathbf{T}} \ \sum_{j=1}^{N}\left(\frac{K_j}{T_j}+\underline{g}_j(T_j)\right)$$

subject to

$$T_{j+1}\geq T_j,$$
$$j=1,\ldots,N-1$$

$$\text{Min}_{\mathbf{T}} \ \sum_{j=1}^{N}\left(\frac{K_j}{T_j}+\overline{g}_j(T_j)\right)$$

subject to

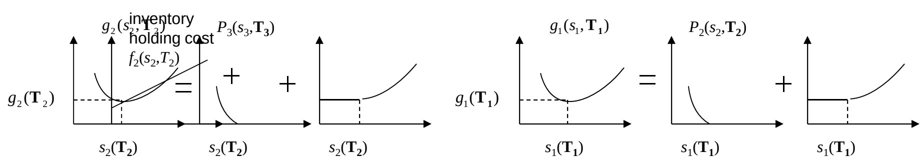
$$T_{j+1}\geq T_j,$$
$$j=1,\ldots,N-1$$

Decomposition of the Total Cost Function

Use the "induced-penalty cost function" P_j to decouple the system cost

stage 2

stage 1

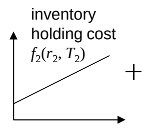


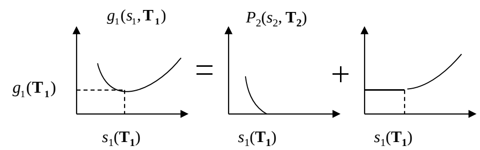
$$G_j(\mathbf{T_j})=\sum_{i=1}^{j} g_i(\mathbf{T_i})\ ,\quad j=1,\ldots,N$$

Bounds on the Stage Cost

stage 2

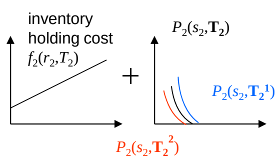
stage 1





Bounds on the Stage Cost

stage 2

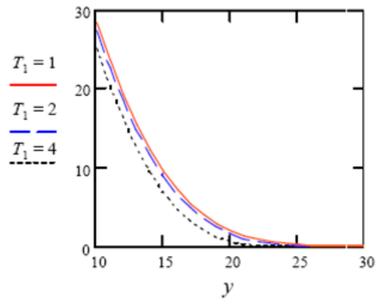


Define
$\mathbf{T}_j^j = (T_j, T_j, \ldots, T_j)$ and $\mathbf{T}_j^1 = (1,1,\ldots,1, T_j)$

When stage j's downstream stages use the same reorder interval as T_j (one), the resulting P_j is a lower (an upper) bound to the original P_j, i.e.,

$$P_j(y,\mathbf{T}_j^j) <= P_j(y,\mathbf{T}_j) <= P_j(y,\mathbf{T}_j^1)$$

Bounds on the Induced-Penalty Cost Function: An Example



A two-stage example with $h_1 = h_2 = 0.5$, $L_1 = L_2 = 1$, $b = 9$.

Heuristic I

(UP)

Let $\bar{g}_j(T_j) = g_j(\mathbf{T}_j^1)$

$$\text{Min}_{\mathbf{T}} \sum_{j=1}^{N} \left(\frac{K_j}{T_j} + \bar{g}_j(T_j) \right)$$

subject to

$$T_{j+1} \ge T_j, \quad j = 1, \dots, N-1$$

(LP)

Let $\underline{g}_j(T_j) = g_j(\mathbf{T}_j^j)$

$$\text{Min}_{\mathbf{T}} \sum_{j=1}^{N} \left(\frac{K_j}{T_j} + \underline{g}_j(T_j) \right)$$

subject to

$$T_{j+1} \ge T_j, \quad j = 1, \dots, N-1$$

- Solve both (UP) and (LP) problems
- Clustering algorithm
- The heuristic solution is whichever yields the smaller cost
- The above decomposition leads to an algorithm to find the optimal reorder intervals; see Shang and Zhou (2010)

Heuristic II

- Approximate the stage cost function $g_j(\mathbf{T_j})$ by the lower-bound cost function

$$g_j^h(s_j, T_j) = \frac{1}{T_j} \sum_{\tau=0}^{T_j-1} \mathsf{E}\left[h_j\left(s_j - D[L_{[1,j]} + \tau]\right) + \left(b + h_{[j,N]}\right)\left(y - D[L_{[1,j]} + \tau]\right)^{-}\right]$$

Let $g_j^h(\bar{T}_j) = g_j^h(s_j(T_j), T_j)$, and $C_j^h(T_j) = K_j/T_j + g_j^h(T_j)$

- Solve

$$\min_{\mathbf{T}} \sum_{j=1}^{N} C_j^h(T_j)$$
$$\text{s.t. } T_{j+1} = n_j T_j, \quad j = 1, \ldots, N-1.$$

- The two-step optimization procedure introduced in the (r,Q) policy would likely work well

NON-STATIONARY DEMAND

Non-stationary Demand Model

- Non-stationary demand is prevalent
  - Product life cycles
  - Seasonality
  - Business cycles
  - Trends
- Non-stationary demand complicates inventory decisions
  - Computation
  - Implementation
- Propose a simple solution to resolve the above issues

The C-S model with non-stationary demand

- Clark and Scarf (1960) show that (time-varying) echelon base-stock policies are optimal
- Let $s_j(t)$ be optimal echelon base-stock level, $j = 1,\ldots, N$
  - Stage j reviews x_j and orders up to $s_j(t)$ if $x_j < s_j(t)$; does not order otherwise
- $s_j(t)$ can be obtained by solving j sets of dynamic programs recursively:
  - Find $s_1(t)$ by solving a DP formulated from a single-stage system
  - With known $s_1(t)$, generate an induced-penalty cost function
  - Charge the penalty cost to stage 2, and formulate a DP to find $s_2(t)$
  - Repeat the procedure recursively
- In short, the optimal value function of an upstream stage depends on all of its downstream stages' optimal base-stock levels

Main results

- Construct an upper bound and a lower bound for $s_j(t)$ in each period t
  - By solving single-stage, finite-horizon problems with original parameters
- Propose a simple heuristic by solving N single-stage problems
  - Reduce the computation time and complexity
- Provide an analytical approximation for optimal base-stock levels and safety stocks
  - Reveal analytical insights on managing inventory
- Propose a simple, time-consistent contract that (nearly) coordinates the supply chain in the finite horizon
  - Facilitate the implementation of the centralized solution

Idea of constructing solution bounds

For each echelon j, $j=1,\ldots, N$

- Construct an upper-bound system by restricting stage i to always order up to the echelon inventory level $x_{i+1,}$ for $i < j$
  - Is the resulting solution from the upper-bound system a lower bound to $s_j(t)$ in each period?
- Construct a lower-bound system by setting h_i=0, c_i=0, for $i < j$
  - Is the resulting solution from the lower-bound system an upper bound to $s_j(t)$ in each period?
- Are the lower and upper bound systems equivalent to a single-stage system? If so, what are the resulting parameters?

Upper-bound system

The lower-bound solution $s_j^\ell(t)$ can be found by solving the upper-bound system, denoted by $S_j^u\left[c_j^u, h_j^u, b_j, L_j\right]$, where

$$c_j^u = c_j + \sum_{k=2}^{j}\left(c_{k-1}\left(\alpha^{\tau_{[k,j]}}\right) + h_{[k,j]}\left(\sum_{i=1}^{\tau_{k-1}} \alpha^{\tau_{[k,j]}+i-1}\right)\right)$$

$$h_j^u = h_{[1,j]}$$

$$b_j = b + h_{[j+1,N]}$$

$$L_j = \tau_{[1,j]}$$

Lower-bound system

The upper-bound solution $s_j^u(t)$ can be found by solving the lower-bound system, denoted by $S_j^\ell\left[c_j^\ell, h_j^\ell, b_j, L_j\right]$, where

$$c_j^\ell = c_j + h_j\left(\sum_{i=\tau_j}^{\tau_{[1,j]}-1} \alpha^i\right)$$

$$h_j^u = h_j$$

$$b_j = b + h_{[j+1,N]}$$

$$L_j = \tau_{[1,j]}$$

Heuristic

- Solving a single-stage system with a weighted average of cost parameters

The heuristic solution $s_j^a(t)$ can be found by solving a single-stage system, denoted by $S_j^a\left[c_j^a, h_j^a, b_j, L_j\right]$, where

$$c_j^a = wc_j^u + (1-w)c_j^\ell$$
$$h_j^a = wh_j^u + (1-w)h_j^\ell$$
$$b_j = b + h_{[j+1,N]}$$
$$L_j = \tau_{[1,j]}$$

Performance of the heuristic under different weights

- Two-stage system, 960 stage 2 solutions

$$\epsilon = \frac{|s_2(t) - s_2^a(t)| \times 100\%}{s_2(t)}$$

w		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Max
$b=50$	#	88	150	217	296	360	402	411	393	346	303	250	480
	$\bar{\epsilon}(\%)$	4.14	3.27	2.53	1.83	1.21	0.81	0.73	0.86	1.35	1.74	1.97	—
$b=15$	#	32	63	96	150	233	302	357	396	397	351	280	480
	$\bar{\epsilon}(\%)$	8.05	6.78	5.64	4.63	3.59	2.52	1.68	1.13	1.04	1.54	2.53	—
Total	#	120	213	313	446	593	704	768	789	743	654	530	960
	$\bar{\epsilon}(\%)$	6.10	5.03	4.09	3.23	2.40	1.67	1.21	1.00	1.20	1.64	2.25	—

Table 2: Number of the heuristic solutions equal to the optimal solutions under different backorder cost rates and weights.

Choosing an effective weight

$b/(b+h[1,N])$	$(0, 0.85]$	$(0.85, 0.925]$	$(0.925, 0.95]$	$(0.95, 0.975]$	$(0.975, 0.99]$	$(0.99, 1)$
w	0.9	0.8	0.7	0.6	0.5	0.4

Table 3: Selecting an effective weight based on the cost ratio.

Four-stage systems (# of s_2, s_3, s_4 solutions = 1632, 1472, 1344)

backorder cost rate	stage 2	stage 3	stage 4
$b = 15$	1.00%	1.60%	1.71%
$b = 50$	0.68%	1.14%	1.31%

Myopic solution

Let the myopic solution for $S_j^a[c_j^a, h_j^a, b_j, L_j]$ be $s_j^m(t)$

Under normal approximation, we can have a closed-form expression for $s_j^m(t)$:

$$s_j^m(t) = \lambda[t, t-L_j] + \sigma[t, t-L_j]\Phi^{-1}(\beta_j), \text{ where } \beta_j = \frac{\alpha^{L_j} b_j - c_j^a(1-\alpha)}{\alpha^{L_j}(b_j + h_j^a)}$$

Myopic solution

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The local base-stock level at stage j can be approximated as

$$s_j^{m'}(t) = s_j^m(t) - s_{j-1}^m(t)$$

$$= \underbrace{\lambda[t-L_{j-1}-1, t-L_j]}_{\text{pipeline inventory}} + \underbrace{\sigma[t, t-L_j]\Phi^{-1}(\beta_j) - \sigma[t, t-L_{j-1}]\Phi^{-1}(\beta_{j-1})}_{\text{safety stock}}$$

Near-optimal coordination scheme

- The optimal solution is time-varying, so in general, it is very difficult to design a contract to coordinate a decentralized control system
  - E.g., Donohue (2000), Parker and Kapuscinski (2008)
- Such a contract, if it existed, would have non-stationary parameters
  - Difficult to implement
- The “separation” feature of our heuristic can lead to a simple time-consistent coordination scheme that achieves the heuristic solution

Summary

We propose single-stage based heuristics that

- Simplify the computation
- Shorten the computation time by allowing parallel processing
- Provide analytical insights into how to manage safety stocks
- Lead to a simple, time-consistent contract that enables a decentralized supply chain to achieve the heuristic solution

Extensions & Future Research Directions

- Assembly systems
- Distribution systems
- Non-stationary demand with fixed batch sizes
- Assemble-to-order systems
- General networks
- Non-stationary demand with fixed reorder intervals
- Supply chain with financial flows [Luo & Shang 2011]