

Evaluation of Cycle-Count Policies for Supply Chains with Inventory Inaccuracy and Implications on RFID Investments

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Inventory record inaccuracy leads to ineffective replenishment decisions and deteriorates supply chain performance. Conducting cycle counts (i.e., periodic inventory auditing) is a common approach to correcting inventory records. It is not clear, however, how inaccuracy at different locations affects supply chain performance and how an effective cycle-count program for a multi-stage supply chain should be designed. This paper aims to answer these questions by considering a serial supply chain that has inventory record inaccuracy and operates under local base-stock policies. A random error, representing a stock loss, such as shrinkage or spoilage, reduces the physical inventory at each location in each period. The errors are cumulative and are not observed until a location performs a cycle count. We provide a simple recursion to evaluate the system cost and propose a heuristic to obtain effective base-stock levels. For a two-stage system with identical error distributions and counting costs, we prove that it is more effective to conduct more frequent cycle counts at the downstream stage. In a numerical study for more general systems, we find that location (proximity to the customer), error rates, and counting costs are primary factors that determine which stages should get a higher priority when allocating cycle counts. However, it is in general not effective to allocate all cycle counts to the priority stages only. One should balance cycle counts between priority stages and non-priority stages by considering secondary factors such as lead times, holding costs, and the supply chain length. In particular, more cycle counts should be allocated to a stage when the ratio of its lead time to the total system lead time is small and the ratio of its holding cost to the total system holding cost is large. In addition, more cycle counts should be allocated to downstream stages when the number of stages in the supply chain is large. The analysis and insights generated from our study can be used to design guidelines or scorecard systems that help managers design better cycle-count policies. Finally, we discuss implications of our study on RFID investments in a supply chain.

(Key words: inventory inaccuracy, cycle counts, multi-echelon systems, RFID)

1. Introduction

Inventory record inaccuracy refers to the discrepancy between physical inventory held in stock and the record of inventory stored in the information system of a firm. Shrinkage, spoilage, misplaced inventories, and transaction errors (e.g., scanning errors and incorrectly counting products) contribute to inaccurate inventory information. Inaccurate inventory information leads to ineffective

replenishment decisions, which, in turn, result in poor service levels and higher inventory costs. This is a major issue affecting supply chain performance in manufacturing, distribution and retail settings. For example, DeHoratius and Raman (2008) found records to be inaccurate 65% of the items stored at a publicly traded retailer. According to ECR Europe (2003), the value of lost inventory due to shrinkage in 2000 was €13.4 billion for retailers and €4.6 billion for manufacturers in Europe.

A common method to mitigate the impact of inventory inaccuracy is to conduct cycle counts. Companies usually implement cycle-count policies according to an ABC classification scheme, i.e., classifying products into A, B, C classes based on product attributes, such as volume, error rate and value, and assigning a count cycle to each class (Jordan 1994). Motivated by empirical studies and business practice, researchers recently have developed various analytical models, aiming to study this problem more rigorously (see §2 below). To our knowledge, all of the existing analytical results are on single-location models. Nevertheless, major apparel retailers and consumer-packaged goods companies have significant inaccuracy problems across their supply chains. Managers have limited knowledge of the extent of inaccuracy at different locations and their impact on overall supply chain performance (Delen et al. 2009, Hardgrave et al. 2009). In this study, we investigate the impact of inventory inaccuracy on the supply chain performance. In particular, we aim to answer the following questions: (1) What is the impact of inventory inaccuracy at a location on the entire supply chain performance? (2) Given the fact that cycle counts are costly, which locations should have more frequent counts? (3) How should different product attributes and system characteristics be taken into account when designing cycle-count policies in a supply chain? The answers to these questions are not readily available in the academic and business literatures. While one may argue that record inaccuracy at downstream locations has a greater impact because of proximity to customers, inaccuracy at an upstream location affects the supply for all downstream locations. Similarly, one may argue that maintaining accurate records at a location with a longer lead time is more important because such locations are slower to respond to demand changes. On the other hand, such locations generally have more pipeline and safety stock, which implies that they would be affected less by record inaccuracy.

In this paper, we consider a periodic-review, N -stage serial supply chain with inventory record inaccuracy. Random customer demand arrives at stage 1. Stage 1 replenishes from stage 2, stage 2 from stage 3, and so on, and stage N from an outside supplier with ample supply. There are constant transportation lead times between stages. Record inaccuracy is caused by a random inventory loss that reduces physical inventory. (See §2 for a discussion on the other causes of inventory inaccuracy.) To describe inventory inaccuracy, we use the term “nominal” to signify inventory records stored in the computer system, and “actual” to signify the physical inventory

levels. Each stage implements a cycle-count policy to correct the inventory records. A fixed stage-specific inspection cost is incurred for each cycle count. Errors at each stage are cumulative and they are not observed by the information system unless the stage conducts a cycle count. Thus, the discrepancy between the nominal inventory and the actual inventory equals the accumulated error since the last cycle count. The material flow is controlled by local base-stock policies. That is, if the local nominal inventory order position at a stage is less than a target base-stock level, the stage places an order to raise the nominal inventory position to the target level. Such a replenishment scheme is commonly seen in practice and most firms have computer systems automate this process. The objective is to minimize the actual average total supply-chain cost per period.

To evaluate the system cost, we derive the actual local inventory variables for any given cycle-count policy. However, obtaining these local inventory variables requires characterizing the local demand for each stage. One key contribution of this paper is that we provide a simple procedure to evaluate the system cost. This procedure recursively evaluates the cost for each echelon (a stage and all of its downstream stages) starting from echelon 1 to echelon N . Moreover, this recursion leads to a heuristic to find local base-stock levels. A numerical study suggests that the heuristic is effective. We then use these results to answer the research questions.

For two-stage systems, regarding the impact of errors, we show that it is more costly to have the same error occur at the downstream stage than at the upstream stage. Furthermore, regarding the cycle-count policies, we show that it is always a better strategy to assign more cycle counts to the downstream stage than to the upstream stage when both stages have identical errors and counting costs. For more general systems, we categorize system parameters into two groups in terms of their effect on cycle-count policy decisions. Primary factors determine which stages should get a higher priority when allocating cycle counts. They include location (position in the supply chain), error rate, and counting cost. All else being equal, downstream locations should be assigned more frequent counts. However, a significantly higher error rate or lower counting cost at an upstream stage may reverse the result. We observe that the marginal benefit of cycle counts is decreasing in its frequency. Thus, one should not allocate all cycle counts to a single location. We therefore suggest using secondary factors to determine whether the policy should strongly favor the high-priority stages or allocate counts in a more balanced way. The secondary factors include lead time and holding cost structure and the supply chain length. We find that more cycle counts should be allocated to a stage if the ratio of the stage's lead time to the total system lead time is small and the ratio of the stage's holding cost to the total holding cost is large. Furthermore, more cycle counts should be allocated to downstream stages if the supply chain is long (i.e., the number of stages in the supply chain is large).

Our study provides useful guidelines for designing cycle-count policies. When allocating a

fixed budget or a fixed number of cycle counts in a supply chain, priority should be given to the downstream locations and locations with significantly higher error rate and lower counting cost for the initial few cycle counts, then to locations with fewer cycle counts to keep a balanced allocation rather than focusing only on high-priority locations. While a more-balanced policy should be chosen in most cases, downstream stages should be given more priority if their share of the total lead time is small or the holding costs are high. In longer supply chains, the policies that slightly favor the few downstream stages should be preferred over policies that are perfectly balanced, unless the upstream error rates are significantly higher.

From an implementation perspective, our study can be used to enhance the functionality of the commonly used ABC classification scheme. The current scheme, which is designed for determining cycle-count frequency for items in a single location, suggests that an item should be counted more frequently if its value (holding cost), volume (demand rate), or error rate is high (Jordan 1994). Our findings suggest that, in addition to the current criteria, a more comprehensive ABC scheme should include the characteristics of an item's corresponding supply chain, such as location (proximity to the customer), local lead time relative to system lead time, the length of its supply chain, and counting costs.

Finally, our research questions are also related to investment decisions on Radio Frequency Identification (RFID) systems. Specifically, if a location installs RFID readers, this will, in principle, eliminate inventory inaccuracy by continuously monitoring the inventory. This has the same effect as conducting a cycle count in each period in our model. In fact, many RFID applications in retail supply chains involve hand-held RFID readers, whose role is to facilitate cycle counting. Because RFID implementation can be very costly (A.T. Kearney 2004), many industry practitioners are concerned about where RFID readers should be installed in a supply chain (Chappell et al. 2002). Labor savings and investment costs of RFID systems can be quantified in each stage in isolation from others (Chopra and Sodhi 2007), but it is not clear how one can quantify the impact of reducing inaccuracy at a particular stage on supply chain performance. The guidelines developed in this paper have implications on how to prioritize RFID investments in a supply chain.

The rest of the paper is organized as follows. §2 discusses major causes of inventory inaccuracy in supply chains and reviews the literature related to each of the causes. §3 presents a single-stage model and discusses the model set-up. §4 introduces the serial system and provides a scheme to evaluate the average total cost per period. §5 presents the lower bound and the heuristic algorithm. §6 and §7 present our analytical and numerical results and provide answers to the research questions. §8 summarizes the implications of our study. §9 concludes. All proofs are provided in the appendix.

2. Causes of Inventory Inaccuracy and Related Literature

Empirical research and industry reports indicate that shrinkage, transaction errors, and misplaced items are the main reasons that cause inventory inaccuracy (Sheppard and Brown 1993, Raman and Ton 2004, ECR Europe 2003). While inventory inaccuracy has been studied in the context of single-stage systems, considering its impact on the multi-stage model leads to complex inventory dynamics and new issues, which have not been discussed in the literature. In the following, we describe each type of inaccuracy and its impact on the supply chain.

Shrinkage

Shrinkage, also known as stock loss, may be due to theft by shoppers or employees, and spoiled and damaged inventory. The impact of shrinkage on the actual inventory is one-sided: it always reduces the actual inventory. Thus, the inventory record is always greater than or equal to the actual inventory when shrinkage is the only reason that causes inventory inaccuracy. Typically, shrinkage could only occur when inventory level is positive. In a supply chain, inventory shrinkages between locations are often independent because losing a unit of inventory at one location is not related to that at another location.

Shrinkage is found to be the dominant cause of inventory inaccuracy in many empirical studies. Kang and Gershwin (2007) report two firms for which the majority of the inventory inaccuracy is due to stock loss. Similarly, ECR Europe (2001, 2003) conduct a comprehensive study over more than 200 companies and suggest that stock loss errors may be more prominent than others. In these studies, shrinkage is estimated to be about 1.7% of the sales for retailers and 0.56% for manufacturers. The National Retail Security Survey of 118 retailers (Hollinger 2003) reports similar results. These studies emphasize the stockout effects of shrinkage. Arguably, shrinkage has the most direct impact on supply chain performance because shrinkage at an upstream location will directly affect the stock availability of its downstream location, which may, in turn, lead to significant stockouts or backlogs for the system. In order to directly address this issue affecting many supply chains, we consider only shrinkage-type inaccuracy in the present paper.

Transaction errors

Transaction errors refer to errors caused by misidentification of the items or miscounting of the items in inbound and outbound processes. For example, consider a warehouse where ten units of product A were sent to a customer according to the computer system, but, in actuality, nine (eleven) units were shipped. This transaction will make the actual inventory larger (smaller) than the nominal inventory. Thus, the impact of transaction error on the actual inventory is two-sided: it may increase or decrease the actual inventory. Transaction errors may also involve multiple products. For instance, if a customer purchases product A, but the sales transaction is recorded

for product B, this will reduce the actual inventory of product A and reduce the nominal inventory of product B. (DeHoratius et al. (2008) call this type of transaction error as invisible demand.)

Transaction errors can lead to complex inventory dynamics between stages in a supply chain. In the above warehouse example, the error of shipping one more (less) unit from the upstream location will lead to one unit more (less) inventory at its downstream location. This error will not be observed by the downstream location's inventory records unless inbound inspection or frequent cycle counts are conducted. Thus, unlike shrinkage, an error occurred at one location often affects another location.

Iglehart and Morey (1972) is the first paper to consider transaction errors in a single-stage model. The inventory system implements a base-stock policy and the objective is to find the cycle-count frequency that minimizes inspection and inventory holding costs subject to a pre-specified inaccuracy probability. In an empirical study of one retailer's records, DeHoratius and Raman (2008) report that the difference between physical inventory and inventory record can be positive or negative. They also find that the extent of errors increases with sales volume, the number of stages in the supply chain, and product variety. Kök and Shang (2007) characterize the optimal inspection and inventory policies in a model in which a fixed cost is incurred for each inspection. They find that a well-designed cycle-count policy is effective. DeHoratius et al. (2008) propose a replenishment policy that utilizes Bayesian updating of the inventory records. In a similar setting, Mersereau (2008) finds that it is optimal to stock less than the myopic stocking level to improve the information content about the actual inventory level. Finally, Deshpande et al. (2009) study the impact of inbound shipping errors on the ordering policies of a two-product, single-stage system. To our knowledge, there has been no work on investigating the impact of transaction errors on a multi-stage supply chain.

Misplaced / inaccessible inventory

Misplaced or inaccessible inventory refers to inventory that is physically at the facility, but its exact location is unknown. For example, if a customer at a bookstore browses a book and leaves it at a different shelf, that unit of inventory is practically not available for sale until the book is returned to its original location, possibly after a cycle count. The impact of misplaced inventory is also one-sided: it reduces actual inventory levels. However, unlike shrinkage, misplaced inventory is not permanently lost, it may be recovered and added to the actual inventory after a cycle count.

Çamdereli and Swaminathan (2009) study inventory decisions and coordination contracts between a supplier and a retailer with misplaced inventory. Similar to the case of transaction errors, we do not know of any work that studies misplaced inventory in the multi-stage setting.

Incorporating all error types in a model without any simplifying assumptions is a difficult task even for single-stage systems. Atali et al. (2005) and Bensoussan et al. (2007) consider multiple

types of errors in a single-stage model. The former assumes that errors occur independently from the current inventory levels in order to derive inventory policies. The latter develops a heuristic feedback policy due to the dimensionality of the problem.

Our research also pertains to the literature with RFID. One of the benefits of RFID systems is greater visibility of material flow and order progress. Gaukler et al. (2008) and Kim et al. (2007) study expediting strategies given the accurate order progress information from upstream stages. Pei and Klabjan (2008) study the optimal inventory policy for a serial system in which RFID systems provide order progress information for replenishments and shipments. They compare the expected costs of the model with different levels of RFID deployment and find that broader deployment reduces costs. This group of papers focuses on gaining visibility into order movements that are usually modeled with a stochastic lead time. Our focus is different in that we compare the benefit of reducing inaccuracy at different stages in a supply chain. For a review of the literature on RFID-related models and future applications, see Zipkin (2006) and Lee and Özer (2007).

The present paper builds on the literature on multi-echelon inventory systems, particularly the classical serial supply chain model studied by Clark and Scarf (1960), Chen and Zheng (1994), and many others. We incorporate inventory record inaccuracy and cycle-count policies in the serial system by introducing unobserved inventory errors at each stage. This leads to an additional dimension in the inventory dynamics and expands the set of decision variables.

Finally, this paper is related to inventory/production systems with random yields. For single-stage yield models under periodic review, Henig and Gerchak (1998) show that base-stock type policies are optimal. Lee and Yano (1988) study a multi-stage model for a single production run to meet a single-period demand. Liu et al. (2008) consider the impact of supply reliability on inventory and pricing decisions. On a related topic, Bollapragada et al. (2004) consider assembly systems with random capacity at each location that are controlled with base-stock policies. In random yield and random capacity models, the uncertainty at every stage is observable in each period after it is realized, which removes any uncertainty about inventory levels in the supply chain. In our model, however, the errors at a stage are not observed and accumulate until a cycle count is performed.

3. Single-Stage System

In this section we consider a single-stage system. We use this model to illustrate the assumptions and the concept of nominal and actual inventory variables. This discussion sets the stage for the serial system.

Consider a periodic-review inventory system that orders according to a base-stock policy. Time is divided into periods of length one and the periods are numbered $0, 1, 2, \dots$. There is a constant

lead time of L periods. Customer demand follows a Poisson process with rate λ . Let $[t, t+r)$ and $[t, t+r]$ denote the time interval over periods $t, t+1, \dots, t+r-1$ and periods $t, t+1, \dots, t+r$, respectively. Let $D[t, t+r)$ and $D[t, t+r]$ denote the total demand that occurs in $[t, t+r)$ and $[t, t+r]$, respectively. We denote $D[r)$ and $D[r]$ the total demand in r and $r+1$ periods for the long-run average cost calculations. We assume that a random loss of inventory occurs in each period that reduces the level of actual inventory. We call the random inventory loss an error and we assume that the errors follow a Poisson process with rate μ . Let $\xi[t, t+r)$ and $\xi[t, t+r]$ denote the accumulated error in $[t, t+r)$ and $[t, t+r]$, respectively, and $\xi[r)$ and $\xi[r]$ denote total error accumulated in r and $r+1$ periods for the long-run average cost calculations. The system implements a cycle-count policy to correct the inventory record. That is, the stage conducts a cycle count every T periods, where T represents the cycle-count interval. The system incurs a fixed cost of K every time a cycle count is conducted. Errors in the system accumulate and are not observed until a cycle count is conducted. As a result, the discrepancy between actual and nominal inventory positions is nondecreasing between two cycle counts. One may view this system as consisting of two processes: (1) The actual inventory dynamics and the cost evaluation process based on the actual inventory, and (2) the ordering process in which orders are placed based on the nominal inventory order position (or inventory record). Below is the sequence of events in each period.

- (1) The stage receives a shipment at the beginning of a period,
- (2) random error and random customer demand occurs during the period,
- (3) if the current period is a cycle-count period, then the stage conducts a cycle count,
- (4) the stage places an order according to the nominal inventory order position,
- (5) costs are evaluated at the end of the period.

We assume that the random demand and error processes during the period are interleaving Poisson processes. The actual inventory level at the end of a period is the beginning actual inventory level minus the sum of the two Poisson random variables, i.e., one period of demand and one period of error, regardless of our assumptions about the order of the two processes.

We now describe how these events affect the nominal and actual inventory variables. We define

the following inventory variables to facilitate the subsequent discussion.

$AIOP^-(t)$ = Actual inventory order position at the beginning of period t ,

$AIOP(t)$ = Actual inventory order position after placing an order at the end of period t ,

$AIL^-(t)$ = Actual inventory level at the beginning of period t after receiving a shipment,

$AIL(t)$ = Actual inventory level after error and demand occur.

$B(t)$ = Actual backlogged consumer demand after error and demand occur.

Let $NIOP^-(t)$, $NIOP(t)$, $NIL^-(t)$, and $NIL(t)$ denote the corresponding nominal inventory variables. In event (1), both $AIL^-(t)$ and $NIL^-(t)$ increase because of the arriving shipment. In event (2), random error reduces the actual inventory variables and random demand reduces the nominal and actual inventory variables. $AIL(t)$ and $AIOP(t)$ are determined by the total demand from the two random processes. In event (3), the nominal inventory variables are aligned with the actual ones if the stage conducts a cycle count. In event (4), the stage places an order according to the nominal inventory order position. That is, at the end of a period, if the nominal inventory order position is less than the local base-stock level s , the stage places an order that raises the nominal inventory order position to s ; otherwise, the stage does not order. Thus, event (4) determines both $NIOP(t)$ and $AIOP(t)$ (or equivalently, $NIOP^-(t+1)$ and $AIOP^-(t+1)$). In event (5), costs are evaluated based on $AIL(t)$. For this purpose, define $[x]^+ = \max\{x, 0\}$ and $[x]^- = \max\{0, -x\}$. A linear holding cost h is incurred for each unit of actual on-hand inventory $[AIL(t)]^+$ per period and a backorder cost b for each unit of actual backlogged consumer demand $B(t)$ per period. The system is backlogged if $AIL(t)$ is negative. However, only a portion of that backlog is caused by the customer demand process, while the rest is due to the error process. The distribution of the actual backlogged demand $B(t)$ depends on the total backlog level $[AIL(t)]^-$ as described later in this section. The objective is to minimize the average total cost per period.

We make the following remarks about the model assumptions. First, the assumption of independent and identically distributed errors implies that the error distribution is independent of the inventory level. While certain types of inventory losses, such as invisible demand may satisfy this assumption, other error types such as shrinkage or spoilage can occur only when inventory level is positive. We make this assumption for tractability of the model. Modeling the dependence on inventory level would require keeping track of periodic demand and error during lead time as a vector. This would cause the size of the state space to grow intractably large, which is similar to the difficulty in the lost-sales system. Second, for tractability, we assume that the discrepancy between the actual inventory and nominal inventory is reconciled only when a cycle count is performed. There is no correction of the records due to the occurrence of a backorder. That is, if the computer shows the item in stock but it is actually not available, the demand is

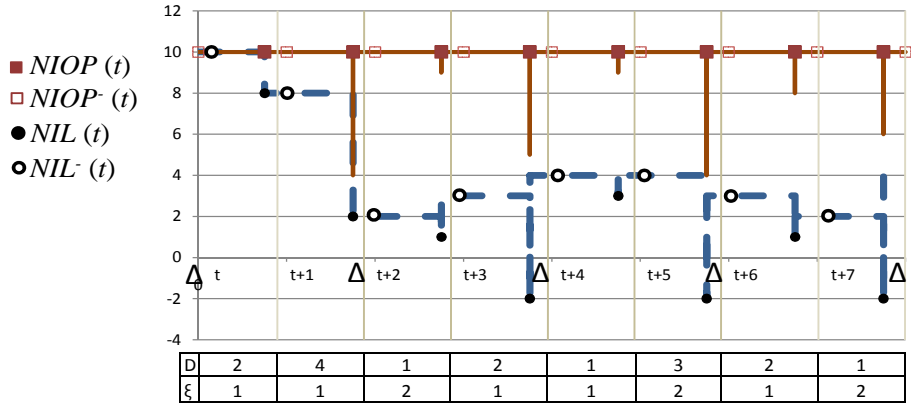
backlogged and the backorder information is not used to correct the records. Because we ignore the occurrence of such corrections in our characterization of the system state, the present model can be seen as an approximation to the systems that utilize the backlog information to correct records when the backlog probability is small. Many companies either do not have proper systems in place to update their computer records or do not allow line workers to change system records to ensure data integrity. Third, demand fulfillment and cost calculation are based only on actual on-hand inventory $(AIL)^+$ and actual backlogged demand B . In daily operations, a company may do its accounting based on the inventory records. However, at the end of a year, the annual cost according to the information system may be different from the actual annual cost, which is realized based on the actual inventory dynamics. This difference is due to the inaccuracy of the records. Therefore, a reconciliation by the accounting department is always necessary.

An Example

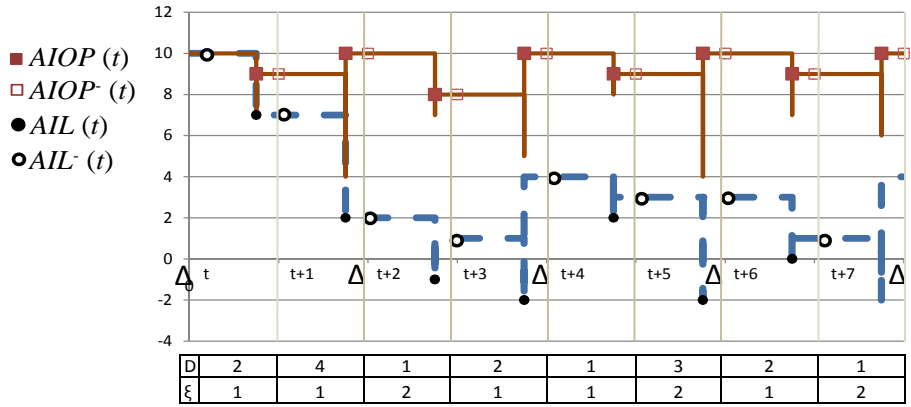
We use an example to illustrate the inventory dynamics under the nominal and actual inventory schemes. Consider a single-stage system with $T = 2$, $L = 2$, and $s = 10$. Figure 1(a) illustrates the dynamics of the nominal inventory order position and inventory level. Figure 1(b) illustrates the corresponding actual inventory variables. The demand and error levels in each period are given in the figure. The system has conducted a cycle count at the end of period $t - 1$, so there is no error in the system at the beginning of period t . The future cycle-count periods are $\{t + 1, t + 3, t + 5, \dots\}$ and the cycle-count periods are marked with the “ Δ ” symbol.

Let us first focus on the nominal inventory variables shown in Figure 1(a). We assume that $NIOP^-(t) = NIL^-(t) = 10$ at the beginning of period t . Because the error in period t is not observed, $NIOP$ and NIL change in the same way as do those in the standard base-stock policy with demand of two units, that is, $NIOP^-(t + 1) = 10$ and $NIL^-(t + 1) = 8$. In period $t + 1$, four units of demand is recorded and an accumulated error of $\xi[t, t + 2] = 2$ is detected during the cycle count. Hence, $NIL(t + 1) = 2$. Finally, the stage places an order of six units to bring $NIOP(t + 1) = 10$, which is equal to $NIOP^-(t + 2)$. The rest of the dynamics can be developed with the same logic. Clearly, the actual inventory dynamics are different because the nominal system does not realize that the error affects the physical inventory level in every period.

Figure 1(b) presents the corresponding actual inventory variables. At the beginning of period t , $AIOP^-(t) = AIL^-(t) = 10$. As a result of two units of demand and one unit of error, we have $AIL(t) = 7$. The stage orders two units according to the nominal inventory order position, and we get $AIOP(t) = AIOP^-(t + 1) = 9$. Now, consider period $t + 5$. At the beginning of the period, the stage receives shipment of one unit sent at the end of period $t + 2$. Hence, $AIL^-(t + 5) = AIL(t + 4) + 1 = 3$. An error of two units and a demand of three units are



(a)



(b)

Figure 1: A single-stage system with $L = 2$, $T = 2$, and $s = 10$. (a) The dynamics of the nominal inventory order position and inventory level. (b) The dynamics of the corresponding actual inventory variables.

realized during the period and we have $AIL(t+5) = -2$. Depending on the sequence of the error and demand realization during the period, some or all of these two units of backlogs may be due to customer demand. We discuss later how we account for backlogs due to demand and error separately. At the beginning of period $t+6$, the stage receives a shipment of five units and $AIL^-(t+6) = -2 + 5 = 3$.

We now show how to evaluate the average system cost per period. The inventory dynamics result in a regenerative process. Specifically, if the system conducts a cycle count at the end of period $t-1$, the nominal inventory order position is set equal to the actual position at the beginning of period t , i.e., $NIOP^-(t) = AIOP^-(t)$. Because the errors are nonnegative, the nominal inventory terms are always greater than or equal to the actual terms. Hence, after a cycle count, the corrected nominal inventory position cannot be more than the target base-stock level s and the system can order up to the target level after aligning the nominal terms with the actual ones. In order to analyze this regenerative process that has a cycle length of T periods, we set the beginning of periods $t, t+T, t+2T, \dots$ as the regenerative epochs. The long-run average cost per period is equal to the expected cost between two consecutive regenerative epochs divided by the cycle length. The $AIOP^-$ in the cycle-count interval $[t, t+T)$ determines AIL^- and AIL in $[t+L, t+L+T)$ as follows.

Proposition 1 For $r = 0, \dots, T-1$,

$$\begin{aligned} NIOP^-(t+r) &= s, \\ AIOP^-(t+r) &= s - \xi[t, t+r] \end{aligned} \tag{1}$$

$$\begin{aligned} AIL^-(t+L+r) &= AIOP^-(t+r) - D[t+r, t+L+r] - \xi[t+r, t+L+r], \\ AIL(t+L+r) &= AIOP^-(t+r) - D[t+r, t+L+r] - \xi[t+r, t+L+r], \end{aligned} \tag{2}$$

and $B(t+L+r)$ is binomially distributed with parameters $[AIL(t+L+r)]^-$ and $\lambda/(\lambda + \mu)$.

The second equation is due to the replenishment policy that orders to cover the demand in the no-cycle-count periods and the demand and the accumulated error in the cycle-count period. The third and fourth equations are similar to those in the standard base-stock models and reflect the effect of the lead time. The last statement gives the distribution of backlogged customer demand given the total backlog level. From the moment AIL falls to zero until the end of the period, two Poisson processes reduce AIL , but only customer arrivals cause a backlog penalty. The process after AIL falls to zero is independent of the demand and error realizations before that point because of the memoryless property of the Poisson process. Hence, the fraction of total backlog that is due to customer demand follows a Binomial distribution with number of trials equal to $[AIL(t+L+r)]^-$ and success probability equal to the ratio of mean demand to mean error

plus mean demand. Thus, the expected value of $B(t + L + r)$ given $AIL(t + L + r)$ is simply $(\lambda/(\lambda + \mu))AIL(t + L + r)^-$. This accounts for the system incurring the backorder penalty for only customer demands when the actual inventory level is less than or equal to zero, but not for the error process. In the example in Figure 1, the actual inventory level in period $t + 5$ is three units after replenishment and minus two units at the end of the period (after two units of error and three units of demand). The actual backlogged demand level, however, depends on the realized sequence of the error and demand occurrences. Proposition 1 tells us that $B(t + 5)$ is distributed according to a binomial distribution with parameters $(2, \lambda/(\lambda + \mu))$.

For expositional simplicity, define $\hat{b} = b\lambda/(\lambda + \mu)$. The average inventory cost per period is given by

$$\begin{aligned} G(s, T) &= \frac{1}{T} \left(\sum_{r=0}^{T-1} \mathbb{E}[hAIL(t + L + r)^+ + bB(t + L + r)] \right) \\ &= \frac{1}{T} \left(\sum_{r=0}^{T-1} \mathbb{E}[hAIL(t + L + r) + (\hat{b} + h)(AIL(t + L + r)^-)] \right). \end{aligned}$$

The distribution of $AIL(t + L + r)$ is determined by $AIOP^-(t + r)$ given by (1). Note that $\xi[t + \lfloor r/T \rfloor T, t + r]$ in that expression yields r periods of error for $r = 0, 1, 2, \dots, T - 1$. Define

$$g(y, r) = \mathbb{E}[h(y - \xi[r] - D[L] - \xi[L]) + (\hat{b} + h)(y - \xi[r] - D[L] - \xi[L])^-].$$

The average inventory cost per period, $G(s, T)$, and the average total cost per period including the counting cost, $C(s, T)$, can be expressed as follows.

Proposition 2 $G(s, T) = (1/T) \sum_{r=0}^{T-1} g(s, r)$ and $C(s, T) = G(s, T) + K/T$.

Note that each $g(y, r)$ function is the same as a single-period inventory cost function with different demand variables. Because each of them is convex, average cost $G(s, T)$ is also convex in the base-stock level s for fixed T . Let $s(T) = \arg \min_s C(s, T)$ be the optimal base-stock level. Define $G(T) = G(s(T), T)$ as the minimum inventory cost and $C(T) = C(s(T), T)$ as the optimal total cost for a given cycle-count policy.

Proposition 3 For $T \geq 2$, (1) $s(T - 1) \leq s(T)$, (2) $G(T - 1) \leq G(T)$.

Proposition 3 states that the optimal base-stock level and the optimal inventory cost increase with the cycle-count interval length T . When the cycle-count interval is longer, more periods of error accumulate in the system. Thus, the system has to maintain a higher stock level to cover the inventory loss and the additional uncertainty. The higher stock level leads to a higher optimal system cost. The impact of the cycle-count interval on $C(T)$ depends on how K/T compares with inventory related costs.

4. Series Systems

We now consider an N -stage series system. Customer demand occurs at stage 1. Stage 1 is replenished by stage 2, stage 2 by stage 3, and so on, and stage N by an outside source with ample supply. The lead time between stage j and $j + 1$ is L_j periods. Let $\bar{L}_j = \sum_{i=j}^N L_i$, the cumulative lead time from stage j to stage N . Each stage orders according to a local base-stock policy. At the beginning of a period, if stage j 's local inventory order position is lower than the base-stock level s_j , the stage places an order to raise its nominal inventory order position to s_j ; otherwise, the stage does not order. Let $S_j \stackrel{def}{=} \sum_{i=1}^j s_i$. Denote $D_j[t, t+r)$ and $D_j[t, t+r]$ the local demand occurring in periods $[t, t+r)$ and $[t, t+r]$ for stage j , respectively. Note that under a local base-stock policy, stage $j - 1$'s order is stage j 's local demand. A non-negative random inventory loss, or error, occurs at each stage every period, reducing the physical inventory level. We assume that the error at stage j is Poisson with rate μ_j and is independent and identically distributed across periods. Also, the error process at a stage is independent of the other stages. Let $\xi_j[t, t+r)$ and $\xi_j[t, t+r]$ denote the accumulated error in periods $[t, t+r)$ and $[t, t+r]$ at stage j . We use $\xi_j[r]$ and $\xi_j[r]$ for the accumulated error of r and $r + 1$ periods in the long-run average cost calculations, respectively. Each stage conducts a cycle-count policy to correct its inventory record. Let T_j be the cycle-count interval for stage j and K_j be the counting cost at stage j . We let variables in bold font denote the vector of that variable for $j = 1, \dots, N$. That is, $\mathbf{s} = (s_1, \dots, s_N)$, $\mathbf{T} = (T_1, \dots, T_N)$. Similarly for $\boldsymbol{\mu}$, \mathbf{L} , \mathbf{h} , and \mathbf{K} .

For stage 1, the sequence of events is the same as that for the single-stage system. For stages $j \geq 2$, the sequence of events is similar to stage 1. Specifically,

- (1) at the beginning of a period, stage j receives a shipment sent L_j periods ago,
- (2) random error occurs during the period,
- (3) downstream order (or local demand) arrives at the end of a period,
- (4) stage j sends a shipment to stage $j - 1$,
- (5) stage j conducts a cycle count, if the current period is a cycle-count period,
- (6) stage j places an order according to the nominal inventory order position at the end of a period,
- (7) costs are evaluated based on the actual inventory level.

Below we discuss how to evaluate the average total cost per period. In §4.1, we show how to evaluate the cost using local inventory variables. In §4.2, we provide an alternative approach, which

evaluates the cost using the corresponding echelon variables. This echelon approach simplifies the computation significantly and sets the stage for the subsequent analysis.

4.1 Evaluation: Local Approach

To explain how these events affect inventory variables, we define the following local inventory variables.

$$\begin{aligned}
AIO P_j'^-(t) &= \text{Actual local inventory order position at stage } j \text{ at the beginning of period } t, \\
AIO P_j'(t) &= \text{Actual local inventory order position after ordering at stage } j \\
&\quad \text{at the end of period } t, \\
AIL_j'^-(t) &= \text{Actual local inventory level at stage } j \text{ at the beginning of period } t \\
&\quad \text{after receiving a shipment placed } L_j \text{ periods ago,} \\
AIL_j'(t) &= \text{Actual local inventory level at stage } j \text{ at the end of period } t.
\end{aligned}$$

The effect of the sequence of events on the actual inventory variables is as follows. In event (1), stage j receives a shipment, which raises the actual inventory level and determines $AIL_j'^-(t)$. In event (2), the error reduces $AIO P_j'^-(t)$ and $AIL_j'^-(t)$. In event (3), a downstream order (local demand) arrives, which reduces $AIO P_j'^-(t)$. In event (4), stage j ships as much as possible to fulfill stage $j - 1$'s order, and $AIL_j'(t)$ is determined. In event (5), the nominal and actual local inventory variables are aligned after a cycle count is conducted. In event (6), $AIO P_j'(t)$ is determined after an order is placed. This $AIO P_j'$ level is carried over to the next period, i.e., $AIO P_j'(t) = AIO P_j'^-(t + 1)$. In event (7), costs are evaluated based on $AIL_j'(t)$. Let h_j denote the echelon holding cost, $h_j' = \sum_{i=j}^N h_i$ the local holding cost for stage j , and b the backorder cost at stage 1. The inventory holding cost for stage j at the end of period t is $h_j'[AIL_j'(t)]^+$, $j = 1, \dots, N$, and the expected cost of backlogged customer demand at stage 1 is $bE[B(t)] = \hat{b}[AIL_1'(t)]^-$, where $\hat{b} = b\lambda/(\lambda + \mu_1)$.

We assume that all stages conduct a cycle count at the end of period $t - 1$. Let $T = lcm\{T_1, \dots, T_N\}$, where $lcm\{\}$ is an operator that generates the least common multiplier. The common cycle-count periods for all stages are $t - 1, t + T - 1, t + 2T - 1, \dots$. After a common cycle-count period, all stages have zero error. Thus, we set the beginning of periods $t, t + T, t + 2T, \dots$ as regenerative epochs. Without loss of generality, we will focus on a regenerative cycle consisting of periods $t + r, r = 0, 1, \dots, T - 1$. For expositional simplicity, define

$$r_j = r + N - j, \quad j = 1, 2, \dots, N.$$

As we shall see, $AIO P_j'^-(t + \bar{L}_{j+1} + r_j)$ and the local demand D_j will jointly determine the actual local inventory level $AIL_j'(t + \bar{L}_j + r_j)$ for stage $j, j = N, N - 1, \dots, 1$.

To illustrate this, we first characterize $AIOP_j'^-$. Similar to the expression for the order position for the single-stage system in (1), we need to adjust the order position down for the error accumulated in the system since the last inspection. Let $\lfloor \cdot \rfloor$ be the round-down operator. Under the considered inventory and cycle-count policy, we have

$$AIOP_j'^-(t + \bar{L}_{j+1} + r_j) = s_j - \xi_j \left[t + \left\lfloor \frac{(\bar{L}_{j+1} + r_j)}{T_j} \right\rfloor T_j, t + \bar{L}_{j+1} + r_j \right). \quad (3)$$

We next characterize the local demand D_j for stage j . Note that the local demand for stage j is the order placed by stage $j - 1$. Thus, we can sequentially characterize these local demands from stage 1, stage 2, ..., up to stage N . Let t be a regenerative epoch and ℓ be any nonnegative integer. Clearly, $D_1[t, t + \ell] = D[t, t + \ell]$. For stage $j = 2, 3, \dots, N$,

$$\begin{aligned} D_j[t, t + \ell] &= D_{j-1}[t, t + \ell] + \xi_{j-1} \left[\left\lfloor \frac{t}{T_{j-1}} \right\rfloor T_{j-1}, \left\lfloor \frac{t + \ell}{T_{j-1}} \right\rfloor T_{j-1} \right). \\ D_j[t, t + \ell] &= D_{j-1}[t, t + \ell] + \xi_{j-1} \left[\left\lfloor \frac{t}{T_{j-1}} \right\rfloor T_{j-1}, \left\lfloor \frac{t + \ell + 1}{T_{j-1}} \right\rfloor T_{j-1} \right). \end{aligned}$$

As seen from these equations, stage $j - 1$ simply passes on all the orders it receives from downstream to upstream stage j and, in the cycle-count periods, the local errors that it observes (if any).

With these local demands and $AIOP_j'^-$, we can characterize $AILL_j'(t + \bar{L}_j + r_j)$ as we did for the single-stage system in (2). This process starts from stage N . Note that

$$\begin{aligned} AILL_N^-(t + L_N + r_N) &= AIOP_N'^-(t + r_N) - D_N[t + r_N, t + L_N + r_N] \\ &\quad - \xi_N[t + r_N, t + L_N + r_N), \end{aligned} \quad (4)$$

$$\begin{aligned} AILL_N'(t + L_N + r_N) &= AIOP_N'^-(t + r_N) - D_N[t + r_N, t + L_N + r_N] \\ &\quad - \xi_N[t + r_N, t + L_N + r_N]. \end{aligned} \quad (5)$$

It is more difficult to characterize $AILL_j'(t + \bar{L}_j + r_j)$, $j < N$, because stage j may not always receive what it orders. To reflect this fact, we define the following inventory terms:

$$\begin{aligned} AIP_j'^-(t) &= \text{actual local inventory in-transit position at the beginning of period } t \\ &\quad \text{after a shipment is received;} \end{aligned}$$

$$AIP_j'(t) = \text{actual local inventory in-transit position at the end of period } t.$$

Now, consider stage $j (< N)$ at the end of period $t + \bar{L}_{j+1} + r_{j+1}$. If stage $j + 1$ has fulfilled stage j 's order in this period,

$$AIP_j'(t + \bar{L}_{j+1} + r_{j+1}) = AIOP_j'(t + \bar{L}_{j+1} + r_{j+1}).$$

On the other hand, if stage $j + 1$ does not have sufficient stock to fulfill stage j 's order, then stage $j + 1$ has a backlog. In this case, $AILL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1}) < 0$ and

$$AIP'_j(t + \bar{L}_{j+1} + r_{j+1}) = AIOP'_j(t + \bar{L}_{j+1} + r_{j+1}) + AILL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1}). \quad (6)$$

Note that the backlog may be due to the error and local demand at stage $j + 1$. The above equation suggests that this backlog reduces the actual inventory in-transit position at stage j ¹.

Combining these two cases together, we have

$$AIP'_j(t + \bar{L}_{j+1} + r_{j+1}) = AIOP'_j(t + \bar{L}_{j+1} + r_{j+1}) + \min\{0, AILL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1})\}, \quad (7)$$

or equivalently,

$$AIP_j^-(t + \bar{L}_{j+1} + r_j) = AIOP_j^-(t + \bar{L}_{j+1} + r_j) + \min\{0, AILL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1})\}. \quad (8)$$

Given the distribution of $AIP_j^-(t + \bar{L}_{j+1} + r_j)$, we can characterize the distribution of future inventory levels similarly to stage N . We have

$$\begin{aligned} AILL_j^-(t + \bar{L}_j + r_j) &= AIP_j^-(t + \bar{L}_{j+1} + r_j) - D_j[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j] \\ &\quad - \xi_j[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j], \end{aligned} \quad (9)$$

$$\begin{aligned} AILL'_j(t + \bar{L}_j + r_j) &= AIP_j^-(t + \bar{L}_{j+1} + r_j) - D_j[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j] \\ &\quad - \xi_j[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j]. \end{aligned} \quad (10)$$

In summary, we can characterize $AILL'_j(t + \bar{L}_j + r_j)$ through (3), (8), (9) and (10) recursively, starting from stage N to stage 1. We are able to characterize $AILL'_j(t)$ for any period t by repeating this process for all regenerative cycles. The average inventory cost per period is

$$G(\mathbf{s}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\left(\sum_{j=1}^N \left(h'_j(AILL'_j(t+r))^+ + h'_j IT_{j-1}(t+r) \right) + \hat{b}(AILL'_1(t+r))^- \right) \right], \quad (11)$$

where $IT_{j-1}(t)$ is the total inventory in transit to stage $j - 1$ at the end of period t , and $IT_0(t) \equiv 0$. Because all customer demand will be fulfilled and all errors will be recovered, $\mathbb{E}[IT_{j-1}(t)] = (\mathbb{E}[D] + \sum_{i=1}^{j-1} \mathbb{E}[\xi_i])L_{j-1}$.

We end this section with a connection to the literature on the bullwhip phenomenon. Lee et al. (1997) demonstrate four reasons that can lead to the bullwhip effect in supply chains. The above characterization of the local demand for each stage indicates that the mean and variance of D_j are larger than those of D_{j-1} . Furthermore, the increase in variance is larger if cycle-count interval of stage $(j - 1)$ is longer. Thus, inventory loss across locations in a supply chain is another factor that may contribute to the bullwhip effect.

¹This is because of the assumption that errors can occur when stage $j + 1$'s actual inventory level is non-positive. When the backlog at stage $j + 1$ is filled after receiving a shipment in subsequent periods, AIP'_j will be restored back to $AIOP'_j$.

4.2 Evaluation: Echelon Approach

The above procedure to obtain the total cost per period and the computation is quite complicated because we need to compute the distribution of the local demand D_j for all stages and periods. We next provide a simple echelon scheme to evaluate the total system cost. This scheme allows us to bypass the computation of the distribution of D_j . Below we define the echelon inventory variables and present the relationships between the local and echelon inventory variables. These relationships are standard and can be found in Zipkin (2000).

$$\begin{aligned}
 AIO P_j^-(t) &= \text{Actual echelon inventory-order position at stage } j \text{ at the beginning of period } t \\
 &= \sum_{i=1}^j AIO P_i'^-(t), \\
 AIL_j^-(t) &= \text{Actual echelon inventory level at stage } j \text{ at the beginning of period } t \\
 &= AIL_j'^-(t) + \sum_{i=1}^{j-1} AIO P_i'^-(t),
 \end{aligned}$$

$$AIL_j(t) = \text{Actual echelon inventory level at stage } j \text{ at the end of period } t,$$

$$\begin{aligned}
 AIP_j^-(t) &= \text{Actual echelon inventory in-transit position at stage } j \text{ at the beginning of period } t, \\
 &= AIP_j'^-(t) + \sum_{i=1}^{j-1} AIO P_i'^-(t).
 \end{aligned}$$

The expression for $AIL_j(t)$ can be similarly established by removing “-” in the variables in the equation for $AIL_j^-(t)$.

With these definitions, the total cost per period $G(\mathbf{s}, \mathbf{T})$ in (11) can be expressed as follows.

$$G(\mathbf{s}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\left(\sum_{j=1}^N h_j AIL_j(t+r) + (\hat{b} + h'_1)(AIL_1(t+r))^- \right) \right]. \quad (12)$$

Note that this is equal to

$$G(\mathbf{s}, \mathbf{T}) = \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\left(\sum_{j=1}^N h_j AIL_j(t + \bar{L}_j + r_j) + (\hat{b} + h'_1)(AIL_1(t + \bar{L}_1 + r_1))^- \right) \right]. \quad (13)$$

The only difference between the above cost functions above is that the time index for stage j is shifted by $\bar{L}_j + r_j - r$ periods in the second equation. Nevertheless, every period is evaluated exactly once and the long-run average total cost per period is the same with either formulation.

Our next task is to provide a simple recursion to obtain $AIL_j(t + \bar{L}_j + r_j)$. From the relationship between the local variables given by (8), (9), and (10) and the definitions of echelon variables, we can derive the following.

Proposition 4 For $j = N, N - 1, \dots, 1$,

$$AIP_j^-(t + \bar{L}_{j+1} + r_j) = \min \left\{ AIOP_j^-(t + \bar{L}_{j+1} + r_j), AIL_{j+1}(t + \bar{L}_{j+1} + r_{j+1}) \right\}, \quad (14)$$

$$\begin{aligned} AIL_j^-(t + \bar{L}_j + r_j) &= AIP_j^-(t + \bar{L}_{j+1} + r_j) - D[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j] \\ &\quad - \sum_{i=1}^j \xi_i[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j], \end{aligned} \quad (15)$$

$$\begin{aligned} AIL_j(t + \bar{L}_j + r_j) &= AIP_j^-(t + \bar{L}_{j+1} + r_j) - D[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j] \\ &\quad - \sum_{i=1}^j \xi_i[t + \bar{L}_{j+1} + r_j, t + \bar{L}_j + r_j], \end{aligned} \quad (16)$$

where $\bar{L}_{N+1} = 0$ and $AAIL_{N+1}^-(t + r) = \infty$.

Given $AIOP_j^-(t + \bar{L}_{j+1} + r_j)$, we can use Proposition 4 to obtain $AIP_j^-(t + \bar{L}_{j+1} + r_j)$, which further determines $AAIL_j^-$ and $AAIL_j$ in period $t + \bar{L}_j + r_j$. From (3) and the definition of the echelon order position, we get

$$\begin{aligned} AIOP_j^-(t + \bar{L}_{j+1} + r_j) &= \sum_{i=1}^j AIOP_i'^-(t + \bar{L}_{j+1} + r_j) \\ &= S_j - \sum_{i=1}^j \xi_i[t + \lfloor (\bar{L}_{j+1} + r_j)/T_i \rfloor T_i, t + \bar{L}_{j+1} + r_j]. \end{aligned} \quad (17)$$

Thus, with Proposition 4 and (17), we can obtain the total cost in Equation (13).

Below we provide a bottom-up recursion to evaluate the total cost per period. The idea is to recursively evaluate the cost for echelon j , for $j = 1, \dots, N$ by assuming that echelon j has ample supply from its upstream stage. Define the following notation to represent the number of errors accumulated since the last cycle count at stage j .

$$\mathbb{M}_{T_j}(\ell) = t + \ell - \left(t + \left\lfloor \frac{\ell}{T_j} \right\rfloor T_j \right).$$

This notation yields the number of periods since the last cycle count at stage j , given a regenerative epoch t and a time length of ℓ periods.

Proposition 5 For $r = 0, 1, 2, \dots, T - 1$, define

$$\begin{aligned} g_1(y, r) &= \mathbb{E}[h_1(y - \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)]) - D[L_1] - \xi_1[L_1]] \\ &\quad + (\hat{b} + h'_1)(y - \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)]) - D[L_1] - \xi_1[L_1]^-. \end{aligned} \quad (18)$$

For $j = 2, \dots, N$ and $r = 0, 1, 2, \dots, T - 1$, define

$$\begin{aligned} g_j(y, r) &= \mathbb{E} \left[h_j \left(y - \sum_{i=1}^j \xi_i[\mathbb{M}_{T_i}(\bar{L}_{j+1} + r_j)] - D[L_j] - \sum_{i=1}^j \xi_i[L_j] \right) \right. \\ &\quad \left. + g_{j-1} \left(\min \left\{ S_{j-1}, y - \sum_{i=1}^j \xi_i[\mathbb{M}_{T_i}(\bar{L}_{j+1} + r_j)] + \sum_{i=1}^{j-1} \xi_i[\mathbb{M}_{T_i}(\bar{L}_j + r_{j-1})] - D[L_j] - \sum_{i=1}^j \xi_i[L_j] \right\}, r \right) \right] \end{aligned}$$

Then, $G(\mathbf{s}, \mathbf{T}) = (1/T) \sum_{r=0}^{T-1} g_N(S_N, r)$ and $C(\mathbf{s}, \mathbf{T}) = \sum_{j=1}^N \frac{K_j}{T_j} + G(\mathbf{s}, \mathbf{T})$.

The recursion in the proposition enables us to easily evaluate the system cost for a given set of base-stock levels. In the next section, we use this recursion to generate a heuristic solution for the optimal base-stock levels and develop a lower bound on the optimal cost.

5. Lower Bound and Heuristic

The above bottom-up recursion is similar to that for the classic multi-echelon inventory problem (e.g., Chen and Zheng 1994, Chen 1999). However, unlike the classic problem, the optimal solution cannot be obtained by minimizing $(1/T) \sum_{r=0}^{T-1} g_j(S_j, r)$ recursively. The reason is as follows. In the first stage, each $g_1(y, r)$ function is convex, but we choose a single S_1 that minimizes $\sum_{r=0}^{T-1} g_1(S_1, r)$. In the second stage, $g_2(y, r)$ calls $g_1(\min\{S_1, y\}, r)$, and g_2 is no longer convex, because S_1 is not the minimizer of individual $g_1(y, r)$ functions. As a result, we cannot guarantee the convexity or unimodality of the functions $g_j(y, r)$ for $j \geq 2$. In §5.1, we derive a lower bound on the total cost per period for any local base-stock policy. In §5.2, we present a heuristic algorithm based on the recursion above. As we shall show in a numerical study, the heuristic solution is near-optimal and therefore can be used to draw insights about supply chains with inventory record inaccuracy.

5.1 Lower Bound

We establish a lower bound to the total cost under any local base-stock policy, i.e., a lower bound to the minimal cost that can be achieved. We compute the lower bound by replacing S_j in (19) with the minimizer of each $g_j(y, r)$ function, which yields convex cost functions at each recursion step. Define a series of \underline{g}_j functions recursively: Let $\underline{g}_1(y, r) = g_1(y, r)$ and

$$S_1(r) = \arg \min_y \underline{g}_1(y, r). \quad (20)$$

Define

$$\underline{g}_{1,1}(y, r) = \begin{cases} \underline{g}_1(y, r), & \text{if } y \leq S_1(r) \\ \underline{g}_1(S_1(r), r), & \text{otherwise.} \end{cases}$$

Note that $\underline{g}_{1,1}(y, r)$ function is decreasing and convex if $y \leq S_1(r)$ and is constant if $y > S_1(r)$. Also, $\underline{g}_{1,1}(y, r) = \underline{g}_1(\min\{S_1(r), y\}, r)$.

With these definitions, the inventory cost part of the total cost in (13) can be expressed as

$$\begin{aligned}
G(\mathbf{s}, \mathbf{T}) &= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\sum_{j=2}^N h_j AIL_j(t + \bar{L}_j + r_j) + \underline{g}_1 \left(\left(AIP_1^-(t + \bar{L}_2 + r_1) + \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)] \right), r \right) \right] \\
&\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\sum_{j=2}^N h_j AIL_j(t + \bar{L}_j + r_j) + \underline{g}_{1,1} \left(\left(AIP_1^-(t + \bar{L}_2 + r_1) + \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)] \right), r \right) \right] \\
&\geq \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\sum_{j=2}^N h_j AIL_j(t + \bar{L}_j + r_j) + \underline{g}_{1,1} \left(\left(AIL_2(t + \bar{L}_2 + r_2) + \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)] \right), r \right) \right] \\
&= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\sum_{j=2}^N h_j AIL_j(t + \bar{L}_j + r_j) \right. \\
&\quad \left. + \underline{g}_1 \left(\min \left\{ S_1(r), \left(AIL_2(t + \bar{L}_2 + r_2) + \xi_1[\mathbb{M}_{T_1}(\bar{L}_2 + r_1)] \right) \right\}, r \right) \right]. \tag{21}
\end{aligned}$$

The first equality follows from (16) and (18). The last inequality holds because $AIL_2(t + \bar{L}_2 + r_2) \geq AIP_1^-(t + \bar{L}_2 + r_1)$ and $\underline{g}_{1,1}$ is a decreasing function.

For $j = 2, \dots, N-1$, suppose that $S_{j-1}(r)$ is known. We substitute $S_{j-1}(r)$ for S_{j-1} in (19), and define the resulting function as $\underline{g}_j(y, r)$. In this case, because we are replacing $S_{j-1}(r)$, the minimizer of each individual function, in $g_{j-1}(\min\{S_{j-1}, y\}, r)$, the resulting $\underline{g}_j(y, r)$ functions are convex in y . Let

$$S_j(r) = \arg \min_y \underline{g}_j(y, r).$$

Define

$$\underline{g}_{j,j}(y, r) = \begin{cases} \underline{g}_j(y, r), & \text{if } y \leq S_j(r) \\ \underline{g}_j(S_j(r), r), & \text{otherwise.} \end{cases}$$

Also, $\underline{g}_{j,j}(y, r) = \underline{g}_j(\min\{S_j(r), y\}, r)$.

With these definitions, (21) can be continued as

$$\begin{aligned}
(21) &= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[\sum_{j=3}^N h_j AIL_j(t + \bar{L}_j + r_j) + \underline{g}_2 \left(\left(AIP_2^-(t + \bar{L}_3 + r_2) + \xi_2[\mathbb{M}_{T_2}(\bar{L}_3 + r_2)] \right), r \right) \right] \\
&\geq \dots \\
&= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_N(AIL_N(t + L_N + r_N)) \right. \\
&\quad \left. + \underline{g}_{N-1} \left(\min \left\{ S_{N-1}(r), \left(AIL_N(t + \bar{L}_N + r_N) + \sum_{i=1}^{N-1} \xi_i[\mathbb{M}_{T_i}(\bar{L}_N + r_{N-1})] \right) \right\}, r \right) \right] \\
&= \frac{1}{T} \sum_{r=0}^{T-1} \underline{g}_N(S_N, r).
\end{aligned}$$

Clearly, $(1/T) \sum_{r=0}^{T-1} \underline{g}_N(S_N, r)$ is convex. Define

$$S_N^* = \arg \min_{S_N} \sum_{r=0}^{T-1} \underline{g}_N(S_N, r).$$

Define $\underline{C}(\mathbf{T}) \stackrel{\text{def}}{=} \sum_{j=1}^N \frac{K_j}{T_j} + \frac{1}{T} \sum_{r=0}^{T-1} \underline{g}_N(S_N^*, r)$. This provides a lower bound to the optimal total supply chain cost given the class of static local base-stock policies, that is, $\underline{C}(\mathbf{T}) \leq C(\mathbf{s}, \mathbf{T})$ for any \mathbf{s} . This lower-bound cost function reduces to the one in Chen and Zheng (1994) when there are no errors in the system. Proposition 6 summarizes these results.

Proposition 6 *For fixed \mathbf{T} ,*

- (1) $\underline{g}_j(y, r)$ is convex in y , for $j = 1, \dots, N$.
- (2) $\underline{C}(\mathbf{T})$ is a lower bound on $C(\mathbf{s}, \mathbf{T})$ for any \mathbf{s} .

In summary, we can use the recursion defined in (18) and (19) to obtain the lower bound. For stage 1, set $\underline{g}_1(y, r) = g_1(y, r)$ and let $S_1(r) = \arg \min_y \underline{g}_1(y, r)$. For $j = 2, \dots, N-1$, suppose that $S_{j-1}(r)$ is known. We substitute $S_{j-1}(r)$ for S_{j-1} in (19), and define the resulting function as $\underline{g}_j(y, r)$ and let $S_j(r) = \arg \min_y \underline{g}_j(y, r)$. We compute $\underline{g}_N(y, r)$ function with $S_{N-1}(r)$ and set $S_N^* = \arg \min_y \frac{1}{T} \sum_{r=0}^{T-1} \underline{g}_N(y, r)$. The lower bound is given by $\underline{C}(\mathbf{T})$.

5.2 Heuristic

The lower bound cost is strictly less than the optimal total cost because we are not able to find a single S_j to represent $S_j(r)$ in each iteration during the construction of the lower bound. We present an algorithm that recursively solves (18) and (19) to obtain a heuristic solution \tilde{S}_j .

Specifically, let $\tilde{g}_1(y, r) = g_1(y, r)$, $\bar{g}_1(y) = \sum_{r=0}^{T-1} \tilde{g}_1(y, r)$, and

$$\tilde{S}_1 = \arg \min_y \bar{g}_1(y).$$

For $j = 2, \dots, N$, suppose that \tilde{S}_{j-1} is known. We substitute \tilde{S}_{j-1} for S_{j-1} in (19), and define the resulting function as $\tilde{g}_j(y, r)$. We define $\bar{g}_j(y) = \sum_{r=0}^{T-1} \tilde{g}_j(y, r)$ and find its first minimizer, \tilde{S}_j . Because $\bar{g}_j(y)$ is not necessarily convex, we find \tilde{S}_j via a line search. The heuristic solution is given by

$$\tilde{s}_1 = \tilde{S}_1, \quad \tilde{s}_j = \tilde{S}_j - \tilde{S}_{j-1}, \quad j = 2, \dots, N.$$

Define $C(\mathbf{T}) = C(\tilde{\mathbf{s}}, \mathbf{T})$, as the heuristic cost. Notice that the proposed heuristic generates the optimal base-stock level for stage 1 because $\bar{g}_1(y)$ is convex, and $\bar{g}_1(\tilde{s}_1) = \sum_r g_1(\tilde{s}_1, r)$ gives the minimum possible cost for stage 1. The heuristic also generates the exact optimal solution when $T_j = 1$ for $j < N$, because the resulting heuristic cost is equal to the lower bound cost.

In the numerical study section, we shall first assess the effectiveness of the heuristic. Then, we provide insights on how to design effective cycle-count policies by using the heuristic solution.

6. Impact of Error and Cycle Counts for Two-Stage Systems

This section analytically characterizes the impact of error and cycle counts at different locations in a two-stage system.

Theorem 1 *Consider a two-stage system with identical counting costs. Let A denote a system with downstream error only (i.e., $\mu_1 > 0, \mu_2 = 0$) and B a system with upstream error only, (i.e., $\mu_1 = 0, \mu_2 > 0$). Then, $C^A(T_1^A, T_2^A) > C^B(T_1^B, T_2^B)$ for any $T_1^A = T_2^B$ and $\mu_1 = \mu_2$.*

Theorem 1 compares two systems with upstream or downstream error only (with the same rate) and the same cycle-count length for the stage that has the error. The impact of downstream errors on the system cost is larger than that of upstream errors. To provide an explanation, let us assume $T_1 = T_2 = 1$ for both systems A and B. Suppose that an error of x units occurs at stage 1 in system A and at stage 2 in system B in some period t . Consider first system A. Since stage 1 conducts a cycle count at the end of period t , stage 1 will order the demand plus the error x units from stage 2 at the end period t . Thus, the error of stage 1 will be part of the local demand to stage 2, which will reduce stage 2's echelon inventory position in the same fashion as if stage 2 had an inventory shrinkage of x units in period t . One can confirm the above statement from equations (23) and (24): The first term in these equations represents the echelon 2 expected holding cost. They are the same for both systems. With this explanation, it is not difficult to see why a downstream error has a bigger impact – both systems have an error of x units at echelon 2 but system A has additional error at stage 1. As a result, system A has a higher system cost than system B.

Note that this result holds for any (L_1, L_2) and (h_1, h_2) . Clearly, the result also holds if $\mu_1 \geq \mu_2$, but it may not hold otherwise. In addition, this result holds for any $K_1 \leq K_2$, but may not hold otherwise.

The next theorem compares the impact of cycle counts at different locations in a system with identical errors at both stages.

Theorem 2 *For a two-stage system with identical error distribution and identical counting costs at both stages (i.e., $\mu_1 = \mu_2$ and $K_1 = K_2$), we have $C(1, T) < C(T, 1)$ for $T > 1$.*

Theorem 2 states that it is more effective to conduct more frequent cycle counts at a downstream stage for a two-stage system with identical error distributions. The reason is similar to that of Theorem 1: By conducting more frequent cycle counts at a downstream stage, one can mitigate the impact of downstream errors to upstream stages through inventory ordering. This result also holds for any (L_1, L_2) and (h_1, h_2) . Again, this result holds for any $K_1 \leq K_2$, but may not hold otherwise, depending on the scale of counting costs relative to the scale of inventory costs.

In fact, both theorems hold for the optimal base-stock levels, because the heuristic yields the optimal base-stock level for stage 1, and the proofs show the result for any echelon inventory level y in the stage 2 cost function.

These results essentially argue from two different perspectives that, all else being equal, more attention should be paid to the downstream location: the first result is on the impact of eliminating errors at different locations and the second result is on the role of cycle counts providing accurate information about the error terms from different locations. While the results tell us that downstream errors are more costly and downstream cycle counts are more beneficial to the system, designing an effective cycle-count policy requires an understanding of how the relative impact changes with supply chain properties such as the lead times and the supply chain length. We conduct a numerical study to answer these questions in §7.

7. Numerical Study

This section provides a numerical study to investigate the impact of inventory inaccuracy on the supply chain performance. Our goal is to provide guidelines for designing effective cycle-count policies. We are interested in the following questions. Which locations should conduct more frequent cycle counts? What are the properties of effective cycle-count policies? How do these decisions change with the structure and parameters of the system?

We use our heuristic base-stock policy to answer these questions through a sensitivity analysis study. We focus our discussion on both two-stage and four-stage systems. The parameters we use in the sensitivity analysis are as follows: For the two-stage system, the parameters are

$$\begin{aligned} \lambda &= 20, \quad (h_1, h_2) \in \{(1, 3), (2, 2), (3, 1)\}, \quad (L_1, L_2) \in \{(1, 5), (3, 3), (5, 1)\}, \\ (\mu_1, \mu_2) &\in \{(1, 0), (0, 1), (1, 1), (2, 2)\}, \quad K_j \in \{2, 6, 10, 14, 18, 22, 26, 30\} \quad \text{for all } j. \end{aligned}$$

These parameters are chosen to represent various supply chain configurations. For example, the lead time vectors represent cases where the share of the downstream lead time to total lead time varies from low to high; the error rates represent the cases of downstream error only, upstream error only, identical errors, and larger identical errors, respectively. In addition, we consider three service levels $\alpha = \{0.8, 0.9, 0.95\}$, and set $\hat{b} = (h_1 + h_2)\alpha/(1 - \alpha)$. Finally, we consider the following 16 cycle-count policies:

$$(T_1, T_2) \in \left\{ \begin{array}{l} (1, 1), (1, 3), (3, 1), (1, 6), (6, 1), (1, 12), (12, 1), (2, 2), \\ (2, 4), (3, 3), (2, 12), (4, 4), (3, 12), (6, 6), (6, 12), (12, 12) \end{array} \right\}.$$

The largest cycle-count interval, 12, corresponds to annual inventory counts if the inventory review period is a month.

For the four-stage system, the parameters are

$$\begin{aligned} \lambda &= 20, \quad \mathbf{h} \in \{(1, 1, 3, 3), (2, 2, 2, 2), (3, 3, 1, 1)\}, \quad \mathbf{L} \in \{(1, 1, 5, 5), (3, 3, 3, 3), (5, 5, 1, 1)\}, \\ \boldsymbol{\mu} &\in \{(1, 1, 1, 1), (1, 1, 2, 1)\}, \quad \mathbf{K} = (10, 10, 10, 10). \end{aligned}$$

In addition, we set $\hat{b} = 72$, representing 90% service level. For this test, we consider 36 cycle-count policies listed in Table 4 ranging from (1, 1, 1, 1) to (12, 12, 12, 12).

We first verify the effectiveness of our heuristic solution under the above chosen parameters. We define the percentage difference between the heuristic cost and the lower bound cost as

$$\frac{C(\tilde{\mathbf{s}}, \mathbf{T}) - \underline{C}(\mathbf{T})}{\underline{C}(\mathbf{T})} \times 100\%.$$

In this study, we find that the average difference is 0.22% and 0.65% for the two-stage systems with error rates (1, 1) and with error rates (2, 2), respectively. The average difference is 0.29% for four-stage systems. We conclude that our heuristic base-stock levels are near optimal. We therefore use it to answer the above research questions.

Sensitivity Analysis and Observations

The purpose of the sensitivity analysis study is to examine the impact of varying one specific parameter on the total cost. Therefore, we need to select a base case as a reference point. We select the following two-stage system as our base case: $\boldsymbol{\mu} = (1, 1)$, $\mathbf{L} = (3, 3)$, $\mathbf{h} = (2, 2)$, $\mathbf{K} = (10, 10)$, and $\hat{b} = 36$. Similarly, we choose the following base case for the four-stage system: $\boldsymbol{\mu} = (1, 1, 1, 1)$, $\mathbf{L} = (3, 3, 3, 3)$, $\mathbf{h} = (2, 2, 2, 2)$, $\mathbf{K} = (10, 10, 10, 10)$, and $\hat{b} = 72$. We measure the impact of cycle-count policies by computing the percentage reduction in the optimal cost relative to the worst case, $C(12, \dots, 12)$, which has minimal correction of errors. Define

$$\Delta(\mathbf{T}) = \frac{C(12, \dots, 12) - C(\mathbf{T})}{C(12, \dots, 12)} \times 100\%.$$

For simplicity, we use $\mathbf{T}' \succ \mathbf{T}''$ to denote that a cycle-count policy \mathbf{T}' achieves more percentage cost reduction than policy \mathbf{T}'' . Clearly, different parameter sets may lead to larger or smaller magnitude of the effects. However, we are interested more in the insights obtained by observing the direction of the change in the total cost in response to the system parameters.

Observations

We summarize main observations from the numerical study below. Notice that a curve in Figures 2-7 is obtained by starting with a base case and changing only the variable specified in the horizontal axis. For example, in Figure 2, the “diamond” shape curve shows the cost reduction for the four-stage base case under different cycle-count policies; the “square” shape curve shows the same when we increase the error rate at stage 3 in the same four-stage case.

1. Figure 2 shows the impact of conducting cycle counts on different locations in four-stage systems. We find that the result in Theorem 2 holds for longer supply chains as well: With identical error rates and counting costs, a downstream cycle count is more effective than an upstream cycle count. A cycle count at an upstream stage could be more beneficial to the system only when the error rate at that location is significantly higher than the downstream locations.

2. Figure 3 shows the benefit of conducting cycle counts under various lead time ratios for the two-stage base case. We particularly consider two cycle-count policies: $(T_1, T_2) = (1, 12)$ and $(12, 1)$. The former (the latter) represents conducting more frequent counts at stage 1 (stage 2). From the top curve, we observe that the benefit of cycle counts at the downstream location decreases as the share of the downstream lead time increases. Since a shorter lead time implies less stock, this result suggests that a leaner downstream stage is more vulnerable to the error. There is a similar lesson in the lean production systems philosophy: By carrying less inventory, one makes deviations from the ideal more costly, but also provides strong incentives for process improvement.

Comparing these two curves, we find that the benefit of downstream cycle count is always larger than that of upstream cycle count. This is consistent with Theorem 2. Nevertheless, this benefit is more prominent when the downstream stage has a shorter lead time share to the system lead time.

3. Figures 4(a) and 4(b) illustrate the two conclusions observed above for the four-stage base system.

4. The benefit of cycle counts at a location increases if the ratio of the holding cost to the total holding cost increases. This can be observed from Tables 3 and 4 in the appendix. Intuitively, when the holding cost is larger, the stage will carry less inventory, making the impact of errors more significant.

5. Figure 5 illustrates the impact of cycle counts on the number of stages for both $N = 2$ and $N = 4$ cases. To make a fair comparison, we change the lead time parameters of the two-stage system by setting $\mathbf{L} = (6, 6)$ and $\mathbf{h} = (4, 4)$. This ensures that total lead time and total holding cost of two-stage system are the same as the four-stage system. We find that the impact of conducting cycle count at the most upstream stage relative to the impact of conducting cycle counts at stage 1 for $N = 4$ (i.e., 0.1% to 1.2%) is smaller than that for $N = 2$ (i.e., 2.1% to 3.5%). This observation suggests that cycle counts should be focused on both stage 1 and stage 2 (on stage 1) when $N = 4$ ($N = 2$). Thus, a rough guideline based on this conclusion for systems with similar error rates and counting costs is that cycle counts should be focused more on the downstream half of the supply chain.

6. Figure 6 shows the marginal benefit of assigning more than one cycle count for the base example. Starting from $\mathbf{T} = (12, 12)$, we change T_1 (T_2) only to increase the downstream (up-

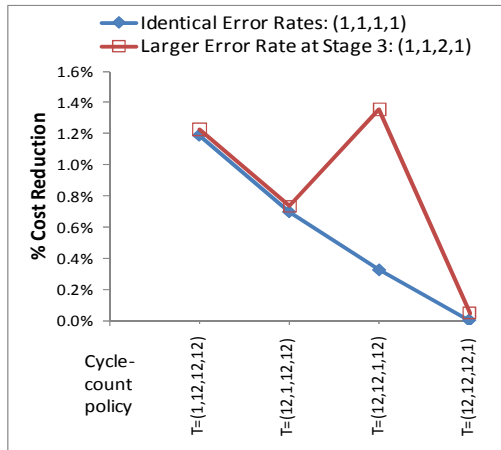


Figure 2

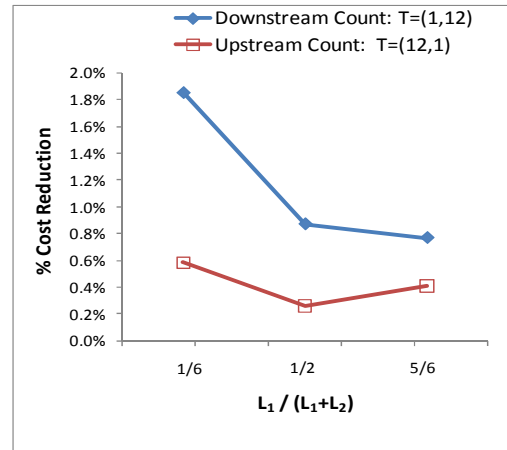


Figure 3

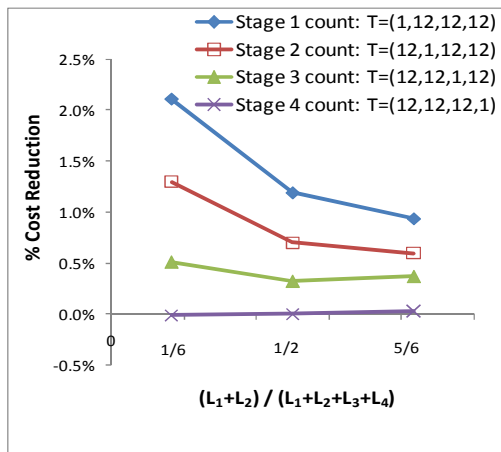


Figure 4(a)

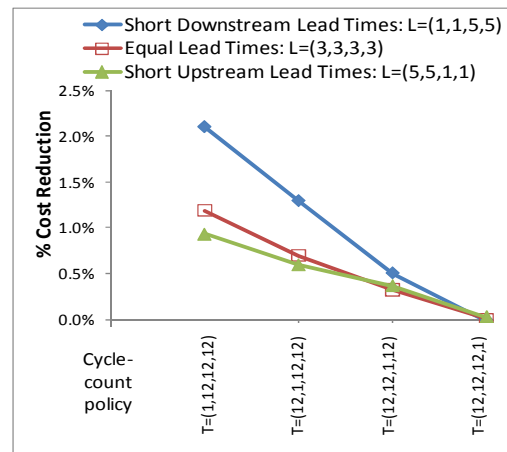


Figure 4(b)

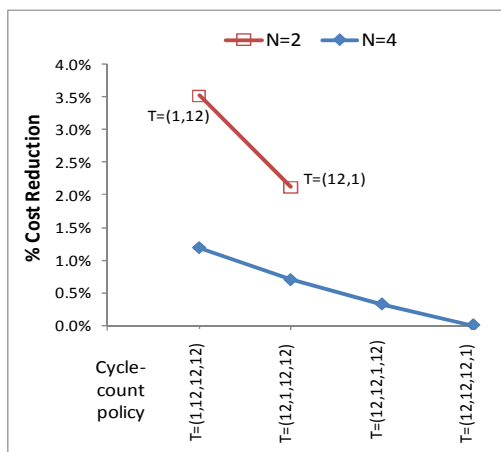


Figure 5

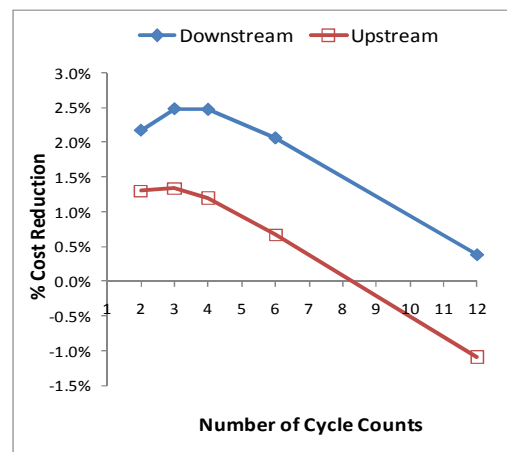


Figure 6

stream) counts. As shown, both upstream and downstream counts exhibit diminishing rates of return. More importantly, the first additional upstream cycle count may be more beneficial to the system than the third additional downstream count. This result immediately suggests that it is not a good strategy to allocate all cycle counts to a single location. Instead, cycle counts should be allocated across locations in a dynamic way.

7. Based on observation 6, an effective cycle-count policy must assign the first few cycle counts downstream but then keep a reasonably balanced allocation of the total cycle counts available. Figure 7(a) shows that $(2, 2) \succ (1, 12)$, where the former has 12 counts and the latter has 13 counts per year. (Note that this comparison holds even when $K_1 = K_2 = 0$, indicating that the difference is mostly due to the inventory costs.) Hence, a balanced policy is usually better than a policy that heavily favors downstream locations. Nevertheless, we know that as the downstream lead time share decreases downstream counts have a larger impact. Figure 7(b) compares policies $(4, 12)$ and $(6, 6)$, both with four counts per year. While $(6, 6) \succ (4, 12)$ when $L_1/(L_1 + L_2) \geq 1/2$, we have $(4, 12) \succ (6, 6)$ as that ratio decreases. Hence, the balance should be tilted slightly towards downstream locations in supply chains with short downstream lead times.

8. The guidelines discussed in observation 7 carry over to longer supply chains. For example, policy $(3, 3, 6, 6)$ always outperforms $(3, 6, 3, 6)$, except when the error rate at the third stage is doubled, and $(6, 6, 3, 3)$ is dominated by both. In general, more balanced cycle-count policies perform better than those that focus only on downstream locations. For example, $(3, 3, 6, 6) \succ (2, 2, 12, 12) \succ (1, 12, 12, 12)$, which have twelve, fourteen, and fifteen cycle counts per year, respectively. (Again, these comparisons hold even when $K_j = 0$ for all j .) Similarly, $(1, 2, 3, 6) \succ (1, 1, 6, 6)$, which have 24 and 28 counts per year, respectively. Note that in a supply chain with a larger number of stages, having a slight preference towards downstream locations when assigning cycle counts outperforms perfectly balanced policies. For example, considering the following policies, both with 24 counts per year, we have $(1, 2, 3, 6) \succ (2, 2, 2, 2)$, as also indicated by observation 5. This is because, cycle counts at the fourth stage do not lead to significant cost reduction. Hence, we shall modify our guidelines to account for this observation: In a longer supply chain, the few downstream stages should be assigned more cycle counts (in a balanced way) than the upstream stages.

9. The above observations are drawn from the base cases, in which each stage has identical counting cost. Clearly, stage counting costs directly affect cycle-count decisions. While observation 1 suggests that downstream counts are more beneficial to the system when counting costs are identical, the result will be reversed if downstream counting costs are significantly higher than upstream counting costs. Table 1 illustrates the optimal cycle-count policies for various counting cost combinations in a two-stage system. The optimal cycle-count policy is obtained by searching

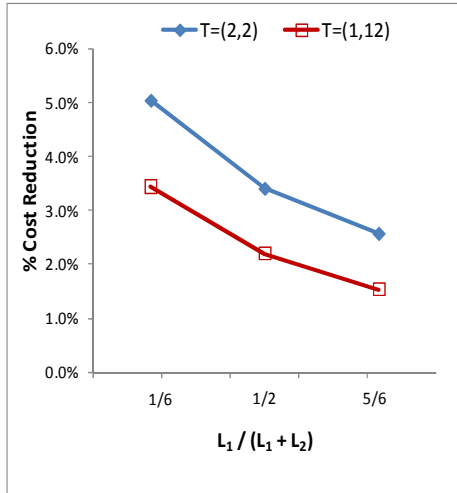


Figure 7(a)

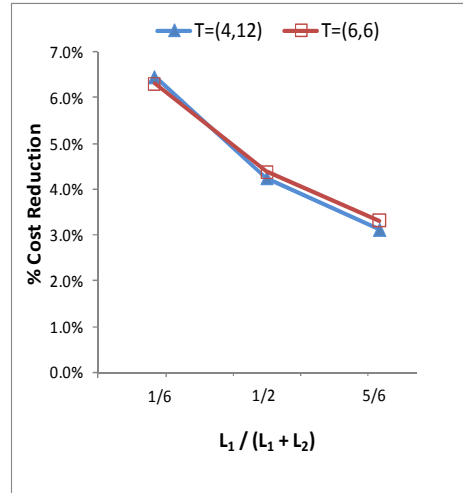


Figure 7(b)

Stage	1	...	i	...	N
Location	$3/2 - 1/N$		$3/2 - i/N$		$1/2$
Error Rate	μ_1 / λ		μ_i / λ		μ_N / λ
Lead time share	$1 - L_1 / \sum_j L_j$		$1 - L_i / \sum_j L_j$		$1 - L_N / \sum_j L_j$
Holding cost	$h_1 / \sum_j h_j$		$h_i / \sum_j h_j$		$h_N / \sum_j h_j$
Counting cost	k_1		k_i		k_N

Figure 8: Scorecard for allocating cycle counts in a supply chain for a single product

over all 36 feasible policies with $T_j \in \{1, 2, 3, 4, 6, 12\}$. For example, as shown in the table, the best cycle-count policy for the base case example with $K_1 = K_2 = 10$ is $T = (4, 6)$. As K_1 increases, downstream cycle-count frequency decreases. To compensate for that, the upstream cycle-count frequency may increase. Similarly, as K_2 increases, upstream cycle-count frequency decreases and downstream cycle-count frequency generally increases. Note that it is optimal to count more frequently at the upstream stage rather than the downstream stage only when K_2 is significantly lower than K_1 . In most cases, even when $K_1/K_2 > 2$, it is optimal to count more frequently downstream, indicating that location in the supply chain should play a very significant role in counting decisions. In addition, the optimal cycle-count policy is $(4, 6)$ for the base case and a wide range of counting costs around that, indicating that optimal cycle-count policy is not very sensitive to the counting cost structure.

		K_2							
		2	6	10	14	18	22	26	30
K_1	2	2,3	2,6	2,6	2,6	2,6	2,6	2,12	2,12
	6	3,4	3,4	4,6	4,6	4,6	4,6	3,12	3,12
	10	4,3	4,6	4,6	4,6	4,6	4,6	4,6	4,12
	14	4,3	4,6	4,6	4,6	4,6	4,6	4,6	4,12
	18	6,4	6,4	4,6	4,6	4,6	4,6	4,6	4,12
	22	6,4	6,4	6,4	4,6	4,6	4,6	4,6	6,12
	26	6,4	6,4	6,4	6,4	6,6	6,6	6,12	6,12
	30	6,4	6,4	6,4	6,4	6,6	6,6	6,12	6,12

Table 1: Optimal cycle-count policy for a two-stage system with base case parameters.

Table 2 presents the best cycle-count policy for four-stage systems with various counting cost combinations. The best policy is chosen among 81 policies with $T_j \in \{1, 3, 6\}$ for $j = 1, \dots, 4$. The results are similar to the two-stage systems. Reducing the fixed cost at a stage results in more frequent cycle counts, and this effect is less significant at an upstream stage. Overall, we can see that the policies presented in Tables 1 and 2 are consistent with observations 1-8 above: Downstream counts reduce costs more than upstream counts (unless an upstream stage has a significantly higher error rate or lower counting cost), balanced cycle-count policies outperform unbalanced ones, and the right degree of balance depends on the system parameters such as the lead time and holding cost structure. For example, for $\mathbf{K} = (16, 12, 8, 4)$, even though $K_1 = 2K_3$, the solution keeps a balanced allocation between first three stages. However, in the case with shorter upstream lead times or higher upstream holding costs, stage 4 counts also become attractive and the solution is a perfectly balanced cycle-count policy.

K_1	K_2	K_3	K_4	Base case	$\mathbf{L} = (1, 1, 5, 5)$	$\mathbf{L} = (5, 5, 1, 1)$	$\mathbf{h} = (1, 1, 3, 3)$	$\boldsymbol{\mu} = (1, 1, 2, 1)$
5	5	5	5	3,3,3,6	1,3,3,6	3,3,3,6	1,3,3,3	1,3,3,6
10	10	10	10	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6
15	15	15	15	3,3,6,6	3,3,6,6	3,3,6,6	3,3,3,6	3,3,3,6
20	20	20	20	3,3,6,6	3,3,6,6	3,3,6,6	3,3,6,6	3,6,3,6
12	11	10	9	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6
16	12	8	4	3,3,3,6	3,3,3,6	3,3,3,3	3,3,3,3	3,3,3,6
32	16	8	4	6,3,3,3	6,3,3,3	6,3,3,3	6,3,3,3	6,3,3,3
10	10	5	5	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6
20	20	10	10	3,3,3,6	3,3,3,6	3,6,3,6	3,3,3,6	3,6,3,6
9	10	11	12	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6	3,3,3,6
4	8	12	16	1,3,3,6	1,3,3,6	1,3,3,6	1,3,3,6	1,3,3,6

Table 2: Optimal cycle-count policy for four-stage systems.

8. Implications

Based on the observations in §7, we categorize the factors that affect cycle-count decisions into two groups. Primary factors determine which stages should get priority (i.e., higher cycle-count frequency). These factors are (i) stage (position in the supply chain) (ii) error rate, and (iii) counting cost. Secondary factors determine whether the policy should strongly favor the high-priority stages or allocate counts in a somewhat balanced way. These are (i) lead time structure, (ii) holding cost structure, and (iii) the length of the supply chain.

These two groups of factors can be used to provide the following guidelines for designing effective cycle-count policies. When allocating a fixed budget or a fixed number of cycle counts in a supply chain, a manager should first consider primary factors by allocating first few cycle counts to downstream locations and locations with significantly higher error rate or low counting cost. Then, the manager should consider the secondary factors to allocate the remaining cycle counts. For example, if the ratio of the lead time of a downstream stage to the total system lead time is large, the manager should allocate the remaining cycle counts to keep a more balanced distribution rather than focusing on the downstream stage. On the other hand, if the supply chain is long, the manager should allocate the remaining counts to favor the few downstream stages.

Our finding also provides new insights that could be useful for companies to design a system-wide cycle-count program. As stated, most companies implement ABC classification scheme to assign a cycle-count frequency to each product. The current criteria used in the scheme focus on product characteristics, such as volume, product value, and error rate (Jordan 1994). From our study, we find that managers should take into account supply-chain characteristics for a product in order to design an efficient system-wide cycle-count program. More specifically, in addition to the existing product characteristics, a more comprehensive ABC classification scheme should include

location (position in the supply chain), counting costs, lead time share of each stage, and holding cost share of each stage. With these criteria, managers may be able to design a scorecard to determine cycle-count frequencies for products. Table 8 is an example for such a scorecard. For the “location” factor, one could choose a linear function that starts at a value larger than 1 at stage 1 and decreases to $1/2$ stage N . This way, the relative importance of the last stage (second stage) decreases (increases) with the supply chain length. Similar explanations based on the observations in §7 can be applied to the other factors.

Finally, our model and analysis also provide implications on RFID investments from the perspective of reducing inventory inaccuracy. Consider the following settings with RFID. (i) In some settings with tagged products, RFID provides perpetual information in locations where readers are installed. This scenario can be evaluated with our framework. For example, for a production line at a factory with a six-stage process, a reader in stages 3 and 5 is equivalent to a cycle-count policy (12, 12, 1, 12, 1, 12). (ii) In some other settings, the cost of readers may be small relative to the cost of RFID tags. In that case, the problem is to decide the at which stage the tags should be put on the products. For example, if stage 3 and its downstream stages have tagged products, this is equivalent to policy (1, 1, 1, 12, 12, 12). (iii) In the apparel industry, it is common to use hand-held RFID readers to do cycle-counts. In this case, the store staff performs cycle counts using the handheld readers near shelves or hangers. These cycle counts are faster (less costly) and possibly more accurate than manual counts. Our results on cycle-count policies are directly relevant for this setting. RFID investment determines which stages have relatively low counting costs and therefore count more often. In any of these scenarios, a supply chain manager needs to do a cost-benefit analysis for the RFID investment. It has been suggested in the literature that from the perspective of labor cost savings, upstream stages (back-end operations) should be RFID-enabled first (Chopra and Sodhi 2007). Our results suggest that, from the perspective of record inaccuracy, all else being equal, downstream stages should get priority.

9. Conclusion

Inventory record inaccuracy is a prevalent issue that affects supply chain performance. While common mitigation approaches such as conducting cycle counts and installing inventory tracking systems have been adopted by many firms, to our surprise, there are no clear guidelines in the literature on how to design cycle-count policies from the perspective of the entire supply chain, i.e., where to conduct counts more frequently, what attributes to consider when designing a cycle-count policy for multiple locations, and how to prioritize RFID investments across a supply chain.

We consider a serial supply chain in which each location implements a cycle-count policy to mitigate the impact of inventory inaccuracy. Specifically, we model record inaccuracy as random

errors that reduce the physical inventory in every period at each stage and that are unobserved by the information system until a cycle count is performed. We characterize the local and echelon inventory variables and develop an evaluation scheme for any given cycle count and base-stock policies. We propose a simple and effective heuristic for finding the base-stock levels. This evaluation scheme and the heuristic enable us to examine the system performance under various cycle-count policies.

For a two-stage system, we prove that it is more costly to have the same error occur at the downstream stage than at the upstream stage. We also prove that with identical errors and counting costs at both stages, it is always more effective to conduct more frequent cycle counts at the downstream stage. In a numerical study, we find that the above conclusion holds for more general systems unless an upstream stage has a significantly higher error rate or lower counting cost. In particular, more cycle counts should be allocated to a downstream stage if the proportion of its lead time to the system lead time is small, the proportion of its holding cost to the total holding cost is large, or the number of stages in the supply chain is large. However, because the marginal benefit of cycle counts is decreasing in its frequency, it is not a good strategy to assign all the cycle counts to the downstream stages. We propose policy design guidelines and a scorecard that may be used in ABC classification schemes for inventory accuracy improvement programs. Based on our analytical and numerical results, we conclude that, in addition to the existing attributes such as volume, error rates, value and counting costs suggested by practitioners, companies should consider the following attributes when designing system-wide cycle-count programs: location in the supply chain, lead times, holding costs, and the number of stages in the supply chain. Finally, the insights and guidelines about cycle counts above are applicable for evaluating the benefits of RFID from the perspective of reducing record inaccuracy.

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Appendix: Proofs

Proposition 3

Let Δ denote the difference of a function, i.e., $\Delta f(y) = f(y+1) - f(y)$. We only show $T = 2$ case as the general T case can be proven similarly. Note that

$$\begin{aligned} C(y, 2) &= \frac{1}{2} \left(g(y, 0) + g(y, 1) \right), \\ C(y, 1) &= g(y, 0) = \frac{1}{2} \left(g(y, 0) + g(y, 0) \right). \end{aligned}$$

To show part (1), we only need to show $\Delta g(y, 0) + \Delta g(y, 0) \geq \Delta g(y, 0) + \Delta g(y, 1)$, or equivalently, $\Delta g(y, 0) \geq \Delta g(y, 1)$ for all y .

$$\begin{aligned} \Delta g(y, 0) &= h_1 - (\hat{b} + h'_1) \mathbf{P}(D[L] + \xi[L] > y), \\ \Delta g(y, 1) &= h_1 - (\hat{b} + h'_1) \mathbf{P}(D[L] + \xi[L] + \xi[1] > y). \end{aligned}$$

Because $D[L] + \xi[L] + \xi[1] \geq_{st} D[L] + \xi[L]$, we have $\mathbf{P}(D[L] + \xi[L] > y) \leq \mathbf{P}(D[L] + \xi[L] + \xi[1] > y)$, for all y . Thus, the inequality $\Delta g(y, 0) \geq \Delta g(y, 1)$ holds.

For part (2),

$$\begin{aligned} g(y, 1) &= \mathbf{E}[h(y - D[L] - \xi[L] - \xi[1])^+ + \hat{b}(y - D[L] - \xi[L] - \xi[1])^-] \\ &\geq \mathbf{E}[h(y - D[L] - \xi[L] - \mathbf{E}[\xi[1]])^+ + \hat{b}(y - D[L] - \xi[L] - \mathbf{E}[\xi[1]])^-] \equiv g'(y) \end{aligned}$$

for all y due to Jensen's inequality. Let s' be the minimizer of $g'(y)$. It is easy to see that $g'(s') = g(s(1), 0)$. Thus, we have

$$\begin{aligned} C(2) &= (1/2)(g(s(2), 1) + g(s(2), 0)) \geq (1/2)(g'(s(2)) + g(s(2), 0)) \geq (1/2)(g'(s') + g(s(2), 0)) \\ &\geq (1/2)(g(s(1), 0) + g(s(1), 0)) = C(1). \end{aligned}$$

Proposition 4

We first show (14). By definition, $AIP_j(t) = AIP'_j(t) + AIOP_{j-1}(t)$. Adding $AIOP_{j-1}(t + \bar{L}_{j+1} + r_{j+1})$ on both sides of (6) yields

$$\begin{aligned} AIP_j(t + \bar{L}_{j+1} + r_{j+1}) &= AIOP'_j(t + \bar{L}_{j+1} + r_{j+1}) + \min\{0, AIL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1})\} \\ &\quad + AIOP_{j-1}(t + \bar{L}_{j+1} + r_{j+1}) \\ &= AIOP_j(t + \bar{L}_{j+1} + r_{j+1}) + \min\{0, AIL'_{j+1}(t + \bar{L}_{j+1} + r_{j+1})\} \\ &= \min\{AIOP_j(t + \bar{L}_{j+1} + r_{j+1}), AIL_{j+1}(t + \bar{L}_{j+1} + r_{j+1})\}. \end{aligned}$$

Equation (14) follows immediately.

We next show (16). Equation (15) can be proved similarly.

For notational simplicity, let $t_{j+1} = t + \bar{L}_{j+1} + r_j$ and $t_j = t + \bar{L}_j + r_j$. By definition, $AiL_j(t) = AiL'_j(t) + \sum_{i=1}^{j-1} AiOP'_i(t)$. Thus,

$$\begin{aligned}
AiL_j(t_j) &= AiL'_j(t_j) + \sum_{i=1}^{j-1} AiOP'_i(t_j) \\
&= AIP'_j{}^-(t_{j+1}) - D_j[t_{j+1}, t_j] - \xi_j[t_{j+1}, t_j] \\
&\quad + \sum_{i=1}^{j-1} \left(AiOP'_i(t_{j+1}) - D_i[t_{j+1}, t_j] + D_{i+1}[t_{j+1}, t_j] - \xi_i[t_{j+1}, t_j] \right) \\
&= AIP'_j{}^-(t_{j+1}) + \sum_{i=1}^{j-1} AiOP'_i{}^-(t_{j+1}) - D[t_{j+1}, t_j] - \sum_{i=1}^j \xi_i[t_{j+1}, t_j] \\
&= AIP_j^-(t_{j+1}) - D[t_{j+1}, t_j] - \sum_{i=1}^j \xi_i[t_{j+1}, t_j].
\end{aligned}$$

Proposition 5

We only prove the $N = 2$ case. The general N case can be proved similarly. When $N = 2$, $\bar{L}_3 = 0, \bar{L}_2 = L_2$ and $\bar{L}_1 = L_1 + L_2$. The total cost per period in (13) becomes

$$\begin{aligned}
&\frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_2 AiL_2(t + \bar{L}_2 + r_2) + h_1 AiL_1(t + \bar{L}_1 + r_1) + (\hat{b} + h'_1)(AiL_1(t + \bar{L}_1 + r_1))^- \right] \\
&= \frac{1}{T} \sum_{r=0}^{T-1} \mathbb{E} \left[h_2 \left(AIP_2^- (t + r_2) - D[t + r_2, t + L_2 + r_2] - \sum_{i=1}^2 \xi_i[t + r_2, t + L_2 + r_2] \right) \right. \\
&\quad \left. + h_1 \left(AIP_1^- (t + L_2 + r_1) - D[t + L_2 + r_1, t + \bar{L}_1 + r_1] - \xi_1[t + L_2 + r_1, t + \bar{L}_1 + r_1] \right) \right. \quad (22) \\
&\quad \left. + (\hat{b} + h'_1) \left(AIP_1^- (t + L_2 + r_1) - D[t + L_2 + r_1, t + \bar{L}_1 + r_1] - \xi_1[t + L_2 + r_1, t + \bar{L}_1 + r_1] \right)^- \right],
\end{aligned}$$

where

$$AIP_1^-(t + L_2 + r_1) = \min\{AIP_1^-(t + L_2 + r_1), AiL_2(t + L_2 + r_2)\}.$$

Equation (22) is due to Proposition 4.

Note that from (17) and Proposition 4,

$$\begin{aligned}
AIP_2^-(t + r_2) &= S_2 - \sum_{i=1}^2 \xi_i \left[t + \left\lfloor \frac{r_2}{T_i} \right\rfloor T_i, t + r_2 \right) = S_2 - \sum_{i=1}^2 \xi_i \left[\mathbb{M}_{T_i}(r_2) \right), \\
AIP_1^-(t + L_2 + r_1) &= s_1 - \xi_1 \left[\mathbb{M}_{T_1}(L_2 + r_1) \right), \\
AiL_2(t + L_2 + r_2) &= S_2 - \sum_{i=1}^2 \xi_i \left[\mathbb{M}_{T_i}(r_2) \right) - D[L_2] - \sum_{i=1}^2 \xi_i[L_2].
\end{aligned}$$

We can substitute these values into (22) and obtain the total cost as a function system parameters. It can be verified that the recursion yields the same total cost.

Theorem 1

Because there is no error upstream (downstream) in system A (B), T_2^A (T_1^B) does not affect the supply chain cost. For simplicity, we only prove the result for $T_1^A = T_2^A = T_1^B = T_2^B = 1$. The proof for the general case follows the same logic.

Define $(a \wedge b) = \min\{a, b\}$, $D_1^A = D[L_1] + \xi_1[L_1]$, $D_1^B = D[L_1]$, $D_2^A = D[L_2] + \xi_1[L_2]$, and $D_2^B = D[L_2] + \xi_2[L_2]$.

For system A, $\mu_2 = 0$ and $\xi_1[M_{T_1}(t)] = 0$ for any t , $\bar{L}_2 = L_2$, $\bar{L}_3 = 0$, $r_2 = r = 0$, $r_1 = r + 1 = 1$. We have

$$\begin{aligned} g_1^A(y, 0) &= \mathbb{E}[h_1(y - \xi_1[M_{T_2}(L_2 + 1)) - D_1^A) \\ &\quad + (b + h_1')(y - \xi_1[M_{T_2}(L_2 + 1)) - D_1^A)^-] \\ &= \mathbb{E}[h_1(y - D_1^A) + (b + h_1')(y - D_1^A)^-]. \end{aligned}$$

Let

$$s_1^A = \arg \min_y \{g_1^A(y, 0)\}.$$

Thus,

$$\begin{aligned} g_2^A(y, 0) &= \mathbb{E}[h_2(y - \xi_1[M_{T_2}(0 + 1)) - D_2^A) \\ &\quad + g_1^A((s_1^A \wedge (y - \xi_1[M_{T_2}(0 + 1)) - D_2^A], 0)] \\ &= \mathbb{E}[h_2(y - D_2^A) + g_1^A(s_1^A \wedge y - D_2^A)]. \end{aligned} \tag{23}$$

For system B, $\mu_1 = 0$ and $\xi_2[M_{T_1}(t)] = 0$ for any t . We have

$$g_1^B(y, 0) = \mathbb{E}[h_1(y - D_1^B) + (b + h_1')(y - D_1^B)^-].$$

Let

$$s_1^B = \arg \min_y \{g_1^B(y, 0)\}.$$

Thus,

$$\begin{aligned} g_2^B(y, 0) &= \mathbb{E}[h_2(y - \xi_2[M_{T_2}(0)]) - D_2^B) \\ &\quad + g_1^B((s_1^B \wedge (y - \xi_2[M_{T_2}(0)]) - D_2^B), 0)] \\ &= \mathbb{E}[h_2(y - D_2^B) + g_1^B(s_1^B \wedge y - D_2^B)] \end{aligned} \tag{24}$$

To show $C^A(T_1, T_2) = C^A(1, 1) > C^B(T_1, T_2) = C^B(1, 1)$, it is equivalent to show

$$g_2^A(y, 0) > g_2^B(y, 0), \quad \forall y.$$

Or equivalently,

$$\mathbb{E}[h_2(y - D_2^A) + g_1^A(s_1^A \wedge y - D_2^A)] > \mathbb{E}[h_2(y - D_2^B) + g_1^B(s_1^B \wedge y - D_2^B)].$$

Because D_2^A and D_2^B follow the same distribution, the first terms on both sides cancel each other and we can compare the second terms pointwise. Define $\hat{y} = y - D_2^A$ for some $D_2^A = D_2^B$. Hence, we need to show the following.

$$g_1^A(s_1^A \wedge \hat{y}, 0) > g_1^B(s_1^B \wedge \hat{y}, 0)$$

The inequality holds, because D_1^A is stochastically larger than D_1^B , which implies that $s_1^A > s_1^B$ and $g_1^A(s_1^A, 0) > g_1^B(s_1^B, 0)$.

Theorem 2

For simplicity, we only prove the result for $T = 2$. The general T case can be proven in a similar manner. Define $(a \wedge b) = \min\{a, b\}$, $D_1^1 = D[L_1] + \xi_1[L_1]$, and $D_2^{1,2} = D[L_2] + \xi_1[L_2] + \xi_2[L_2]$. We use superscripts ‘‘A’’ and ‘‘B’’ to represent the cost functions and the optimal base-stock levels associated with $C(1, 2)$ and $C(2, 1)$, respectively. Define

$$\begin{aligned} s_1^A &= \arg \min_y \{g_1^A(y, 0) + g_1^A(y, 1)\}, \\ s_1^B &= \arg \min_y \{g_1^B(y, 0) + g_1^B(y, 1)\}. \end{aligned}$$

Showing $C(2, 1) > C(1, 2)$ is equivalent to showing

$$g_2^B(y, 0) + g_2^B(y, 1) \geq g_2^A(y, 0) + g_2^A(y, 1), \quad \forall y. \quad (25)$$

For A, $\xi_1[M_{T_1}(t)] = 0$ for any t , $\bar{L}_2 = L_2$, $\bar{L}_3 = 0$, $r_2 = r$, $r_1 = r + 1$. We have

$$\begin{aligned} g_1^A(y, 0) &= \mathbb{E}[h_1(y - D_1^1) + (b + h_1')(y - D_1^1)^-], \\ g_1^A(y, 1) &= \mathbb{E}[h_1(y - D_1^1) + (b + h_1')(y - D_1^1)^-], \\ g_2^A(y, 0) &= \mathbb{E}[h_2(y - D_2^{12} - \xi_2[M_{T_2}(L_3 + r)]) + g_1^A((s_1^A \wedge (y - D_2^{12} - \xi_2[M_{T_2}(L_3 + r)])), 0)] \\ &= \mathbb{E}[h_2(y - D_2^{12}) + g_1^A((s_1^A \wedge (y - D_2^{12})), 0)], \\ g_2^A(y, 1) &= \mathbb{E}[h_2(y - D_2^{12} - \xi_2[1]) + g_1^A((s_1^A \wedge (y - D_2^{12} - \xi_2[1])), 1)], \end{aligned}$$

For B, $T_1 = 2$, $T_2 = 1$, $\bar{L}_2 = L_2$, $\bar{L}_3 = 0$, $r_2 = r$, $r_1 = r + 1$. Hence, $\xi_2[M_{T_2}(t)] = 0$ for any t . We

have

$$\begin{aligned}
g_1^B(y, 0) &= \mathbb{E}[h_1(y - D_1^1 - \xi_1[M_{T_1}(L_2 + 1)]) + (b + h'_1)(y - D_1^1 - \xi_1[M_{T_1}(L_2 + 1)])^-], \\
g_1^B(y, 1) &= \mathbb{E}[h_1(y - D_1^1 - \xi_1[M_{T_1}(L_2 + 2)]) + (b + h'_1)(y - D_1^1 - \xi_1[M_{T_1}(L_2 + 2)])^-], \\
g_2^B(y, 0) &= \mathbb{E}[h_2(y - D_2^{12} - \xi_1[M_{T_1}(L_2 + r_2)]) \\
&\quad + g_1^B((s_1^B \wedge (y - D_2^{12} - \xi_1[M_{T_1}(L_2 + r_2)]) + \xi_1[M_{T_1}(L_2 + r_1)]), 0)] \\
&= \mathbb{E}[h_2(y - D_2^{12}) + g_1^B((s_1^B \wedge (y - D_2^{12} + \xi_1[\mathbb{M}_2(L_2 + 1)]), 0)], \\
g_2^B(y, 1) &= \mathbb{E}[h_2(y - D_2^{12} - \xi_1[1]) + g_1^B((s_1^B \wedge (y - D_2^{12} - \xi_1[1] + \xi_1[\mathbb{M}_2(L_2 + 2)]), 1)].
\end{aligned}$$

Plugging these into (25), we get

$$\begin{aligned}
&\mathbb{E}[h_2(y - D_2^{12}) + \mathbb{E}[h_1((s_1^B \wedge (y - D_2^{12} + \xi_1[\mathbb{M}_2(L_2 + 1)])) - D_1^1 - \xi_1[\mathbb{M}_2(L_2 + 1)]) \\
&\quad + (b + h'_1)((s_1^B \wedge (y - D_2^{12} + \xi_1[\mathbb{M}_2(L_2 + 1)])) - D_1^1 - \xi_1[\mathbb{M}_2(L_2 + 1)])^-] \\
&\quad + \mathbb{E}[h_2(y - D_2^{12} - \xi_1[1]) + \mathbb{E}[h_1((s_1^B \wedge (y - D_2^{12} - \xi_1[1] + \xi_1[\mathbb{M}_2(L_2 + 2)])) - D_1^1 - \xi_1[\mathbb{M}_2(L_2 + 2)]) \\
&\quad + (b + h'_1)((s_1^B \wedge (y - D_2^{12} - \xi_1[1] + \xi_1[\mathbb{M}_2(L_2 + 2)])) - D_1^1 - \xi_1[\mathbb{M}_2(L_2 + 2)])^-] \\
\geq &\mathbb{E}[h_2(y - D_2^{12}) + \mathbb{E}[h_1((s_1^A \wedge (y - D_2^{12})) - D_1^1) + (b + h'_1)((s_1^A \wedge (y - D_2^{12})) - D_1^1)^-]] \\
&\quad + \mathbb{E}[h_2(y - D_2^{12} - \xi_2[1]) + \mathbb{E}[h_1((s_1^A \wedge (y - D_2^{12} - \xi_2[1])) - D_1^1) \\
&\quad + (b + h'_1)((s_1^A \wedge (y - D_2^{12} - \xi_2[1])) - D_1^1)^-]
\end{aligned}$$

Canceling the common terms from both sides and noting that 1) $\xi_2[1]$ and $\xi_1[1]$ follow the same distribution, 2) the two $\xi_1[\mathbb{M}_2(L_2 + 1)]$ terms in the first line and the second line and the two $\xi_1[\mathbb{M}_2(L_2 + 2)]$ terms in the third line and fourth line of the inequality come from the same time period (i.e., $t + L_2 + r_1$), we get the following:

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[h_1(((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 1)]) \wedge (y - D_2^{12})) - D_1^1) \\
&\quad + (b + h'_1)((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 1)]) \wedge (y - D_2^{12})) - D_1^1)^-] \\
&\quad + \mathbb{E}[\mathbb{E}[h_1(((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 2)]) \wedge (y - D_2^{12} - \xi_1[1])) - D_1^1) \\
&\quad + (b + h'_1)((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 2)]) \wedge (y - D_2^{12} - \xi_1[1])) - D_1^1)^-] \\
> &\mathbb{E}[\mathbb{E}[h_1((s_1^A \wedge (y - D_2^{12})) - D_1^1) + (b + h'_1)((s_1^A \wedge (y - D_2^{12})) - D_1^1)^-] \\
&\quad + \mathbb{E}[\mathbb{E}[h_1((s_1^A \wedge (y - D_2^{12} - \xi_2[1])) - D_1^1) + (b + h'_1)((s_1^A \wedge (y - D_2^{12} - \xi_2[1])) - D_1^1)^-]
\end{aligned}$$

This is, in turn, equivalent to

$$\begin{aligned}
&\mathbb{E}[g_1^A((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 1)]) \wedge (y - D_2^{12}), 0)] + \mathbb{E}[g_1^A((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 2)]) \wedge (y - D_2^{12} - \xi_1[1]), 0)] \\
> &\mathbb{E}[g_1^A(s_1^A \wedge (y - D_2^{12}), 0)] + \mathbb{E}[g_1^A(s_1^A \wedge (y - D_2^{12} - \xi_2[1]), 0)]
\end{aligned}$$

Note that $s_1^A = \arg \min (g_1^A(y, 0) + g_1^A(y, 1)) = \arg \min (g_1^A(y, 0))$ because $g_1^A(y, 0) = g_1^A(y, 1)$. We also have that $s_1^B \geq s_1^A$ because $s_1^B = \arg \min (g_1^B(y, 0) + g_1^B(y, 1))$.

For the first terms on both side of the inequality, fixing D_2^{12} and $\xi_1[\mathbb{M}_2(L_2 + 1)]$, we get $g_1^A(s_1^A \wedge y - D_2^{12}, 0) \leq g_1^A(s_1^B - \xi_1[\mathbb{M}_2(L_2 + 1)] \wedge y - D_2^{12}, 0)$, because $g_1^A(y, 0)$ is a decreasing function for $y < s_1^A$ and ξ_1 is nonnegative.

For the second terms on both side of the inequality, because $\xi_1[1]$ and $\xi_2[1]$ follow the same distribution and they are in separate expectations, we can compare the terms for fixed $\xi_1[1] = \xi_2[1]$. We have $g_1^A((s_1^B - \xi_1[\mathbb{M}_2(L_2 + 2)]) \wedge (y - D_2^{12} - \xi_1[1]), 0) \geq g_1^A(s_1^A \wedge (y - D_2^{12} - \xi_2[1]), 0)$, because $g_1^A(y, 0)$ is a decreasing function for $y < s_1^A$. This completes the proof.