Resource and Revenue Management in Nonprofit Operations[1]

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Abstract

Nonprofit firms sometimes engage in for-profit activities for the sole purpose of generating revenue to subsidize their mission activities. The organization is then confronted with a consumption vs. investment tradeoff. Investment corresponds to the allocation of capacity for revenue customers, while consumption corresponds to serving mission customers. We model this problem as a multi-period stochastic dynamic program. In each period, the organization must decide how much of the current assets should be invested in revenue-customer service capacity, and at what price the service should be sold. We provide sufficient conditions under which the optimal capacity allocation and pricing decisions are of threshold type. Similar results are derived when the selling price is fixed but banking of assets is allowed. We compare the performance of the optimal threshold policies with heuristics that may be more appealing to managers of non-profit organizations. Numerical experiments indicate that, while banking appears to only have a marginal effect, dynamic pricing can provide a significant benefit.

1 Introduction

Nonprofit organizations often engage in a mix of activities, some of which are designed primarily to generate excess revenue to subsidize other activities that more directly serve the organization's charitable mission. This has led nonprofit researchers to address the strategic question of how to construct an optimal mix or portfolio of activities (Gruber and Mohr 1982; Oster 1995). However, few have paid attention to the practical decisions about resource and capacity allocation that must be made once a given portfolio of activities has been constructed.[1] As far as we know, no one has applied operations research methods to these decisions. In this paper, we use a modeling approach to develop insights regarding the optimal allocation of resources among revenue-generating and mission-serving activities over time in an organization whose objective is to maximize mission impact.

In order to isolate the tradeoffs involved in balancing revenue-generating and mission-serving activities across many time periods, we have constructed a simplified model of an organization with just two activities and two corresponding customer groups. Adapting a convention introduced by Weisbrod (1998), we refer to *R-activities* as those that generate more revenue than costs. These serve *R-customers*. R-activities may, but need not, also generate mission impact. We refer to *M-activities* as those that generate positive mission impact, but require a financial subsidy.[2] They serve *M-customers*. The organization begins with an endowment of resources but is attempting to be self-sufficient, with its R-activities funding its M-activities. It does not rely on philanthropic or government subsidies. When we determine the optimal resource allocation decision between these two activities, we find a threshold effect. Under a set of plausible assumptions, resources should be allocated first to R-activities until the threshold is met. This pattern holds up for scenarios that involve fixed pricing with banking as well as dynamic pricing without banking. The threshold increases when we allow R-activities to generate mission impact.

[1] One exception is Young (2004). He addresses a broad range of resource allocation issues, within and across activities, using a marginal contribution framework. The main differences between his analysis and ours are 1) we capture the potential tradeoff between investing in mission activities now versus in the future, 2) we allow for randomness is demand for the revenue-producing activities, and 3) our model does not assume a diminishing social return.

[2] It is important to note that on the financial side, we are only concerned with net financial contribution. An M-activity may generate some revenue that covers part of its operational costs. As long as the net financial contribution of producing a unit of this good or service is negative, we can model it as a pure mission good with costs equal to the net negative contribution. Similarly, an R-activity must produce a net positive financial contribution that can be used in future periods to subsidize M-activities.

We anticipate that this finding will run counter to the intuitions of nonprofit managers whom we expect would want to give priority to M-activities. According to our model, this allocation of resources to M-activities will result in a reduced total mission impact when future periods are considered, even given that future mission impacts are discounted. An allocation to M-activities before the threshold is met must be justified for reasons that are external to our model. By identifying this crucial tradeoff between current and future mission impact, we hope to stimulate further research on this topic.

Before explaining our model in detail, it may be helpful to explain the kind of allocation decision we have in mind. In writing about nonprofit strategy, Oster (1995, p. 93) says, "Organizations should balance their mix of activities to assure a focus on mission and economic viability." We could add that once a balanced portfolio of activities is created, nonprofits should allocate their resources so as to generate the greatest social impact in the long run. Economic viability is a means to that end. To visualize the balancing act, consider the following matrix (based on Gruber and Mohr (1982, Oster (1995)).

<table>
<tr><td rowspan="3">Net Financial Impact</td><td>Positive</td><td colspan="2">R-activities</td><td></td></tr>
<tr><td>Neutral</td><td></td><td></td><td></td></tr>
<tr><td>Negative</td><td></td><td colspan="2">M-activities</td></tr>
<tr><td></td><td></td><td>Low</td><td>Medium</td><td>High</td></tr>
<tr><td></td><td></td><td colspan="3">Mission Impact</td></tr>
</table>

This matrix classifies activities in a nonprofit according to their mission and financial impacts on the organization. R-activities are those with a positive financial impact, regardless of mission impact. M-activities are those with a medium or high mission impact and a negative financial impact. The ideal activity would be positioned in the upper right hand corner, an R-activity with high mission impact. Unfortunately, few nonprofits find themselves in the enviable position of having their activities in this space. The activities are generally provided by for-profit organizations. Most nonprofits have important M-activities that need to be subsidized. With increased competition for philanthropic and government funding over the past two decades, nonprofits are under increased pressure to generate more subsidy from their own R-activities, even if they are relatively low on mission contribution. This allows for cross subsidy within the nonprofit. As a result of this pressure, the creation of revenue-generating ventures has become a major topic of discussion in the nonprofit field over the past two decades. See, for instance, Skloot (1983, Skloot (1987, Skloot (1988, Dees (1998, Young (2002, Oster, Massarsky, and Beinhacker (1995, Foster and Bradach (2005, Weisbrod (2005). Numerous 'how to' guides have been produced to help nonprofits develop profit-making ventures, see for instance Steckel, Simons, and Lengsfelder (1989, Alter (2000, Boschee (2001, Anderson, Dees, and Emerson (2002, Larson and Forth (2002, Robinson (2002, Dees (2004).

This kind of nonprofit cross-subsidy can take different forms. The examples we are particularly interested in involve activities to which common resources might be allocated at the beginning of a budget period and not be reallocated until the next period. For instance, Aravind Eye Hospital in India offers free cataract surgery to the poor using subsidies from its cataract operations for the rich (Pande 1998). It must allocate medical staff to the different facilities to optimize the mix. In the same way, the Boston Symphony Orchestra runs its 'Boston Pops' series using some of the same musicians and facilities as the regular Orchestra, and relies on the revenues to subsidize other Orchestra activities that are artistically important but less lucrative (Oster 1995). This pattern is common among performing arts organizations that schedule more popular performances to subsidize more artistically important, but less popular ones. Consider the reliance of ballet companies on seasonal performances of 'The Nutcracker' or theatre companies that rely on popular musicals or plays. The mix is often decided at the start of each season or year. Sometimes social service agencies will use key resources, such as their staff, facilities, or equipment, to deliver marketable products that subsidize more charitable operations. For instance, CIPO Productions, a Brazilian organization, provides impoverished young people with training in photography, video production, and Web design, and helps subsidize these activities with revenue from selling the use of its production and computer equipment to customers who can pay (Elstrodt, Schindler, and Waslander 2004). Brinckerhoff (2000, p. 16) describes an Illinois school for behaviorally-challenged adolescents that has created a business using its staff expertise to deliver workshop on 'Dealing with Difficult Teenagers', for which it charges admission. The net proceeds help to fund the school's core mission activities. Any social service provider that includes a practice focused on paying clients in order to subsidize its mission-related services faces the kind of resource allocation decision we are discussing.

Once the strategic decision is made to engage in activities that produce a net financial contribution to subsidize other activities that primarily contribute to the mission, nonprofit managers are faced with budgeting and resource allocation decisions. How much of their organization's resources and capacity in a given period do they devote to the activities with profit potential, as opposed to activities that serve the mission, but at a cost? Our model focuses on this resource-allocation decision. Since social impact is the ultimate objective, this decision essentially involves making tradeoffs between achieving mission impact now versus generating additional financial resources for mission impact in the future.[3]

To make this problem tractable we have created a simple model of a very complex world. Nonprofits are complex and diverse, with widely varying economic structures. They include, for instance, homeless shelters, ballet companies, hospitals, environmental advocacy groups, social service agencies, museums, sports and athletic associations, schools and colleges, and grassroots political groups. For any given model, including ours, it will be easy to think of many nonprofit organizations that do not fit. We understand that most of them are more complex than our simple two-activity model, and that some nonprofits are not aiming to be economically self-sufficient. Nonetheless, we believe our model, by isolating this tradeoff, highlights a dynamic that is relevant to many nonprofit organizations as they search for the kind of balanced portfolio recommended by Oster (1995). The underlying dynamic in our model captures an increasingly common tension that nonprofit managers face. We offer it as, we hope, a meaningful first step in what we anticipate will be a larger, longer-term project of applying modeling techniques to decision making in nonprofit settings. This paper is written primarily for the operations science community, with the goal of stimulating further research on this topic. We hope that it also stimulates more interest by the operations community in a large section of the economy (7% to 10%, depending on the metric) to which it has dedicated little attention.[4] It should also inform the thinking of nonprofit managers regarding the potential implications of their resource allocation decisions as they pursue that

[3]This allows us to compare R- and M-activities in a way that Young's marginal-contribution model does not address.

[4]McCardle (2005) suggests that the operations community, through its professional societies, set standards for *pro bono* work, similarly to what is done in the legal profession.

balance between financial and mission activities moving forward.

Key assumptions and results

We now state more precisely and discuss in turn the key assumptions and results in the paper.

Periodic Capacity Shifting and Resource Allocation. Our model assumes that capacity can be reallocated only periodically at decision times and not between these periods. This may appear artificially rigid, but it clearly fits with nonprofits that 1) are using common core assets in both R-activities and M-activities, and 2) have to schedule the use of these assets in advance. The first condition fits with the recommendations of strategists in this area. According to Oster (1995, p. 91) the most promising new activity extensions, including those created to generate revenue, are ones that can 'best share in the core assets of the organization'. The second condition, that these core assets cannot be shifted between activities during the period may not apply in some cases, but it is not uncommon. Consider performing arts groups that schedule their seasons in advance. Shifting capacity mid-season is extremely rare. They make their bets and live with them. Even social service agencies that attempt to balance paying clients (R-customers) with charitable clients (M-customers) may need to decide in advance how much staff time to make available for paid and charitable work. It is important to keep in mind that the decision period can be short. Aravind Eye Hospital may be able to shift resources (doctors and nurses) frequently, but is unlikely to do so mid-shift. In fact, given that Aravind maintains separate hospitals for its R- and M-customers, the organization is likely to make its resource commitments weeks or even months in advance.

Of course, some organizations may not be constrained in this way. In many instances, resources can actually be shared during the production period by the R- and M-activities. We briefly explore in this paper the possibility of re-allocating the resources during a decision period. Under simplifying assumptions, we find that the threshold effect still holds. However, further work is needed towards a complete study of this variant of the problem.

Linearity of Social Return. We assume that nonprofit organizations can expand their M-activities without experiencing diminishing social returns. Given that many nonprofit organizations are serving only a small share of the need for their mission activities, this seems like a reasonable assumption in those cases. Aravind can address only a small portion of the need for cataract surgery among the poor in India. A social service agency could usually serve many more charitable clients, if it simply had the capacity, without any dilution of impact. The case is less clear with performing-arts organizations, if we think of customer demand for the artistic productions. However, if we think in terms of valuable artistic pieces (ballets, plays, symphonies, etc.) that could be performed, it is conceivable that the potential number of artistic productions that the company might like to perform far exceeds its current capacity. Additional capacity for M-performances may not lead to diminishing returns on this artistic level.[5]

Discounting Future Social Impacts. We assume that projected future social outcomes should be discounted. There are two rationales for this. The first is mathematical. If we fail to use a mission-impact discount rate and also allow for the possibility of earning real interest (above the inflation rate of operating costs) on banked funds, the mathematical model will result in no investment in mission goods in the current period, ever (Keeler and Cretin 1984). It will always be superior to bank the funds in order to purchase more mission impact in later periods. This immediate result would defeat the purpose of modeling. The second reason is that, in most cases, addressing social problems (global warming, poverty, spread of disease, etc.) or delivering social goods (health care, education, the arts) is better done sooner than later. A discount factor captures this degree of urgency, which can vary widely in the sector. It is also often used in practice to reflect the uncertainty surrounding the delivery and value of projected future benefits. Using a discount rate implies that a more certain (immediate) impact should be valued more highly than a less certain (future) impact of the same magnitude. We recognize that the use of discount rates with regard to social impacts is controversial (Klausner 2003), and it certainly oversimplifies the complex considerations that should inform time preferences in the nonprofit sector, such as considerations of intergenerational equity or the potential value of intervening at strategic times. However, we believe that the use of some discounting, some preference for immediate impacts versus later impacts, is reasonable for our purposes.

Absence of Fundraising. We explicitly leave fundraising out of our resource allocation model and reserve it as a topic for future extensions. Our model isolates the conditions of a nonprofit organization that aims to have a self-sufficient portfolio of activities, making it independent of

[5]For the non-linear case, we expect that the threshold effect holds at least when the rate at which the marginal social return decreases is not too high relative to the 'thickness' of the tail of the distribution or R-customer demand.

fundraising. One option for including fundraising would be to treat it, as Weisbrod (1998) suggests, simply as an R-activity. Fundraising costs could be deducted from donations to indicate a net financial yield, which presumably would be positive, but with low or zero direct mission impact. This approach is acceptable provided that operating capacity and resources could be allocated to fundraising, just as well as they could be allocated to other R-activities that more resemble operating programs. We are not yet convinced that this is the right way to conceive of fundraising for resource allocation purposes and prefer to reserve this as a matter for future exploration. We would like to explore alternative ways to incorporate the option of allocating capacity to fundraising activities in a way that respects the difference between fundraising and activities that generate fee or sales income.

Sorting Customers. Our model distinguishes R-customers from M-customers. In some cases, it is the nature of the customers (*e.g.*, income level) that determines the degree of mission impact of an activity. We assume that, when it matters (with regard to mission impact), these customers will sort themselves to the appropriate activities. In particular, R-customers will not simply switch to being M-customers if a comparable M-activity is available as a substitute. In most cases, this is not a problem because the M-activities are often not good substitutes for the R-activities. Even when the activities are similar, various barriers are likely to lead to appropriate sorting. Few of Aravind's wealthy patients would pretend to be poor to get the free version of the surgery. In some cases, R-customers can also be M-customers without disrupting the model because the mission impact is not tied to the nature of the customers served. This is the case with the performing-arts organizations. The same person can attend a Pops concert and a regular symphony without affecting the model results.

Possibility of Banking. To test the robustness of our model, we add the option of banking funds for the current period to earn interest. With banking in the model, managers have three options in the allocation of funds: to the revenue goods, to the mission goods, or banking them. This modification should not be controversial. It is used only with the fixed price model in which the nonprofit is a price taker with regard to its R-activities. We provide sufficient conditions under which the optimal capacity allocation and pricing decisions are of threshold type. We also provide sufficient conditions on the number of periods-to-go under which banking is never optimal. That is, any banking is suboptimal toward the end of the time horizon. Our numerical results also suggest that there are only small expected gains from including the option of banking in problems with reasonable parameters.

Dynamic Pricing. Since nonprofits may want to set their prices strategically to serve mission and revenue objectives, we have created a version of the model that allows for dynamic pricing. In this model, we do not allow banking. The threshold effect still holds. We also find that, under reasonable assumptions for the probability distribution of the demand from R-customers (but not for all distributions), the optimal price that the firm should charge is decreasing in the level of the asset.

Finally, when the price is fixed and banking is not allowed we also show, perhaps surprisingly, that the optimal service-capacity allocation is myopic. The optimal threshold is not time-dependent and can be analytically derived.

No doubt many nonprofit leaders will want to allocate some resources to M-activities in all periods simply because that is the organization's mission. Their instincts may be based on factors that fall outside our model. To drop all M-activities could affect the culture of the organization, the motivation of staff, board members, volunteers, and donors, and it could even jeopardize the nonprofit legal status of the organization, with the consequent tax implications. Also, for some organizations, especially for some types of medical services, if there are no alternative organizations that can provide a replacement for critical services, the impact of M-activities can be significantly non-linear. The marginal social value of the first few M-activities will be very high. These may be sufficient reasons not to allocate all resources solely to R-activities, even when resources are below the threshold. Our numerical studies show, however, that under reasonable parameters in our model, a proportional allocation heuristic performs poorly compared to the optimal threshold policies. The important thing is for nonprofit leaders to understand that the gains from their decision to support M-activities must offset the potential loss of future social impact that is predicted by our model. Our analysis should help nonprofit managers understand the cost (in terms of future mission impact) of diverting resources away from revenue-generating activities when the resource level is below the optimal threshold. They should make this decision with their eyes open.

For-profit firms: the investment vs. dividend problem with incomplete markets

While our work has been motivated by an interest in the management of nonprofit organizations, our model is also relevant to pricing and financial decisions in other problems. In a for-profit firm, a corresponding investment vs. consumption tradeoff arises in the dividend payment problem, where the firm must decide how much of the available cash to pay out as dividends, and how much to re-invest into the operation.

The dynamics in our model, which determine the cash constraints in the following periods, are similar to the interaction between financial and operational decisions explored in recent work. The large literatures on cash-flow management and corporate finance do not consider the interaction of operational and financial decisions. When firms operate in a complete market Modigliani and Miller (1958) show that these decisions can be made independently. As has often been noted, however, many firms do not operate in a perfect market, and the cash constraints determined by a firm's financial decisions do interact with its operational decisions. This is especially true for small firms that do not have access to reasonably priced capital (such as through the equity or bond market), either because of their small size or because they operate in an economy without a well-developed financial system. Buzacott and Zhang (2001) consider a manufacturer with limited funds who needs an asset-based loan from a bank to finance future growth. The objective is to maximize the retained earnings at the end of the time horizon. Babich and Sobel (2004) study production, sales, and loan size decisions to optimize the expected discounted proceeds from an initial public offering of stock. In this stream of research the control of dividends problem studied by Li, Shubik, and Sobel (2003), which includes inventory-replenishment decisions, seems to be the most closely related to our model. In particular, they establish that the optimal policy is myopic for linear holding and backorder costs. Their model is similar to our problem with banking, but with the organization having unlimited access to borrowing. Demand is backordered in their model, while we assume lost-sales (lost-sales is better suited to the problem faced by many firms, especially in service industries).

We consider pricing decisions, which generally has not been addressed in the literature on joint operational and financial decisions. The prices set by a firm affect its revenue, which in turns affects the cash constraints and future revenue and growth. Our model is most closely linked to the vast literature on dynamic inventory models with pricing and stochastic demand. Elmaghraby and Keskinocak (2003) and Bitran and Caldentey (2003) present comprehensive reviews of this literature. For more recent results, see also Chen, Ray, and Song (2003) who consider the lost sales case, and Chen and Simchi-Levi (2004a) for the backorder case. Chen, Wu, and Yao (2004) model the system with a Brownian motion and provide a worst-case bound for the value of dynamic pricing. In this stream of research, the pricing strategy and the inventory stock replenishment decisions determine the inventory levels available for future periods. This differs fundamentally from the cash constraints that follow from the capacity allocation decisions in our model. For instance, Huh and Janakiraman (2005) propose an elegant framework based on sample path analysis to retrieve and extend most of the existing results for multi-period dynamic pricing and inventory problems. Because in our setting the pricing decision directly affects the budget constraint of the following period, their approach cannot be applied to our model (see Appendix A.1 for more detail).

For the single period case, our model can be shown to be equivalent to a newsvendor problem with pricing and with a constraint on the capacity decision. For the classical newsvendor problem with pricing (without a limit on the capacity), there exist conditions on the hazard rate of the demand distribution for the additive and multiplicative pricing models which guarantee the uniqueness of the optimal price and stock level (Petruzzi and Dada 1999). More recent articles have proposed other conditions on the generalized hazard rate of the demand distribution for there to exist a unique local optimum for the price and stock level decisions. Wang, Jiang, and Shen (2004) show that this is the case if the generalized hazard rate of the demand distribution is increasing. Bernstein and Federgruen (2005) prove that if the generalized hazard rate is bounded below by one half, the expected profit is log-concave.

However, stronger conditions than unimodality (or log-concavity) are usually required when analyzing a multi-period dynamic problem. The condition we derive for an optimal threshold policy to exist bounds the generalized failure rate below, and allows us to propagate concavity of the optimal value function in the multi-period case. We also discuss bounding the elasticity of revenue from below by one. The elasticity of revenue corresponds in our context to the elasticity of sales recently introduced by Kocabiyikoğlu and Popsescu (2005) for the newsvendor problem with pricing. For the single period case, we retrieve some of the results developed in Kocabiyikoğlu and

Popsescu (2005).

Organization of the paper

The rest of the paper is organized as follows. Section 2 addresses the problem of optimal capacity and banking decisions, with a fixed selling price to R-customers. We also discuss conditions under which it is optimal not to bank. Section 3 then considers the problem without banking where the organization makes a pricing decision in each period. Section 4 presents numerical studies. We first investigate the relative merit of the optimal capacity policy relative to a proportional-allocation policy. We then explore the expected value of including banking and pricing decisions in the problem specification. Section 5 discusses extensions and directions for future research. We also briefly explore the case of flexible resources. We conclude in Section 6.

2 Resource Allocation with Banking

2.1 The Model

Consider a non-profit organization whose mission is to deliver a service to a specific group of people, which we refer to as its mission customers, or M-customers. Serving one M-customer increases the social impact of the organization by s (in social return units) but does not generate revenue. In order to fund this non-profit activity, the organization offers its services to a market of revenue customers, or R-customers. Revenue-generating activities are also sometimes mission related in practice. This may for instance be the case when the organization's mission involves education- or health-related objectives. We denote by τ the social return of serving one R-customer. The unit service capacity costs for M- and R-customers are equal to c_M and c_R, respectively. Note again that we assume lost-sales, and that the costs for R-customers are incurred as a function of the capacity allocated (for example the number of scheduled musical performances) even if that capacity is not fully utilized. Without loss of generality, we assume $c_R = 1$ (this is simply a change of asset, or monetary units, and we will assume that the selling price and the cost of servicing M-customers have also been modified so as to be specified in these units).

We assume ample demand of M-customers, that is, service capacity allocated to this group will always be consumed. On the other hand, the R-customer demand is random. We consider discrete-time problems where demand in each time period is a random variable Θ with a finite mean (to keep the notations simple we assume stationary demands, but our results extend directly to the case of independent Θ). We represent by $f(\cdot) : [\underline{\theta}, \bar{\theta}] \mapsto \mathbf{R}^+$ the probability density function of Θ, where $\underline{\theta}$ is non-negative and $\bar{\theta}$ can possibly be equal to $+\infty$. We will also consider the corresponding cumulative distribution $F(\cdot)$, and the tail distribution $G(\cdot) = 1 - F(\cdot)$.

In this section, we assume the price is fixed, that is the organization is a price-taker, so that there is no dependence of the R-customer demand on the unit selling price p, which is fixed. The price may be fixed by the market in which the organization is competing for R-customers or, due to marketing or organizational constraints, the organization may decide to propose its service to R-customers at the same price in all time periods.

Also for this section, we assume that the organization has an alternative financial application for its assets in a risk-free investment with return β. The organization needs to determine what portion of its assets should be 'banked', what portion should be invested in capacity for R-customers, and what portion should be allocated to capacity for M-customers. In addition to analyzing efficient capacity allocation and banking strategies for this problem, we will briefly discuss the relative importance of including banking in the problem formulation. We will show that if the time horizon is short enough, banking is never optimal.[6]

We begin by considering discrete-time problems, with a finite horizon of T periods. This may arise, for instance, if the initial assets of the organization are from a grant that specifies a duration of time in which to achieve its objectives. The social discount factor for delaying service to an M-customer to the next period is α, with $0 \leq \alpha < 1$. This discount factor measures the urgency of the social need that the organization addresses. The overall social return from the organization's activities is the total discounted number of M-customers served. The objective is then to determine a banking and capacity-allocation policy that maximizes the expected social return over the time horizon T. We formulate the problem of jointly determining the best banking and capacity-allocation decisions as a finite-horizon Markov decision process.

The system state is the assets a_t held by the organization at the beginning of period $t \in [1, \ldots T]$. At the beginning of each period, the organization decides how much service capacity y_t to provide

[6]We conjecture, but have not proved, that similar results exist for the price-dependent demand models of Section 3, where we will ignore banking.

for R-customers, and sets the amount of current assets which are banked, $z_t \geq 0$. The choices of y_t and z_t are limited by the current resources, according to the constraint $y_t + z_t \leq a_t$. The remaining resources, $x_t = a_t - y_t - z_t$, are allocated to serving M-customers. Demand Θ is then realized, and the number of R-customers served by the organization is the smallest of demand and capacity, which we write $y_t \wedge \theta_t$. The service is perishable so that any unused capacity is lost. The resources available to the organization at the beginning of the following period are

$$a_{t+1} = p\,(y_t \wedge \theta_t) + \beta\, z_t.$$

The organization contributes to its mission by serving $(a_t - y_t - z_t)/c_M$ M-customers and $y_t \wedge \theta_t$ R-customers, yielding a social return of $s/c_M(a_t - y_t - z_t) + \tau(y_t \wedge \theta_t)$. Without loss of generality, we set in the rest of this paper $s/c_M = 1$ (this is simply a change in social return units, and we assume that the units of τ, the social return from serving an R-custormer, have also been modified so as to be specified in these units; the units of the value-to-go function also become changed as a consequence).

Denote by $v_t(a)$ the maximum social-impact-to-go at period t given current resources $a \geq 0$. For the last period, all assets are allocated to the service of M-customers, so that $v_T(a) = a$. For $t < T$, $v_t(a)$ can be shown to satisfy the optimality equations (see, for instance, Heyman and Sobel 1984)

$$\begin{aligned} v_t(a) &= \max_{\substack{0 \leq y \quad 0 \leq z \\ y + z \leq a}} \quad (a - y - z) + \tau\, \mathbf{E}_\Theta(\, y \wedge \Theta) + \alpha H^{v_{t+1}}(y, z) && (1) \\ &= a + \max_{\substack{0 \leq y \quad 0 \leq z \\ y + z \leq a}} \quad J^{v_{t+1}}(y, z), && (2) \end{aligned}$$

where the operators J^v and H^v are defined for any real-valued function $v(\cdot)$ as

$$\begin{aligned} J^v(y, z) &= -y - z + \tau\, \mathbf{E}_\Theta(y \wedge \Theta) + \alpha H^{v_{t+1}}(y, z) && (3) \\ H^v(y, z) &= \mathbf{E}_\Theta\ v\,(p(y \wedge \Theta) + \beta z\,)\,, && (4) \end{aligned}$$

and $\mathbf{E}_\Theta$ is the expectation over Θ.

We denote by $(y_t^*(a), z_t^*(a))$ the optimal capacity allocation and banking decision at period t given current assets a. The optimal policy $(y_t^*(a), z_t^*(a))$ corresponds to the maximizer of $J^{v_{t+1}}$ subject to the constraints $0 \leq y + z \leq a$. (In case of multiple optima, $(y_t^*(a), z_t^*(a))$ designates the minimum optimal decision using, say, the lexicographic order.)

As a side result, note that if $\alpha\beta \geq 1$ the organization always prefers to bank rather than to serve M-customers (except in period T). Likewise, conditions on p and τ can also ensure that serving R-customers is never optimal.

Proposition 1

- *If $\alpha\beta > 1$, allocating capacity to mission customers only in the last time period is optimal.*
- *If $\alpha p + \tau < \alpha\beta$ or $\alpha p + \tau < 1$, serving R-customers is never optimal.*
- *If $\alpha\beta < 1$ and $\alpha p + \tau < 1$, using all assets to serve M-customers in the first period is optimal.*

Proof. This is shown from a sample-path argument. Suppose a policy is optimal, and it is such that $x_t = a_t - y_t - z_t$ M-customers are served at time t, with $x_t > 0$. Denote by x_s and z_s the optimal decision for $s = 1, \ldots, T$. Consider now an alternative policy, with decisions $\bar{x}_s$, $\bar{z}_s$ such that $\bar{x}_s = x_s$ for $s \neq t, T$ and $\bar{x}_t = 0$, $\bar{x}_T = x_T + \beta^T x_t$. This corresponds to $\bar{z}_s = z_s$ for $s = 1, \ldots, t-1$, and $\bar{z}_s = z_s + \beta^{t-s} x_t$ for $s = t, \ldots, T-1$. Note that before time t, the policies are identical. With the new policy, for any realization of the Θ_s, $s = 1, \ldots, T$, the objective is improved by $x_t((\alpha\beta)^{T-t} - 1) > 0$. The second and third parts of the property can be shown using a similar argument.

■

2.2 Optimal Resource Allocation

The interesting case is then $\alpha\beta < 1$ and $\alpha p + \tau > 1$, which we assume for the rest of this section. This also implies $\alpha p + \tau > \alpha\beta$. The following lemma states that the concavity of the social-impact-to-go can be iteratively propagated to all time periods.

Lemma 1 *If $v_{t+1}(\cdot)$ is differentiable, non-decreasing and concave, then $v_t(\cdot)$ is also differentiable, non-decreasing and concave.*

Proof. It is assumed that $v_{t+1}(\cdot)$ is differentiable, non-decreasing and concave. Then $v_{t+1}(p\,(y \wedge \theta) + \beta z)$ and hence $H^{v_{t+1}}(y, z)$ are also differentiable and jointly concave in y and z. It follows that $J^{v_{t+1}}(y, z)$ is also differentiable and jointly concave, so that $v_t(\cdot)$ is the sum of a linear function and the maximum of a differentiable concave function subject to convex constraints increasing in a. As a result, $v_t(a)$ is non-decreasing, concave (Heyman and Sobel 1984), and differentiable (Bonnans and Shapiro 2000, Theorem 4.16).

■

The next result states that the optimal policy for R-customer capacity plus banking is of threshold type.

Theorem 1 *For each time period t, there exists a threshold (a_t^*, y_t^*, z_t^*), with $a_t^* = y_t^* + z_t^*$, such that the optimal capacity decision $y_t^*(a)$ and the optimal banking decision $z_t^*(a)$ satisfy $y_t^*(a) + z_t^*(a) = a_t^* \wedge a$ for all a, and $y_t^*(a) = y_t^*$, $z_t^*(a) = z_t^*$ for $a > a_t^*$. Furthermore, the optimal banking decision $z_t^*(a)$ is non-decreasing in a.*

Proof. Since $v_T(a) = a$, $v_t(\cdot)$ is differentiable, non-decreasing, concave for all $t \leq T$ from Lemma 1 and by iterating on the value function. It follows that $J^{v_{t+1}}(y, z)$ is jointly concave for $t < T$ and we can define y_t^*, z_t^* the optimum of $J^{v_{t+1}}(y, z)$. The first part of the theorem follows directly from the concavity of $J^{v_{t+1}}(y, z)$ and the linear constraint $0 \leq y + z \leq a$ in (1).

To prove the monotonicity, when $a > a_t^*$ the optimal banking decision is constant and equal to z_t^*. When $a < a_t^*$, the optimal decisions verify $y_t^* = a - z_t^*$ and the social impact-to-go $v_t(a)$ is equal to $\max_{0 \leq z \leq a} \alpha H^{v_{t+1}}(a - z, z)$. Defining $h(z) = H^{v_{t+1}}(a - z, z)$, the first order condition $h(z)' = 0$ is equivalent to

$$\beta \int_{\underline{\theta}}^{a-z} v_{t+1}'(p\theta + \beta z) f(\theta) d\theta + \left[-\tau + \alpha(\beta - p) v_{t+1}'((\beta - p)z + pa)\right] G(a - z) = 0. \tag{5}$$

The first term in this equation is increasing in a and decreasing in z since $v_{t+1}'(\cdot)$ is a positive decreasing function. For the same reason, and for $p > \beta$, the second term is also increasing in a and decreasing in z. We must then have that that $z_t^*(a)$ increases in a.

■

As a side note, the equivalent first order condition for y with $J^{v_{t+1}}(y, a - y)$ yields

$$-\alpha\beta \int_{\underline{\theta}}^{y} v'(p\theta + \beta(a - y)) f(\theta) d\theta + \left[\tau + \alpha(p - \beta) v'((p - \beta)y + \beta a)\right] G(y) = 0. \tag{6}$$

The first term increases in a, while the second one decreases, so that the monotonicity of $y_t^*(\cdot)$ remains indeterminate.

This result extends to the infinite-horizon case by letting T approach $+\infty$, as stated in the following result.

Corollary 1 *The optimal policy for the infinite horizon case is of threshold type. Further, the optimal banking decision $z^*(\cdot)$ is non-increasing in the current assets.*

Proof. In the following, the value function is indexed by the number of periods to go $n \triangleq T - t$. In this notation, $v_0(\cdot)$ is the social impact of the last period, that is $v_0(a) = a$. Note then that $H^{v_0}(y, z) \leq p\mathbf{E}_\Theta \Theta + \beta z$, and

$$\max_{y \geq 0, z \geq 0} \tau\, \mathbf{E}_\Theta(y \wedge \Theta) + \alpha H^{v_0}(y, z) - y - z \leq (\tau + \alpha p)\mathbf{E}_\Theta \Theta + \max_{y \geq 0, z \geq 0} (\alpha\beta - 1)z - y.$$

Since $\alpha\beta < 1$, this expression is bounded from above by a finite $M = (\tau + \alpha p)\mathbf{E}_\Theta \Theta$. It follows that $v_1(a) \leq a + M$. For $n = 2$ we then have

$$\begin{aligned}
\max_{y \geq 0, z \geq 0} \tau\, \mathbf{E}_\Theta(y \wedge \Theta) + \alpha H^{v_1}(y, z) - y - z \quad &\leq \quad \max_{y \geq 0, z \geq 0} \tau\, \mathbf{E}_\Theta \Theta + \alpha \mathbf{E}_\Theta \left(py \wedge p\Theta + \beta z + M\right) - y - z \\
&\leq \quad (1 + \alpha)M,
\end{aligned}$$

where the first equality comes from $v_1(a) \leq a + M$, and the second from the definition of M. As a result, $v_2(a) \leq a + (1 + \alpha)M$. Iterating on n we conclude, for $n \geq 1$,

$$v_n(a) \leq a + \sum_{i=0}^{n-1} \alpha^i M = a + \frac{1 - \alpha^n}{1 - \alpha} M. \tag{7}$$

Since $v_n(a)$ is increasing in n, $v_n(a)$ has a finite limit $v^*(a)$ as n approaches $+\infty$ where $v^*(a) \leq a + M/(1 - \alpha)$. Furthermore, from Theorem 1, the two dimensional action set $0 \leq y + z \leq a$ can be reduced to the subset of the real line $0 \leq z \leq a_n^* \wedge a$, with $y = (a_n^* \wedge a) - z$. The result then follows directly from Theorem 1 and the application of Heyman and Sobel (1984, Theorem 8-15).

■

For $\beta = 0$ the organization should never bank, which also corresponds to the case considered in Section 3. Furthermore, if $p > 1$ (i.e. p is larger than one asset unit) the optimal policy is myopic. That is, the optimal threshold does not depend on t and corresponds to the optimal threshold of the single-period problem. This is shown in the following result.

Corollary 2 *When the price is fixed such that $p > 1$ and no banking is possible, the myopic policy is optimal. At time t, the optimal capacity allocation is $y_t^*(a) = a^* \wedge a$ where*

$$a^* = \begin{cases} F^{-1}\left(1 - \frac{1}{\tau+\alpha p}\right), & \text{if } \tau + \alpha p > 1 \\ 0, & \text{if } \tau + \alpha p \leq 1 \end{cases} \tag{8}$$

Proof. Assume a^* is the optimal threshold at $t+1$. The threshold at time t is the maximand of $J^{v_{t+1}}(y, 0)$ and from Theorem 1, $v_{t+1}(\cdot)$ is differentiable and the first order condition yields

$$\left[\tau + \alpha p v'_{t+1}(pa)\right] G(a) = 1 \tag{9}$$

For $p > 1$, we have that $pa^* > a^*$, and therefore $v'_{t+1}(pa^*) = 1$ (since a^* is the optimal threshold at time $t+1$). It follows that a^* as defined in (8) satisfies equation (9). Similarly, since $v'_T(a) = 1$, a^* is the optimal threshold at time $T-1$ and the result follows by backward iteration.

■

Sobel (1984) derived very general sufficient conditions for myopic policies to be optimal. In our model these conditions would require that the set $S(a^*) = [a^*, +\infty[$ be consistent, that is, for all $\theta \in [\underline{\theta}, \bar{\theta}]$, $p(y \wedge \theta) \in S(a^*)$. This is not the case here for $\theta < a^*/p$. The organization should never invest in capacity for R-customers when $\tau + \alpha p \leq 1$. When $\alpha p + \tau > 1$, the optimal policy is not myopic in general and the thresholds depend on the time period. If furthermore $p > 1$, the optimal threshold corresponds to the optimal capacity level of a newsvendor model with overage unit cost equal to 1 (corresponding to an M-customer), and underage unit cost αp. For general values of c_R and s/c_M, the previous results directly extend to

$$a^* = F^{-1}\left(1 - \frac{c_R s}{c_M(\tau + \alpha p)}\right),$$

with $(\alpha p + \tau)/c_R \geq 1$ and $p/c_R > 1$. The optimal policy for $y_t^*(\cdot)$ will also be overall increasing (since $y_t^*(0) = 0$ and $y_t^*(a) = y^*$ for $a \geq a^*$), but note that, for $\beta > 0$, Theorem 1 does not address how $y_t^*(\cdot)$ evolves with a.

Corollary 2 allows us to derive conditions for banking never to be optimal when the price is fixed. Assume in the following that $\alpha\beta < 1$, $\alpha p + \tau > 1$ and $p > 1$. We first give sufficient conditions to propagate an upper bound on the derivative of the social-impact-to-go. With $n = T - t$ the number of periods to go this upper bound is equal to

$$u(n) = \frac{\tau + (1 - \alpha p - \tau)(\alpha p)^n}{1 - \alpha p}.$$

Lemma 2 *Assume that the following three conditions hold at time* t*,*

- **(C1)** *the optimal banking decision yields* $z_t^*(a) = 0$ *for all* $a \geq 0$*,*
- **(C2)** $v_t'(a) \leq u(T-t)$*.*

If in addition

$$u(T-t) \leq (\alpha\beta)^{-1}, \tag{10}$$

then (C1) and (C2) also hold at time $t-1$*.*

Proof. From Lemma 1 and $v_T(a) = a$, we know that $v_t(\cdot)$ is differentiable for all t. Then, to show (C1), fix y and consider the derivative of $J(y, z)$ with respect to the second variable,

$$\begin{aligned} \frac{dJ}{dz}(y, z) &= -1 + \alpha\beta \mathbf{E}_\Theta \ v_t'\left((py \wedge p\Theta) + \beta z\right) \\ &\leq -1 + \alpha\beta u(T-t) \\ &\leq 0, \end{aligned}$$

where the inequalities follow from (C3) and from (10). As a result, for any given y, $J(y, z)$ is maximized $z = 0$, and $v_{t-1}(a)$ satisfies (C1). Finally, using the chain rule at the optimal decision and (C2), we obtain for $y = a$ (below the threshold)

$$v_{t-1}'(a) = (\tau + \alpha p v_t'(a))G(a) \leq \tau + \alpha p u(T-t) = u(T-t+1)$$

Since $v_{t-1}'(\cdot)$ is non-increasing, the inequality in (C2) also holds for $v_{t-1}'(\cdot)$ at every a.

■

We can now show that there exists a time period starting from which the organization should stop banking and follow the threshold policy described in Corollary 2.

Proposition 2 *Define* n_0 *as*

$$n_0 = \left\lfloor \frac{\ln\left[\tau + (\alpha p - 1)(\alpha\beta)^{-1}\right] - \ln(\tau + \alpha p - 1)}{\ln(\alpha p)} \right\rfloor.$$

For $T - n_0 \leq t \leq T$*, the optimal policy never involves banking,* i.e., $z_t^*(a) = 0$*. If furthermore* $p > 1$*, the optimal capacity allocation is* $y_t^*(a) = a^* \wedge a$ *where* a^* *is given by (8).*

Proof. For $n = 0$, $v_T(a) = a$ satisfies (C1) and (C2) of Lemma 2. We can then recursively apply Lemma 2 as long as $n = T - t$ satisfies (10) which is equivalent to $n \leq n_0$. The second part of the proposition holds from Theorem 1.

■

In particular, if the time horizon is short enough ($T \leq n_0$) the optimal policy is myopic and banking is never optimal.

Clearly, with only one time period, the organization should never bank. Under the stated conditions, the risk-free asset has the worst return of all applications for the assets. It is of use as a hedge against uncertain demand, as a low realization may lead to having to rebuild capital from a very low position and to having to go through many periods unable to serve M-customers. As the value-function becomes more concave due to there being more time periods to go, the optimal policy becomes more 'risk-averse', as there is a higher opportunity cost of having to the start the next period with a low cash position, and banking becomes more attractive.

An interesting practical situation corresponds to the case where the revenue customers do not contribute to the mission, *i.e.*, $\tau = 0$. Based on the previous analysis we can summarize the results in the following corollary for this case,

Corollary 3 *Assume that R-customers do not contribute to the mission ($\tau = 0$).*

- *If $\alpha\beta > 1$, allocating capacity to mission customers only in the last time period is always optimal.*
- *If $p < \beta$ or $p < 1$, serving R-customers is never optimal.*
- *If $\alpha\beta < 1$ and ($p < \beta$ or $\alpha p < 1$), using all assets to serve M-customers in the first period is optimal.*
- *If $\alpha\beta < 1$ and $\alpha p > 1$, the optimal policy is of threshold type.*
- *If $\beta = 0$ and $\alpha p > 1$, the optimal policy is of threshold type and myopic.*
- *If $\alpha\beta < 1$ and $\alpha p > 1$, for $T - n_0 \leq t \leq T$ the optimal policy is of threshold type, myopic and never involves banking,* i.e., *$z_t^*(a) = 0$ with*

$$n_0 = \left\lfloor -\frac{\ln(\alpha\beta)}{\ln(\alpha p)} \right\rfloor .$$

3 Resource Allocation with Pricing

3.1 The Model

We have assumed above that the organization is a price-taker, in that it proposes its service at a fixed price to the revenue customers. Non-profit organizations do however often have market power in their revenue-generating activities. In such a setting, the organization competes in a market as a price-setter, and observes a demand which is a function of that price. The organization needs to develop a pricing strategy jointly with its capacity allocation and banking strategy. Characterizing the optimal policy for these joint decisions for the general model appears, however, to be far more challenging. To be able to provide some insight into the pricing problem, we make two key simplifications in this section. This allows us to explore the structure of the optimal pricing and capacity allocation decisions in more detail, while still maintains the more interesting characteristics of the problem. First, we consider the case where that R-customers have no social impact, *i.e.*, $\tau = 0$. This is a reasonable assumption if alternative for-profit providers for paying customers are readily available. Second, we assume that the organization never banks, or the banked amount is negligible compared to the assets allocated to capacity for the M- or R-customers. Although we do not formally show an equivalent to Proposition 2 for the price-dependent case, we expect that similar conditions on the time horizon exist for which banking is never optimal.

We consider a demand function with multiplicative uncertainty, that is $D = \gamma(p)\Theta$, where $\gamma(\cdot) : [0, +\infty[\mapsto [0, +\infty[$ is the price response function and Θ is a random variable with finite mean. The response function $\gamma(\cdot)$ is differentiable and decreasing. The price elasticity $e(\cdot)$ is equal to $e(p) = -p\gamma'(p)/\gamma(p)$ where $\gamma'(\cdot)$ is the first order derivative of $\gamma(\cdot)$. We will also refer to the revenue function, defined as $r(p) = p\gamma(p)$ which is assumed to be non-monotone and strictly concave in p. The maximand $\underline{p} \triangleq \text{argmax}_p\, r(p)$ as well as the intercept $\overline{p}$ such that $r(\overline{p}) = 0$ are then well defined. As before, $f(\cdot)$ denotes the probability density function of Θ, and $F(\cdot)$ and $G(\cdot)$ the corresponding cumulative and tail distributions. We also introduce the generalized failure rate (GFR), defined as $g(\cdot) : [\underline{\theta}, \bar{\theta}] \mapsto \mathbf{R}^+$ such that

$$g(\theta) = \theta \frac{f(\theta)}{G(\theta)}. \tag{11}$$

At the beginning of each period, the organization sets the selling price p_t and decides how much service capacity y_t to provide for R-customers. The choice of y_t is limited by the current resources, which corresponds to the constraints $0 \leq y_t \leq a_t$. The remaining resources, $a_t - y_t$, are allocated to serving M-customers. Demand D is then realized, and the number of R-customers served by the organization is $y_t \wedge \gamma(p)\theta$. The resources available to the organization at the beginning of the following period are

$$a_{t+1} = p_t(y_t \wedge \gamma(p_t)\theta).$$

The goal is then to determine a pricing and capacity allocation policy that maximizes the social return over the time horizon T.

For $t < T$, social-impact-to-go at period t can then be shown to satisfy the optimality equations

$$v_t(a) = \max_{\substack{0 < p \\ 0 \leq y \leq a}} \quad a - y + \alpha H^{v_{t+1}}(p, y), \tag{12}$$

where the operator $H^v(p, y)$ is defined for any real-valued function $v(\cdot)$ as

$$H^v(p, y) \;=\; \mathbf{E}_\Theta \; v\,(\,py \,\wedge\, r(p)\Theta\,)\,. \tag{13}$$

We denote by $(p_t^*(a), y_t^*(a))$ the optimal pricing and capacity allocation decision at period t given the current assets a. The optimal policy $(p_t^*(a), y_t^*(a))$ corresponds to the maximizer of $\alpha H^{v_t}(p, y) - y$ subject to the constraints $0 < p$, $0 \leq y \leq a$.

Solving (12) however can be challenging, even for the single period case (in the following we refer to the single period problem as the case where $T = 2$, since no decision is made in the last period). When $T = 2$, the objective can be written as

$$v(a) = a + \max_{\substack{0 < p \\ 0 \leq y \leq a}} \quad \alpha p \mathbf{E}_\Theta\,\left(\,y \,\wedge\, \gamma(p)\Theta\,\right) - y \tag{14}$$

and the optimization problem is equivalent to a newsvendor problem with pricing and with capacity constraint. The unconstrained newsvendor problem with pricing has been the subject of much research (see Petruzzi and Dada (1999) for a complete survey). More recently, Wang, Jiang, and Shen (2004) provide sufficient conditions for the uniqueness of the optimal decisions y^* and p^*. They show that if Θ has increasing generalized failure rate (*i.e.*, $g(\cdot)$ is increasing over $[\underline{\theta}, \bar{\theta}]$), then y^* and

p^* exist and are unique. To that end, they introduce the "stocking factor" $y/\gamma(p)$ in Equation (14). They show that for any given z, an optimal $p(z)$ exists and the function $H(z)$ defined as

$$H(z) = \alpha \mathbf{E}_\Theta \, r(p) \, (\, z \wedge \Theta \,) - \gamma(p) z$$

is unimodal in z. Nothing is said however about the concavity of the objective function, even along the optimal price $p^*(z)$. Similarly, Bernstein and Federgruen (2005) propose a sufficient condition on $g(\cdot)$ which guaranties that the previous function $H(\cdot)$ is log-concave and hence unimodal.

The analysis of multi-period dynamic problems however typically requires stronger properties than unimodality or log-concavity. In particular, most of the previous approaches to address dynamic joint pricing-inventory problems (for general probability distributions) rely on the joint concavity of $\alpha H^v(p, y) - y$ in (p, y) for any concave value function $v(\cdot)$ (see, for instance, Federgruen and Heching (1999)). However, even for the single period case, the py term in the definition of H^v in (13) is not jointly concave. Changes of variable such as $z = y/\gamma(p)$, or the condition on the generalized failure rate $g(\cdot)$ proposed by Wang, Jiang, and Shen (2004) or Bernstein and Federgruen (2005) do not circumvent this problem. Also, sample path approaches as developed by Huh and Janakiraman (2005) cannot be applied to our model (see Appendix A.1).

In the following, we assume that the general failure rate at the stocking factor $y/\gamma(p)$ is not smaller than the inverse of the price elasticity, that is $\forall p \in [\underline{p}, \overline{p}[$ and $\forall y$, $y/\gamma(p) \in [\underline{\theta}, \bar{\theta}]$,

$$g\left(\frac{y}{\gamma(p)}\right) \geq \frac{1}{e(p)}. \tag{15}$$

For instance, when $e(\cdot)$ is decreasing, condition (15) holds when the generalized hazard rate is bounded by one, *i.e.*, $g(\theta) \geq 1$ (since $e(\underline{p}) = 1$ with $\underline{p}$ satisfying the first order condition $r'(p) = 0$). Concave response functions as in Federgruen and Heching (1999) (which include linear functions or power functions of the form $\gamma(p) = c - kp^b$, with $b \geq 1$ for some positive constants $c, k > 0$), and some convex functions (such as $\gamma(p) = c - k \ln(1 + p)$ for some positive constants $c, k > 0$) have decreasing price elasticities.

For the single period case, condition (15) is very similar to the condition proposed by Bernstein and Federgruen (2005) for which the generalized hazard rate is bounded below by one half. In our case however, the bound depends on the price elasticity which makes the condition more restrictive. Condition (15) is also neither more general nor more restrictive than the increasing generalized failure rate condition proposed by Wang, Jiang, and Shen (2004). Condition (15) allows for non-monotonic generalized failure rates. If Θ is uniformly distributed over $[\underline{\theta}, \bar{\theta}]$, condition (15) holds for decreasing $e(\cdot)$ and $\underline{\theta}/\bar{\theta} \geq 1/2$. More generally, for any distribution defined over $[0, +\infty[$ with an increasing generalized failure rate, $g(\theta) > 1$ and hence condition (15) holds for decreasing $e(\cdot)$ if the distribution is truncated at $\underline{\theta}$ such that $g(\underline{\theta}) > 1$, and scaled accordingly.

More generally condition (15) states that the elasticity of revenues is larger than one. The elasticity of revenues represents the percentage change in the rate of lost revenues with respect to price and is equivalent to the elasticity of sales introduced very recently by Kocabiyikoğlu and Popsescu (2005) for the newsvendor problem with pricing. Following Kocabiyikoğlu and Popsescu (2005), the elasticity of revenues is equal to

$$-\frac{pL'(p,y)}{L(p,y)} = -\frac{p\gamma'(p)}{\gamma(p)}\, g\left(\frac{y}{\gamma(p)}\right) = e(p)\, g\left(\frac{y}{\gamma(p)}\right)$$

where $L(p, y) \triangleq P(\gamma(p)\Theta \geq y)$ and $L'(\cdot, y)$ its derivative with respect to p. Kocabiyikoğlu and Popsescu (2005) show furthermore that if the elasticity of sales is bounded by one, *i.e.*, condition (15) holds, an optimal solution exists for the newsvendor problem with pricing and the corresponding optimal price is decreasing with the inventory level. In our context, these concavity and monotonicity properties can actually be propagated across time periods under condition (15).

More precisely, we will show that if condition (15) holds then $H^v(p, y)$ is concave along the optimal price, that is,

$$\tilde{H}^v(y) = \max_{0<p} H^v(p, y) \tag{16}$$

is concave in y. $\tilde{H}^v(\cdot)$ is well defined and $\tilde{p}(y)$ the maximizer $H^v(\cdot, y)$ is interior ($\tilde{p}(y) \in]0, \overline{p}[$) for any continuous positive function $v(\cdot)$ since

$$H^v(0, y) = H^v(\overline{p}, y) = 0$$

for any $y \in \Re^+$. Concavity of $\tilde{H}^v(\cdot)$ guaranties in turn that the social-impact-to-go function $v_t(\cdot)$ is concave and that the optimal policy is of threshold type. Furthermore, since $r(\cdot)$ is non-monotone strictly concave, $r(\cdot)$ and hence $H^v(\cdot, y)$ are increasing in p for $p < \underline{p}$ (recall that $\underline{p}$ is the maximand of $r(\cdot)$). The optimal price $\tilde{p}(y)$ is hence bounded from below by $\underline{p}$.

A threshold policy in our context is characterized by $T - 1$ thresholds $(\hat{p}_t, \hat{a}_t)$, $t \in [1 \ldots T - 1]$, such that the decisions $(p(a), y(a))$ given assets a at the beginning of period t are

$$
\begin{aligned}
y(a) &= a \wedge \hat{a}_t \\
p(a) &= \hat{p}_t \quad \text{if} \;\; a \geq \hat{a}_t.
\end{aligned}
$$

Under a threshold policy, the organization tries to guarantee a particular service capacity for R-customers. Only when this threshold is assured are M-customers served.

Finding the optimal price, however, is not a trivial problem, and solutions can be counter-intuitive. One could indeed reasonably expect the optimal price to be decreasing in the capacity allocated to R-customers. However, we can see from a simple counter-example that this is not always the case. For the model described here and with a distribution $F(\cdot)$ with two mass points, the results of a numerical simulation are presented in Figure 1 (with probability mass 1/2 at $\theta = 0.1$ and 2). The optimal price is not monotonic. An informal way of interpreting this result is as follows: If the price is set with full information about demand, the optimal price for the high-demand outcome is higher than the optimal price for the low-demand outcome. The uncertain demand case is a compromise between the two. At some point, as capacity increases, more "high-demand customers" are expected to be served, and the optimal price moves towards the higher price. Similar examples can be constructed with a continuous $F(\cdot)$. We will see, however, that with condition (15) the optimal price is non-increasing.

3.2 Optimal Capacity Allocation and Pricing Strategy

In the following, we explore the optimal capacity allocation and pricing strategy. To that end, we make use of the first and second order derivatives of $\tilde{H}^v(y)$. However, $\tilde{H}^v(y)$ may not be twice differentiable due to the constraint $y \leq a$ in its definition. For this reason, we introduce a family of unconstrained dynamic problems parameterized by $\epsilon > 0$. We show that the corresponding operators $\tilde{H}^{v^\epsilon}(y)$ are concave, and likewise for the optimal value functions v_t^ϵ. We then show that $v_t^\epsilon \to v_t$ pointwise when $\epsilon \to 0$, where the $v_t(\cdot)$ satisfy the optimality equations.

Consider $\mathbf{R}_{-\infty} = \mathbf{R} \cup \{-\infty\}$ and the extension of any function $\varphi(\cdot) :]\underline{x}, \overline{x}[\mapsto \mathbf{R}$ such that $\varphi(x) = -\infty$ when $x \notin]\underline{x}, \overline{x}[$. (In the following we use the same notation for a given function and its extension.) For instance, we will consider the extension of the logarithm function such that

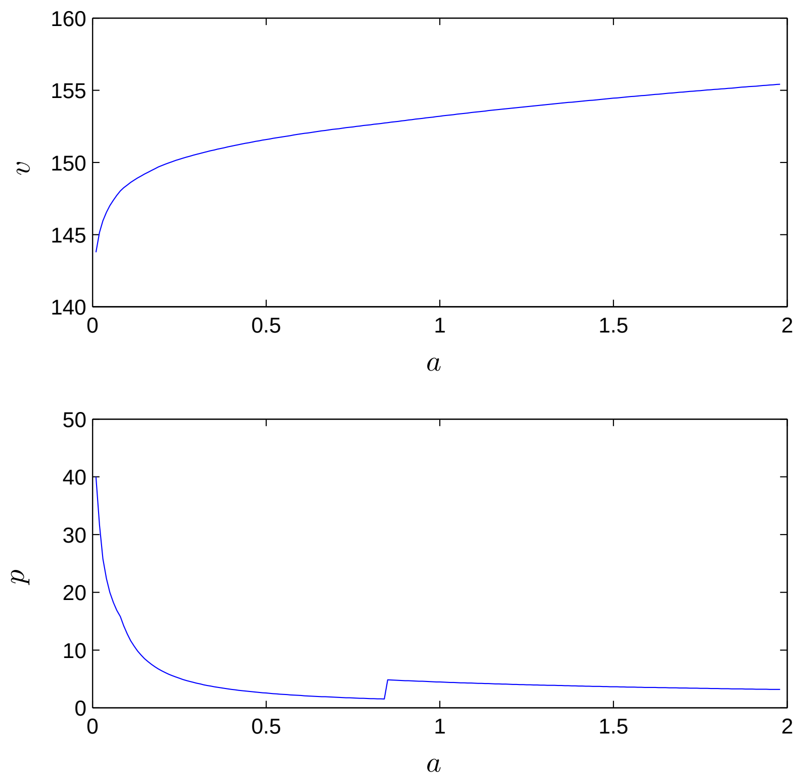

Figure 1: Example of non-monotonic price: value function and optimal price.

$\varphi(x) = -\infty$ when $x \leq 0$. The family of unconstrained problems is obtained by omitting the constraints $0 \leq y \leq a$, and introducing logarithmic barrier functions in the objective function.

More precisely, for any $\epsilon > 0$ we consider the dynamic problem,

$$v_t^\epsilon(a) \;=\; \max_y \left\{a - y + \epsilon \log[(a-y)y] + \alpha \tilde{H}^{v_{t+1}^\epsilon}(y)\right\} \tag{17}$$

$$v_T^\epsilon(a) \;=\; a. \tag{18}$$

The optimal capacity decision never equals the bounds (that is, $0 < \tilde{y}^\epsilon(a) < a$). Backward iteration then shows that $v_t^\epsilon(a)$ is twice differentiable.

Using condition (15) on the generalized failure rate, we can show that $\tilde{H}^{v_t^\epsilon}(\cdot)$ is concave if $v_t^\epsilon(\cdot)$ is also concave.

Lemma 3 *Assume that condition (15) holds. For any $\epsilon > 0$, if $v_{t+1}^\epsilon(\cdot)$ is twice differentiable and concave, then $\tilde{H}^{v_{t+1}^\epsilon}(\cdot)$ is also twice differentiable and concave. Furthermore, the pricing decision $\tilde{p}^\epsilon(y)$ maximizing $H^{v_{t+1}^\epsilon}(p, y)$ is non-increasing in y.*

Proof. For clarity, we drop the subscript of $v_{t+1}^\epsilon(\cdot)$ in the following, writing simply $v^\epsilon(\cdot)$. Similarly we suppress the subscript ϵ from $\tilde{p}^\epsilon(y)$ in the following.

We check the conditions for joint strict concavity of H^{v^ϵ}, locally at the optimal decision $(\tilde{p}(y), y)$. By definition,

$$H^v(y,p) = \int_{\underline{\theta}}^{z(y,p)} v^\epsilon(r(p)\theta)\theta f(\theta)d\theta + v^\epsilon(py)G(z(y,p)), \tag{19}$$

where $z(y,p) \triangleq y/\gamma(p)$ which we sometimes denote simply by z. Since $v^\epsilon(\cdot)$ is twice differentiable, each term of (19), and hence H^v, are twice differentiable. Furthermore, since H^{v^ϵ} is concave in each of the variables, we only need to show

$$\nabla_{py}^{v^\epsilon}(p,y) \triangleq H_{pp}^{v^\epsilon}(p,y)H_{yy}^{v^\epsilon}(p,y) - [H_{py}^{v^\epsilon}(p,y)]^2 > 0$$

where $H_{pp}^{v^\epsilon}$, $H_{yy}^{v^\epsilon}$, and $H_{py}^{v^\epsilon}$ represent the second derivatives of H^{v^ϵ}. Likewise, we denote by $H_p^{v^\epsilon}$ and $H_y^{v^\epsilon}$ the first derivative of H^{v^ϵ} with respect to p and y. For any given y, optimal price $\tilde{p}(y)$ exists and is an interior point. Therefore, the first-order optimality condition for p applies, that is

$$H_p^{v^\epsilon}(p,y) = 0, \tag{20}$$

where

$$H_p^{v^\epsilon}(p,y) = r'(p)\int_{\underline{\theta}}^{z}(v^\epsilon)'(r(p)\theta)\theta f(\theta)d\theta + y(v^\epsilon)'(py)G(z(y,p)). \quad (21)$$

Since we are concerned with concavity at the optimal $\tilde{p}$, we may use the optimality conditions (20-21) to obtain

$$\begin{aligned} H_{pp}^{v^\epsilon}(p,y) &= -y\left(\frac{r''(p)}{r'(p)} + p\frac{(\gamma'(p))^2}{\gamma^2(p)}g(z)\right)G(z)(v^\epsilon)'(py) + y^2G(z)(v^\epsilon)''(py) + \\ & (r'(p))2\int_{\underline{\theta}}^{z}(v^\epsilon)''(r(p)\theta))\theta^2 g(\theta)d\theta \end{aligned} \quad (22)$$

Similarly, the other derivatives can be written as

$$H_y^{v^\epsilon}(p,y) = p(v^\epsilon)'(py)G(z) \quad (23)$$

$$H_{yy}^{v^\epsilon}(p,y) = p^2(v^\epsilon)''(py)G(z) - \frac{p}{\gamma(p)}(v^\epsilon)'(py)f(z) \quad (24)$$

$$H_{py}^{v^\epsilon}(p,y) = (v^\epsilon)'(py)\left[1 + p\frac{\gamma(p)'}{\gamma(p)}g(z)\right]G(z) + py(v^\epsilon)''(py)G(z) \quad (25)$$

We show next that $H_{py}^{v^\epsilon} < 0$ which implies that the derivative of $\tilde{p}(\cdot)$ is negative with $\tilde{p}'(\cdot) = -H_{py}^{v^\epsilon}/H_{pp}^{v^\epsilon}$. From the concavity of $v^\epsilon(\cdot)$, $H_{py}^{v^\epsilon}$ is negative if $1 + p\gamma(p)'/\gamma(p)\, g(z) = 1 - g(z)/e(p) < 0$ which is equivalent to condition (15).

We can now study the concavity of H^{v^ϵ} locally at the optimal decision $(\tilde{p}(y), y)$. From the previous derivatives (22), (24) and (25) we obtain, after algebraic simplification,

$$\begin{aligned} \nabla_{py}^{v^\epsilon}(p,y) &= H_{pp}^{v^\epsilon}(p,y)H_{yy}^{v^\epsilon}(p,y) - [H_{py}^{v^\epsilon}(p,y)]^2 \\ &= ((v^\epsilon)'(py))^2G^2(z)\left[p\frac{r''(p)}{r'(p)}g(z) - 1 - 2p\frac{\gamma'(p)}{\gamma(p)}g(z)\right] \\ &- (v^\epsilon)'(py)\,(v^\epsilon)'(py)G(z)^2py\left[p\frac{r''(p)}{r'(p)} + 1 + \left(p\frac{\gamma'(p)}{\gamma(p)} + 1\right)^2 g(z)\right] \\ &+ p(r'(p))^2\int_{\underline{\theta}}^{z}(v^\epsilon)''(r(p)\theta))\theta^2 f(\theta)d\theta\left(p(v^\epsilon)''(py)G(z) - \frac{1}{\gamma(p)}(v^\epsilon)'(py)f(z)\right) \end{aligned}$$

Recall that the optimal price is larger than or equal to $\underline{p}$. Hence from the concavity of $r(\cdot)$, $r'(p) \le 0$ and $r''(p)/r'(p) \ge 0$ at the optimal price. From the concavity of $v^\epsilon(\cdot)$ and using the previous definition of $h(\cdot)$, we deduce

$$\nabla_{py}^{v^\epsilon}(p,y) \ge -((v^\epsilon)'(py))^2G^2(z)\left[1 - 2e(p)g(z)\right]$$

But from conditions (15), $1-2e(p)g(z) < -1$ and $\nabla^{v^\epsilon}_{py}(p,y) > 0$. $H^{v^\epsilon}(p,y)$ is then locally strictly concave at $(\tilde{p}(y), y)$ and from Lemma 7 (in appendix), $\tilde{H}^{v^\epsilon}(y)$ is twice differentiable and concave. To complete the condition of $\tilde{p}(y)$ interior for Lemma 7, note from (21) that, for any y, $H^{v^\epsilon}_p(p,y)$ is strictly positive for p small enough (both terms become positive with $r\prime(p) > 0$ for $p < \overline{p}$), and strictly negative for p close enough to $\overline{p}$ (the first term becomes increasingly negative from concavity of $v^\epsilon(\cdot)$ and since $r\prime(p) > 0$ for $p > \overline{p}$, while the second term goes to zero since $z(p,y)$ goes to infinity).

■

It follows that equation (17) preserves concavity, which we establish in the next result.

Lemma 4 *Assume that condition (15) holds. For any $\epsilon > 0$, if $v^\epsilon_{t+1}(\cdot)$ is twice differentiable and concave, then $v^\epsilon_t(\cdot)$ is also twice differentiable and concave. Also, the optimal pricing decision $p^{*\epsilon}_t(a)$ is non-increasing in the current assets a.*

Proof. Take $\phi(y,a) \triangleq a - y + \epsilon \log((a-y)y) + \tilde{H}^{v^\epsilon}(y)$ in (17). From Lemma 3, $\phi(y,a)$ is twice differentiable and jointly strictly concave (from the $\log(a-y)y$ term). We can hence apply Lemma 7 (the optimal y is interior, since the barrier goes to $-\infty$ at both 0 and a). to show that $v^\epsilon_t(\cdot)$ is also twice differentiable and concave. Note that $p^{*\epsilon}_t(a) = \tilde{p}^\epsilon(y^{*\epsilon}(a))$. The second part then follows from Lemma 3 and the fact that $y^{*\epsilon}(a)$ increases in a.

■

Starting from $v^\epsilon_T(a) = a$ is concave, Lemma 4 ensures that $v^\epsilon_t(\cdot)$ is concave for $t = 0, 1, \ldots, T$. Now, by letting $\epsilon \to 0$, we obtain the same result for $v_t(\cdot)$ in the original problem.

Lemma 5 *If (15) holds, then for $\epsilon \to 0$ the $v^\epsilon_t(\cdot)$ converge pointwise to the $v_t(\cdot)$ which solve the optimality equations (12). Further, the $v_t(\cdot)$ are concave and the optimal pricing decision $\tilde{p}_t(\cdot)$ is non-increasing.*

Proof. We show the result by iterating on t. Assume that for period t, $v^\epsilon_t(\cdot)$ converges pointwise to the optimal value function $v_t(\cdot)$ of the original problem, which is increasing and concave. We propagate pointwise convergence to the next time period (Lemmas 8, 9, 10, and 11 are in appendix):

- From Lemma 8, for ϵ sufficiently small, $|v_t^\epsilon(\cdot)| \leq l(\cdot)$ where $l(\cdot)$ is integrable. By Lebesgue's dominated convergence theorem (see, *e.g.*, Billingsley 1995, p.209, thm. 16.4), this implies that $H^{v_t^\epsilon}(\cdot,\cdot)$ converges point-wise to $H^{v_t}(\cdot,\cdot)$.

- Since $H^{v_t^\epsilon}(p,y)$ is concave in $p \in]0,\overline{p}[$, using Lemmma 10 at each y shows that $\tilde{H}^{v_t^\epsilon}(y)$ converges to $\tilde{H}^{v_t}(y)$. Which is to say, $\tilde{H}^{v_t^\epsilon}(\cdot)$ converges pointwise to $\tilde{H}^{v_t}(\cdot)$.

- Since $a - y + \epsilon \log(a-y)y + \alpha\tilde{H}^{v_t^\epsilon}(y)$ is concave in $y \in]0,a[$ and converges pointwise (for each a and y) to $\phi(y) = a - y + \alpha\tilde{H}^{v_{t-1}}(y)$, we can apply Lemma 10 at each a to show that $v_{t-1}^\epsilon(a)$ converges to $v_{t-1}(a)$. Which is to say, $v_{t-1}^\epsilon(\cdot)$ converges pointwise to $v_{t-1}(\cdot)$. Lemma 10 also shows that $v_{t-1}(a)$ is optimal since $v_{t-1}(a) = \max_{0\leq y\leq a}\phi(y)$ corresponds to the optimality equations (12).

Concavity of $v_{t_1}(\cdot)$ is now immediate from the concavity of $v_{t-1}^\epsilon(\cdot)$, which is preserved through pointwise convergence.

To establish that $\tilde{p}(\cdot)$ is non-increasing we need to show that, for each y, $\tilde{p}_{t-1}^\epsilon(y) = \text{argmax}_p H^{v_t^\epsilon}(p,y)$, which is non-increasing in y from Lemma 3, converges to $\text{argmax}_p H^{v_t}(\cdot,y)$. We do this by checking that conditions for Lemma 11 apply. For each $y > 0$, we use Lemma 9 to verify that $H^{v_t}(\cdot,y)$ is locally strictly concave at the maximum and therefore has a unique maximizer. Since $H^{v_t}(\cdot,y)$ is concave and goes to zero at $p = 0$ and at $p = \overline{p}$, it has an interior maximum. The function $py \wedge r(p)\theta$ satisfies the conditions for Lemma 9 and, since the next period's $v_t(\cdot)$ is increasing and concave, $\phi(p,\theta) = v_t(py \wedge r(p)\theta)f(\theta)$ also satisfies the conditions for Lemma 9 (in $[\underline{p},\overline{p}] \times [\underline{\theta},\bar{\theta}]$).

■

We are now ready to characterize the optimal capacity allocation and pricing decisions.

Theorem 2 *If condition (15) holds, the optimal policies are of threshold type and the optimal pricing decisions are non-increasing in the current assets.*

Proof. Immediate from Lemma 4 and the concavity of the $\tilde{H}^{v_t}(\cdot)$.

■

A threshold policy may appear somewhat counter-intuitive to many organizations. To our knowledge, no empirical studies exist on how resources are actually allocated in practice between mission and revenue customers in non-profit organizations. However, strategies based on proportional allocations may be more appealing to managers. Under such policies, a fixed percentage of the current assets is allocated to the M-customers, so that the organization always contributes to its mission. In contrast, under a thresholds policy, no M-customers are served if $a_t \leq a^*$. This may, in the long run, jeopardize the culture of the organization as non-profit oriented, which in turn can affect its ability to recruit volunteers and to provide high service quality to the M-customers, among other concerns (see Dees 1998; Dees and Anderson 2003). From Theorem 2, the optimal threshold policy has a higher social impact than proportional allocation strategies. If this gain is significant, the organization may benefit from obtaining buy-in from its stakeholders on the benefit of a threshold policy, while acting in other ways to maintain a non-profit oriented culture. This is discussed further in Section 4.

Theorem 2 also shows that the optimal price is decreasing in the current assets. However, the theorem does not give any more information regarding the shape of $p_t^*(\cdot)$. The following proposition states properties satisfied by $ap_t^*(a)$. This quantity represents the total revenue that the organization generates if its current assets are fully allocated to R-customers and demand exceeds capacity. These properties can be of use in developing effective heuristics.

Proposition 3 *If condition (15) holds, $ap_t^*(a)$ is increasing in a. Further, $\lim_{a\to 0} ap_t^*(a) = 0$.*

Proof. To show the first part, recall that the derivative of the optimal price for the ϵ-problems is equal to $\tilde{p}' = -H_{py}^{v^\epsilon}/H_{pp}^{v^\epsilon}$. From (22), (25), and $H_{pp}^{v^\epsilon} \leq 0$, it follows that

$$
\begin{aligned}
(y\tilde{p}(y))' \geq 0 \quad &\Leftrightarrow \quad y\tilde{p}'(y) + \tilde{p}(y) \geq 0 \\
&\Leftrightarrow \quad -H_{py}^{v^\epsilon} y + H_{pp}^{v^\epsilon}\tilde{p}(y) \leq 0 \\
&\Leftrightarrow \quad -(v^\epsilon)'(\tilde{p}(y)y)G(z)y\left(p\frac{r''(\tilde{p}(y))}{r'(\tilde{p}(y))} + 1 + p\frac{\gamma'(\tilde{p}(y))}{\gamma^2(\tilde{p}(y))}(\gamma(\tilde{p}(y)) + \tilde{p}(y)\gamma'(\tilde{p}(y)))g(z)\right) \\
&\qquad + (r'(\tilde{p}(y)))^2\int_{\underline{\theta}}^{z}(v^\epsilon)'(r(\tilde{p}(y))\theta))\theta^2 g(\theta)d\theta \leq 0 \\
&\Leftarrow \quad -(v^\epsilon)'(\tilde{p}(y)y)G(z)y\left(p\frac{r''(\tilde{p}(y))}{r'(\tilde{p}(y))} + 1 + p\frac{\gamma'(\tilde{p}(y))}{\gamma^2(\tilde{p}(y))}r'(\tilde{p}(y))g(z)\right) \leq 0
\end{aligned}
$$

where the last inequality holds since $r'() \leq 0$ at the optimal price and the result follows from Lemma 5.

■

We conclude this section by exploring the infinite horizon case. We extend our results by letting T approach $+\infty$.

Corollary 4 *If condition (15) holds, the optimal policy for the infinite horizon case is of threshold type. Further, the optimal pricing decision $p^*(\cdot)$ is non-increasing in the current assets*

Proof. The proof is similar to the proof of Corollary 1 with $M = \alpha r(\underline{p})\mathbf{E}_\Theta \Theta$ and $v_n(a) \leq a + \frac{1-\alpha^n}{1-\alpha} M$ where $n = T - t$.

■

4 Numerical Studies

4.1 The Value of Optimal Capacity Decisions

We begin by studying the value of optimal capacity decisions, by considering the case where a banking option is not available and the price is fixed in a typical example. The parameters for this example are $T = 8$, $\alpha = 0.85$, θ uniform in $[1, 2]$ and $p = 8.76$. The distribution satisfies the conditions for theorem 2, and the price was selected as the price with is optimal above the threshold when a pricing decision is included with the same problem parameters.

The optimal policy for this problem is of threshold type, as shown in corollary 2. Nevertheless, a proportional allocation policy may be appealing because of its simplicity. Such a policy, where the same fraction of assets is allocated to mission customers independently of the asset, is also advantageous in other respects. Serving mission customers in all periods may be important in maintaining the mission culture of the organization, especially if is also includes volunteer staff. We compare the performance of the optimal policy with a proportional allocation policy. This tradeoff should be taken in consideration in deciding whether an otherwise sub-optimal policy is to be followed.

In practice, it may be difficult to find the best fraction for the proportional allocation policy, that is, what fraction of a is allocated to y. The best value y/a, in term of expected discounted social impact, will depend on the initial endowment. Figure 2 plots the value function for three different y/a proportions (where, by definition of fixed-proportion policy, each proportion remains

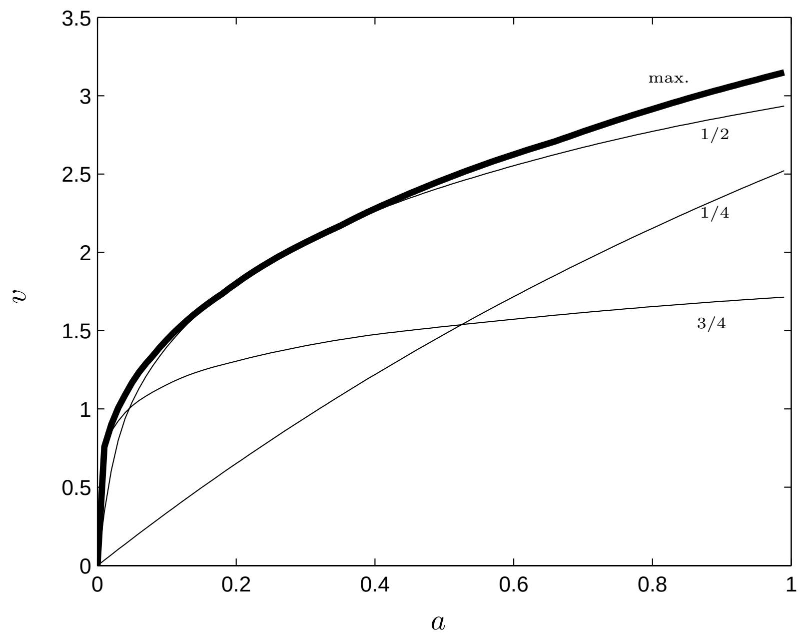

Figure 2: Capacity allocation to each class of customers. Value funtions for fixed-proportion policy with different allocation ratios, and maximum over all ratios.

constant throughout all periods). Also plotted is the maximum of the value functions associated with every possible choice for the fixed proportion. This corresponds to the organization making an initial decision on the proportion optimally based on the asset level at begining of the first period.

Figure 3 compares the value function associated with the optimal policy (where the y/a ratio can change in each period) with the maximum over different ratios of the value functions for the proportional allocation policy. Note that relative difference between the two curves is substantial. For higher values of a (*e.g.*, $a = 1$) it is on the order of 30%. Near zero it becomes extremely large (*e.g.*, higher than 100% for $a < 0.02$). The value of the optimal policy relative to the best fixed proportion is very high if the organization starts with few assets. In this case, the optimal policy allows the organization to grow more aggressively and more quickly reach a substantial asset level. This allows the organization to then fully exploit the demand from revenue customers, and use the resulting resources to serve mission customers.

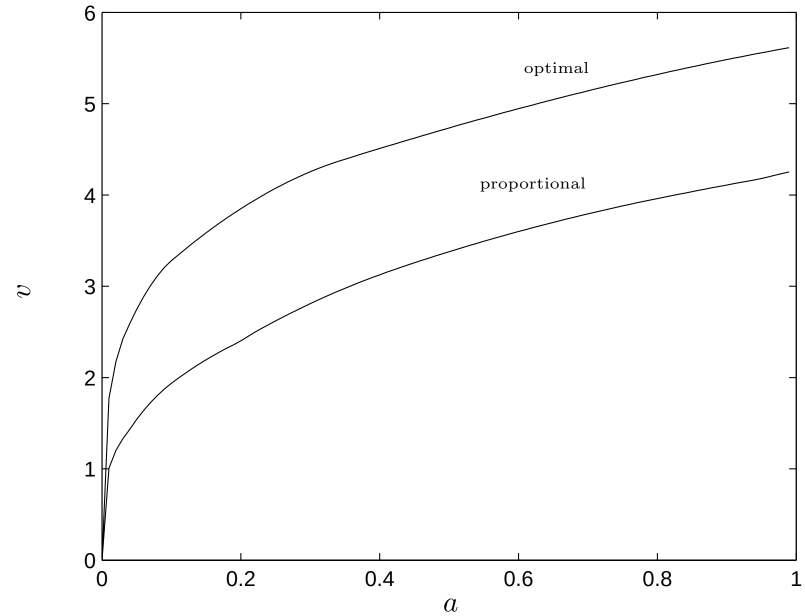

Figure 3: Capacity allocation to each class of customers. Max. of value functions for proportional-allocation policies, and value function for optimal policy.

4.2 The Value of Banking

We begin by considering the problem with a fixed price and banking decision. Figure 4 shows the optimal policy, for a typical case. The parameters are $T = 8$ $\beta = 1.15$, $\alpha = 0.85$, $p = 5.56$, and θ uniform in $[0, 1]$. In agreement with theorem 1, the optimal policy is of threshold type for $z+y$, and z is non-decreasing. In our numerical examples we have also always found y to be non-decreasing.

Note that banking becomes attractive when the organization has more assets. For a low asset level, the risk of unmet demand is lower. The organization's focus is then on growing the available cash, and all assets are allocated to servicing mission customers, which generates higher expected profit. At a higher asset level, the organization is at a higher risk from realizations with very low demand, and banking is then used to hedge the outcome and avoid transitioning to a very low cash positions.

The value of banking depends on the price being charged. To investigate this we compute the value function for the same return on the banked assets and different prices. We also compute, for the same set of prices, the value function without banking (or, equivalently, with zero return on banking). For each price, we compute the ratio of the two value functions, and the maximum of this ratio over all possible states at the initial period (*i.e.*, the initial asset level). Figure 5 plots this maximum ratio as a function of price, with T, β and α as above. The main observation is that banking is of relatively limited value, providing for a gain of about 4% or less (and, again, note that this is the highest gain over all initial asset levels).

The expected gain from having the banking decision available is not monotonic in the price. Banking is more valuable when allocating capacity to R-customers is profitable but the profit margins are small. For small prices, R-customers are never profitable and it is optimal to spend all the capital on M-customers in the first period. The option of banking is therefore of no value. At the other end of the plot, when the profit margin is large, there is also less value in having the banking option available. For higher prices (and with the same demand distribution) the return rate from banking is, by comparison, less attractive. The organization has less of an incentive to hedge the outcome by banking since, given the high profit margin, it is easier to recover quickly from a period with unusually low demand. (For very large prices, we found the percentage gain to converge to a non-zero value.)

On the other hand, note that the value of the option of taking a loan, would be substantial for low asset-levels (given a reasonable rate). In particular, the value function would no longer be zero at zero net capital.

4.3 The Value of Pricing Decisions

We now investigate the value of optimal pricing decisions. The model used is that of section 3, which excludes the option of banking.

Figure 6 shows a typical example of the optimal policy. The parameters are $T = 8$ $\beta = 1.15$, $\alpha = 0.85$, and θ uniform in $[1, 2]$ (which satisfies conditions for theorem 2). The structure of the optimal policy is consistent with theorem 2. The optimal capacity policy is of threshold type, and the optimal pricing policy is non-increasing.

We compare the optimal policy with a fixed-price policy. For the fixed-price policy, we use the price which is optimal above the threshold in the optimal price policy. A number of recent articles investigating different models in for-profit firms have found that the addition of dynamic pricing

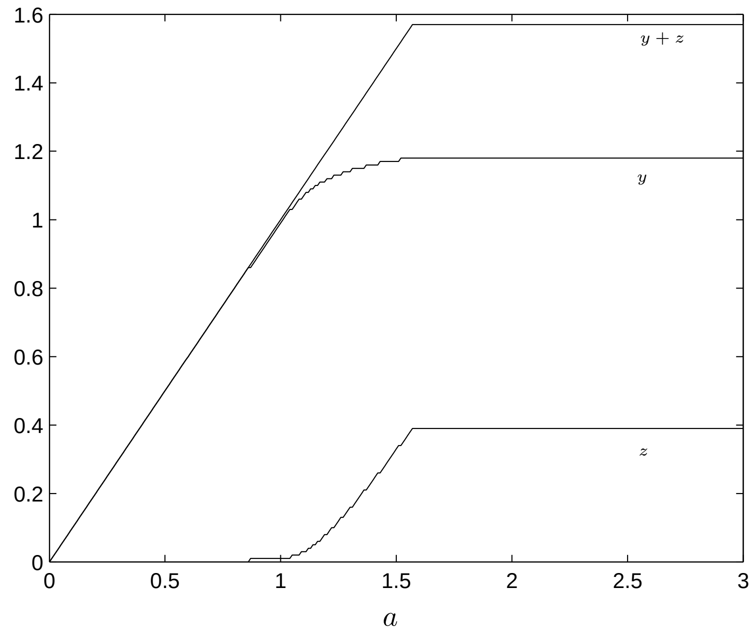

Figure 4: Optimal policy with banking.

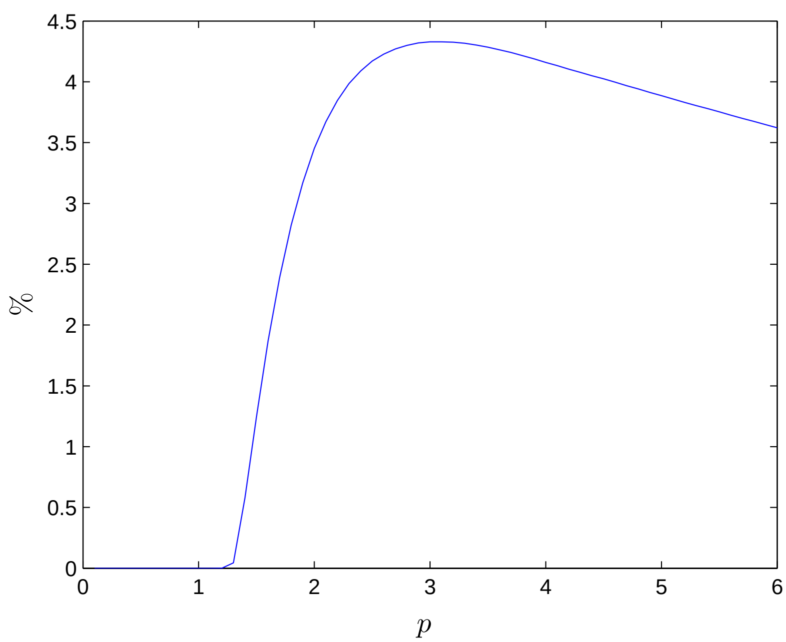

Figure 5: Maximum percentage expected gain from banking. Maximum ratio of value functions over different prices (and same demand distributions).

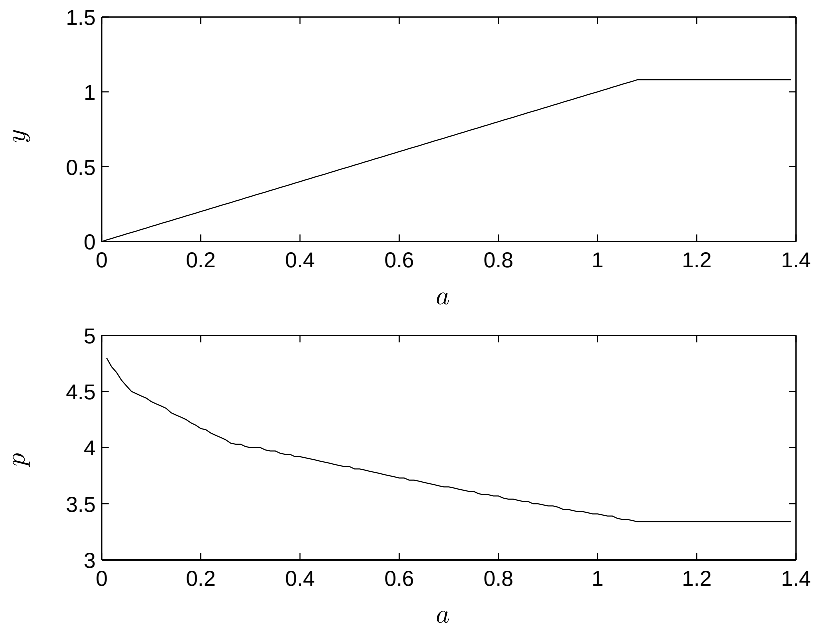

Figure 6: Optimal policy with dynamic pricing ($y(a)$ and $p(a)$ for first period).

provides only small perceptual gains in expected revenue (usually 5% or less; see for instance Gallego and van Ryzin 1994; Chen, Wu, and Yao 2004). However, in this case, and as seen in Figure 7, the difference in expected value can be substantial. This is driven by the fact that the capacity constraint is different from period to period. When the asset level is small, there will be a large amount of unmet demand if the threshold price is charged. Note that the difference in expected value between the two policies is larger for a small initial endowment. This is when the organization is sure to benefit from charging a significantly higher price in the initial periods.

Note, however, that if loans are available at a reasonable rate there will be little or no value in dynamic pricing. Even when the asset-level is low, the optimal policy will be to borrow capital in order to provide optimal capacity for the given demand distribution.

5 Extensions and Future Research

5.1 Other Demand Functions

It is worth noting that condition (15) is used in the proofs only *at the optimal price*. That is, if we assume that for all a

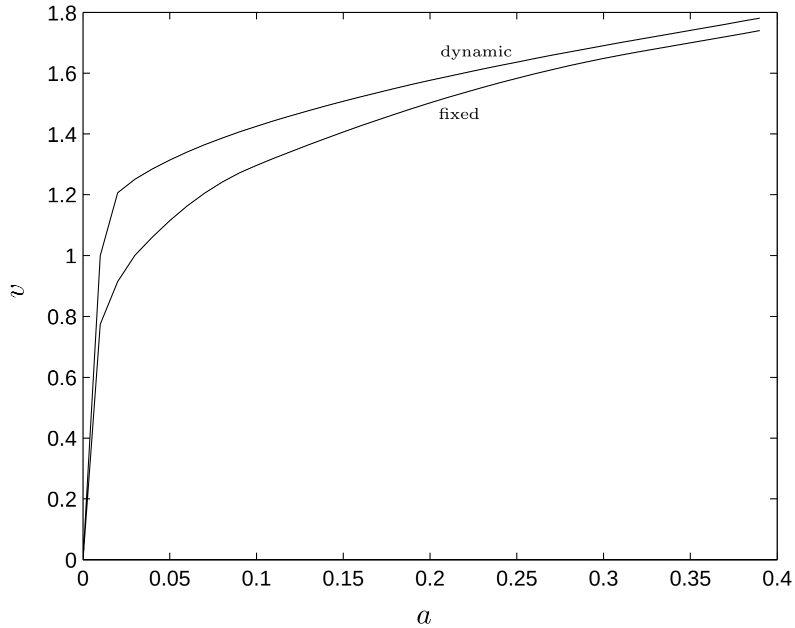

Figure 7: Value functions with dynamic price and with fixed price.

$$g\left(\frac{a}{\gamma(p^*(a))}\right) \geq \frac{1}{e(p^*(a))} \tag{26}$$

then Theorem 2 and Proposition 2 still hold.

One approach to extend our results would then be to find sufficient conditions on $f(\cdot)$ guarantying (15). Note however that showing (15) for more general $f(\cdot)$ would require propagating properties stronger than concavity on the social-impact-to-go $v_t(\cdot)$. We haven't done this, but we believe it is an interesting direction for future research.

Also, we have restricted our analysis to the multiplicative demand case. For more general demand functions of the form $D = f(p, \Theta)$ for some function $f(\cdot)$, the elasticity of revenues is given by, following Kocabiyikoğlu and Popsescu (2005),

$$\varepsilon(p, y) = p\frac{L'(p, y)}{1 - L(p, y)},$$

where $L(p, y) \triangleq P(D \geq y)$, with L' the derivative of L with respect to p. Condition (15) becomes then $\varepsilon(p, y) \geq 1$. This condition can be used to extend our results to the case of more general demand functions.

5.2 Objective Functions

We assumed linear social returns, $s\,x_t$, where x_t is the capacity allocated to M-customers in Period t. A natural extension of our model is to the case of decreasing marginal social return, where the impact of serving x mission customers is described by a concave function $s(\cdot)$. A special case of particular interest corresponds to the situation where demand for M-customers is also random. If the demand from M-customers is described by a random variable Φ, the organization contributes to its mission by $s(x) = \mathbf{E}_{\Phi}\left(x \wedge \Phi \right)$. The optimal policy is however no longer of threshold type. It should be possible, and of interest, to to derive simple and effective heuristics for this case based on the results presented here.

5.3 Flexible Capacity

We assumed that the service capacity for R-customers needs to be committed at the beginning of each period, before demand is realized. In this section we briefly explore the case where the organization has the flexibility to allocate its service capacity between the two classes of customer after the actual demand of the current period is known.

Without loss of generality, we assume that $c_R = c_M = s = 1$. At the beginning of each period, the organizations makes an investment decision and chooses how much total service capacity $c_t \leq 0$ to reserve and sets the amounts of current assets which are banked, $z_t \leq 0$. The choices of c_t and z_t are limited by the current resources, *i.e.*, $z_t + c_t = a$. Demand is then realized and the organization must decide how the capacity c_t should be allocated between the M- and R-customers given the realization of R-customers θ. In other words, $c_t = x_t + y_t$ where x_t and y_t represent the numbers of served M-customer and R-customers, respectively. The optimality equation of the maximum social-impact-to-go at period t, can then be written as

$$v_t(a) = \max_{\substack{0 \leq c, \quad 0 \leq z, \\ c + z = a}} \mathbf{E}_{\Theta} \max_{\substack{0 \leq x, \quad 0 \leq y, \\ x + y = c}} x + \tau(y \wedge \Theta) + \alpha v_{t+1}\left(p(y \wedge \Theta) + \beta z\right), \quad (27)$$

with $v_T(a) = a$. This equation can be simplified by writing $c = a - z$ and $x = c - y = a - y - z$, so that

$$v_t(a) = a + \max_{0 \le z \le a} \mathbf{E}_\Theta \max_{0 \le y \le a-z} J^{v_{t+1}}(y, z, \Theta), \tag{28}$$

where the operator J^v is defined as

$$J^v(y, z, \theta) = -z - y + \tau\,(y \wedge \theta) + \alpha v_{t+1}\,(p(y \wedge \theta) + \beta z\,). \tag{29}$$

Using a similar approach, Proposition 1 can be shown to hold when the service capacity is shared. Again, the interesting case is $\alpha\beta < 1$ and $\alpha p + \tau > 1$. We first address the capacity allocation decision after demand is realized. The next result characterizes the optimal service capacity decision for R-customers, $\tilde{y}(c, \theta)$, given the capacity choice c and the total number of R-customers θ.

Proposition 4 *When capacity is shared, it is optimal to serve as many R-customers as possible and*

$$\tilde{y}(c, \theta) = c \wedge \theta.$$

Proof. This is shown from a sample-path argument. Consider a supposedly optimal policy with optimal decisions (y_t, z_t), $t < T$ and let x_t, $t < T$ be the corresponding number of served M-customers. Suppose there is a time instant s such that $y_s < c \wedge \theta$, with $x_s > 0$. Consider now an alternative policy with decision $\bar{y}_s = c \wedge \theta$ at time s, and $\bar{y}_t = y_t$, $s \neq t$. Assume also that the banking decision is such that $\bar{z}_t = z_t$ for all t. It follows that the corresponding number of served M-customers $\bar{x}_t$, $t < T$ is such that $\bar{x}_s = x_s - (\bar{y}_s - y_s)$, $\bar{x}_{s+1} = p(\bar{y}_s - y_s) + x_{s+1}$ and $\bar{x}_t = \bar{x}_t$ for $t < s$ and $t > s + 1$. For $t < s$ the policies are coupled and have the same social impact in each period. The difference between the social impact of the modified policy and the supposedly optimal is $\tau(\bar{y}_s - y_s) + (\bar{x}_s - x_s) = (\bar{y}_s - y_s)$ at time s, and to $p(\bar{y}_s - y_s)$ at time $s+1$. The policies are coupled for later periods and since $(\tau - 1)(\bar{y}_s - y_s) + \alpha p(\bar{y}_s - y_s) = (\tau + \alpha p - 1)(\bar{y}_s - y_s) > 0$ the modified policy outperforms the supposedly optimal policy.

■

Hence, the optimal allocation decision is an infinite threshold policy. Proposition 4 allows us to simplify the optimality equations to

$$v_t(a) \quad = \quad a + \max_{0 \le z \le a} \quad \mathbf{E}_\Theta J^{v_{t+1}}((a-z) \wedge \Theta, z, \Theta). \tag{30}$$

The following lemma states that the concavity of the social-impact-to-go can be propagated.

Lemma 6 *If $v_{t+1}(\cdot)$ is differentiable, non-decreasing and concave, then $v_t(\cdot)$ is also differentiable, non-decreasing and concave.*

Proof. The proof is similar to the proof of Lemma 1, noting that $J^{v_{t+1}}((a-z) \wedge \theta, z, \theta)$ in (30) is jointly concave in a and z for any increasing concave function $v_{t+1}(\cdot)$.

■

Since $v_T(a) = a$, the previous Lemma implies that the optimal value function is concave. The corresponding optimal banking decision $z^*(a)$ (defined as the minimum optimal decision in case of multiple optima) is not necessarily of threshold type since the constraint a appears in the maximand. The optimal decisions $z^*(a)$ and $c^*(a) \triangleq a - z^*(a)$ can be some other function of a. On the other hand, if $\tau \ge 1$, $z^*(a)$ is monotone in the assets as stated in the following proposition.

Proposition 5 *When capacity is shared, the optimal banking decision $z^*(a)$ is non-decreasing in a if $\tau \ge 1$.*

Proof. From Lemma 6, the optimal $v_t(\cdot)$ is differentiable, non-decreasing and concave. The optimal banking decision satisfies the first order condition which, after some algebra, is equivalent to

$$(1-\tau)G(a-z) + \alpha\beta \int_0^{+\infty} v'(p\theta + \beta z) f(\theta) d\theta + \alpha(\beta - p) v'(pa + (\beta - p)z) G(a-z) = 0 \tag{31}$$

The first term of Equation (31) increases in a and decreases in z, since a $G(\cdot)$ is decreasing function and $1-\tau \le 0$. The second term of Equation (31) also increases in a and decreases in z since $v'_{t+1}(\cdot)$ is a positive decreasing function. Similarly, for $p > \beta$, the last term is increases in a and decreases in z, so that $z_t^*(a)$ increases in a.

■

Based on these results, we can now consider the corresponding problem with dynamic pricing. Following the approach developed in Section 3, consider a demand function with multiplicative uncertainty, $D = \gamma(p)\Theta$. When banking is ignored ($\beta = 0$) and R-customers do not have any impact ($\tau = 0$), the optimality equation of the maximum social-impact-to-go at period t can be written as

$$\begin{aligned} v_t(a) &= a + \max_{0 \le p \le \overline{p}} \mathbf{E}_\Theta J^{v_{t+1}}(a \wedge \gamma(p)\Theta\,, 0, \gamma(p)\Theta) \\ &= a + \max_{0 \le p \le \overline{p}} -\mathbf{E}_\Theta[a \wedge \gamma(p)\Theta] + \alpha H^{v_{t+1}}(p, a), \end{aligned} \tag{32}$$

where $H^v(\cdot,\cdot)$ is defined as in (13).

However, the term $-\mathbf{E}_\Theta[a \wedge \gamma(p)\Theta]$ is, in general, not concave in p for decreasing $\gamma(\cdot)$ (whether $\gamma(\cdot)$ is convex, concave or even linear). Condition (15), for instance, does not guaranty concavity of this term. In fact, the optimal social-impact-to-go may not be concave for reasonable demand functions. Characterizing the optimal policies for this problem would then appear then to be challenging.

6 Conclusion

The problem faced by a non-profit organization that also runs a for-profit operation to generate resources differs in fundamental ways from the profit-maximization problem of commercial ventures. To our knowledge, we provide the first analytical approach for examining resource management issues in this context. This is likely to become a more topical question, as a decrease in the number of grants and donations has put increased pressure on non-profit organizations to become self-sustaining. As noted in the introduction, nonprofit organizations represent a large section of the economy (7% to 10%).[7] There has been little or no research on this subject in the field of operations management. We believe the approach proposed in this paper to be a productive initial framework to pursue such work.

We have investigated how a nonprofit organization should dynamically allocate its assets over time, between its revenue generating activities and its mission, in order to maximize the organiza-

[7] A recent study (Bradley, Jansen, and Silverman 2003) estimates that the nonprofit sector could earn an additional USD 100 billion from improved management processes.

tion's social impact. Theoretical analysis and numerical studies suggest that there is limited value in the option of banking assets from one period to the next. Dynamic pricing in the for-profit side of the operation, on the other hand, can in some circumstances be of significant value.

This paper does not address strategic infrastructure investment decisions. Our concern is to provide insight into operational decisions. We show that the optimal capacity allocation policy is of threshold type. In practical terms this means that, should an adverse cash situation arise in a given period which limits the organization's ability to provide services, it is best for the organization not to compromise its revenue source. If the asset-level is low, the organization should first re-build its asset-base by servicing exclusively revenue customers, and only then act towards its mission. Over all periods, this policy will allow the organization to have a higher expected social impact, serving more mission customers. While this needs to be balanced with other considerations such as maintaining the organization's mission culture (which will likely become an issue if the organization only services revenue customers for several consecutive periods), the analysis allows us to quantify the tradeoff.

A Appendix

A.1 Sample Path Approach

In Huh and Janakiraman (2005), the authors propose a novel approach that simplifies existing analysis of periodic-review inventory systems with pricing. In particular, their elegant framework includes and extends all existing results for models with backorders (Federgruen and Heching 1999, Chen and Simchi-Levi 2004a; Chen and Simchi-Levi 2004b) or with lost sales Chen, Ray, and Song (2003).

In the single period version of these systems, given the current on-hand inventory x, the manager must choose the price p and the inventory level $y \geq x$ that maximize the corresponding expected profit $\pi(p, \theta)$, where Demand $D(p, \Theta)$ is price-sensitive. In the multiple period case, after demand is realized, the remaining net inventory (equal to $[y - D(p, \theta)]^+$ for the lost sales case) is carried over to the next period.

The core of the approach developed by Huh and Janakiraman (2005) is to propose conditions for $\pi(p, \theta)$ (the "Unifying Assumption") guarantying the classical (s, S)-policy to be optimal for *all possible realizations* of Θ. More precisely, denote y^* the maximizer of $\pi(p, \theta)$ and consider time period t. This unifying assumption implies that for any decision (y_t^2, p_t^2) for which $y_t^2 \neq y^*$, there is a choice (y_t^1, p_t^1) such that, *for all* θ:

1. $\pi(y_t^1, p_t^1) > \pi(y_t^2, p_t^2)$, *i.e.*, (y_t^1, p_t^1) increases the profit generated in the current period,

2. $[y_t^1 - D(p_t^1, \theta)]^+ \leq [y_t^2 - D(p_t^2, \theta)]^+$, *i.e.*, the inventory levels in the following period are such that $x_{t+1}^1 \leq x_{t+1}^2$.

(where $[y_t^1 - D(p_t^1, \theta)]^+$ is replaced by $[y_t^1 - D(p_t^1, \theta)]$ in the backorder case, see the proof of Proposition 3.2 in Huh and Janakiraman (2005)).

The fact that $x_{t+1}^1 \leq x_{t+1}^2$ allows System 1 to be coupled with System 2 starting from Period $t+1$. Indeed the inventory level of System 1, y_{t+1}^1, can be chosen to be equal to the inventory level of System 2, y_{t+1}^2, since $y_{t+1}^1 = y_{t+1}^2 \geq x_{t+1}^2 \geq x_{t+1}^1$ (recall that we need $y \geq x$). Since System 1 generates a higher profit in Period t (with no ordering cost), the performance of System 2 can always be improved as long as $y_t^2 \neq y^*$.

Our capacity allocation problem with pricing can also be formulated with a profit function $\pi(p, y)$ incurred in each period. By looking one period ahead, we can set $\pi(p, y) = \alpha E[py \wedge r(p)\Theta] - y$ and rewrite the optimality equations as follows

$$v_T = 0 \tag{33}$$

$$v_t(a) = \max_{\substack{0 < p \\ 0 \le y \le a}} \pi(p, y) + \alpha H^{v_{t+1}}(p, y), \ 1 < t < T \tag{34}$$

$$v_1(a) = a + \max_{\substack{0 < p \\ 0 \le y \le a}} \pi(p, y) + \alpha H^{v_{t+1}}(p, y). \tag{35}$$

In order to proceed with an approach similar to Huh and Janakiraman (2005), we need to find conditions on $\pi(p, y)$ such that two systems with different decisions at time period t can be coupled starting from time period $t+1$. This means that at least for some values of (y_t^2, p_t^2), we should be able to find (y_t^1, p_t^1), $y_t^1 > y_t^2$, such that in the next period the assets under System 1, a_{t+1}^1, is larger than a_{t+1}^2, the assets of System 2. This would allow coupling both systems by setting $y_{t+1}^1 = y_{t+1}^2$ which would be possible since $y_{t+1}^1 = y_{t+1}^2 \le a_{t+1}^2 \le a_{t+1}^1$ (recall that we need $y \le a$ in our model). In particular, this coupling should hold at least when $y_t^2 > y_t^1 > \hat{a}_t$ with $p_t^1 \ge p_t^2$ and $p_t^2 y_t^2 > p_t^1 y_t^1$ (wich also covers the important case where $y_t^2 > y_t^1 > \hat{a}_t$ and $p_t^2 = p_t^1 = \hat{p}_t^*$), if one were to show the optimality of $\hat{a}_t$-threshold policies with decreasing price $p^*(a)$ and increasing $ap^*(a)$ (which we know to be true from Theorem 2 and Proposition 3 of this paper). The following proposition however states that such a coupling is impossible.

Proposition 6 *For any (y_t^2, p_t^2) and (y_t^1, p_t^1), such that $y_t^2 > y_t^1$, $p_t^1 \ge p_t^2 \ge \underline{p}$ and $p_t^2 y_t^2 > p_t^1 y_t^1$, there exists a realization θ such that $a_{t+1}^1 < a_{t+1}^2$.*

Proof. In the following, we drop the subscripts t and $t+1$. Choose θ such that $\theta < p^1 y^1 / r(p^1)$. Recall that $\underline{p} = argmax_p r(p)$ and $r(p)$ decreases for $p \ge p^*$ so that $r(p^2) \ge r(p^1)$. Thus $r(p^1)\theta \le r(p^2)\theta$ and since $p_t^1 y_t^1 < p_t^2 y_t^2$ we deduce $a^1 = p^1 y^1 \wedge r(p_2) < p^2 y^2 \wedge r(p_2) = a^2$.

■

A.2 Support Lemmas

Lemma 7 *Consider $\phi(\cdot,\cdot) :]\underline{x},\overline{x}[\times]\underline{y},\overline{y}[\mapsto \mathbf{R}$, with $\underline{x},\overline{x},\underline{y},\overline{y} \in \mathbf{R}$. Assume that ϕ is twice differentiable and that, for each y, $\phi(\cdot,y)$ has a unique, interior maximum for all $y \in]\underline{y},\overline{y}[$, denoted by $\tilde{x}(y) = \operatorname{argmax}_x \phi(x,y)$. If locally at $(\tilde{x}(y),y)$ for all $y \in]\underline{y},\overline{y}[$, ϕ is strictly concave, then $\tilde{\phi}(y) = \max_x \phi(x,y) = \phi(\tilde{x}(y),y)$ is twice differentiable and strictly concave.*

Proof. ϕ_x and ϕ_{xx} denote the first and second order derivatives with respect to the first variable. The meanings of the notations ϕ_y, ϕ_{yy} and ϕ_{xy} follow naturally.

From the implicit function theorem on the first-order condition $\phi_x = 0$, $\tilde{x}$ is differentiable and $\tilde{x}' = -\phi_{xy}/\phi_{xx}$. Since ϕ is twice differentiable and $\tilde{x}$ is an interior point, the derivatives of $\tilde{\phi}$ exist and are equal to

$$\tilde{\phi}'(y) = \tilde{x}'(y)\phi_x(\tilde{x}(y),y) + \phi_y(\tilde{x}(y),y) = \phi_y(\tilde{x}(y),y),$$

where we use the first order condition in $\tilde{x}$ that is $\phi_x(\tilde{x}(y),y) = 0$,

and

$$\begin{aligned}
\tilde{\phi}''(y) &= \tilde{x}'(y)\phi_{xy}(\tilde{x}(y),y) + \phi_{yy}(\tilde{x}(y),y) \\
&= -\frac{\phi_{xy}^2(\tilde{x}(y),y)}{\phi_{xx}(\tilde{x}(y),y)} \; \phi_{xy}(\tilde{x}(y),y) + \phi_{yy}(\tilde{x}(y),y) \\
&= \frac{\phi_{xx}(\tilde{x}(y),y)\; \phi_{yy}(\tilde{x}(y),y) - \phi_{xy}^2(\tilde{x}(y),y)}{\phi_{xx}(\tilde{x}(y),y)} < 0.
\end{aligned}$$

where the last inequality comes from the strict concavity of $\phi(\cdot,\cdot)$ locally at $(\tilde{x}(y),y)$ which implies that both the numerator and denominator are positive.

■

Lemma 8 *For $0 < \epsilon < 1$, the optimal $v_t^\epsilon(\cdot)$ satisfying (17) is bounded as follows,*

$$c_t^0 + c_t^1(\log a)^- \;\leq\; v_t^\epsilon(a) \;\leq\; c_t^2 + a \tag{36}$$

where we defined $(x)^- = 0 \wedge x$ and c_t^1 and c_t^2 are non-negative finite constants.

Proof. The statement is true for $t = T$ with $v_T(a) = a$. Assume the result holds at time t. We first show that

$$c_t^3 + c_t^4(\log y)^- \leq \tilde{H}^{v_t^\epsilon}(y) \leq c_t^5 \tag{37}$$

where c_t^4 and c_t^5 are non-negative finite constants. We set p equal to any p_0 to get a lower bound,

$$\begin{aligned}\tilde{H}^{v_t^\epsilon}(y) &\geq \int_0^{y/\gamma(p_0)} v_t^\epsilon(r(p_0)\theta)f(\theta)d\theta + v_t^\epsilon(p_0 y)G(y/\gamma(p_0)) \\ &\geq \int_0^{y/\gamma(p_0)} (c_t^0 + c_t^1(\log\theta)^-)f(\theta)d\theta + v_t^\epsilon(p_0 y)G(y/\gamma(p_0)) \\ &\geq c_t^0 G(y/\gamma(p_0)) + c_t^1\int_0^{y/\gamma(p_0)} \log\theta^- f(\theta)d\theta + v_t^\epsilon(p_0 y)G(y/\gamma(p_0)) \\ &\geq (c_t^0)^- + c_t^1\int_0^{y/\gamma(p_0)} (\log\theta)^- f(\theta)d\theta + (v_t^\epsilon(p_0 y))^-\end{aligned}$$

where the second inequality comes from (36) and the last one from the fact that $0 \leq G(\theta) \leq 1$ for all θ. Note that

$$\int_0^{y/\gamma(\underline{p})} (\log\theta)^- f(\theta)d\theta \geq \sup_\theta f(\theta)\int_0^1 \log\theta\, d\theta = -\sup_\theta f(\theta)$$

and since c_t^1 is non-negative we obtain from (36) (along with $(x+y)^- \geq x^- + y^-$),

$$\begin{aligned}\tilde{H}^{v_t^\epsilon}(y) &\geq (c_t^0)^- - c_t^1 \sup_\theta f(\theta) + (c_t^0)^- + c_t^1(\log(\underline{p}y))^- \\ &\geq 2(c_t^0)^- + c_t^1(\log\underline{p} - \sup_\theta f(\theta)) + c_t^1(\log y)^-,\end{aligned}$$

which shows the lower bound in (37). We are now ready to show (36) a time $t-1$. Set $y = a/2$ for which

$$\begin{aligned}v_{t-1}^\epsilon(a) &\geq a/2 + 2\epsilon\log(a) - 2\epsilon\log 4 + \alpha\tilde{H}^{v_t^\epsilon}(a/2) \\ &\geq 2(\log a)^- - 2\log 4 + \alpha\tilde{H}^{v_t^\epsilon}(a/2) \\ &\geq 2(\log a)^- - 2\log 4 + \alpha\left(c_t^3 + c_t^4(\log a/2)^-\right) \\ &\geq \alpha c_t^3 - 2\log 4 - \alpha c_t^4 \log 2 + (2 + \alpha c_t^4)(\log a)^-\end{aligned}$$

where the second inequality holds since $0 < \epsilon < 1$, the third one from (37), and the last one from $(x+y)^- \geq x^- + y^-$.

An equivalent, somewhat simpler, proof structure leads to the upper bound on $v_t^\epsilon(a)$: 1) The expectation is bounded above by a constant, and therefore so is the maximum over p; 2) The maximum over y of $a-y+\log(a-y)y+\alpha\tilde{H}^{v_{t+1}^\epsilon}(y)$ is bounded by the maximum of $a-y+\log(a-y)y$ plus the maximum $\alpha\tilde{H}^{v_{t+1}^\epsilon}(y)$; 3) Finally, use $a/2 + 2\epsilon\log a < a$, for $0 < \epsilon < 1$.

■

Lemma 9 *Consider $\phi(\cdot,\cdot) : [\underline{x}, \overline{x}] \times \Theta \mapsto \mathbf{R}$, where Θ is a subset of $\mathbf{R}$ with non-zero measure. Assume $\phi(x,\theta)$ (integrable in θ for all x) is such that for all θ and for all x, $\phi(x,\theta)$ is locally either 1) concave and strictly increasing in x, or 2) strictly concave in x. Then if $\psi(x) = \int_\Theta \phi(x,\theta)d\theta$ has and interior maximum, $\psi(\cdot)$ is strictly concave at $x^* = \operatorname{argmax}_x \psi(x)$.*

Proof. If $\phi(\cdot,\theta)$ is locally strictly concave at x^* in a set of θ with non-zero measure, then $\psi(\cdot)$ is locally strictly concave at x^*. This is the same as saying that if $\psi(\cdot)$ is not locally strictly concave at x^*, then, a. e. in θ, $\phi(\cdot,\theta)$ is not locally strictly concave at x^*. But this would then imply that condition 1) holds a. e. in θ, which would imply $\psi(\cdot)$ increasing at x^*, contradicting the optimality of x^*.

■

Lemma 10 *(Convergence of the* sup*) Consider functions $\psi^\epsilon :]\underline{x}, \overline{x}[\mapsto \mathbf{R}$, parameterized by $\epsilon > 0$, such that, for every ϵ, ψ^ϵ is concave. Given pointwise convergence to some function $\psi :]\underline{x}, \overline{x}[\mapsto [0, +\infty[$, that is*

$$\lim_{\epsilon \to 0} \psi^\epsilon(x) = \psi(x),\ \forall x \in]\underline{x}, \overline{x}[\,,$$

then ψ is concave and can be continuously extended to be $\psi : [\underline{x}, \overline{x}] \mapsto [0, \infty[$. Further, $\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x)$ exists and is equal to $\max_x \psi(x)$.

Proof. The concavity of ψ is immediate by the propagation of inequalities through pointwise convergence. It follows that ψ is continuous, bounded over $]\underline{x}, \overline{x}[$ and can therefore continuously be extended to the closed interval $[\underline{x}, \overline{x}]$. In the following ψ designates the extended function.

For the convergence of $\sup_x \psi^\epsilon(x)$, consider any maximizer of the (extended) limit function, $\tilde{x} = \operatorname{argmax}_x \psi(x)$. First, suppose $\tilde{x}$ is interior. From concavity, for any $\delta > 0$,

$$\psi^\epsilon(x) \leq u_1^\epsilon(x) = \psi^\epsilon(\tilde{x}) + (x - \tilde{x})\frac{\psi^\epsilon(\tilde{x}) - \psi^\epsilon(\tilde{x} - \delta)}{\delta},\ \forall x \notin]\tilde{x} - \delta, \tilde{x}[$$

and

$$\psi^\epsilon(x) \leq u_2^\epsilon(x) = \psi^\epsilon(\tilde{x}) + (x - \tilde{x})\frac{\psi^\epsilon(\tilde{x} + \delta) - \psi^\epsilon(\tilde{x})}{\delta},\ \forall x \notin]\tilde{x}, \tilde{x} + \delta[.$$

The bounds above imply

$$\psi^\epsilon(x) \leq \begin{cases} \max\{u_1^\epsilon(\underline{x}), u_1^\epsilon(\tilde{x}-\delta)\} & \text{for} \quad x \leq \tilde{x}-\delta \\ \max\{u_2^\epsilon(\tilde{x}-\delta), u_2^\epsilon(\tilde{x})\} & \text{for} \quad \tilde{x}-\delta \leq x \leq \tilde{x} \\ \max\{u_1^\epsilon(\tilde{x}), u_1^\epsilon(\tilde{x}+\delta)\} & \text{for} \quad \tilde{x} \leq x \leq \tilde{x}+\delta \\ \max\{u_2^\epsilon(\tilde{x}+\delta), u_2^\epsilon(\overline{x})\} & \text{for} \quad \tilde{x}+\delta \leq x. \end{cases}$$

Taking the limit as $\epsilon \to 0$ and using pointwise convergence of $\psi^\epsilon(\cdot)$ we have that, should the limit of the supremum exist,

$$\begin{aligned} \lim_{\epsilon\to 0} \sup_x \psi(x) &\leq \max\{u_1(\underline{x}), u_1(\tilde{x}-\delta), u_2(\tilde{x}-\delta), \psi(\tilde{x}), u_1(\tilde{x}+\delta), u_2(\tilde{x}+\delta), u_2(\overline{x})\} \\ &= \psi(\tilde{x}) + \max\left\{\psi(\tilde{x}) - \psi(\tilde{x}+\delta), \psi(\tilde{x}) - \psi(\tilde{x}-\delta)\right\}, \end{aligned}$$

(where $u_1(\cdot)$ and $u_2(\cdot)$ are defined like $u_1^\epsilon(\cdot)$ and $u_2^\epsilon(\cdot)$ but with $\psi(\cdot)$ instead of $\psi^\epsilon(\cdot)$). The equality is from $u_1(\cdot)$ non-decreasing and $u_2(\cdot)$ non-increasing, a consequence of the concavity of $\psi(\cdot)$. Letting $\delta \to 0$, and by continuity of $\psi(\cdot)$, we obtain

$$\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \leq \psi(\tilde{x}).$$

Since $\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \geq \lim_{\epsilon\to 0} \psi^\epsilon(\tilde{x}) = \psi(\tilde{x})$, the limit of the supremum of ψ_ϵ exists and is equal to $\max_x \psi(x)$.

Second, suppose $\tilde{x} = \underline{x}$. This implies from concavity that ψ is non-increasing. We apply an argument similar to the above, based on pointwise convergence at $\underline{x}+\delta$ and $\underline{x}+2\delta$. Since the limit function is concave and non-increasing, and by the same steps as above, we must have

$$\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \leq \psi(\underline{x}+\delta) + \left\{\psi(\underline{x}+\delta) - \psi(\underline{x}+2\delta)\right\}.$$

Again letting $\delta \to 0$ and using continuity of $\phi(\cdot)$,

$$\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \leq \psi(\underline{x}).$$

On the other hand, from pointwise convergence we have

$$\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \geq \psi(\underline{x}+\delta),$$

which, letting $\delta \to 0$ and by continuity of $\psi(\cdot)$, implies

$$\lim_{\epsilon\to 0} \sup_x \psi^\epsilon(x) \geq \psi(\underline{x}).$$

The lower and upper bounds show that the limit exists and is as stated. The same argument applies when $\tilde{x} = \overline{x}$.

■

Lemma 11 *(Convergence of the* argsup*) Consider ψ and ψ^ϵ as in Lemma 10. Further, assume that* $\operatorname{argmax}_x \psi(x)$ *is unique. Define $\tilde{x} = \operatorname{argmax}_x \psi(x)$ and $\tilde{x}^\epsilon = \operatorname{argsup}_x \psi^\epsilon(x)$. Then* $\lim_{\epsilon\to 0} \tilde{x}^\epsilon$ *exists and is equal to $\tilde{x}$.*

Proof. If $\tilde{x}^\epsilon$ does not converge, there is a sub-series of decreasing $\epsilon_k,\ k = 1, 2, \ldots$ such that $\tilde{x}_{\epsilon_k}$ converges to some x_1 other than $\tilde{x}$. Define $x_2 = (\tilde{x} + x_1)/2$. For ϵ_k small enough, x_2 is strictly between $\tilde{x}$ and $\tilde{x}_{\epsilon_k}$. But from pointwise convergence and uniqueness of the argmax of ψ, for ϵ_k small enough, $\psi_{\epsilon_k}(\tilde{x}) > \psi_{\epsilon_k}(x_2)$. But, from optimality of $\tilde{x}_{\epsilon_k}$, $\psi_{\epsilon_k}(x_2) \leq \psi_{\epsilon_k}(\tilde{x}_{\epsilon_k})$. These two inequalities violate concavity of ψ^ϵ.

■

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References

Alter, S. K. (2000). *Managing the Double Bottom Line: A Business Planning Guide for Social Enterprises*. Alter Paperback.

Anderson, B. B., J. G. Dees, and J. Emerson (2002). *Developing Viable Earned Income Strategies in Strategic Tools for Social Entrepreneurs: Enhancing the Performance of Your Enterprising Nonprofits*. John Wiley & Sons.

Babich, V. and M. J. Sobel (2004). Pre-ipo operational and financial decisions. *Management Science 50*(7), 935–948.

Bernstein, F. and A. Federgruen (2005). Decentralized supply chains with competing retailers under demand uncertainty. *Management Science 51*(1), 18–29.

Billingsley, P. (1995). *Probability and Measure* (3 ed.). John Wiley & Sons.

Bitran, G. and R. Caldentey (2003). An overview of pricing models and revenue management. *Manufacturing and Service Operations Management 5*, 203–229.

Bonnans, F. and A. Shapiro (2000). *Perturbation analysis of optimization problems*. Springer.

Boschee, J. (2001). *The Social Enterprise Sourcebook: Profiles of Social Purpose Businesses Operated by Nonprofit Organizations*. Northland Institute.

Bradley, B., P. Jansen, and L. Silverman (2003, May). The nonprofit sector's $100 billion opportunity. *HBR 5*(1).

Brinckerhoff, P. C. (2000). *Social Entrepreneurship: The Art of Mission-Based Venture Development*. John Wiley & Sons.

Buzacott, J. and R. Zhang (2001). Inventory management with asset-based financing. *Management Science 50*(9), 1274–1292.

Chen, H., O. Wu, and D. Yao (2004). Optimal pricing and replenishment in a single-product inventory system. Technical report, IEOR Columbia.

Chen, X. and D. Simchi-Levi (2004a). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research 52*, 887–896.

Chen, X. and D. Simchi-Levi (2004b). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The ininite horizon case. *Mathematics of Operations Research 29*, 698–723.

Chen, Y., S. Ray, and Y. Song (2003). Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. Technical report, The Chinese Univ. of Hong Kong.

Dees, G. (1998). Enterprising non-profit. *Harvard Business Review, January-February.*

Dees, G. (2004). *in Generating and Sustaining Nonprofit Earned Income: A Guide to Successful Enterprise Strategies, S. M. Oster, C. Massarsky and S. Beinhacker*, Chapter : Putting Nonprofit Business Ventures in Perspective. Jossey-Bass.

Dees, G. and B. Anderson (2003). Sector-bending: Blurring the lines between nonprofit and for-profit. *Society 40*(4).

Elmaghraby, W. and P. Keskinocak (2003). Dynamic pricing in the presence of inventory considerations: Research overview, current practices and future directions. *Management Science 10*(7), 1287–1309.

Elstrodt, H.-P., A. Schindler, and A. C. Waslander (2004). Helping nonprofits do business. *The McKinsey Quarterly, Web exclusive December*.

Federgruen, A. and A. Heching (1999). Combined pricing and inventory control under uncertainty. *Operations Research 47*(3), 454–475.

Foster, W. and J. Bradach (2005, February). Should nonprofits seek profits? *Harvard Business Review*.

Gallego, G. and G. van Ryzin (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science 40*, 999–1020.

Gruber, R. E. and M. Mohr (1982, Spring). Strategic management for multiprogram nonprofit organizations. *California Management Review 3*, 15–22.

Heyman, D. and M. Sobel (1984). *Stochastic Models In Operations Research (II)*. Dover Publications.

Huh, W. and G. Janakiraman (2005). Optimality results in inventory-pricing control: An alternative approach. Technical report, IEOR Columbia.

Keeler, E. and S. Cretin (1984). Discounting of life-saving and other nonmonatory effects. *Management Science 29*(3), 300–306.

Klausner, M. (2003, Spring). When time isnt money: Foundation payouts and the time value of money. *California Management Review 1*(1), 50–59.

Kocabiyikoğlu, A. and I. Popsescu (2005). Joint pricing and revenue management with general stochastic demand. Technical report, INSEAD.

Larson, R. and V. Forth (2002). *The Essential Guide to Starting a Moneymaking Business in Your Nonprofit Organization*. Amherst H. Wilder Foundation.

Li, L., M. Shubik, and M. J. Sobel (2003). Control of dividends, capital subscriptions, and physical inventories. Technical report, Yale University and Case Western Reserve University.

McCardle, K. (2005, October). Pro bono o. r.: O. r. for the public good. *OR/MS Today 32*(6), 32–36.

Modigliani, F. and H. Miller (1958). The cost of capital, corporation finance, and the theory of investment. *American Economic Review 48*, 261–267.

Oster, S. M. (1995). *Strategic Management for Nonprofit Organizations: Theory and Cases*. Oxford University Press.

Oster, S. M., C. Massarsky, and S. Beinhacker (1995). *Generating and Sustaining Nonprofit Earned Income: A Guide to Successful Enterprise Strategies*. Jossey-Bass.

Pande, T. (1998). A passion for restoring lost vision. *India Abroad (New York edition) XXIX*(1).

Petruzzi, N. C. and M. Dada (1999). Pricing and the newsvendor problem: A review with extensions. *Operations Research 47*(2), 183–194.

Robinson, A. (2002). *Selling Social Change (Without Selling Out): Earned Income Strategies for Nonprofits*. Jossey-Bass.

Skloot, E. (1983, January-February). Should nonprofits seek profits. *Harvard Business Review 83*(1).

Skloot, E. (1987). *The Nonprofit Sector: A Research Handbook*, Chapter : Enterprise and Commerce in Nonprofit Organizations. Yale.

Skloot, E. (1988, January-February). Should not-for-profits go into business? *Harvard Business Review 83*(1).

Sobel, M. (1984). Myopic solutions of markovian decision processes and stochastic games. *Operations Research 26*, 1394–1398.

Steckel, R., R. Simons, and P. Lengsfelder (1989). *Filthy Rich and Other Nonprofit Fantasies: Changing the Way Nonprofit Do Business in the 90s*. Ten Speed Press.

Wang, Y., L. Jiang, and Z. Shen (2004). Channel performance under consignment contract with revenue sharing. *Management Science 50*(1), 34–47.

Weisbrod, B. (1998). *To Profit or Not to Profit: the Commercial Transformation of the Nonprofit Sector*. Cambridge University Press.

Weisbrod, B. A. (2005, Winter). The pitfalls of profits. *Stanford Social Innovation Review 2*(3), 40–47.

Young, D. Y. (2002). *in The State of Nonprofit America*, Chapter : Commercialization, Social Ventures, and For-Profit Competition, Lester M. Salamon. Brookings Institution Press.

Young, D. Y. (2004). *in Generating and Sustaining Nonprofit Earned Income: A Guide to Successful Enterprise Strategies, S. M. Oster, C. Massarsky and S. Beinhacker*, Chapter : Deploying Resources Effectively. Jossey-Bass.