

Ph.D. Seminar in Corporate Finance

## Multitask Principal-Agent Model

(Hölmstrom-Milgrom, Salanié)

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### Hölmstrom-Milgrom (1987) view of Linear Contracts

Suppose the outcome is produced by a diffusion process described by the following stochastic differential equation:

$$dx_t = a_t dt + \sigma dW_t,$$

where  $W_t$  is Brownian motion,  $t \in [0, 1]$  and the Agent controls trend  $a_t$ . The utilities of both the Agent and the Principal depend on the final outcome  $x_1$ . The Agent's utility function is

$$u(x_1 - \int_0^1 a_t dt),$$

where  $u$  is CARA utility function  $u(x) = -\exp\{-kx\}$ . The Principal is risk-neutral. Hölmstrom-Milgrom (1987) show that the optimal contract is linear in  $x_1$ .

This result is intuitive. Recall that Binomial discrete-time process converges to Brownian motion. Assume that the outcome may increase or decrease by a fixed amount in each period and the Agent controls the probabilities of these changes. As CARA utility function exhibits no wealth effect, the optimal contract consists in repeating the contract that is optimal in each period. But these one-period contracts pay to the Agent a fixed wage and a bonus if the outcome has increased. Therefore, the optimal contract must give the Agent the bonus that is proportional to the number of periods for which the outcome has increased. The result of Hölmstrom-Milgrom (1987) is obtained by taking the continuous-time limit.

## The Multitask Model (Hölmstrom-Milgrom 1991)

Assume that the Agent has two tasks to complete and controls two effort variables,  $a_1$  and  $a_2$ . His utility function is

$$-\exp\{-r(w - C(a_1, a_2))\},$$

where  $r$  is a coefficient of the Agent's absolute risk aversion and  $C$  is a convex function. The Principal is risk-neutral and observes the joint output,

$$\begin{cases} x_1 = a_1 + \epsilon_1 \\ x_2 = a_2 + \epsilon_2, \end{cases}$$

where

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right).$$

Given the results of Hölmstrom-Milgrom (1987), we can limit out attention to linear wage contracts,

$$w(x_1, x_2) = \alpha'x + \beta = \alpha_1x_1 + \alpha_2x_2 + \beta.$$

Under this contract the Principal's expected profit is

$$a_1 + a_2 - \alpha_1a_1 - \alpha_2a_2 - \beta,$$

while the Agent's certain equivalent is

$$\alpha_1a_1 + \alpha_2a_2 + \beta - C(a_1, a_2) - \frac{r}{2}\alpha'\Sigma\alpha.$$

Observe that  $\beta$  does not affect the Agent's optimal choice and is a pure transfer between the Principal and the Agent. The Principal is solving the following optimization problem:

$$\max_{\alpha_1, \alpha_2} \alpha_1 + a_2 - C(a_1, a_2) - \frac{r}{2}\alpha'\Sigma\alpha,$$

where

$$(a_1, a_2) \in \arg \max_{a'_1, a'_2} \alpha_1a_1 + \alpha_2a_2 - C(a_1, a_2).$$

Incentive constraint for the Agent gives

$$\alpha_i = C'_i(a_1, a_2), \quad (*)$$

for  $i = 1, 2$ . By differentiating (\*) with respect to  $\alpha_1$ , we get

$$\begin{cases} 1 = C''_{11} \frac{\partial a_1}{\partial \alpha_1} + C''_{12} \frac{\partial a_2}{\partial \alpha_1} \\ 0 = C''_{12} \frac{\partial a_1}{\partial \alpha_1} + C''_{22} \frac{\partial a_2}{\partial \alpha_1}, \end{cases}$$

or

$$\begin{cases} \frac{\partial a_1}{\partial \alpha_1} = \frac{C''_{22}}{D''} \\ \frac{\partial a_2}{\partial \alpha_1} = -\frac{C''_{12}}{D''}, \end{cases}$$

where

$$D'' = C''_{11}C''_{22} - (C''_{12})^2 > 0.$$

Similarly, we obtain

$$\begin{cases} \frac{\partial a_1}{\partial \alpha_2} = -\frac{C''_{12}}{D''} \\ \frac{\partial a_2}{\partial \alpha_2} = \frac{C''_{11}}{D''}. \end{cases}$$

Therefore, for  $i \neq j$  we find

$$\frac{\partial a_i}{\partial \alpha_j} > 0,$$

and

$$\frac{\partial a_i}{\partial \alpha_j} = \begin{cases} < 0, & \text{if } C''_{12} > 0 \\ > 0, & \text{if } C''_{12} < 0. \end{cases}$$

Observe that  $C''_{12} > 0$  and  $C''_{12} < 0$  correspond to perfect substitutes and perfect complements cases, respectively.

Let us go back to the Principal's maximization problem. By differentiating the objective function with respect to  $a_i$ , we get

$$1 - C'_i - r\alpha'\Sigma \frac{\partial \alpha}{\partial a_i} = 0, \quad (*)$$

for  $i = 1, 2$ . Differentiating (\*) with respect to  $a_j$  gives

$$\frac{\partial \alpha_i}{\partial a_j} = C''_{ij}, \quad (\diamond)$$

for  $i, j = 1, 2$ . Plugging (\*) and ( $\diamond$ ) into (\*) gives

$$\alpha = (I + rC''\Sigma)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (\bullet)$$

## Discussion

Let us study some consequences of the above formulas. First, assume that the tasks are independent ( $C''$  is diagonal) and that the signals are independent ( $\sigma_{12} = 0$ ). Then  $(\bullet)$  becomes

$$\alpha_i = \frac{1}{1 + rC''_{ii}\sigma_i^2},$$

for  $i = 1, 2$ . This is the same formula as when the Principal considers the two tasks separately.

Now suppose that  $C''$  is not diagonal,  $\sigma_{12} = 0$  and  $\sigma_2$  goes to infinity. In other words, only the first task is observable. Formula  $(\bullet)$  becomes

$$\begin{cases} \alpha_1 = \frac{1 - (C''_{12}/C''_{22})}{1 + r\sigma_1^2[C''_{11} - ((C''_{12})^2/C''_{22})]} \\ \alpha_2 = 0. \end{cases}$$

If the two tasks are complements ( $C''_{12} < 0$ : an increase in  $a_1$  makes  $a_2$  less costly), then  $\alpha_1$  will be higher when  $C''_{12}$  is more negative. If the two tasks are substitutes ( $C''_{12} > 0$ : an increase in  $a_1$  makes  $a_2$  more costly), then  $\alpha_1$  will be lower when  $C''_{12}$  is more positive.

Under the same assumption that  $\sigma_2$  is infinite, assume that  $C(a_1, a_2) = c(a_1 + a_2)$ . In other words, only total effort is relevant for the Agent's utility. Then  $C''_{11} = C''_{12} = C''_{22}$ , and  $(\bullet)$  becomes

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0. \end{cases}$$

In this case inducing the Agent to perform in task 1 discourages him to perform in task 2. The Principal gives up on incentives.

Hölmstrom-Milgrom (1991) conclude that the multitask model may explain why real-world incentives are less powered than their theoretical counterparts.

## References

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