On-Line Appendix for

Optimal and Heuristic Echelon (r, nQ, T) Policies in Serial Inventory Systems with Fixed Costs

Kevin H. Shang • Sean X. Zhou

Fuqua School of Business, Duke University, Durham, North Carolina 27708, USA
Systems Engineering & Engineering Management, Chinese University of Hong Kong, China
khshang@duke.edu • zhoux@se.cuhk.edu.hk

In the following proofs, we approximate demand and control variables as continuous when we prove the convexity of cost functions. In addition, unless otherwise noted, we use $(f(x))'$ to denote the derivative with respect to x and $(g(x,y))'$ to denote the partial derivative with respect to the first argument x for any continuous and differentiable functions $f(x)$ and $g(x,y)$. Finally, we need the following result which is due to Lemma 1 of Chen (2000). For $j = 1, \ldots, N$,

$$\frac{1}{Q_j}\sum_{x=1}^{Q_j} G_j(O_j^*[y+x], \mathbf{Q_{j-1}}, \mathbf{T_j})) = \widehat{G}_j(\min\{y, r_j(\mathbf{Q_j}, \mathbf{T_j})\}, \mathbf{Q_j}, \mathbf{T_j}),$$

where G_j and $\widehat{G}_j$ are defined in (10). The same equation holds when we replace G_j and $\widehat{G}_j$ with g_j and $\widehat{g}_j$ defined in (12) and (13). We shall omit the reference when this result is used in the following proofs.

Proposition 1 *Define*

$$G_1(y) \;=\; \frac{1}{T_1}\left(\sum_{\tau=0}^{T_1-1} \mathsf{E}[h_1(y - D[L_1+\tau]) + (b + h_{[1,N]})(y - D[L_1+\tau])^-]\right).$$

For $j = 2, ..., N$, define recursively

$$G_j(y) \;=\; \frac{1}{T_j}\sum_{\tau=0}^{T_j-1} \mathsf{E}\left[h_j(y - D[L_j+\tau]) + G_{j-1}\left(O_{j-1}\left[y - D\left[L_j + \left\lfloor \frac{\tau}{T_{j-1}} \right\rfloor T_{j-1}\right)\right]\right)\right].$$

Then, $G(\mathbf{r}, \mathbf{Q}, \mathbf{T}) = \widehat{G}_N(r_N)$, where $\widehat{G}_N(r_N) = (1/Q_N)\sum_{x=1}^{Q_N} G_N(r_N + x)$.

Proof of Proposition 1

Let t be an order period for stage N, and for $\tau = 0, 1, ..., T_N - 1$, recall that

$$IL_N^-(t + L_N + \tau) = IP_N(t) - D[t, t + L_N + \tau),$$

and

$$IL_N(t+L_N+\tau) = IP_N(t) - D[t, t+L_N+\tau].$$

Given $IP_N(t) = y$, we have

$$\begin{aligned}
G_N(y) &= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\left[h_N(y-D[L_N+\tau]) + G_{N-1}\left(O_{N-1}\left[y - D\left[L_N + \left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}\right)\right]\right)\right] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\left[h_N(IL_N(t+L_N+\tau)) + G_{N-1}\left(O_{N-1}\left[IL_N^-\left(t+L_N+\left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}\right)\right]\right)\right] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\left[h_N(IL_N(t+L_N+\tau)) + G_{N-1}\left(IP_{N-1}\left(t+L_N+\left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}\right)\right)\right] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\Bigg[h_N(IL_N(t+L_N+\tau)) \\
&\quad + \frac{1}{T_{N-1}}\sum_{\ell=0}^{T_{N-1}-1}\Bigg(h_{N-1}IL_{N-1}\left(t+L_{[N-1,N]}+\left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}+\ell\right) \\
&\quad + G_{N-2}\left(O_{N-2}\left[IL_{N-1}^-\left(t+L_{[N-1,N]}+\left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}+\left\lfloor\frac{\ell}{T_{N-2}}\right\rfloor T_{N-2}\right)\right]\right)\Bigg)\Bigg] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\Bigg[h_N(IL_N(t+L_N+\tau)) + h_{N-1}(IL_{N-1}(t+L_{[N-1,N]}+\tau)) \\
&\quad + G_{N-2}\left(O_{N-2}\left[IL_{N-1}^-\left(t+L_{[N-1,N]}+\left\lfloor\frac{\tau}{T_{N-2}}\right\rfloor T_{N-2}\right)\right]\right)\Bigg] \\
&= \vdots \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\left[\sum_{j=2}^{N} h_j(IL_j(t+L_{[j,N]}+\tau)) + G_1\left(IP_1\left(t+L_{[2,N]}+\left\lfloor\frac{\tau}{T_1}\right\rfloor T_1\right)\right)\right] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\Bigg[\sum_{j=2}^{N} h_j(IL_j(t+L_{[j,N]}+\tau)) + \frac{1}{T_1}\sum_{\ell=0}^{T_1-1} h_1\left(IL_1\left(t+L_{[1,N]}+\left\lfloor\frac{\tau}{T_1}\right\rfloor T_1+\ell\right)\right) \\
&\quad + (h_{[1,N]}+b)\left(IL_1\left(t+L_{[1,N]}+\left\lfloor\frac{\tau}{T_1}\right\rfloor T_1+\ell\right)\right)^-\Bigg] \\
&= \frac{1}{T_N}\sum_{\tau=0}^{T_N-1} \mathsf{E}\left[\sum_{j=1}^{N} h_j(IL_j(t+L_{[j,N]}+\tau)) + (h_{[1,N]}+b)(IL_1\left(t+L_{[1,N]}+\tau\right))^-\right],
\end{aligned}$$

where the third equality follows from (1), the fourth equality follows from (4) and (7), the fifth equality follows from

$$\frac{1}{T_N}\sum_{\tau=0}^{T_N-1}\left[\frac{1}{T_{N-1}}\sum_{\ell=0}^{T_{N-1}-1} IL_{N-1}\left(t+L_{[N-1,N]}+\left\lfloor\frac{\tau}{T_{N-1}}\right\rfloor T_{N-1}+\ell\right)\right]$$

$$= \frac{1}{T_N} \sum_{\tau=0}^{T_N-1} IL_{N-1}\left(t + L_{[N-1,N]} + \tau\right)$$

and

$$\frac{1}{T_N} \sum_{\tau=0}^{T_N-1} \left[\frac{1}{T_{N-1}} \sum_{\ell=0}^{T_{N-1}-1} G_{N-2}\left(O_{N-2}\left[IL_{N-1}^{-}\left(t + L_{[N-1,N]} + \left\lfloor \frac{\tau}{T_{N-1}} \right\rfloor T_{N-1} + \left\lfloor \frac{\ell}{T_{N-2}} \right\rfloor T_{N-2} \right) \right] \right) \right]$$

$$= \frac{1}{T_N} \sum_{\tau=0}^{T_N-1} \left[G_{N-2}\left(O_{N-2}\left[IL_{N-1}^{-}\left(t + L_{[N-1,N]} + \left\lfloor \frac{\tau}{T_{N-2}} \right\rfloor T_{N-2} \right) \right] \right) \right].$$

Since $IP_N(t)$ is uniformly distributed between $\{r_N + 1, r_N + 2, ..., r_N + Q_N\}$, $\mathsf{E}[G_N(IP_N(t))] = \widehat{G}_N(r_N) = G(\mathbf{r}, \mathbf{Q}, \mathbf{T})$.

Proposition 2 *For fixed* $\mathbf{Q}$ *and* $\mathbf{T}$*, the optimal reorder point* $r_j(\mathbf{Q_j}, \mathbf{T_j})$ *minimizes* $\widehat{g}_j(r_j, \mathbf{Q_j}, \mathbf{T_j})$*,* $j = 1, ..., N$*.*

Proposition 3 $\widehat{G}_j(\mathbf{Q_j}, \mathbf{T_j}) = \sum_{i=1}^{j} \widehat{g}_i(\mathbf{Q_i}, \mathbf{T_i})$*, for* $j = 1, ..., N$*,*

Proof of Propositions 2 and 3

Propositions 2 and 3 are related, so we show them together. To show these results, it suffices to show $\widehat{G}_j(y, \mathbf{Q_j}, \mathbf{T_j}) = \widehat{g}_j(y, \mathbf{Q_j}, \mathbf{T_j}) + \sum_{i=1}^{j-1} \widehat{g}_i(\mathbf{Q_i}, \mathbf{T_i})$ for all j. We prove this result by induction.

For notational simplicity, let $r_j = r_j(\mathbf{Q_j}, \mathbf{T_j})$ for all j. For $j = 1$, $\widehat{g}_1(y, Q_1, T_1) = \widehat{G}_1(y, Q_1, T_1)$ by definition. Suppose that the result holds true for $j = i$, that is, $Q_i T_i \widehat{G}_i(y, \mathbf{Q_i}, \mathbf{T_i}) = Q_i T_i (\widehat{g}_i(y, \mathbf{Q_i}, \mathbf{T_i}) + \sum_{k=1}^{i-1} \widehat{g}_k(\mathbf{Q_k}, \mathbf{T_k}))$. Then, for $j = i + 1$ and letting $D_{i+1}(\ell) = D[L_{i+1} + \lfloor \ell / T_i \rfloor T_i)$, we have

$$\begin{aligned}
& Q_{i+1} T_{i+1} \widehat{G}_{i+1}(y, \mathbf{Q_{i+1}}, \mathbf{T_{i+1}}) \\
= & \sum_{x=1}^{Q_{i+1}} \sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\left[h_{i+1}(y + x - D[L_{i+1} + \ell]) + G_i\left(O_i^*\left[y + x - D_{i+1}(\ell) \right], \mathbf{Q_{i-1}}, \mathbf{T_i} \right) \right] \\
= & \sum_{z=0}^{q_i-1} \sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\left[\sum_{x=1}^{Q_i} \left(h_{i+1}(y + zQ_i + x - D[L_{i+1} + \ell]) \right.\right. \\
& \left.\left. + G_i\left(O_i^*\left[y + zQ_i + x - D_{i+1}(\ell) \right], \mathbf{Q_{i-1}}, \mathbf{T_i} \right) \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{z=0}^{q_i-1}\sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\Bigg[\sum_{x=1}^{Q_i} h_{i+1}(y+zQ_i+x-D[L_{i+1}+\ell]) \\
&\qquad\qquad +Q_i\widehat{G}_i\left(\min\left\{y+zQ_i-D_{i+1}(\ell),r_i\right\},\mathbf{Q_i},\mathbf{T_i}\right)\Bigg)\Bigg] \\
&= \sum_{z=0}^{q_i-1}\sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\Bigg[\sum_{x=1}^{Q_i} h_{i+1}(y+zQ_i+x-D[L_{i+1}+\ell]) \\
&\quad +Q_i\widehat{g}_i\left(\min\left\{y+zQ_i-D_{i+1}(\ell),r_i\right\},\mathbf{Q}_i,\mathbf{T_i}\right)-Q_i\widehat{g}_i(\mathbf{Q}_i,\mathbf{T}_i)+Q_i\sum_{k=1}^{i}\widehat{g}_k(\mathbf{Q_k},\mathbf{T_k})\Bigg] \\
&= \sum_{x=1}^{Q_{i+1}}\sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\Bigg[\Bigg(h_{i+1}(y+x-D[L_{i+1}+\ell])+g_i\left(O_i^*\left[y+x-D_{i+1}(\ell)\right],\mathbf{Q}_i,\mathbf{T_i}\right) \\
&\quad -g_i(y+x-mQ_i,\mathbf{Q}_i,\mathbf{T}_i)\Bigg)\Bigg]+T_{i+1}Q_{i+1}\sum_{k=1}^{i}\widehat{g}_k(\mathbf{Q_k},\mathbf{T_k}) \\
&= \sum_{x=1}^{Q_{i+1}}\sum_{\ell=0}^{T_{i+1}-1} \mathsf{E}\Bigg[h_{i+1}(y+x-D[L_{i+1}+\ell])+g_{i,i+1}\left(y+x-D_{i+1}(\ell),\mathbf{Q}_i,\mathbf{T_i}\right)\Bigg] \\
&\quad +Q_{i+1}T_{i+1}\sum_{k=1}^{i}\widehat{g}_k(\mathbf{Q}_k,\mathbf{T_k}) \\
&= Q_{i+1}T_{i+1}\left(\widehat{g}_{i+1}(y,\mathbf{Q_{i+1}},\mathbf{T_{i+1}})+\sum_{k=1}^{i}\widehat{g}_k(\mathbf{Q_k},\mathbf{T_k})\right),
\end{aligned}
$$

where m in the fifth equality is an integer such that $r_i+1 \le y+x-mQ_i \le r_i+Q_i$. In the above derivation, the fourth equality follows from the inductive assumption on $\widehat{G}_i(y,\mathbf{Q_i},\mathbf{T_i})$, and the penultimate equality follows from the definition of $g_{i,i+1}(y,\mathbf{Q}_i,\mathbf{T}_i)$. This completes the proof.

Proposition 4

(1) With fixed batch sizes $\mathbf{Q}$, $P_j(y,\mathbf{Q_j},\mathbf{T_j^j}) \le P_j(y,\mathbf{Q_j},\mathbf{T_j}) \le P_j(y,\mathbf{Q_j},\mathbf{T_j^1})$, *for all* y *and* $j = 2,...,N$.

(2) With fixed reorder intervals $\mathbf{T}$, $P_j(y,\mathbf{Q_j^j},\mathbf{T_j}) \le P_j(y,\mathbf{Q_j},\mathbf{T_j}) \le P_j(y,\mathbf{Q_j^1},\mathbf{T_j})$, *for all* y *and* $j = 2,...,N$.

Proof of Proposition 4

To prove this proposition, we need the following lemma. For convenience, we approximate demand and inventory variables as continuous. With additional technicalities, the result holds true for the case where demand and inventory variables are discrete.

Lemma 7 *For $j = 1, \ldots, N$,*

(1) For fixed $\mathbf{Q_j}$ and $\mathbf{T_{j-1}}$, $r_j(\mathbf{Q_j}, \mathbf{T_j})$ is nondecreasing in T_j.

(2) For fixed $\mathbf{T_j}$ and $\mathbf{Q_{j-1}}$, $r_j(\mathbf{Q_j}, \mathbf{T_j})$ is nonincreasing in Q_j.

Proof. We first prove part (1). We omit $\mathbf{Q_j}$ in the related functions because it is given. From Topkis (1998), it is sufficient to show that $\widehat{G}_j(y, \mathbf{T_j})$ is submodular in y and T_j because $\widehat{G}_j(y, \mathbf{T_j})$ is convex in y (See Lemma 2 of Chao and Zhou 2005). As $\mathbf{T_{j-1}}$ is given, to show the submodularity of $\widehat{G}_j$ on y and T_j, we need to show the submodularity of $\widehat{G}_j$ on y and n_{j-1} as the change of T_j has to be an integer multiple of T_{j-1}. Consider a set of reorder intervals $\mathbf{T_j} = (T_1, T_2, ..., T_{j-1}, T_j)$ and $T_j = n_{j-1}T_{j-1}$. Let $\mathbf{T_j} + 1 \stackrel{def}{=} (T_1, T_2, ..., T_{j-1}, (n_{j-1}+1)T_{j-1})$.

$$
\begin{aligned}
& (\widehat{G}_j(y, \mathbf{T_j}+1))' - (\widehat{G}_j(y, \mathbf{T_j}))' \\
= \; & \frac{1}{Q_j((n_{j-1}+1)T_{j-1})} \int_0^{Q_j} \sum_{\tau=0}^{(n_{j-1}+1)T_{j-1}-1} \mathsf{E}\left[G_{j-1}\left(O^*_{j-1}\left[y + x - D\left[L_j + \left\lfloor \frac{\tau}{T_{j-1}} \right\rfloor T_{j-1}\right)\right], \mathbf{T_{j-1}}\right)\right]' dx \\
& - \frac{1}{Q_j n_{j-1} T_{j-1}} \int_0^{Q_j} \sum_{\tau=0}^{n_{j-1}T_{j-1}-1} \mathsf{E}\left[G_{j-1}\left(O^*_{j-1}\left[y + x - D\left[L_j + \left\lfloor \frac{\tau}{T_{j-1}} \right\rfloor T_{j-1}\right)\right], \mathbf{T_{j-1}}\right)\right]' dx \\
= \; & \frac{1}{Q_j(n_{j-1}+1)T_j} \int_0^{Q_j} \mathsf{E}\Bigg[n_{j-1}T_{j-1}G_{j-1}\left(O^*_{j-1}\left[y + x - D\left[L_j + n_{j-1}T_{j-1}\right]\right], \mathbf{T_{j-1}}\right) \\
& - \sum_{\tau=0}^{n_{j-1}T_{j-1}-1} G_{j-1}\left(O^*_{j-1}\left[y + x - D\left[L_j + \left\lfloor \frac{\tau}{T_{j-1}} \right\rfloor T_{j-1}\right)\right], \mathbf{T_{j-1}}\right)\Bigg]' dx \\
= \; & \frac{1}{Q_j(n_{j-1}+1)T_j} \sum_{z=0}^{q_{j-1}-1} \mathsf{E}\Bigg[T_j Q_{j-1} \widehat{G}_{j-1}\left(\min\left\{r_{j-1}(\mathbf{T_{j-1}}), y + zQ_{j-1} - D\left[L_j + n_{j-1}T_{j-1}\right)\right\}, \mathbf{T_{j-1}}\right) \\
& - \sum_{\tau=0}^{T_j-1} Q_{j-1}\widehat{G}_{j-1}\left(\min\left\{r_{j-1}(\mathbf{T_{j-1}}), y + zQ_{j-1} - D\left[L_j + \left\lfloor \frac{\tau}{T_{j-1}} \right\rfloor T_{j-1}\right)\right\}, \mathbf{T_{j-1}}\right)\Bigg]' \\
\leq \; & 0.
\end{aligned}
$$

The inequality holds because $D\left[L_j + n_{j-1}T_{j-1}\right)$ is stochastically larger than $D\left[L_j + \lfloor(\tau/T_{j-1})\rfloor T_{j-1}\right)$ for $\tau < T_j$ and $\widehat{G}_{j-1}(y, \mathbf{T_{j-1}})$ is convex in y. Hence, $\widehat{G}_j(y, \mathbf{T_j})$ is submodular in y and T_j, which implies that $r_j(\mathbf{T_j})$ is nondecreasing in T_j.

We next prove part (2). We omit $\mathbf{T_j}$ in $r_j(\mathbf{Q_j}, \mathbf{T_j})$ and $\widehat{G}_j$ function for all j because $\mathbf{T_j}$ is fixed. Since $\widehat{G}_j(y, \mathbf{Q_j})$ is convex in y, it is sufficient to show that $\widehat{G}_j(y, \mathbf{Q_j})$ is supermodular in y and Q_j. For simplicity, we use $D_j(\tau)$ to represent $D\left[L_j + \lfloor\tau/T_{j-1}\rfloor T_{j-1}\right)$. Note that

$$
\begin{aligned}
& (\widehat{G}_j(y, \mathbf{Q_j}))' \\
= \; & \frac{1}{Q_j T_j}\left(\sum_{\tau=0}^{T_j-1}\int_{x=0}^{Q_j} \mathsf{E}[h_j(y+x-D[L_j+\tau]) + G_{j-1}\left(O_{j-1}^*\left[y+x-D_j(\tau)\right], \mathbf{Q_{j-1}}\right) dx\right)' \\
= \; & \frac{1}{Q_j T_j}\sum_{\tau=0}^{T_j-1}\left(h_j Q_j + \sum_{z=0}^{q_{j-1}-1} Q_{j-1}\mathsf{E}\left[\widehat{G}_{j-1}\left(\min\left\{y+zQ_{j-1}-D_j(\tau), r_{j-1}(\mathbf{Q_{j-1}})\right\}, \mathbf{Q_{j-1}}\right)\right]'\right).
\end{aligned}
$$

Since $\mathbf{Q_{j-1}}$ is given, to show the supermodularity of $\widehat{G}_j$ on y and Q_j, we need to show the supermodularity of $\widehat{G}_j$ on y and q_{j-1}

Let $\mathbf{Q_j} + 1 \stackrel{def}{=} (Q_1, ..., Q_{j-1}, (q_{j-1}+1)Q_{j-1})$ and $Q_j = q_{j-1}Q_{j-1}$

$$
\begin{aligned}
& (\widehat{G}_j(y, \mathbf{Q_j}+1))' - (\widehat{G}_j(y, \mathbf{Q_j}))' \\
= \; & \frac{1}{(q_{j-1}+1)Q_j T_j}\sum_{\tau=0}^{T_j-1}\left(\mathsf{E}\left[(q_{j-1}+1)Q_{j-1}\left(\widehat{G}_{j-1}\left(\min\left\{y+q_{j-1}Q_{j-1}-D_j(\tau), r_{j-1}(\mathbf{Q_{j-1}})\right\}, \mathbf{Q_{j-1}}\right)\right)'\right.\right. \\
& \left.\left. - \sum_{z=0}^{q_{j-1}-1} Q_{j-1}\left(\widehat{G}_{j-1}\left(\min\left\{y+zQ_{j-1}-D_j(\tau), r_{j-1}(\mathbf{Q_{j-1}})\right\}, \mathbf{Q_{j-1}}\right)\right)'\right]\right) \\
\geq \; & 0,
\end{aligned}
$$

where the inequality follows from the convexity of $\widehat{G}_{j-1}(y, \mathbf{Q_{j-1}})$ in y and for any sample path $D_j(\tau) = d$,

$$
\min\left\{y+q_{j-1}Q_{j-1}-d, r_{j-1}(\mathbf{Q_{j-1}})\right\} \geq \min\left\{y+zQ_{j-1}-d, r_{j-1}(\mathbf{Q_{j-1}})\right\}
$$

for $z < q_{j-1}$. Thus, $\widehat{G}_j(y, \mathbf{Q_j})$ is supermodular in y and Q_j and the result follows. □

We continue the proof of Proposition 4. Here, we only show detailed steps for a two-stage system because a complete proof for the general N-stage system is quite lengthy. An interested reader could request the complete proof from the authors.

Let's first prove part (1). Again, we omit $\mathbf{Q}_j$ in the related cost functions. With some algebra, we can show

$$\begin{aligned} P_j(y, \mathbf{T_j^j}) &= \frac{1}{Q_j T_j} \sum_{z=0}^{q_{j-1}-1} \sum_{\ell=0}^{T_j-1} \mathsf{E}\left[Q_{j-1}\widehat{g}_{j-1}\left(\min\left\{ y + zQ_{j-1} - D[L_j), r_{j-1}(\mathbf{T_{j-1}^j}) \right\}, \mathbf{T_{j-1}^j} \right) \right] \\ &\quad -\widehat{g}_{j-1}(\mathbf{T_{j-1}^j}) \end{aligned}$$

where $\mathbf{T_{j-1}^j} = (T_j, \ldots, T_j)$ that has $j-1$ dimensions, and

$$\begin{aligned} & P_j(y, \mathbf{T_j}) \\ = \; & \frac{1}{Q_j T_j} \sum_{z=0}^{q_{j-1}-1} \sum_{\ell=0}^{T_j-1} \mathsf{E}\left[Q_{j-1}\widehat{g}_{j-1}\left(\min\left\{ y + zQ_{j-1} - D\left[L_j + \left\lfloor \frac{\ell}{T_{j-1}} \right\rfloor T_{j-1} \right), r_{j-1}(\mathbf{T_{j-1}}) \right\}, \mathbf{T_{j-1}} \right) \right] \\ & -\widehat{g}_{j-1}(\mathbf{T_{j-1}}). \end{aligned}$$

To show $P_j(y, \mathbf{T_j^j}) \le P_j(y, \mathbf{T_j})$, it is equivalent to showing that, for $j = 2, \ldots, N$,

$$\begin{aligned} & \sum_{z=0}^{q_{j-1}-1} \sum_{\ell=0}^{T_j-1} \mathsf{E}\left[\widehat{g}_{j-1}\left(\min\{y + zQ_{j-1} - D[L_j), r_{j-1}(\mathbf{T_{j-1}^j})\}, \mathbf{T_{j-1}^j} \right) \right] \\ & - \sum_{z=0}^{q_{j-1}-1} \sum_{\ell=0}^{T_j-1} \mathsf{E}\left[\widehat{g}_{j-1}\left(\min\left\{ y + zQ_{j-1} - D\left[L_j + \left\lfloor \frac{\ell}{T_{j-1}} \right\rfloor T_{j-1} \right), r_{j-1}(\mathbf{T_{j-1}}) \right\}, \mathbf{T_{j-1}} \right) \right] \\ \le \; & T_j q_{j-1}\widehat{g}_{j-1}(\mathbf{T_{j-1}^j}) - T_j q_{j-1}\widehat{g}_{j-1}(\mathbf{T_{j-1}}). \end{aligned} \tag{1}$$

Note that the right-hand side of the inequality in (1) is a constant for fixed $\mathbf{T_j}$, and is independent of y. We use the two-stage system with $T_1 = 1$ and $T_2 = 2$ to illustrate the idea. For convenience, let r_1 and $\underline{r}_1$ denote $r_1(1)$ and $r_1(2)$, respectively. Recall that

$$\widehat{g}_1(y, 1) = \frac{1}{Q_1} \sum_{x=1}^{Q_1} \mathsf{E}[h_1(y + x - D[L_1]) + (b + h_{[1,2]})(y + x - D[L_1])^-],$$

and

$$\begin{aligned} \widehat{g}_1(y, 2) &= \frac{1}{2Q_1} \sum_{x=1}^{Q_1} \sum_{\ell=0}^{1} \mathsf{E}[h_1(y + x - D[L_1 + \ell]) + (b + h_{[1,2]})(y + x - D[L_1 + \ell])^-] \\ &= \frac{1}{2}[\widehat{g}_1(y, 1) + \mathsf{E}\widehat{g}_1(y - D, 1)]. \end{aligned}$$

The inequality (1) can be written as

$$\sum_{z=0}^{q_1-1} \Big(\mathsf{E}\widehat{g}_1(\min\{y + zQ_1 - D[L_2), \underline{r}_1\}, 1) - \mathsf{E}\widehat{g}_1(\min\{y + zQ_1 - D[L_2), r_1\}, 1)$$

$$
\begin{aligned}
& +\mathsf{E}\widehat{g}_1(\min\{y+zQ_1-D[L_2)-D,\underline{r}_1-D\},1)-\mathsf{E}\widehat{g}_1(\min\{y+zQ_1-D[L_2)-D,r_1\},1)\Big) \\
\leq\ & q_1\Big(\widehat{g}_1(\underline{r}_1,1)-\widehat{g}_1(r_1,1)+\mathsf{E}\widehat{g}_1(\underline{r}_1-D,1)-\widehat{g}_1(r_1,1)\Big). \qquad (2)
\end{aligned}
$$

Before we proceed, we need the result that both $\widehat{g}_j(y,\mathbf{T_j})$ and $\widehat{g}_j(y,\mathbf{T_j^j})$ are convex in y, for $j=1,\ldots,N$. This can be easily shown by induction, so we omit the detailed steps here.

Now, we are ready to show (2). We first show that

$$
\begin{aligned}
& \sum_{z=0}^{q_1-1}\Bigg(\mathsf{E}\widehat{g}_1(\min\{y+zQ_1-D[L_2),\underline{r}_1\},1)-\mathsf{E}\widehat{g}_1(\min\{y+zQ_1-D[L_2),r_1\},1)\Bigg) \\
\leq\ & q_1(\widehat{g}_1(\underline{r}_1,1)-\widehat{g}_1(r_1,1)). \qquad (3)
\end{aligned}
$$

¿From Lemma , we have $\underline{r}_1 \geq r_1$. Conditioning on $D[L_2)=d(L_2)$, if $y+zQ_1-d(L_2)\leq r_1$ or $y+zQ_1-d(L_2)\geq \underline{r}_1$ for any $0\leq z\leq q_1$, then

$$
\widehat{g}_1(\min\{y+zQ_1-d[L_2),\underline{r}_1\},1)-\widehat{g}_1(\min\{y+zQ_1-d[L_2),r_1\},1)\leq \widehat{g}_1(\underline{r}_1,1)-\widehat{g}_1(r_1,1),
$$

because r_1 minimizes $\widehat{g}_1(y,1)$. If $r_1<y+zQ_1-d(L_2)\leq \underline{r}_1$, then $\widehat{g}_1(y+zQ_1-d(L_2),1)-\widehat{g}_1(r_1,1)\leq \widehat{g}_1(\underline{r}_1,1)-\widehat{g}_1(r_1,1)$, because $\widehat{g}_1(y,1)$ is convex and is minimized at r_1. Thus, the inequality holds for any sample path of $D[L_2)$ and (3) follows.

We next show that

$$
\begin{aligned}
& \sum_{z=0}^{q_1-1}\Bigg(\mathsf{E}\widehat{g}_1(\min\{y+zQ_1-D[L_2)-D,\underline{r}_1-D\},1)-\mathsf{E}\widehat{g}_1(\min\{y-D[L_2)-D,r_1\},1)\Bigg) \\
\leq\ & q_1(\mathsf{E}\widehat{g}_1(\underline{r}_1-D,1)-\widehat{g}_1(r_1,1)). \qquad (4)
\end{aligned}
$$

For any sample path $D=d$ and $D[L_2)=d(L_2)$ (note that these two random variables are independent), we consider the following four different cases for any given $0\leq z\leq q_1$.

Case 1. If $y+zQ_1-d(L_2)-d\leq \min\{\underline{r}_1-d,r_1\}$,

$$
\begin{aligned}
& \widehat{g}_1(\min\{y+zQ_1-d(L_2)-d,\underline{r}_1-d\},1)-\widehat{g}_1(\min\{y+zQ_1-d(L_2)-d,r_1\},1) \\
=\ & 0 \\
\leq\ & \widehat{g}_1(\underline{r}_1-d,1)-\widehat{g}_1(r_1,1),
\end{aligned}
$$

which follows from that r_1 minimizes $\widehat{g}_1(y,1)$.

Case 2. If $y + zQ_1 - d(L_2) - d > \max\{\underline{r}_1 - d, r_1\}$,

$$\begin{aligned} & \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, \underline{r}_1 - d\}, 1) - \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, r_1\}, 1) \\ = \; & \widehat{g}_1(\underline{r}_1 - d, 1) - \widehat{g}_1(r_1, 1). \end{aligned}$$

Case 3. If $\underline{r}_1 - d \le y + zQ_1 - d(L_2) - d \le r_1$,

$$\begin{aligned} & \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, \underline{r}_1 - d\}, 1) - \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, r_1\}, 1) \\ = \; & \widehat{g}_1(\underline{r}_1 - d, 1) - \widehat{g}_1(y + zQ_1 - d(L_2) - d, 1) \le \widehat{g}_1(\underline{r}_1 - d, 1) - \widehat{g}_1(r_1, 1), \end{aligned}$$

where the inequality follows from the fact that r_1 minimizes $\widehat{g}_1(y, 1)$ and the convexity of $\widehat{g}_1(y, 1)$.

Case 4. If $r_1 \le y + zQ_1 - d(L_2) - d \le \underline{r}_1 - d$,

$$\begin{aligned} & \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, \underline{r}_1 - d\}, 1) - \widehat{g}_1(\min\{y + zQ_1 - d(L_2) - d, r_1\}, 1) \\ = \; & \widehat{g}_1(y + zQ_1 - d(L_2) - d, 1) - \widehat{g}_1(r_1, 1) \le \widehat{g}_1(\underline{r}_1 - d, 1) - \widehat{g}_1(r_1, 1), \end{aligned}$$

where the inequality follows from $r_1 \le y + zQ_1 - d(L_2) - d \le \underline{r}_1 - d$. Thus, (4) is true. With (3) and (4), (2) immediately follows. This completes the proof for the two-stage system with $T_2 = 2$.

Below we sketch the proof for the general T_1 and T_2 case. We need to show

$$\begin{aligned} & \sum_{z=0}^{q_1-1} \sum_{\ell=0}^{T_2-1} \mathsf{E}\left(\widehat{g}_1\left(\min\{y + zQ_1 - D[L_2), \underline{r}_1\}, \mathbf{T_1^2} \right) - \widehat{g}_1\left(\min\left\{ y + zQ_1 - D\left[L_2 + \left\lfloor \frac{\ell}{T_1} \right\rfloor T_1 \right), r_1 \right\}, T_1 \right) \right) \\ \le \; & q_1 T_2 (\widehat{g}_1(\underline{r}_1, \mathbf{T_1^2}) - \widehat{g}_1(r_1, T_1)). \end{aligned} \tag{5}$$

The key to the proof is the following result:

$$\widehat{g}_1(y, \mathbf{T_1^2}) = \frac{1}{T_2} \sum_{\tau=0}^{T_2-1} \mathsf{E}\widehat{g}_1\left(y - D\left[\left\lfloor \frac{\tau}{T_1} \right\rfloor T_1 \right), T_1 \right). \tag{6}$$

Based on this, (5) can be equivalently written as

$$\begin{aligned} & \sum_{z=0}^{q_1-1} \sum_{\ell=0}^{T_2-1} \mathsf{E}\Bigg[\widehat{g}_1\left(\min\{y + zQ_1 - D[L_2), \underline{r}_1\} - D\left[\left\lfloor \frac{\ell}{T_1} \right\rfloor T_1 \right), T_1 \right) \\ & \qquad - \widehat{g}_1\left(\min\left\{ y + zQ_1 - D\left[L_2 + \left\lfloor \frac{\ell}{T_1} \right\rfloor T_1 \right), r_1 \right\}, T_1 \right) \Bigg] \\ \le \; & q_1 \sum_{\ell=0}^{T_2-1} \mathsf{E}\widehat{g}_1\left(\underline{r}_1 - D\left[\left\lfloor \frac{\ell}{T_1} \right\rfloor T_1 \right), T_1 \right) - q_1 T_2 \widehat{g}_1(r_1, T_1). \end{aligned} \tag{7}$$

It should be noted that both sides of the inequality are based on the same function $\widehat{g}_1(y, T_1)$. Hence, the proof of (7) can be carried out similarly to that of the case of $T_2 = 2$, except that we have to validate T_2 inequalities here.

For the upper bound, we have actually verified it for $T_2 = 2$ and one can follow an analogous procedure to show $P_2(y, \mathbf{Q_2}, \mathbf{T_2}) \leq P_2(y, \mathbf{Q_2}, \mathbf{T_2^1})$ for general T_2.

We next prove part (2). We only prove the result with $T_j = 1$ for all j because the general T_j case can be done in a similar way. In the following proof, without confusion, we omit $\mathbf{T_j}$ in the cost functions.

It is sufficient to show the following:

$$\frac{1}{Q_j}\sum_{x=1}^{Q_j} \mathsf{E}\left[g_{j-1,j}\left(y + x - D[L_j), \mathbf{Q^j_{j-1}}\right)\right] \leq \frac{1}{Q_j}\sum_{x=1}^{Q_j} \mathsf{E}\left[g_{j-1,j}\left(y + x - D[L_j), \mathbf{Q_{j-1}}\right)\right],$$

or

$$\begin{aligned} & \frac{1}{Q_j}\sum_{x=1}^{Q_j}\left(\mathsf{E}\left[g_{j-1}\left(O_{j-1}[y + x - D[L_j)], \mathbf{Q^j_{j-1}}\right)\right] - \mathsf{E}\left[g_{j-1}\left(O_{j-1}[y + x - D[L_j)], \mathbf{Q_{j-1}}\right)\right]\right) \\ \leq \; & \frac{1}{Q_j}\sum_{x=1}^{Q_j} g_{j-1}(\underline{r}_{j-1} + x, \mathbf{Q^j_{j-1}}) - \frac{1}{Q_{j-1}}\sum_{x=1}^{Q_{j-1}} g_{j-1}(r_{j-1} + x, \mathbf{Q_{j-1}}), \end{aligned} \tag{8}$$

where $\mathbf{Q^j_{j-1}} = (Q_j, \ldots, Q_j)$ that has $j - 1$ dimensions. Here, $\underline{r}_j$ is the optimal reorder point when Q_i is set to be equal to Q_j for $i < j$.

We use a two-stage system with $Q_1 = 1$ and $Q_2 = 2$ to illustrate the idea of the proof. The inequality (8) reduces to the following:

$$\begin{aligned} & \frac{1}{2}\sum_{x=1}^{2} \mathsf{E}\left[g_1\left(O_1[y + x - D[L_2)], 2\right)\right] - \frac{1}{2}\sum_{x=1}^{2} \mathsf{E}\left[g_1\left(O_1[y + x - D[L_2)]\right), 1\right] \\ \leq \; & \frac{1}{2}\sum_{x=1}^{2} g_1(\underline{r}_1 + x, 2) - g_1(r_1 + 1, 1), \end{aligned}$$

or

$$\begin{aligned} & \mathsf{E}\widehat{g}_1\left(\min\{y - D[L_2), \underline{r}_1\}, 2\right) - \frac{1}{2}\sum_{x=0}^{1} \mathsf{E}g_1\left(\min\{y - D[L_2) + x, r_1\}, 1\right) \\ \leq \; & \widehat{g}_1(\underline{r}_1, 2) - g_1(r_1 + 1, 1). \end{aligned} \tag{9}$$

Recall from Lemma that $r_1 + 1$ minimizes $g_1(y, 1)$ and $\underline{r}_1 \leq r_1$. We consider the following three different cases for the sample path of $D[L_2) = d(L_2)$.

Case 1. For $y - d(L_2) \leq \underline{r}_1 \leq r_1$. In such a case, (9) becomes

$$\begin{aligned} & g_1(y - d(L_2) + 1, 1) + g_1(y - d(L_2) + 2, 1) - g_1(y - d(L_2), 1) - g_1(\min\{y - d(L_2) + 1, r_1\}, 1) \\ \leq \quad & g_1(\underline{r}_1 + 1, 1) + g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1) - g_1(r_1 + 1, 1). \end{aligned}$$

Because $y - d(L_2) + 1 < r_1 + 1$, we have $g_1(y - d(L_2) + 1, 1) - g_1(y - d(L_2), 1) \leq 0 \leq g_1(\underline{r}_1 + 1, 1) - g_1(r_1 + 1, 1)$. So we only need to show $g_1(y - d(L_2) + 2, 1) - g_1(\min\{y - d(L_2) + 1, r_1\}, 1) \leq g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1)$. To prove this, we further consider two subcases:

- if $y - d(L_2) + 1 \leq r_1$, then $y - d(L_2) + 2 \leq r_1 + 1$. Thus, $g_1(y - d(L_2) + 2, 1) - g_1(\min\{y - d(L_2) + 1, r_1\}, 1) \leq 0 \leq g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1)$.
- If $y - d(L_2) + 1 > r_1$, then $r_1 + 1 < y - d(L_2) + 2 \leq \underline{r}_1 + 2$. Thus, $g_1(y - d(L_2) + 2, 1) - g_1(\min\{y - d(L_2) + 1, r_1\}, 1) \leq g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1)$, where the inequality holds because $r_1 + 1$ is the minimizer of $g_1(y, 1)$.

Therefore, (9) holds true in this case.

Case 2. For $\underline{r}_1 < y - d(L_2) \leq r_1$, (9) becomes

$$\begin{aligned} & g_1(\underline{r}_1 + 1, 1) + g_1(\underline{r}_1 + 2, 1) - g_1(y - d(L_2), 1) - g_1(\min\{y - d(L_2) + 1, r_1\}, 1) \\ \leq \quad & g_1(\underline{r}_1 + 1, 1) + g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1) - g_1(r_1 + 1, 1). \end{aligned}$$

This inequality clearly holds since $r_1 + 1$ is the minimizer of $g_1(y, 1)$.

Case 3. For $\underline{r}_1 \leq r_1 < y - d(L_2)$, (9) becomes

$$\begin{aligned} & g_1(\underline{r}_1 + 1, 1) + g_1(\underline{r}_1 + 2, 1) - g_1(r_1, 1) - g_1(r_1, 1) \\ \leq \quad & g_1(\underline{r}_1 + 1, 1) + g_1(\underline{r}_1 + 2, 1) - g_1(r_1 + 1, 1) - g_1(r_1 + 1, 1). \end{aligned}$$

This inequality clearly holds since $r_1 + 1$ is the minimizer of $g_1(y, 1)$. This completes the proof for this special two-stage system.

Proposition 5 *$\bar{c}_j(T_j)$ is convex in T_j.*

Proof of Proposition 5

It suffices to show that $\widehat{g}_j(y, \mathbf{T_j^1})$ is jointly convex in y and T_j for all j. Again, we omit $\mathbf{Q_j}$ in $r_j(\mathbf{Q_j}, \mathbf{T_j})$ for simplicity. For $j = 1$,

$$\begin{aligned}
\widehat{g}_1(y, T_1) &= \frac{1}{Q_1 T_1}\left(\sum_{x=1}^{Q_1}\sum_{\tau=0}^{T_1-1} \mathsf{E}[h_1(y + x - D[L_1 + \tau]) + (b + h_{[1,N]})(y + x - D[L_1 + \tau]^-]\right) \\
&= \frac{1}{Q_1}\sum_{x=1}^{Q_1} \mathsf{E}[h_1(y + x - D[L_1 + U(T_1)]) + (b + h_{[1,N]})(y + x - D[L_1 + U(T_1)]^-], \quad (10)
\end{aligned}$$

in which $U(T_1)$ is a discrete uniform random variable over $[0, T_1 - 1]$. By using Lemma 3.8 in Shaked and Shanthikumar (1988), we can show that $U(T_1)$ is stochastically increasing and linear in sample path (SIL(sp)) in T_1 . Furthermore, since $D[t]$ is the sum of $t + 1$ i.i.d. random demand, $D[t]$ is SIL(sp) in t (Shaked and Shanthikumar (1988), Example 4.3). Due to the preservation property (Shaked and Shanthikumar (1994), Theorem 6 B.11), these two results together imply that $D[L_1 + U(T_1)]$ is SIL(sp) in T_1.

Under continuous approximations of demand and T_1, the cumulative demand term $D[L_1 + U(T_1)]$ becomes a continuous random variable $D[L_1 + u(0, 1)T_1)$, where $u(0, 1)$ denotes the uniform distribution over $[0, 1]$. It follows immediately that the function within summation of (10) is jointly convex in y and T_1 by Lemma 5 and Theorem 6 in Rao (2003) and so is $\widehat{g}_1(y, T_1)$.

For $j > 1$, let $\mathbf{1_{j-1}} = (1, 1, ..., 1)$ with $j - 1$ dimensions. Note that

$$\begin{aligned}
\widehat{g}_j(y, \mathbf{T_j^1}) &= \frac{1}{Q_j T_j}\left(\sum_{x=1}^{Q_j}\sum_{\tau=0}^{T_j-1} \mathsf{E}[h_j(y + x - D[L_j + \tau]) + g_{j-1,j}(y + x - D[L_j + \tau), \mathbf{1_{j-1}})]\right) \\
&= \frac{1}{Q_j}\sum_{x=1}^{Q_j} \mathsf{E}[h_j(y + x - D[L_j + U(T_j)]) + g_{j-1,j}(y + x - D[L_j + U(T_j)), \mathbf{1_{j-1}})] \\
&= \frac{1}{Q_j}\sum_{z=0}^{q_{j-1}-1} \mathsf{E}\Bigg[\sum_{x=1}^{Q_{j-1}} h_j(y + zQ_{j-1} + x - D[L_j + U(T_j)]) \\
&\qquad + \widehat{g}_{j-1}(\min\{y + zQ_{j-1} - D[L_j + U(T_j), r_{j-1}(\mathbf{1_{j-1}})\}, \mathbf{1_{j-1}})\Bigg].
\end{aligned}$$

Again, $D[L_j + U(T_j))$ is SIL(sp) in T_j. Under continuous approximations, $\mathsf{E}[\widehat{g}_{j-1}(\min\{y + zQ_{j-1} - D[L_j + U(T_j), r_{j-1}(\mathbf{1_{j-1}}))\}, \mathbf{1_{j-1}})]$ is jointly convex since $\widehat{g}_{j-1}(y, \mathbf{1_{j-1}})$ is convex in y and minimized at $r_{j-1}(\mathbf{1_{j-1}})$. Thus, $\widehat{g}_j(y, \mathbf{T_j^1})$ is jointly convex in y and T_j.

Proposition 6 *For $j = 1, ..., N$,*

(1) $\widehat{G}_j^\ell(y, Q_j, T_j) + \pi_j \le \widehat{G}_j(y, \mathbf{Q_j}, \mathbf{T_j}) \le \widehat{G}_j^u(y, Q_j, T_j) + \pi_j$.

(2) $r_j^\ell(Q_j, T_j) \le r_j(\mathbf{Q_j}, \mathbf{T_j}) \le r_j^u(Q_j, T_j)$.

(3) $\widehat{G}_j^\ell(Q_j, T_j) + \pi_j \le \widehat{G}_j(\mathbf{Q_j}, \mathbf{T_j}) \le \widehat{G}_j^u(Q_j, T_j) + \pi_j$.

(4) $\widehat{G}_j^\ell(Q_j, T_j)$ *and* $\widehat{G}_j^u(Q_j, T_j)$ *are jointly convex and increasing in* Q_j *and* T_j.

Proof of Proposition 6

Parts (1)-(3) are a generalization of Theorem 4 in Shang and Song (2007). One can apply the same technique to show these results. We only show part (4).

$$\begin{aligned}\widehat{G}_j^\ell(y, Q_j, T_j) &= \frac{1}{Q_j T_j} \sum_{\tau=0}^{T_j-1} \sum_{x=1}^{Q_j} \mathsf{E}[h_j(y + x - D[L_{[1,j]} + \tau])^+ + (b + h_{[j+1,N]})(y + x - D[L_{[1,j]} + \tau])^-] \\ &= \frac{1}{Q_j} \sum_{x=1}^{Q_j} \mathsf{E}[h_j(y + x - D[L_{[1,j]} + U(T_j)])^+ + (b + h_{[j+1,N]})(y + x - D[L_{[1,j]} + U(T_j)])^-],\end{aligned}$$

where $U(T_j)$ is a discrete uniform random variable over $[0, T_j - 1]$. Applying the same analysis in the proof of Proposition 5, $D[L_{[1,j]} + U(T_j)]$ is SIL(sp) in T_j.

We approximate y, Q_j, and T_j as continuous variables. That is,

$$\begin{aligned}\widehat{G}_j^\ell(y, Q_j, T_j) &= \mathsf{E}[h_j(y + v(0,1)Q_j - D[L_{[1,j]} + u(0,1)T_j]) \\ &\quad +(b + h_{[j,N]})(y + v(0,1)Q_j - D[L_{[1,j]} + u(0,1)T_j])^-],\end{aligned}$$

where both $v(0,1)$ and $u(0,1)$ are uniform random variables over $(0,1)$.

To show this result, we only need to show $\widehat{G}_j^\ell(y, Q_j, T_j)$ is jointly convex. With any given sample path of $v(0,1) = v$, $y + vQ_j$ is linear. Again, analogous to the proof of Proposition 5, the function is convex in y, Q_j and T_j. After taking expectation over $v(0,1)$, the resulting $\widehat{G}_j^\ell(y, Q_j, T_j)$ is also convex. Similarly, we can prove the convexity of $\widehat{G}_j^u(y, Q_j, T_j)$ in the same way.

For the monotonicity in Q_j (T_j), it can be proved by taking the difference of $\widehat{G}_j^\ell(y, Q_j, T_j)$ over Q_j (T_j). We below show that $\widehat{G}_j^\ell$ is increasing in T_j. In such case, Q_j is given, so we omit it in the cost function.

$$\begin{aligned}&\widehat{G}_j^\ell(r_j^u(T_j), T_j) - \widehat{G}_j^\ell(r_j^u(T_j - 1), T_j - 1) \\ \ge\ &\widehat{G}_j^\ell(r_j^u(T_j), T_j) - \widehat{G}_j^\ell(r_j^u(T_j), T_j - 1)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{Q_j}\sum_{x=1}^{Q_j}\Bigg\{\frac{1}{T_j}\sum_{\ell=0}^{T_j-1}\mathsf{E}\Big[h_j(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^+ + (b+h_{[j+1,N]})(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^-\Big] \\
&\quad -\frac{1}{T_j-1}\mathsf{E}\sum_{\ell=0}^{T_j-2}\Big[h_j(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^+ + (b+h_{[j+1,N]})(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^-\Big]\Bigg\} \\
&= \frac{1}{Q_jT_j(T_j-1)}\sum_{x=1}^{Q_j}\mathsf{E}\Bigg[T_j(h_j(r_j^u(T_j)+x-D[L_{[1,j]}+T_j-1])^+ \\
&\qquad +(b+h_{[j+1,N]})(r_j^u(T_j)+x-D[L_{[1,j]}+T_j-1])^-) \\
&\quad -\sum_{\ell=0}^{T_j-1}\Big[h_j(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^+ + (b+h_{[j+1,N]})(r_j^u(T_j)+x-D[L_{[1,j]}+\ell])^-\Big]\Bigg] \geq 0.
\end{aligned}
$$

The last inequality is due to the fact that $r_j^u(T_j)$ is the minimizer of $\widehat{G}_j^\ell$. By taking the difference of $\widehat{G}_j^\ell(y,Q_j,T_j)$ over Q_j, one can prove that $\widehat{G}_j^\ell(y,Q_j,T_j)$ increases in Q_j.

The monotonicity for $\widehat{G}_j^u(y,Q_j,T_j)$ can be similarly proved.