Optimal Auction Design

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Many different kinds of auctions:

- Sealed bid second price – Highest bidder wins, pays second highest bid as price;
- Sealed bid first price – Highest bidders wins, pays bid;
- English – Sequential bidding in the ipen, highest bidder wins, pays bid as price;
- Dutch – Descending auction, first bidder to stop the descend wins (Adjustable preferred).

Examples: Oil development leases, Treasury bills, Natural gas, Art, Securities.

Let there be n potential buyers, so the set of bidders is

$$N = \{1, 2, \cdots, n\}.$$

Seller does not know potential buyer's valuation. Buyer i has valuation

$$t_i \in [a_i, b_i], \quad with -\infty < a_i < b_i < \infty$$

Density $f_i(t_i) > 0$ on support, and $f_i(\bullet)$ is continuous. The **c**.**d**.**f** is

$$F_i : [a_i, b_i] \to [0, 1]$$

$$F_i(t_i) = \int_{a_i}^{t_i} f_i(s_i) ds_i$$

$$T = [a_1, b_1] \times [a_2, b_2], \cdots \times [a_n, b_n]$$

$$T_{-i} = X_{j \in N, j \neq i}[a_j, b_j]$$

Independant, private values model:

The independant assumption is important.

$$f_{-i}(t_{-i}) = \prod_{j \in N, j \neq i} f_j(t_j)$$

Since potential buyer i's valuation t_i is known to him, about is the relevant density.

Seller has personal estimate t_0 — public information.

Private value — My estimate t_i does not change on knowing t_j, $j \neq i$

Common value — My estimate t_i does not change on knowing t_j, $j \neq i$.

We will focus on private values case.

What is a auction?

A sealed bid auction is a rule which given bids $b_1, b_2, \cdots, b_n$, allocates the object according to a function $A(b_1, b_2, \cdots, b_N) \in \{1, 2, \cdots, n\}$ at a price $P(b_1, b_2, \cdots, b_N)$. (Other autions have more compliacted game structures).

Given this sealed bid auction, each individual picks a bidding rule $b_i^*(t_i)$. This is a function of his information t_i.

REVELATION PRINCIPLE

Lemma1: Suppose $(b_1(\bullet), \cdots, b_N(\bullet))$ is a Nash equilibrium of the sealed bid auction $(A(b_1, \cdots, b_N), P(b_1, \cdots, b_N))$. Then there exists a *direct revelation mechanism* which has the same equilibrium outcome as the given equilibrium and in which truthtelling is a Nash equilibrium.

Proof :

A emphrevelation game \ direct revelation mechanism is one in which :

1. Bidders simutaneously and confidentially announce their value estimates to the seller;
2. The seller then determine who gets the object $\hat{A}(t_1, \cdots, t_N)$ and at what price $\hat{P}(t_1, \cdots, t_N)$.

   Notice that the information communicated is the "type" t_i.

To continue with the proof, consider the following mechanism.

Allocate the object according to

$$\hat{A}(t_1, \cdots, t_N) = A(b_1^*(t_1), \cdots, b_n^*(t_n))$$

$$\hat{P}(t_1, \cdots, t_N) = P(b_1^*(t_1), \cdots, b_n^*(t_n))$$

Let $\nu(P, t_i, A)$ be the utility to type i. Then

$$\mathbf{E}_{-i}[\nu(P(b_{-i}^*(t_{-i}), b_i^*(t_i)), t_i, A(b_{-i}^*(t_{-i}), b_i^*))] \geq \mathbf{E}_{-i}[\nu(P(b_{-i}^*(t_{-i}), b_i), t_i, A(b_{-i}^*(t_{-i}), b_i))] \;\; \forall b_i$$

Thus

$$\nu(P(b_{-i}^*(t_{-i}), b_i^*(t_i)), t_i) \geq \nu(P(b_{-i}^*(t_{-i}), b_i^*(\hat{t}_i)), t_i) \;\; \forall b_i$$

Given others tell the truth, it is Nash for i to tell the truth. The equlibrium outcome is the same as before

$$\hat{A}(t_1, \cdots, t_N) = A(b_1^*(t_1), \cdots, b_n^*(t_n))$$

$$\hat{P}(t_1, \cdots, t_N) = P(b_1^*(t_1), \cdots, b_n^*(t_n))$$

Q.E.D.

The idea that it suffice to analyse *direct revelation mechanism* is then follows. This idea is called the REVELATION PRINCIPLE by *Myerson*. It was first pointed by *Alan Gibbard* in 1973.

Thus for the sealed bid mechanisms, at least, the revelation principle ensures there is no loss of generality in looking at direct mechanisms. One can prove it for general auction games. Here Θ is the strategy space and $\theta \in \Theta$is the strategy chosen. Then an auction with strategy space Θ allocates the object according to function$A(\theta_1, \cdots, \theta_n)$ and price $P(\theta_1, \cdots, \theta_n)$. It is easy to prove the revelation principle with this definition.

Direct Revelation Mechanism

Let $t = (t_1, \cdots, t_n)$ be the announced value estimates. A *direct revelation mechanism* is a pair (p, x)

$$p\colon T \longrightarrow \mathbf{R^n}$$

$$x\colon T \longrightarrow \mathbf{R^n}$$

$p_i(t) \equiv$ probability of i getting the object

$x_i(t) \equiv$ expected price paid. (We allow for payment even if the object is not won)

This definition allows for randomized schemes.

The expected utility of a buyer is

$$U_i(p, x, t_i) = \int_{T_{-i}} (\nu_i(t_i)p_i(t) - x_i(t))f_{-i}(t_i)dt_{-i}$$

where $dt_{-i} = dt_1 \cdots dt_{i-1}dt_{i+1} \cdots dt_n$.

The seller's expected utility is

$$U_0(p, x) = \int_T (\nu_0(t)(1 - \sum_{j\in N} p_j(t)) + \sum_{j\in N} x_j(t))f(t)dt$$

where $dt_{-i} = dt_1 \cdots dt_n$.

[**A**] Probability Constraints :

$$\sum_{j\in N} p_j(t) \le 1 \;\; and \;\; p_i(t) \ge 0, \quad \forall i \in N, \forall t \in T.$$

[**B**] Individual Rationality :

$$U_i(p, x, t_i) \ge 0 \;\; i = 1, \cdots, N \;\; t_i \in [a_i, b_i]$$

Truthtelling is equilibrium of revelation game

[**C**] Incentive Compatibility :

$$U_i(p, x, t_i) \ge \int_{T-i} (\nu_i(t)p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i))f_{-i}(t_{-i})dt_{-i}$$

$$i = 1, \cdots, N; \;\; \forall t_i \in [a_i, b_i]; \;\; \forall s_i \in [a_i, b_i].$$

A feasible auction mechanism satifies [**A**], [**B**], [**C**].

Analysis : Given (p,x), we define

$$Q_i(p,t_i) = \int_{T-i} p_i(t) f_i(t_{-i}) dt_{-i}$$

Q_i is the conditional probability that bidder i will get the object given t_i.

Lemma2 :

$$(p,x) \ \ is \ \ feasible \iff s_i \leq t_i \ \ then \ \ Q_i(p,s_i) \leq Q_i(p,t_i);$$

$$i = 1, \cdots, N; \ \forall s_i, t_i \in [a_i, b_i]$$

$$U_i(p,x,t_i) = U_i(p,x,a_i) + \int_a^{t_i} Q_i(p,s_i) ds_i \ \ \forall i \in N; \ \ \forall t_i \in [a_i, b_i]$$

$$U_i(p,x,a_i) \geq 0; \ \ \forall i \in N;$$

$$\sum_{j \in N} p_i(t) \leq 1 \ \ and p_i(t) \geq 0 \ \ \forall i \in N$$

Proof :

In the independant values case $\nu_i(t_i) = t_i$

Hence, Incentive Compatibility (IC)

$$\begin{aligned}
\Longrightarrow U_i(p,x,t_i) &= \int_{T_{-i}} (t_i p_i(t_{-i}, t_i) - x_i(t_{-i}, t_i)) f_{-i}(t_{-i}) dt_i \\
&\geq \int_{T-i} (t_i p_i(t_{-i}, s_i) - x_i(t_i, s_i)) f_i(t_i) dt_{-i} \\
&= \int_{T-i} (s_i p_i(t_{-i}, s_i) - x_i(t_i, s_i)) f_i(t_i) dt_{-i} + (t_i - s_i) Q_i(p, s_i). \\
&= U_i(p,x,s_i) + (t_i - s_i) Q_i(p,s_i).
\end{aligned}$$

Let $s_i < t_i$. Then

$$\begin{aligned}
& U_i(p,x,t_i) && \geq U_i(p,x,s_i) + (t_i - s_i) Q_i(p,s_i) \\
\text{and} \quad & U_i(p,x,s_i) && \geq U_i(p,x,t_i) + (s_i - t_i) Q_i(p,t_i)
\end{aligned}$$

$$\Longrightarrow 0 \geq (t_i - s_i) Q_i(p,s_i) + (s_i - t_i) Q_i(p,t_i)$$

$$\Longrightarrow Q_i(p,t_i) \geq Q_i(p,s_i)$$

Also,

$$U_i(p,x,t_i) - U_i(p,x,s_i) \geq (t_i - s_i)Q_i(p,s_i) \qquad [\mathbf{D}]$$

$$\begin{aligned}
t_i > s_i \Longrightarrow \frac{U_i(p,x,t_i) - U_i(p,x,s_i)}{t_i - s_i} &\geq Q_i(p,s_i) \\
Let\ t_i \rightarrow s_i, \frac{dU_i(p,x,s_i)}{ds_i} &\geq Q_i(p,s_i) \\
t_i < s_i \Longrightarrow \frac{U_i(p,x,t_i) - U_i(p,x,s_i)}{t_i - s_i} &\leq Q_i(p,s_i) \\
Let\ t_i \rightarrow s_i, \frac{dU_i(p,x,s_i)}{ds_i} &\leq Q_i(p,s_i) \\
\Longrightarrow \frac{dU_i(p,x,s_i)}{ds_i} &\leq Q_i(p,s_i)
\end{aligned}$$

Now $U_i(p,x,t_i)$ is monotone in t_i by $[\mathbf{D}]$.

Monotone functions are *a.s.* differentiable. So the derivative *a.s* exits and

$$U_i(p,x,t_i) - U_i(p,x,a_i) = \int_a^{t_i} Q_i(p,s_i)ds_i$$

Q.E.D.

The only thing to show is Individual Rationality. The rest follow.

But if $t_i \geq a_i$,

$$\begin{aligned}
U_i(p,x,t_i) &= U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)ds_i \\
&\geq U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)ds_i \\
&\geq U_i(p,x,a_i)
\end{aligned}$$

I.R. follows.

What is the optimal auction? The optimal auction satisfies

$$Max \;\; U_0(p,x) \quad s.t.[\mathbf{A}],[\mathbf{B}],[\mathbf{C}].$$

This is characterized in the next theorem.

Lemma3 : Suppose that $p\colon T \to \mathbf{R^n}$ maximizes

$$\int_T \Big(\sum_{i\in N}(t_i - \frac{1-F_i(t_i)}{f_i(t_i)} - t_0)p_i(t)\Big)f(t)dt$$

$$s.t. \quad s_i \le t_i \Longrightarrow Q_i(p,s_i) \le Q_i(p,t_i) \quad and \quad \sum p_j(t) \le 1, \;\; p_i(t) \ge 0, \;\; \forall i$$

Also

$$x_i(t) = p_i(t)t_i - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)ds_i$$

Then (p, x) reprensts an optimal auction.

Proof:

$$U_0(p,x) = \int_T \nu(t_0)f(t)dt + \sum_{i\in N}\int_T p_i(t)(\nu_i(t) - \nu_0(t))f(t)dt$$
$$+ \sum_{i\in N}\int_T (x_i(t) - p_i(t)\nu_i(t))f(t)dt$$

$$\begin{aligned}
\int_T (x_i(t) - p_i(t)v_i(t))f(t)dt &= -\int_{a_i}^{b_i} U_i(p,x,t_i)f_i(t_i)dt_i \\
&= -\int_{a_i}^{b_i} U_i(p,x,a_i)f_i(t_i)dt_i - \int_{a_i}^{b_i}\int_{a_i}^{t_i}(Q_i(p,s_i)ds_i)f_i(t_i)\,dt_i \\
&= -U_i(p,x,a_i) - \int_{a_i}^{b_i}\int_{s_i}^{b_i} f_i(t_i)Q_i(p,s_i)dt_i\,ds_i \\
&= -U_i(p,x,a_i) - \int_{a_i}^{b_i}(1 - F_i(s_i))Q_i(p,s_i)ds_i \\
&= -U_i(p,x,a_i) - \int_T (1 - F_i(t_i))p_i(t)f_{-i}(t_{-i})dt
\end{aligned}$$

Also $\nu_i(t) - \nu_0(t) = t_i - t_0$

Substituting this mess,

$$U_0(p,x) = \int_T \Big(\sum_{i\in N}(t_i - \frac{1-F_i(t_i)}{f_i(t_i)} - t_0)p_i(t)\Big)f(t)dt + \int_T \nu_0(t)f(t)dt - \sum_{i\in N} U_i(p,x,a_i)$$

Notice that x appears only in the last term, hence, given p, choose x s.t. the last term is equal to zero.

Now the IC constrain is

$$U_i(p,x,t_i) = U_i(p,x,a_i) + \int_a^{t_i} Q_i(p,s_i)ds_i \qquad [*]$$

IR is

$$U_i(p, x, a_i) \geq 0$$

And recall

$$U_i(p, x, t_i) = \int_{T_{-i}} (\nu_i(t_i)p_i(t) - x_i(t))f_{-i}(t_{-i})dt_{-i}$$

$$Q_i(p, t_i) = \int_{T-i} p_i(t)f_i(t_{-i})dt_{-i}$$

$$[*] \Longrightarrow \int_{T_{-i}} (\nu_i(t_i)p_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)ds_i - x_i(t))f_{-i}(t_{-i})dt_{-i} = U(p, x, a_i) \geq 0$$

Set $U(p, x, a_i) = 0$ i.e.

$$x_i(t) = p_i(t)\nu_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)ds_i$$

REVELATION EQUILIBRIUM THEOREM:

The seller's Expected utility from a feasible auction mechanism is completely determined by the function p and the number $U_i(p, x, a_i)$.

Suppose :

1. the seller goes to the bidder with highest estimate t_i, i.e., $A(t_i, \cdots, t_n) = max \;\; t_i$ (my notation);
2. $U_i(p, x, a_i) = 0$;
3. bidders are symmetric ;
4. $a_i = 0$;

then the English, Dutch, First Price Dealed Bid and Second Price Sealed Bid all yield the same revenue. (This result for First and Second Price was proved by Vickery in 1961) However, the First or Second Price auction need not be optimal.

A Detour :

Let us take a detour from Myerson's general paper and study some examples :

$$t_i \quad \sim \quad uniform[0, \bar{t}] \quad \forall i$$

Second Price Sealed Bid

Let the other $N-1$ bidders bid according to $b^*(t_j)$.

If $b \geq max_{j \neq i} \quad b^*(t_j)$ i wins, pays $max_{j\neq i} \quad b^*(t_j)$;

If $b < max_{j\neq i} \quad b^*(t_j)$ i gets nothing.

$$\nu(b, t_i) = \int_{T_{-i}} \mathbf{1}_{\{b \geq max_{j\neq i} \ b^*(t_j)\}} (t_i - max_{j\neq i} \ b^*(t_j)) f(t_{-i}) dt_{-i}$$

Lemma :

It is a dominant strategy fo the second price auction to tell the truth. i.e. set $b = t$ (A dominant strategy is one which is optimal to carry out irresponsive of what others do).

Proof :

If $b < t_i$, increasing b_i increases the probability of winning but does affect the price;

If $b = t_i$, increasing b_i has no effect on the price but you may win the item and pay $> t_i$

$$\Longrightarrow Optimal \quad to \quad set \quad b_i = t_i$$

Formally, if $b_i > t_i$

$$\nu(b, t_i) = \int_{T_{-i}} \mathbf{1}_{\{t_i \geq max_{j\neq i} \ b^*(t_j)\}} (t_i - max_{j\neq i} \ b^*(t_j)) f(t_{-i}) dt_{-i}$$

$$+ \int_{T_{-i}} \mathbf{1}_{\{b > max \ b^*(t_j) > t_i\}} (t_i - max \ b^*(t_j)) f(t_{-i}) dt_{-i}$$

The second term is negative $\Longrightarrow set \ \ b = t_i$

Thus, in the Second Price auction one tells the truth. The seller gets the second higher bid.

Denote this by $\tilde{t}^*$ i.e. $\tilde{t}^* \equiv$ second highest bid.

To computer $\mathbf{E}[\tilde{t^*}]$, we need to know its density.

$$\{\tilde{t^*} \leq r\} \iff \{\text{at least } n-1 \text{ of the } t_i \text{ are } \leq r\}$$

$$\begin{aligned}
\iff \{t_1, \cdots, t_{n-1}, t_n \leq r\} \quad & \cup \quad \{t_1, \cdots, t_{n-1} \leq r, t_n > r\} \\
& \cup \quad \{t_1, \cdots, t_{n-2}, t_n \leq r, t_{n-1} > r\} \\
& \cdots \\
& \cup \quad \{t_2, \cdots, t_n \leq r, t_1 > r\}
\end{aligned}$$

$$p\{\tilde{t^*} \leq r\} = (\frac{r}{\overline{t}})^n + n(\frac{r}{\overline{t}})^{n-1}(\frac{\overline{t} - r}{\overline{t}})$$

$$\implies f(r) = \frac{nr^{n-1}}{\overline{t}^n} + n(n-1)\frac{r^{n-2}}{\overline{t}^{n-1}}(\frac{\overline{t} - r}{\overline{t}}) - n\,\frac{1}{\overline{t}}(\frac{r}{\overline{t}})^{n-1}$$

$$\begin{aligned}
\mathbf{E}[\tilde{t^*}] &= \int_0^{\overline{t}} (\frac{nr^{n-1}}{\overline{t}^n} + n(n-1)\frac{r^{n-2}}{\overline{t}^{n-1}}(\frac{\overline{t} - r}{\overline{t}}) - n\,\frac{1}{\overline{t}}(\frac{r}{\overline{t}})^{n-1})dr \\
&= \int_0^{\overline{t}} (\frac{nr^n}{\overline{t}^n} + \frac{n(n+1)r^{n-1}}{\overline{t}^{n-1}}(\frac{\overline{t} - r}{\overline{t}}) - \frac{n}{\overline{t}}(\frac{r^n}{\overline{t}^{n-1}}))dr \\
&= \int_0^{r} \{n(\frac{r}{\overline{t}})^n - n(n-1)(\frac{r}{\overline{t}})^n + n(n-1)(\frac{r}{\overline{t}})^{n-1} - n(\frac{r}{\overline{t}})^n)dr \\
&= \int_0^{r} \{(\frac{r}{\overline{t}})^{n-1} n(n-1) - n(n-1)(\frac{r}{\overline{t}})^n\}dr \\
&= \frac{(n-1)r^n}{\overline{t}^{n-1}} - \frac{n(n-1)}{n+1}\,\frac{r^{n+1}}{\overline{t}^n} \\
&= \frac{r^n}{\overline{t}^{n-1}}\{(n-1) - \frac{n(n-1)r}{(n+1)\overline{t}}\} \\
&= \frac{r^n}{\overline{t}^n}\{\frac{(n-1)(n+1)\overline{t} - n(n-1)r}{(n+1)}\} \\
&= \frac{(n-1)r^n}{(n+1)\overline{t}^n}((n+1)\overline{t} - nr) \\
&= \frac{n-1}{n+1}\,\overline{t}
\end{aligned}$$

Last equality obtained by setting $r = \overline{t}$

First Price Sealed Bid

Now truth telling is not optimal.

Suppose the other $N-1$ bidders bid a monotone function of their information $b^*(t_i)$.

Then

If $b \geq max_{j\neq i} \ \ b^*(t_j)$ i wins, pays b;

If $b < max_{j\neq i} \ \ b^*(t_j)$ i gets nothing.

$$\begin{aligned} U(b,t_i) &= \int_{T_{-i}} \mathbf{1}_{\{b\geq max_{j\neq i} b^*(t_j))\}}(t_i-b)f(t_{-i})dt_{-i} \\ &= (t_i-b)\, Prob[b\geq max_{j\neq i} \ \ b^*(t_j)] \end{aligned}$$

Now

$$\begin{aligned} b \quad & \geq max_{j\neq i} \ \ b^*(t_j) \\ \Longleftrightarrow \quad & b\geq b^*(t_j) \quad \forall j\neq i \\ \Longleftrightarrow \quad & b^{*-1}(b)\geq t_j \quad \forall j\neq i \\ \Longleftrightarrow \quad & b^{*-1}(b)\geq max_{j\neq i} \ \ t_j \end{aligned}$$

$$\Longrightarrow U(b,t_i) = (t_i-b) \ \ Prob\,[b\geq max_{j\neq i}\ b^*(t_j)].$$

In our uniform example, let $t^* = max_{j\neq i} \ \ b^*(t_j)$

$$\begin{aligned} Prob[t^*<k] &= Prob[t_j<k \ \ \forall j\neq i \\ &= (\frac{k}{\bar{t}})^{n-1} \\ \Longrightarrow U(b,t_i) &= (t_i-b)(\frac{b^{*-1}(b)}{\bar{t}})^{n-1} \end{aligned}$$

The FOC w.r.t. b is

$$-(\frac{b^{*-1}(b)}{\bar{t}})^{n-1} + (t_i-b)(n-1)(\frac{b^{*-1}(b)}{\bar{t}})^{n-2}\,\frac{d(b^{*-1}(b))}{db} = 0$$

Notice that if all others do not shade their bids, i.e. $b^*(t_j)=t_j$, we have

$$\begin{aligned} & -(\frac{b}{\bar{t}})^{n-1} + (t_i-b)(n-1)(\frac{b}{\bar{t}})^{n-2} = 0 \\ \Longleftrightarrow \quad & -\frac{b}{\bar{t}} + (t_i-b)(n-1) = 0 \\ \Longleftrightarrow \quad & \frac{b}{\bar{t}} = (t_i-b)(n-1) \end{aligned}$$

$$\begin{aligned} \iff \qquad & b = \bar{t}t_i(n-1) - \bar{t}b(n-1) \\ \iff \qquad & b = \frac{\bar{t}(n-1)}{1+\bar{t}(n-1)}\, t_i \quad < \, t_i \end{aligned}$$

So, truthtelling is not a Nash equilibrium.

Going back to the FOC

$$-\frac{b^{*-1}(b)}{\bar{t}} + (t_i - b)(n-1)\,\frac{d(b^{*-1}(b))}{db} = 0$$

We are searching for a <u>symmetric Nash equlibrium</u>

So

$$b^{*-1}(b^*) = t_i$$

and

$$\frac{d(b^{*-1}(b))}{db} = \frac{1}{\frac{db^*}{dt_i}}$$

$$\begin{aligned} -\frac{t_i}{t} + (t_i - b^*(t_i))(n-1)\,\frac{1}{\frac{db^*}{dt_i}} &= 0 \\ (t_i - b^*(t_i))(n-1) &= \frac{t_i}{\bar{t}}\,\frac{db^*(t_i)}{dt_i} \\ (1 - \frac{b^*(t_i}{t_i})(n-1)\bar{t} &= \frac{db^*(t_i)}{dt_i} \end{aligned}$$

Try a solution of the form $\quad b^*(t_i) = a\,t_i + c\,t_i^k + constant, \qquad k \neq 0$

$$\begin{aligned} \frac{db^*(t_i)}{dt_i} &= a + kc\,t_i^{k-1} \\ &= a + \frac{k}{t_i}\,c\,t_i^k \\ &= a + \frac{k}{t-i}(b^*(t_i) - at_i) \\ &= (1-k)a + k\frac{b^*(t_i)}{t_i} \end{aligned}$$

Matching coefficients

$$\begin{aligned} k &= -(n-1)\bar{t} \quad < 0 \\ (1-k)\,a &= (n-1)\bar{t} \quad \Longrightarrow \quad a = \frac{(n-1)\bar{t}}{1+(n-1)\bar{t}} \quad < 1 \end{aligned}$$

So

$$b^*(t_i) = \frac{(n-1)\bar{t}}{1+(n-1)\bar{t}}\, t_i + c\, t_i^{-(n-1)\bar{t}} + const.$$

$$\begin{aligned}
\frac{b^*(t_i)}{t_i} &= \frac{(n-1)\bar{t}}{1+(n-1)\bar{t}} + \frac{c}{t_i^{(n-1)\bar{t}+1}} \\
(1-\frac{b^*(t_i)}{t_i}) &= \frac{1}{1+(n-1)\bar{t}} - \frac{c}{t_i^{(n-1)\bar{t}+1}} \\
\frac{db^*(t_i)}{dt_i} &= \frac{(n-1)\bar{t}}{1+(n-1)\bar{t}} - \frac{(n-1)\bar{t}c}{t_i^{(n-1)\bar{t}+1}} \qquad (\ it \ \ works! \ )
\end{aligned}$$

Also

$c < 0$ to ensure $\frac{db^*(t_i)}{dt_i} > 0$

$b^*(0) = 0 \Longrightarrow \quad constant = 0$

c indeterministic $\Longrightarrow$ More than one solutions ?

The seller receives

$$E[\quad max_i \ \ b^*(t_i)]$$

$$P[\quad max_i \ \ t_i < k] = P[\forall t_i < k] = \frac{k}{\bar{t}}^n$$

$$\Longrightarrow f(k) = n\frac{k^{n-1}}{\bar{t}^n}$$

$$\begin{aligned}
& E[\quad max_i \ \ b^*(t_i)] \\
= \ & \int_0^{\bar{t}} atn\frac{t^{n-1}}{\bar{t}^n}dt + \int_0^{\bar{t}} n\frac{k^{n-1}}{\bar{t}^n}kct^{n-1}dt \\
= \ & \frac{an}{n+1}\bar{t} + \frac{kcn}{2n-1}\bar{t}^{n-1}
\end{aligned}$$

Something seems to be wrong here. Crosscheck it!

The Optimal Auction In The Regular Case

$$c_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

c_i increases strictly in $t_i \quad \forall i$, i.e.

$$s_i < t_i \iff c_i(s_i) < c_i(t_i)$$

Consider an auction where :

If $t_0 > max_{i \in N} \; c_i(t_i)$, the seller keeps the good;

If $t_0 \leq max_{i \in N} \; c_i(t_i)$, highest $c_i(t_i)$ gets it.

$$p_i(t) > 0 \Longrightarrow c_i(t_i) = max_{j \in N} \; c_j(t_j) \geq \; t_0$$

For all $t \in T$, this mechanism maximizes the sum

$$\sum_{i \in N} (c_i(t_i) - t_0) p_i(t)$$

$$s.t. \sum_{j \in N} p_j(t) \leq 1 \;\; \text{and} \;\; p_i(t) \leq 0 \quad \forall i$$

It is optimal except we need to check that

$$Q_i(p, s_i) \leq Q_i(p, t_i) \;\; \text{for} \;\; s_i \leq t_i$$

But

$$\begin{aligned} s_i < t_i \;\; & \Longrightarrow \;\; c_i(s_i) < c_i(t_i) \\ & \Longrightarrow \;\; p_i(t_{-i}, s_i) \leq p_i(t_{-i}, t_i) \;\; \forall t_{-i} \\ & \Longrightarrow \;\; Q_i(p, s_i) \leq Q_i(p, t_i) \;\; \forall t_i \end{aligned}$$

Finally

$$x_i(t) = p_i(t) t_i - \int_{a_i}^{t_i} p_i(t_{-i}, s_i) \, ds_i$$

Let

$$Z_i(t_{-i}) = inf\{s | c_i(s_i) \geq t_0 \& c_i(s_i) \geq c_j(t_j) \;\; \forall j \neq i$$

i.e. $Z_i(t_{-i})$ is the highest of all other bids.

Then

$$p_i(t_i, s_i) = \begin{cases} 1 & \text{if } s_i > Z_i(t_{-i}); \\ 0 & \text{if } s_i < Z_i(t_{-i}). \end{cases}$$

$$\int_{a_i}^{t_i} p_i(t_i, s_i) ds_i = \begin{cases} t_i - Z_i(t_{-i}) & \text{if } t_i \geq Z_i(t_{-i}); \\ 0 & \text{if } t_i < Z_i(t_{-i}). \end{cases}$$

$$\Longrightarrow x_i(t) = \begin{cases} Z_i(t_{-i}) & \text{if } p_i(t) = 1; \\ 0 & \text{if } p_i(t) = 0. \end{cases}$$

This is an modified second price auction!

i.e.

$$Z_i(t_{-i}) = max \ \ \{c_i^{-1}(t_0), \ \ max_{j \neq i} \ \ t_j\}$$

The modification is in the reservation price $c_i^{-1}(t_0)$.

Rest of Myerson has to do with correlated values.