

Addenda to Growth Optimal Portfolio Restrictions on Asset Pricing Models

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Appendix A: Growth Optimal Bond Portfolios and the Pricing of Risky Assets

We begin with the general no-arbitrage approach to the pricing of risky assets. Consider an arbitrary set of N assets (where N may be unbounded) and their nominal gross returns $\{R_{it+1}, i=1, \dots, N\}$ between two arbitrary dates t and $t+1$ (inclusive of dividends and capital gains).¹ The only requirement we place on the asset menu is that it includes default-free nominal discount bonds, perhaps of all maturities. Then in a frictionless market, there exists a strictly positive random variable m_{t+1} that prices these N risky assets in that:

$$E[R_{it+1}m_{t+1}|I_t] = 1 \quad i, t \tag{1}$$

where $E[\cdot|I_t]$ denotes the conditional expectation of its argument given an unspecified set of information I_t available at time t . The pricing kernel m_{t+1} , which is generally not unique unless markets are complete, takes the ratio form:

$$m_{t+1} = q_{t+1}/q_t \tag{2}$$

where q_{t+1} denotes the time $t+1$ state price per unit probability implicit in the pricing kernel.

Equilibrium models restrict the pricing kernel m_{t+1} . For example, q_t is given by $e^{-\rho t} u'(c_t)/p_{ct}$ where ρ is the discount factor, $u(\cdot)$ is the period utility function of the representative agent, and p_{ct} is the nominal price of consumption at time t . Hence, m_{t+1} is the inflation-translated aggregate intertemporal marginal rate of substitution in these circumstances. The goal

¹In what follows, we do not take a stand on whether trading takes place in continuous time or at discrete intervals. What matters is that frictionless trading can take place at the two dates t and $t+1$.

of the bulk of asset pricing theory is to identify m_{t+1} , which in most cases means finding portfolios maximally (or perfectly) correlated with m_{t+1} . Hansen and Jagannathan (1991) show how the search for empirical counterparts to m_{t+1} can be restricted by the conditional first and second moments of returns.

Consider now the application of the no-arbitrage pricing operator to default-free nominal discount bonds. The pricing of fixed income securities reflects only expectations about state prices per unit of probability and not the distribution of cash flows across states of nature. Accordingly, let P_t^J denote the price of a J period pure discount bond at time t. Its no-arbitrage price is given by:

$$P_t^J = \frac{E[q_{t+J} | I_t]}{q_t} \quad (3)$$

The one period simple return of this bond R_{t+1}^J is given by:

$$\begin{aligned} R_{t+1}^J &= \frac{P_{t+1}^{J-1}}{P_t^J} = \frac{q_t}{q_{t+1}} \frac{E[q_{t+J} | I_{t+1}]}{E[q_{t+J} | I_t]} = m_{t+1}^{-1} \frac{E[q_{t+J} | I_{t+1}]}{E[q_{t+J} | I_t]} \\ &\equiv m_{t+1}^{-1} (1 + u_{Jt+1}); \quad u_{Jt+1} = \frac{E[q_{t+J} | I_{t+1}] - E[q_{t+J} | I_t]}{E[q_{t+J} | I_t]} \end{aligned} \quad (4)$$

so that u_{Jt+1} equals the percentage change in the expected time t+J state price per unit probability implicit in the pricing kernel, a martingale difference sequence with zero conditional mean.

Similarly, the continuously compounded return of this bond is given by:

$$\ln R_{t+1}^J = \ln m_{t+1}^{-1} + \ln E[q_{t+J} | I_{t+1}] - \ln E[q_{t+J} | I_t] = \ln(1 + u_{Jt+1}) - \ln m_{t+1} \quad (5)$$

which is especially relevant for continuous time models in which m_{t+1} and $1+u_{Jt+1}$ often take exponential forms. In order to avoid repeatedly refer to both the linear and logarithmic models, we will systematically refer to the linear model (4) since the arithmetic for both formulations is

essentially identical.²

Inspection of (4) and (5) reveals a simple, but potentially powerful, observation: m_{t+1}^{-1} or its logarithm can be extracted from bond returns alone if it is possible to perfectly hedge or diversify away the percentage changes in the perceived future price of risk u_{Jt+1} . That is, there exists a portfolio of these discount bonds with returns equal to m_{t+1}^{-1} if the joint distribution of the expected future risk prices u_{Jt+1} is singular. This observation is interesting because term structure models typically assume that bond prices are spanned by a smaller number of common factors, typically permitting the pricing kernel to be measured from bond returns and applied to the pricing of other risky assets. The empirical evidence also suggests that bond returns are nearly perfectly linearly dependent.

If the conditional distribution of discount bond returns is singular, discount bond returns will follow a multiple factor model of the form:³

$$R_{t+1}^J = m_{t+1}^{-1}(1 + b_{Jt}' f_{t+1}); \quad u_{Jt+1} = b_{Jt}' f_{t+1}; \quad b_{Jt} \in I_t \quad (6)$$

where b_{Jt} is a vector of factor loadings and f_{t+1} is the corresponding vector of zero conditional mean common factors of dimension less than the number of discount bonds. In continuous time models, it is the conditional covariance matrix of geometric bond returns $\ln R_{t+1}^J$ that is typically singular so that $\ln(1+u_{Jt+1})$ (i.e., $\ln E[q_{t+J} | I_{t+1}] - \ln E[q_{t+J} | I_t]$) follows a factor structure in these circumstances.

Two additional conditions are required to construct a portfolio that recovers m_{t+1}^{-1} from

²Note that we are treating the geometric return of a bond portfolio as the portfolio weighted average of the geometric returns of the component bonds, which is only precisely correct in the limit of continuous trading or if bond returns are continuously compounded.

bond returns (or its logarithm from geometric bond returns): (1) a vector of ones does not lie in the column span of the factor loadings b_{jt} and (2) the factor loading matrix has full column rank (these two conditions can be combined as the requirement that the matrix consisting of a column of ones and the factor loadings has full column rank). Under these conditions, there are numerous portfolios that cost a dollar and that have weights orthogonal to the factor loading matrix. The return of each such portfolio is m_{t+1}^{-1} (or $-\ln m_{t+1}$ for the case of geometric bond returns), the multiplicity again being a consequence of the existence of redundant assets in this setting.

These conditions amount to the following proposition:

Proposition: Suppose that a (possibly countably infinite) N vector of asset returns \underline{R}_{t+1} satisfy the no-arbitrage pricing relation (1) and its underlying assumptions. Let \underline{R}_{t+1}^b be a finite dimensional M vector of bond returns that is a subset of \underline{R}_{t+1} . Then the random variable m_{t+1}^{-1} can be recovered from a portfolio of bond returns if and only if bond returns have the stochastic structure:

$$\underline{R}_{t+1}^b = \iota m_{t+1}^{-1} + B_t^b \psi_{t+1}; \quad |B_t^{b'} B_t^b| > 0; \quad B_t^{b'} \iota \neq \iota \quad (7)$$

where ι is a suitably conformable vector of ones.

Proof: Sufficiency follows from the discussion leading from (3) to (6). Necessity follows from an equally straightforward manipulation of those equations. Suppose m_{t+1}^{-1} can be recovered from a bond portfolio p with weights $w_{pt} \in I_t$. From (4), its returns are given by:

$$R_{pt+1}^b = w_{pt}' m_{t+1}^{-1} (\iota + u_{t+1}) = m_{t+1}^{-1} + w_{pt}' u_{t+1} m_{t+1}^{-1} \quad (8)$$

³All multifactor models of which we are aware do not include any maturity specific noise which can, however, be accommodated easily so long as these risks are diversifiable (i.e., satisfy a weak law of large numbers) and there are a large number of maturities.

In order for \underline{R}_{pt+1}^b to be equal to m_{t+1}^{-1} , the random vector $u_{t+1}m_{t+1}^{-1}$ must have a singular covariance matrix. Denote the covariance matrix as $B_t^b V[\psi_{t+1}] B_t^{b'}$ where B_t^b has full column rank (i.e., $|B_t^{b'} B_t^b| > 0$). Let ψ_{0t+1} denote the minimum variance linear combination of ψ_{t+1} with weights summing to one so that ψ_{t+1} has the representation:

$$\psi_{t+1} = \iota \psi_{0t+1} + \phi_{t+1}; \text{Cov}[\psi_{0t+1}, \phi_{t+1} | I_t] = |V[\phi_{t+1} | I_t]| = 0 \quad (9)$$

For example, if $V[\phi_{t+1} | I_t]$ is normalized to be the identity matrix, ψ_{0t+1} is just the average of the elements of ψ_{t+1} . Hence, bond returns possess the stochastic structure:

$$\underline{R}_{t+1}^b = \iota m_{t+1}^{-1} + B_t^b \iota \psi_{0t+1} + B_t^b \phi_{t+1} \quad (10)$$

If $B_t^b \iota = \iota$, $\underline{R}_{pt+1}^b = m_{t+1}^{-1} + \psi_{0t+1}$, establishing a contradiction. Accordingly, $B_t^b \iota \neq \iota$. This completes the proof.⁴

It is worth reflecting on the empirical content of (7). It is unsurprising that the distribution of bond returns must be singular and, as we noted earlier, the empirical evidence suggests that bond returns are nearly perfectly linearly dependent. Less obvious, perhaps, is the requirement that bond returns can possess only one equicorrelated component m_{t+1}^{-1} ; that is, bond returns cannot possess an additional equicorrelated component like ψ_{0t+1} .

Bond returns will possess the requisite stochastic structure in a number of simple but somewhat plausible settings. For example, suppose that q_t is a sequence of independently and identically distributed strictly positive random variables. In this case, the returns of all discount bonds with maturities greater than one period will equal m_{t+1}^{-1} . Similarly, if the q_t sequence is M -dependent—that is, q_{t+M+1} is distributed independently of $\{q_{t-j}; j \geq 0\}$ —the returns of all discount

bonds with maturities greater than M periods will equal m_{t+1}^{-1} . Finally, suppose that q_t is sufficiently weakly dependent so that $E[q_{t+J}|I_t]$ converges to a constant as J grows without bound. Then the returns of discount bonds converge to m_{t+1}^{-1} as their maturities grow without bound.⁵ The manifest nonuniqueness of the growth optimal portfolio in these cases arises from the existence of redundant assets.

The required relation will not occur – that is, $B_t^b = 1$ – when there is an equicorrelated component in the random variables $1+u_{Jt+1}$ (or their logarithms) across maturities. This is not merely an abstract possibility –the recent model of Constantinides (1992) builds an equicorrelated component into $\ln 1+u_{Jt+1}$ (see his equation (4) for the analogue of q_{t+1}).

To see the basic problem engendered by an equicorrelated component like ψ_{0t+1} , suppose that q_t consists of two independent components, one that is persistent and one that is transitory. In particular, suppose that the transitory component is simply an independently and identically distributed positive random variable (it could be stationary or M -dependent with no substantive complications) and that the permanent component takes the form $E[\tilde{q}|I_t]$ where \tilde{q} is a strictly positive random variable that will be realized in the indefinite future (i.e., $\tilde{q} \notin I_t \forall t$). Clearly, $E[\tilde{q}|I_t]$ forms a martingale sequence by construction.⁶ If q_t is the sum of these components, the

⁴Another way of stating these results is that m_{t+1}^{-1} from discount bond portfolio returns if and only if a weighted average of $E[q_{t+J}|I_{t+1}]$ lies in the span of I_t .

⁵This result may be found in Kazemi (1992). Note that m_{t+1}^{-1} can be recovered in general from the returns of portfolios of finite maturity bonds in general in these circumstances. That is, one need not observe the limiting infinite maturity discount bond required by Kazemi (1992).

⁶This is not an entirely random choice. Consider an aggregative equilibrium model with no inflation so that q_t is given by $e^{-\rho t}u'(c_t)$. As J grows without bound, q_{t+J} converges to a martingale (as is apparent from the investor Euler equation—see Foldes (1978)) and, hence, converges to a finite random variable under weak assumptions by the martingale convergence theorem. The random variable \tilde{q} plays the role of this random variable and the example

returns of all discount bonds with maturities greater than one period will equal $m_{t+1}^{-1} E[\tilde{q} | I_{t+1}] / m_{t+1}^{-1}$ so that m_{t+1}^{-1} cannot be identified from bond returns alone. If q_t is the product of these components, the returns of all discount bonds with maturities greater than one period will equal the inverse of the ratio of the two transitory components, not m_{t+1}^{-1} as required. The latter case is the sort that occurs in Constantinides (1992).

This basic idea generalizes beyond this example. Any persistent component of q_t must have differential short and long run effects on the pricing kernel if it is possible to extract m_{t+1}^{-1} or its logarithm from discount bond returns. This would happen in the example sketched above if $E[\tilde{q} | I_t]$ played a more important role in long maturity discount bond prices than in those of shorter term bonds. That is, $B_t^b \neq t$ in this example if the factors affecting interest rates, irrespective of their degrees of persistence, have differential impact on long and short rates. In these circumstances, m_{t+1}^{-1} is the sole equicorrelated component of nominal discount bond returns.

Another interpretation of this characterization of m_{t+1} is that our typical specifications of term structure models permit the identification of the so-called growth optimal portfolio first studied by Kelly (1956), Latané (1959), Markowitz (1959), and Breiman (1960).⁷ This portfolio

sketched in the text is a particularly simple combination of this long run effect with a transient short run factor. Of course, an example can easily be rigged by appropriate specification of the stochastic process of the consumption deflator p_{ct} .

⁷Growth optimal portfolios have been discussed in three contexts in the literature. The first is as a normative prescription for investor behavior—see Merton and Samuelson (1977) in addition to the papers cited above. The second is as a model of capital market equilibrium as in the papers by Roll (1973), Kraus and Litzenberger (1975), Rubinstein (1976a), and Rubinstein (1976b). Finally, Cox and Huang (1989) have exploited the 'risk neutralizing' properties of growth optimal portfolios to facilitate the analysis of dynamic portfolio optimization (see Merton (1990) and Long (1990) for additional applications and discussion).

solves the programming problem:⁸

$$\max_{\underline{w}_{gt}} E[\ln \underline{w}'_{gt} \underline{R}_{t+1} | I_t] \text{ s. t. } \underline{w}'_{gt} \iota = 1 \quad (11)$$

where \underline{w}_{gt} is the N vector of growth optimal portfolio weights at time t on N arbitrary assets.

Given frictionless markets, a necessary and sufficient condition for the solution to this problem is the absence of arbitrage, a solution that is not necessarily unique in incomplete markets or in the presence of redundant assets (which is discussed further below). The Euler equation for this problem is given by:

$$E \left[\frac{R_{it+1}}{R_{gt+1}} \middle| I_t \right] = 1 \quad (12)$$

so that R_{gt+1} , the return of the growth optimal portfolio (or any such portfolio if uniqueness does not obtain), equals m_{t+1}^{-1} .

Procedures for identifying the growth optimal portfolio perform identify the pricing kernel as well. What the analysis leading to (7) adds is the circumstances under which this portfolio can be constructed from bond returns alone. That is, m_{t+1}^{-1} (or $-\ln m_{t+1}$ for the case of geometric bond returns) is effectively a traded asset if state prices per unit probability q_t have the stochastic structure described above.

Finally, consider the link between mean-variance efficient and growth optimal bond portfolios. Let a vector of discount bond returns (or a vector of discount bond portfolios like coupon bonds) satisfy the factor structure:

$$\underline{R}_{t+1}^b = m_{t+1}^{-1} (\iota + B_t^b f_{t+1}) = \iota m_{t+1}^{-1} + B_t^b \psi_{t+1} \quad (13)$$

⁸There is nothing sacred about the growth optimal portfolio; one can represent state prices as the marginal conditions of arbitrary smooth concave functions in the absence of arbitrage. The role

Consider the returns along with the first and second moments of an arbitrary portfolio w_t of discount bond returns:

$$\begin{aligned}
R_{wt+1}^b &= m_{t+1}^{-1} + b_{wt}^b \psi_{t+1} \\
E[R_{wt+1}^b | I_t] &= E[m_{t+1}^{-1} | I_t] + b_{wt}^b E[\psi_{t+1} | I_t] \\
V[R_{wt+1}^b | I_t] &= V[m_{t+1}^{-1} | I_t] + 2b_{wt}^b \text{Cov}[\psi_{t+1}, m_{t+1}^{-1} | I_t] + b_{wt}^b V[\psi_{t+1} | I_t] b_{wt}^b
\end{aligned} \tag{14}$$

A straightforward calculation reveals that portfolio w is a mean-variance efficient bond portfolio if its factor loading vector takes the form:

$$\begin{aligned}
b_{wt}^b &= V[\psi_{t+1} | I_t]^{-1} \{ \lambda_{wt} E[\psi_{t+1} | I_t] - \text{Cov}[\psi_{t+1}, m_{t+1}^{-1} | I_t] \} \\
\lambda_{wt} &= \frac{E[R_{wt+1}^b | I_t] - E[m_{t+1}^{-1} | I_t] + E[\psi_{t+1} | I_t] V[\psi_{t+1} | I_t]^{-1} \text{Cov}[\psi_{t+1}, m_{t+1}^{-1} | I_t]}{E[\psi_{t+1} | I_t] V[\psi_{t+1} | I_t]^{-1} E[\psi_{t+1} | I_t]}
\end{aligned} \tag{15}$$

In our setting, a particularly interesting mean-variance efficient bond portfolio is the one with the same expected return as the growth optimal bond portfolio, $E[m_{t+1}^{-1} | I_t]$. The returns of this mean-variance efficient bond portfolio are characterized by:

$$\begin{aligned}
R_{mt+1}^b &= m_{t+1}^{-1} + b_{mt}^b \psi_{t+1} \\
b_{mt}^b &= \frac{E[\psi_{t+1} | I_t] V[\psi_{t+1} | I_t]^{-1} \text{Cov}[\psi_{t+1}, m_{t+1}^{-1} | I_t]}{E[\psi_{t+1} | I_t] V[\psi_{t+1} | I_t]^{-1} E[\psi_{t+1} | I_t] - V[\psi_{t+1} | I_t]^{-1} \text{Cov}[\psi_{t+1}, m_{t+1}^{-1} | I_t]} V[\psi_{t+1} | I_t]^{-1} E[\psi_{t+1} | I_t]
\end{aligned} \tag{16}$$

Then growth optimal bond portfolio returns $R_{gt+1}^b = m_{t+1}^{-1}$ are linked with the returns of this mean-variance efficient portfolio given by (16) via:

$$\begin{aligned}
R_{gt+1}^b &= R_{mt+1}^b + \varepsilon_{gt+1}^b \\
\varepsilon_{gt+1}^b &= -b_{mt}^b \psi_{t+1}; \text{Cov}[R_{mt+1}^b, \varepsilon_{gt+1}^b | I_t] = 0
\end{aligned} \tag{17}$$

Most modern term structure models are cast in terms of continuous time models driven

played by m_{t+1}^{-1} in the stochastic structure of discount bond returns in the absence of arbitrage given by (4) makes this focus on the growth bond optimal portfolio is natural in this application.

by a finite number of Wiener processes. As is well-known (cf., Long (1990) for a recent discussion), the growth optimal portfolio of risky assets with returns driven by diffusion processes is mean-variance efficient. It is straightforward but tedious to verify that ε_{gt+1}^b converges to zero and m_{t+1}^{-1} converges to $-\ln m_{t+1}$ in the continuous time limit when bond prices are driven by diffusion processes.

Analyses of the term structure of interest rates typically focus on the assumptions necessary to explicitly compute bond prices. In the process, researchers commonly place sufficient structure on the problem so that discount bond returns or their logarithms follow a conditional factor structure. Under reasonably weak assumptions, the growth optimal bond portfolio generally has the same returns as the growth optimal portfolio of the ensemble of risky assets that may be freely traded in frictionless markets. Hence, tests for the 'risk neutralizing' properties of growth optimal bond portfolios implicitly challenge a broad class of asset pricing models, not just particular parametric examples.

Appendix B: Term Structure Models and Growth Optimal Bond Portfolios

In the previous section, we provided an abstract stochastic process characterization of the conditions under which m_{t+1}^{-1} can be extracted from nominal discount bond portfolio returns alone. In this section, we tie this discussion to conventional diffusion-based term structure models.

We first examine real bond pricing in the standard model in which the representative household maximizes the discounted expected present value of isoelastic utility $\exp(-\rho t)c_t^{1-\gamma}/(1-\gamma)$. If consumption is conditionally lognormally distributed over all time intervals, the return of a J period bond over the interval between times t and t+Δt is given by:

$$\begin{aligned}
R_{t+\Delta t}^J &= m_{t+\Delta t}^{-1} \frac{E[q_{t+J} | I_{t+\Delta t}]}{E[q_{t+J} | I_t]} = \frac{\exp(-\rho t) c_t^{-\gamma}}{\exp[-\rho(t+\Delta t)] c_{t+\Delta t}^{-\gamma}} \frac{E[c_{t+J}^{-\gamma} | I_{t+\Delta t}]}{E[c_{t+J}^{-\gamma} | I_t]} \\
&= \exp(-\rho \Delta t) \left(\frac{c_{t+\Delta t}}{c_t} \right)^\gamma \exp \left\{ \gamma E[\ln c_{t+J} | I_{t+\Delta t}] - \frac{\gamma^2}{2} \text{Var}[\ln c_{t+J} | I_{t+\Delta t}] \right\} \\
&\quad \times \exp \left\{ - \left[\gamma E(\ln c_{t+J} | I_t) - \frac{\gamma^2}{2} \text{Var}(\ln c_{t+J} | I_t) \right] \right\}
\end{aligned} \tag{18}$$

Taking the limit as Δt becomes arbitrarily small yields:

$$\begin{aligned}
d \ln P_t^J &= -\gamma d \ln c_t + \gamma d E(\ln c_{t+J} | I_t) - \frac{\gamma^2}{2} dV(\ln c_{t+J} | I_t) \\
dE(\ln c_{t+J} | I_t) &\equiv \lim_{\Delta t \rightarrow 0} \{ E(\ln c_{t+J} | I_{t+\Delta t}) - E(\ln c_{t+J} | I_t) \} \\
dV(\ln c_{t+J} | I_t) &\equiv \lim_{\Delta t \rightarrow 0} \{ V(\ln c_{t+J} | I_{t+\Delta t}) - V(\ln c_{t+J} | I_t) \}
\end{aligned} \tag{19}$$

We can produce all diffusion models of which we are aware by assuming that the conditional means and variances of the log of consumption at different future dates are driven by a finite number of state variables driven by Wiener processes.

As is apparent from examining (19), it is not possible to recover $\ln m_t = \gamma d \ln c_t$ from instantaneous bond portfolio returns if there are common zero mean components in the conditional means and variances of the log of consumption at different future dates. The simplest case where this case arises is when consumption growth is homoskedastic and follows a time invariant lognormal diffusion. While this case might not be that realistic because of the implication that bond returns are perfectly correlated, the problem illustrated by this simple case generalizes – it is not possible to recover real intertemporal marginal rates of substitution when consumption growth has a component which is a time invariant lognormal diffusion. In this case, there is a persistent component of q_t that has identical short and long run effects, eliminating the possibility of extracting the inverse of the intertemporal marginal rate of substitution from discount bond returns alone.

Finally, this setting can be broadened to accommodate the nominal term structure.

Following Feenstra (1986), we add money to the economy as a mechanism for facilitating transactions. Accordingly, we assume that the representative agent must forego $\varphi(c_t, \bar{m}_t)$ units of consumption where consumption is c_t and real money balances are \bar{m}_t with $\varphi_c(c_t, \bar{m}_t) > 0$ and $\varphi_{\bar{m}}(c_t, \bar{m}_t) < 0$. The inflation-translated marginal rate of substitution in this model is given by:

$$m_{t+\Delta t} = \frac{u'(c_{t+\Delta t}, t + \Delta t)}{u'(c_t, t)} \frac{1}{1 + \pi_{t+\Delta t}} \left(\frac{1 + \varphi_c(c_t, \bar{m}_t)}{1 + \varphi_c(c_{t+\Delta t}, \bar{m}_{t+\Delta t})} \right); 1 + \pi_{t+\Delta t} = \frac{p_{ct+\Delta t}}{p_{ct}} \quad (20)$$

while the marginal rate of substitution between money and other assets ensures that:

$$E \left[\frac{u'(c_{t+\Delta t}, t + \Delta t)}{u'(c_t, t)} \left(\frac{[1 + \varphi_c(c_t, \bar{m}_t)][1 + \varphi_{\bar{m}}(c_t, \bar{m}_t)]}{1 + \varphi_c(c_{t+\Delta t}, \bar{m}_{t+\Delta t})} \right) \middle| I_t \right] = 1 \quad (21)$$

which, given exogenous processes for real consumption and nominal money balances, can be used to solve for the price level and rate of inflation. Taking the limit as Δt goes to zero yields the corresponding family of nominal term structure models.

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