Uncertainty and Risk in Financial Markets*

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Abstract

This paper considers a general equilibrium model in which the distinction between uncertainty and risk is formalized by assuming agents have incomplete preferences over state-contingent consumption bundles, as in Bewley (1986). Without completeness, individual decision making depends on a *set* of probability distributions over the state space. A bundle is preferred to another if and only if it has larger expected utility for all probabilities in this set. When preferences are complete this set is a singleton, and the model reduces to standard expected utility. In this setting, we characterize Pareto optima and equilibria, and show that the presence of uncertainty generates robust indeterminacies in equilibrium prices and allocations for any specification of initial endowments. We derive comparative statics results linking the degree of uncertainty with changes in equilibria. Despite the presence of robust indeterminacies, we show that equilibrium prices and allocations vary continuously with underlying fundamentals. Equilibria in a standard risk economy are thus robust to adding small degrees of uncertainty. Finally, we give conditions under which some assets are not traded due to uncertainty aversion.

JEL Codes: D0, D5, D8, G1

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1 Introduction

In his classic work, Knight (1921) argues that there is an important difference between uncertainty and risk, where risk is characterized by randomness that can be measured precisely. Knight also argues that this difference is important in markets. If risk were the only relevant feature of randomness, well-organized financial institutions should be able to price and market insurance contracts that only depend on risky phenomena. Uncertainty, however, creates frictions that these institutions may not be able to accommodate.

Ellsberg (1961) suggests a more precise definition of uncertainty, in which an event is uncertain or ambiguous if it has unknown probability. In particular, Ellsberg's paradox illustrates important consequences of this distinction by showing that individuals may prefer gambles with precise probabilities to gambles with unknown odds. Uncertainty and risk are distinct characteristics of random environments, and they can also affect individuals' behavior very differently. Such behavior is inconsistent with the expected utility model, and this observation has inspired a significant amount of recent research in economics.

Since uncertainty, as distinct from risk, can exert a significant influence on individual behavior, it should also be a significant determinant of equilibrium outcomes. For example, Knight (1921) claims that risk is insurable through exchange while uncertainty is not. Uncertainty should arguably lead to two notable departures from standard risk-sharing behavior in expected utility models. When uncertainty is prevalent, some insurance markets might break down, resulting in equilibria with no trade. Moreover, indeterminacy may also arise in this setting. Without uncertainty, the probabilities of risky events are known and frictionless markets can precisely price contracts contingent on risky events, at least generically. Even well-functioning markets, however, may not be able to precisely price contracts conditional on uncertain events, since the probabilities of these events are unknown. Such indeterminacy in equilibrium outcomes can generate excess price volatility and predictions that are extremely sensitive to small measurement errors. In this paper, we show that the presence of uncertainty leads directly to these twin predictions.

We consider a general equilibrium model in which the distinction between uncertainty and risk is formalized by assuming agents have incomplete preferences over state-contingent consumption bundles, as in Bewley (1986). Without completeness, individual decision-making depends on a *set* of probability distributions over the state space. A bundle is preferred to another if and only if it has larger expected utility for all probabilities in this set. When preferences are complete this set is a singleton, and the model reduces to standard expected utility. Since incompleteness is reflected by multiple probabilities, this approach provides one way to formalize the distinction between risk and uncertainty.

We consider an otherwise standard Arrow-Debreu exchange economy with a complete set of state-contingent security markets in which preferences might be incomplete.[1] In this setting, we characterize Pareto optima and equilibria, and show that the presence of uncertainty generates robust indeterminacies in equilibrium prices and allocations for any specification of initial en-

[1] Our model is an example of the general equilibrium models with unordered preferences developed by Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer and Sonnenschein (1975), and Fon and Otani (1979).

dowments. We derive comparative statics results linking the degree of uncertainty with changes in equilibria. Despite the presence of robust indeterminacies, we show that equilibrium prices and allocations vary continuously with underlying fundamentals. Equilibria in a standard risk economy are thus robust to adding small degrees of uncertainty. Finally, we give conditions under which some assets are not traded due to uncertainty aversion.

The basic intuition of our analysis can be traced back to a result in Bewley (1989). He shows that uncertainty makes opportunities for mutually satisfactory trades difficult to find in an exchange economy since individuals might be unwilling to insure each other. This aversion to trade is counterbalanced by the presence of risk aversion, which makes mutual insurance attractive. We show how equilibria can be characterized by the interplay between uncertainty and risk. For example, uncertainty is sometimes so large that no trade results; other times, the desire to insure prevails and there is trade. This trade-off is not captured by the standard expected utility model, where only risk aversion has a role.

The basic tool of our analysis is a characterization of Pareto optimal allocations that is the natural generalization of the corresponding characterization for complete preferences. In the expected utility setting, an (interior) allocation is Pareto optimal when marginal rates of substitution between any two states are equal across individuals. Without completeness, comparisons are carried out "one probability distribution at a time", with one bundle preferred to another if and only if it has higher expected utility for every probability distribution. Corresponding to each bundle, there is then a *set* of marginal rates of substitution for each individual, and we show that an allocation is Pareto optimal when there exists at least one common element in these sets. In addition, every element in the intersection of these sets yields a price supporting this Pareto optimal allocation.

These results lead naturally to a characterization of equilibria that illustrates the fundamental role uncertainty plays in hampering trading opportunities and fostering indeterminacies. Roughly, more uncertainty induces a larger set of beliefs, and thus may induce a larger overlap in marginal rates of substitution at a given allocation. For example, the larger is agents' perception of uncertainty, the easier it is for a given initial endowment to be Pareto optimal, and for a range of prices to support it as an equilibrium. Furthermore, even when perceived uncertainty is small and equilibrium involves trade, there may be infinitely many equilibrium allocations and prices. Thus, in our model, there is always a trade-off between uncertainty and risk.

Recently, many authors have studied equilibrium models with uncertainty, where uncertainty is modelled using either Choquet expected utility (CEU), or maxmin expected utility (MEU) (see Schmeidler (1989) and Gilboa and Schmeidler (1989) respectively).[2] Decision makers with CEU preferences evaluate a consumption bundle using its expected utility computed according to a capacity (a non-additive probability), while decision makers with MEU preferences evaluate a consumption bundle using the minimum expected utility over some set of probabilities. Although different in general, these models coincide in an important special case used in most of this work, CEU with a convex capacity. In this case, the Choquet expected utility is equal to the minimum

[2] Anderson, Hansen, and Sargent (2001), Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), Dana (2000) and Dana (2001), Dow and Werlang (1992), Epstein (2001), Epstein and Wang (1994), Hansen, Sargent, and Tallarini (1999), Hansen and Sargent (2001), Liu (1999), Mukerji and Tallon (2001), and Tallon (1998) are some notable examples.

expected utility over a particular set of probabilities. These models are thus similar to incomplete preferences in that uncertainty enters individual decision making through a set of probabilities.[3]

The equilibrium implications of these models, however, can be very different. Since individuals make decisions based on the minimum expected utility over this set of priors, their behavior displays pessimism. In equilibrium, this pessimism creates strong incentives for mutual insurance. In particular, indeterminacies and no trade in equilibrium are typically absent. For example, Dana (2000) shows that in the CEU model with common convex capacity, equilibria are generically determinate and coincide with equilibria in a standard expected utility model with fixed priors; indeterminacies arise robustly only in the case of no aggregate uncertainty, and only in this case are there observable differences between this model and expected utility in equilibrium. In Rigotti and Shannon (2003), we show that the generic determinacy of equilibria obtains for general MEU preferences, and in particular for CEU preferences with convex but differing capacities.

In our framework, on the other hand, indeterminacies arise for any initial endowments, thus indeterminacy is due solely to uncertainty. The reason for this difference is easy to illustrate in a model with two states. With CEU and MEU preferences, the agent's indifference curves have a kink along the certainty line and nowhere else, because on different sides of this line the minimum expected utility occurs at different probabilities. This kink makes indeterminacies possible, but only in very particular circumstances. With incompleteness, better-than sets have a kink at the consumption bundle that is being evaluated, *regardless* of where this bundle is. This kink is a fundamental consequence of uncertainty, and only disappears when preferences are complete and there is no uncertainty.

The remainder of the paper is organized as follows. The next section describes the basic decision-theoretic framework we use. Section 3 introduces the general equilibrium model and characterizes Pareto optima and equilibria. Section 4 presents the results on indeterminacy and comparative statics. Section 5 considers the case in which some events are uncertain while others are risky. Section 6 compares our results to the related literature. Section 7 concludes.

2 Preliminaries: Incomplete Preferences and Uncertainty

In this section we briefly describe individual behavior under uncertainty when preferences are not necessarily complete. Incompleteness in decision making under uncertainty was first studied by Aumann (1962).[4] In a series of papers, Bewley (1986), (1987), and (1989), further developed this model, which he called Knightian decision theory.[5] The basic result of Bewley's approach is to modify the standard expected utility framework by replacing the unique subjective probability distribution used in expected utility with a *set* of probability distributions. When an individual's preferences satisfy the completeness axiom, she can compare any two state-contingent consumption bundles; she decides which one is preferred based on their respective expected utilities. If

[3] A more detailed discussion of the decision theoretic alternatives to incompleteness and of the results derived in these different models is contained in Section 6.

[4] Recently, Aumann's work has been extended and clarified by Dubra, Maccheroni, and Ok (2001) and Shapley and Baucells (1998).

[5] Bewley's original paper has been published recently as Bewley (1964).

an individual's preferences are not complete, she is not necessarily able to compare every pair of consumption bundles. Because incompleteness is reflected by multiplicity of beliefs, she computes many expected utilities for each consumption bundle, and these might not be ranked uniformly.

To formalize this discussion, suppose the state space Ω is finite, and index the states by $s = 1, \ldots, S$. Let $x = (x_1, \ldots, x_S)$ and $y = (y_1, \ldots, y_S)$ be two consumption vectors in R^S_+. We assume an individual's preference relation $\succ$ for consumption bundles is represented by a unique closed, convex set of probability distributions Π and a continuous function $u : \mathsf{R}_+ \to \mathsf{R}$, unique up to positive affine transformations, such that

$$x \succ y \qquad \text{if and only if} \qquad \sum_{s=1}^{S} \pi_s u(x_s) > \sum_{s=1}^{S} \pi_s u(y_s) \text{ for all } \pi \in \Pi.$$

Abusing notation slightly, we can rewrite this as

$$x \succ y \qquad \text{if and only if} \qquad E_\pi [u(x)] > E_\pi [u(y)] \text{ for all } \pi \in \Pi$$

where $E_\pi [\cdot]$ denotes the expected value with respect to the probability distribution π, and $u(x)$ denotes the vector $(u(x_1), \ldots, u(x_S))$. We call this *assumption A0*.[6] Preferences of this kind have been characterized by Bewley (1986) in the Anscombe-Aumann framework, and by Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001) in a Savage setting.[7] Following Bewley, we say $\succ$ is *complete* if for all $x \in \mathsf{R}^S_+$, $\text{cl}\ \{y \in \mathsf{R}^S_+ : x \succ y \text{ or } y \succ x\} = \mathsf{R}^S_+$. The set of probabilities Π reduces to a singleton whenever the preference ordering $\succ$ is complete, in which case the usual expected utility representation obtains. Without completeness, comparisons between alternatives are carried out "one probability distribution at a time", with one bundle preferred to another if and only if it is preferred under every probability distribution considered by the agent.[8]

Bewley (1986) notes that the above representation captures the distinction Knight (1921) draws between risk and uncertainty. An event is risky when the probability is known, and uncertain otherwise. Similarly, the decision maker perceives only risk when Π is a singleton, and uncertainty otherwise. Incompleteness and uncertainty are equivalent measures of the same phenomenon. That is, the amount of uncertainty the decision maker perceives is equivalently reflected by the size of the set of priors Π and the degree of incompleteness of the preference order $\succ$. This idea can be formalized as follows. Given any preference order $\succ$ on R^S_+, let

$$C_\succ = \{(x, y) \in \mathsf{R}^S_+ \times \mathsf{R}^S_+ : x \text{ and } y \text{ are comparable under } \succ\}.$$

[6] While we work in Bewley's framework, in which each agent has a family of expected utility functions characterizing his incomplete preference order, our results should carry over to a more general setting in which agents' incomplete preferences are generated by a family of utility functions $v : \Delta \times \mathsf{R}^S_+ \to \mathsf{R}$ over probabilities and contingent consumption bundles (π, x) that are not necessarily linear or separable in the probability π. For example, it should be possible to incorporate features like probabilistic sophistication, as in Machina and Schmeidler (1992).

[7] Similar models have also been studied in statistical decision theory. See Nau (1992), Nau (2003), and Seidenfeld, Schervish, and Kadane (1995).

[8] The natural notion of indifference in this setting says two bundles are indifferent whenever they have the same expected utility for each probability distribution in Π.

We say the preference order $\succ'$ is *more complete* than the preference order $\succ$ if $C_{\succ} \subset C_{\succ'}$ and for every $(x, y) \in C_{\succ}$, $x \succ y \iff x \succ' y$. Less completeness in terms of the decision maker's preference order then corresponds to the decision maker's perception of more uncertainty.[9]

Proposition 1 *If $\succ$ and $\succ'$ are two preference orders over R_+^S satisfying assumption A0 and represented by (u, Π) and (u', Π') respectively, then $\succ'$ is more complete than $\succ$ if and only if u' is a positive affine transformation of u and $\Pi' \subset \Pi$.*

Proof: If u' is a positive affine transformation of u and $\Pi' \subset \Pi$, then the preference order $\succ'$ generated by (u', Π') is clearly more complete than the preference order $\succ$ generated by (u, Π).

Suppose $\succ'$ is more complete than $\succ$, where $\succ$ and $\succ'$ are represented by (u, Π) and (u', Π') respectively. Consider two bundles $\overline{x}$ and $\overline{y}$ (with $\overline{x} \neq \overline{y}$) such that $\overline{x}_s = \overline{x}_{s'}$ and $\overline{y}_s = \overline{y}_{s'}$ for any two states s, s'. Since $\succ$ and $\succ'$ are complete over constant bundles, $(\overline{x}, \overline{y}) \in C_{\succ'}$ and $(\overline{x}, \overline{y}) \in C_{\succ}$. Without loss of generality assume $\overline{x} \succ \overline{y}$. Because $\succ'$ is more complete than $\succ$ we must have $\overline{x} \succ' \overline{y}$ as well. Thus $\succ$ and $\succ'$ agree on all constant bundles. Since u (respectively, u') is simply a member of the affine family representing the preference $\succ$ (respectively, $\succ'$) restricted to constant bundles, u must be a positive affine transformation of u'. Then without loss of generality, take $u = u'$.

Let Π and Π' be sets of probabilities such that (u, Π) represents $\succ$ and (u, Π') represents $\succ'$. Consider any two bundles x and y such that $(x, y) \in C_{\succ}$. Since $\succ'$ is more complete than $\succ$, $(x, y) \in C_{\succ'}$ and $x \succ y \iff x \succ' y$. Thus:

$$E_\pi [u(x)] > E_\pi [u(y)] \text{ for all } \pi \in \Pi \quad \iff \quad E_\pi [u(x)] > E_\pi [u(y)] \text{ for all } \pi \in \Pi'$$

and

$$E_\pi [u(x)] < E_\pi [u(y)] \text{ for all } \pi \in \Pi \quad \iff \quad E_\pi [u(x)] < E_\pi [u(y)] \text{ for all } \pi \in \Pi'.$$

Now we claim $\Pi' \subset \Pi$. If not, there exists $\pi' \in \Pi' \setminus \Pi$. By the separating hyperplane theorem, there exists $v \neq 0$ such that $\pi' \cdot v > 0 > \pi \cdot v$ for all $\pi \in \Pi$. Choose bundles x and y such that $u(x_s) - u(y_s) = v_s$ for each s. Since $E_\pi[u(x)] - E_\pi[u(y)] = \pi \cdot v < 0$ for all $\pi \in \Pi$, $(x, y) \in C_{\succ}$ and $y \succ x$. But $E_{\pi'}[u(x)] - E_{\pi'}[u(y)] = \pi' \cdot v > 0$, which is a contradiction. Thus $\Pi' \subset \Pi$. ■

A graph may help clarify how Bewley's representation works. In Figure 1 the axes measure consumption in each of the two possible states. Given a probability distribution over the two states, a standard indifference curve through the bundle y represents all the bundles that have the same expected utility as y according to this distribution. As the probability distribution changes, we obtain a family of these indifference curves representing different expected utilities according to different probabilities. The thick curves represent the most extreme elements of this family, while thin curves represent other possible elements.

A bundle like x is preferred to y since it lies above all of the indifference curves corresponding to some expected utility of y. Also, y is preferred to w since w lies below all of the indifference curves through y. Finally, z is not comparable to y since it lies above some indifference curves through y

[9] A similar result can be found in Ghirardato, Maccheroni, and Marinacci (2002).

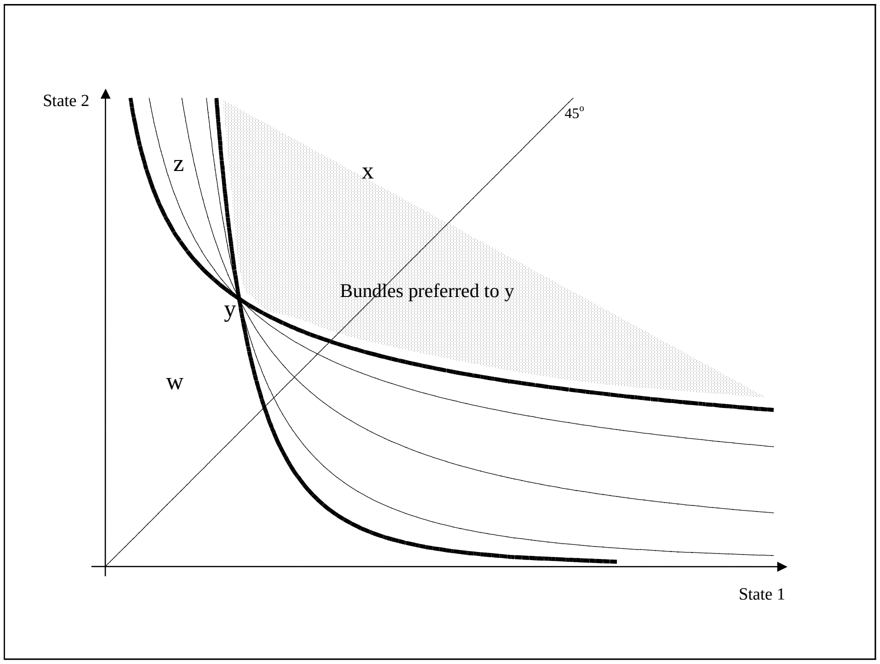

Figure 1: Incomplete Preferences

and below others. Incompleteness induces three regions: bundles preferred to y, dominated by y, and incomparable to y. This last area is empty only if there is a unique probability distribution over the two states and the preferences are complete. Therefore, for any bundle y, the better-than-y set has a kink at y whenever there is uncertainty. This kink is a direct consequence of the multiplicity of probability distributions in Π, and vanishes only when Π is a singleton.

When preferences are not complete, the usual revealed preference arguments do not apply. If x is chosen when y is available, we cannot say x is revealed preferred to y, we can only say y is not revealed preferred to x. In other words, one cannot explain choice among incomparable alternatives. To address this problem, Bewley (1986) introduces a behavioral assumption based on the idea of *inertia*. The inertia assumption posits the existence of a *status quo* that is abandoned only for alternatives that are preferred to it. When the inertia assumption holds, if x is chosen when y is available *and* y is the status quo, then x is revealed preferred to y. If the only alternatives available are not comparable to y, the individual sticks with y. In Figure 1, the inertia assumption implies that if y is the status quo, alternatives like z will not be chosen since they are incomparable to y.

Formally, the inertia assumption as stated in Bewley (1986) requires that if y is the status quo, then x is chosen when y is available only if $E_\pi [u(x)] > E_\pi [u(y)]$ for all $\pi \in \Pi$. If Π is a singleton, and therefore there is no uncertainty, this notion of inertia implies bundles indifferent to y are not chosen. This type of behavior seems unduly restrictive, since it introduces reluctance to abandon the status quo even when uncertainty is absent. On the other hand, one can easily modify the inertia assumption to correct this problem as follows. If y is the status quo, a weaker inertia assumption states that x is chosen when y is available only if $E_\pi [u(x)] \geq E_\pi [u(y)]$ for all $\pi \in \Pi$.

The inertia assumption is a behavioral statement, rather than a property derived from preferences. Work in both economics and psychology provides evidence of behavior under uncertainty that is consistent with such a status quo bias. A classic reference is Samuelson and Zeckhauser (1988). In an experimental setting, they find significant status quo bias in investment decisions regarding portfolio composition following a hypothetical inheritance. Moreover, suggestive of the trade-off between uncertainty and risk embedded in our model, this bias varies with the strength of preference and the number of alternatives: the status quo bias is weaker the stronger is the subjects' preference for the alternatives, and stronger in the face of more alternatives. They also find similar evidence in field data on health plan choices and portfolio division in TIAA-CREF plans among Harvard employees. More recently, Madrian and Shea (2001) document inertia in both participation and portfolio composition following auotmatic enrollments in 401(k) plans in a large American corporation. Ameriks and Zeldes (2000) find similar evidence in data from TIAA-CREF and Surveys of Consumer Finance documenting a significant relationship between age and portfolio choices. Almost half of their sample made no change in portfolio composition over the course of the 9 years they observe, while the same period saw drastic changes in the returns to bond and equity holdings. Einhorn and Hogarth (1985) find evidence supporting a status quo bias in initial probability assessments in a number of experiments.[10]

Although there is significant evidence of endowment effects consistent with inertia in a variety of settings, a limitation of this assumption in some models may stem from the difficulty of defining a plausible status quo. In a general equilibrium model, however, there is a natural candidate for the status quo: an individual's initial endowment. In our analysis, as the next section makes clear, the inertia assumption plays the role of a natural equilibrium refinement device. With incompleteness, there may be many equilibria and inertia may select among them.

3 Uncertainty and Complete Markets

In this section, we describe a simple exchange economy in which agents' preferences over contingent consumption are incomplete. Except for this innovation, we consider the standard Arrow-Debreu model of complete contingent security markets. The main result of this section is a characterization of Pareto optimal allocations and equilibria. This characterization is based on the fact that incomplete preferences are described by a family of utility functions. The usual condition for Pareto optimality in terms of equating marginal rates of substitution extends naturally using a set-valued notion of marginal rates of substitution. Having obtained a characterization of Pareto optimal allocations, we use a standard argument to characterize equilibria.

There are two dates, 0 and 1. At date 1, there are S possible states of nature, indexed by $s = 1, \ldots, S$. There is a single consumption good available at date 1; for simplicity we assume there is no consumption at date 0. At date 0 agents can trade in a complete set of Arrow securities for contingent consumption at date 1. We let Δ denote the standard simplex in R^S_+.

[10]See also Fox and Tversky (1995) and Heath and Tversky (1991).

There are finitely many agents, indexed by $i = 1, \ldots, I$. Each agent has an endowment $\omega^i \in \mathsf{R}^S_+$ of contingent consumption at date 1, and a preference order $\succ^i$ over R^S_+. We use superscripts to denote agents and subscripts to denote states. To model the distinction between risk and uncertainty we follow Bewley (1986) and allow each agent's preference order to be incomplete. We maintain the following assumptions regarding agents' characteristics:

(A1) for each $i = 1, \ldots, I$, $\omega^i \in \mathsf{R}^S_{++}$

(A2) for each $i = 1, \ldots, I$, there exists a closed, convex set $\Pi^i \subset \Delta$ and a C^1, concave, strictly increasing function $u^i : \mathsf{R}_+ \to \mathsf{R}$ such that for any $x, y \in \mathsf{R}^S_+$, $x \succ^i y$ if and only if

$$\sum_{s=1}^{S} \pi_s u^i(x_s) > \sum_{s=1}^{S} \pi_s u^i(y_s) \text{ for all } \pi \in \Pi^i.$$

Although preferences are potentially incomplete, there are natural notions of Pareto optimality and equilibrium in this model. Indeed, our model is a special case of equilibrium models with unordered preferences developed in Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer and Sonnenschein (1975), and Fon and Otani (1979). Our definitions of Pareto optimality and equilibrium follow theirs.

We call an allocation $(x^1, \ldots, x^I)$ a *feasible allocation* if $\sum_i x^i \leq \sum_i \omega^i$. Then as in standard models, a feasible allocation is Pareto optimal if there is no other feasible allocation that each agent strictly prefers.[11]

Definition A feasible allocation $\left(x^1, \ldots, x^I\right)$ is *Pareto optimal* if there is no other feasible allocation $\left(y^1, \ldots, y^I\right)$ such that $y^i \succ^i x^i$ for each $i = 1, \ldots, I$.

Similarly, the definition of equilibrium is standard: each agent chooses an element that is maximal in his budget set and all markets clear.

Definition A feasible allocation $\left(x^1, \ldots, x^I\right)$ and a non-zero price vector $p \in \mathsf{R}^S_+$ are an *equilibrium* if

1. $x \succ^i x^i \Rightarrow p \cdot x > p \cdot \omega^i$ for all i,
2. $p \cdot x^i = p \cdot \omega^i$ for all i.

Although the definition of equilibrium is standard, many allocations may be equilibrium allocations when preferences are incomplete. This happens precisely because no assumptions

[11] If each individual's beliefs have full support, this notion of Pareto optimality is equivalent to one which requires instead that there exists no feasible allocation $\left(y^1, \ldots, y^I\right)$ such that for each agent i

$$\sum_{s=1}^{S} \pi_s u^i(y^i_s) \geq \sum_{s=1}^{S} \pi_s u^i(x^i_s) \text{ for all } \pi \in \Pi^i,$$

with strict inequalities for some i. This follows by standard arguments, using the continuity and monotonicity of preferences.

are made about agents' choices between incomparable alternatives. In particular, equilibria are not necessarily individually rational. To address this issue, we use the inertia assumption as a possible equilibrium refinement device. Inertia says alternatives incomparable to the status quo are not chosen. We take an individual's initial endowment as his status quo, and apply this reasoning. Given his budget set, the inertia assumption implies that an agent will not choose a bundle different from the initial endowment if it is incomparable, which leads naturally to the following definition.

Definition A feasible allocation $\left(x^1, \ldots, x^I\right)$ and a non-zero price vector $p \in \mathsf{R}_+^S$ are an *equilibrium with inertia* if

1. $x \succ^i x^i \Rightarrow p \cdot x > p \cdot \omega^i$ for all i,
2. $p \cdot x^i = p \cdot \omega^i$ for all i,
3. for each i, either $x^i = \omega^i$, or $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for each $\pi^i \in \Pi^i$.

Equilibrium with inertia considers focal equilibrium allocations in which (some) agents do not trade. With inertia, an equilibrium involves trade only if it is preferred to the status quo by all individuals who engage in trading. Inertia has particularly sharp consequences when the initial endowment allocation is an equilibrium. In that case, it is also the unique equilibrium allocation with inertia provided agents are risk averse and their beliefs have full support, as the following simple lemma demonstrates.

Lemma 1 *Suppose assumptions A1-A2 hold, and that for each i, u^i is strictly concave and $\Pi^i \subset \text{rint } \Delta$.*[12] *If $(\omega^1, \ldots, \omega^I)$ is an equilibrium allocation, then it is the unique equilibrium allocation with inertia.*

Proof: Let p be a price vector supporting $(\omega^1, \ldots, \omega^I)$ as an equilibrium. Suppose by way of contradiction that there is another equilibrium allocation with inertia, $(x^1, \ldots, x^I) \neq (\omega^1, \ldots, \omega^I)$. Let T be the set of agents who trade in this equilibrium, so $T = \{i : x^i \neq \omega^i\}$. Then $T \neq \emptyset$, and by inertia we know that for each $i \in T$, $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for all $\pi^i \in \Pi^i$. Since $(\omega^1, \ldots, \omega^I)$ is an equilibrium supported by p, we claim that

$$p \cdot x^i > p \cdot \omega^i \text{ for all } \ i \in T$$

To see this, fix $i \in T$ and note that for each $n \geq 1$,

$$x^i + \frac{1}{n}\omega^i \succ^i \omega^i$$

[12]Given a set X, rint X denotes the relative interior of X. For Δ and any subset of Δ, by relative interior we mean relative to $\{q \in \mathsf{R}^S : \sum_s q_s = 1\}$. Thus rint $\Delta = \{\pi \in \Delta : \pi_s > 0 \; \forall s\}$, and for $C \subset \Delta$,

$$\text{rint } C = \{q \in C : \exists \text{ a neighborhood } V \text{ of } q \text{ in } \mathsf{R}^S \text{ s.t. } V \cap \text{rint } \Delta \subset C\}.$$

by the strict monotonicity of u^i. Then for each $n \geq 1$,

$$p \cdot \left(x^i + \frac{1}{n} \omega^i \right) > p \cdot \omega^i$$

which implies that

$$p \cdot x^i \geq p \cdot \omega^i$$

Now suppose there exists $i \in T$ such that $p \cdot x^i = p \cdot \omega^i$. Fix $\alpha \in (0,1)$. By strict concavity of u^i and the fact that $\Pi^i \subset \text{rint } \Delta$,

$$E_{\pi^i}[u^i(\alpha x^i + (1-\alpha)\omega^i)] > E_{\pi^i}[u^i(\omega^i)] \quad \forall \pi^i \in \Pi^i$$

But

$$p \cdot (\alpha x^i + (1-\alpha)\omega^i) = \alpha p \cdot x^i + (1-\alpha) p \cdot \omega^i = p \cdot \omega^i$$

which is a contradiction. Thus

$$p \cdot x^i > p \cdot \omega^i \text{ for all } \ i \in T.$$

Adding over i yields

$$p \cdot \left(\sum_{i \in T} x^i \right) > p \cdot \left(\sum_{i \in T} \omega^i \right).$$

But this is a contradiction, as

$$\sum_{i \in T} x^i = \sum_{i \in T} \omega^i$$

by feasibility and the definition of T. ■

Our first basic result shows that, as in the standard model with risk alone, equilibria exist, and every equilibrium allocation is Pareto optimal. These results are straightforward applications of work on equilibrium analysis with unordered preferences; see Mas-Colell (1974), Gale and Mas-Colell (1975), and Fon and Otani (1979).

Theorem 1 *Under assumptions A1 and A2, every economy has an equilibrium and every equilibrium allocation is Pareto optimal.*

Proof: See appendix. ■

Next we establish a parallel result regarding the existence of equilibria with inertia. While this result may at first seem a trivial consequence of individual rationality in equilibrium, recall that in our framework equilibrium consumption is not necessarily individually rational. Because preferences are incomplete, some equilibrium consumption bundles might be incomparable to initial endowments. Indeed, the inertia assumption is added precisely to restore individual rationality in equilibrium. The idea behind the argument is then to modify Gale and Mas-Colell's original existence proof appropriately by adding conditions corresponding to the inertia assumption.

Theorem 2 *Under assumptions A1 and A2, every economy has an equilibrium with inertia.*

Proof: See appendix. ∎

Next we give several useful results. The first characterizes Pareto optimal allocations using a no-trade result in Bewley (1989). The main idea is to use the set of supports of the better-than set to characterize Pareto optimal allocations and equilibria. For this we require the additional assumption that each agent's von Neumann-Morgenstern utility index has no critical points.

(A3) For each i, $u^{i\prime}(c) > 0$ for every $c > 0$

Now for any consumer i and each $x \in \mathsf{R}^S_{++}$, let

$$\Pi^i(x) = \left\{ \left(\frac{\pi_1 u^{i\prime}(x_1)}{\sum_t \pi_t \; u^{i\prime}(x_t)}, \ldots, \frac{\pi_S u^{i\prime}(x_S)}{\sum_t \pi_t \; u^{i\prime}(x_t)} \right) : \pi \in \Pi^i \right\}.$$

An element $q \in \Pi^i(x)$ is an element of the simplex Δ, i.e. is itself a probability measure. Therefore this set represents the vectors of "marginal beliefs" induced by the bundle x. As we show below, the set $\Pi^i(x)$ is the set of supports of the better-than-x set. Notice that if x is constant across states, $\Pi^i(x) = \Pi^i$. Also, if Π^i is a singleton, so is $\Pi^i(x)$, so when preferences are complete this construction just yields the ordinary (normalized) gradient.

Since the set $\Pi^i(x)$ also represents the possible supports of the better-than-x set, an (interior) allocation $(x^1, \ldots, x^I)$ is Pareto optimal if and only if there is a common element of the sets $\Pi^i(x^i)$. This result is established formally in the following theorem, adapted from Bewley (1989).[13]

Theorem 3 *Under assumptions A1-A3, an interior allocation* $\left(x^1, \ldots, x^I\right)$ *is Pareto optimal if and only if*

$$\bigcap_{i=1}^{I} \Pi^i\left(x^i\right) \neq \emptyset.$$

Proof: Define the sets K^i and K as follows:

$$K^i = \left\{ z \in \mathsf{R}^S : \; \sum_{s=1}^{S} \pi_s u^i\left(z^i_s + x^i_s\right) > \sum_{s=1}^{S} \pi_s u^i\left(x^i_s\right) \text{ for all } \pi \in \Pi^i \right\}$$

and

$$K = \left\{ z \in \prod_{i=1}^{I} K^i : \; \sum_{i=1}^{I} z^i = 0 \right\} \cup \{0\}.$$

Then $\left(x^1, \ldots, x^I\right)$ is a Pareto optimal allocation if and only if $K = \{0\}$. So we need only show that

$$\bigcap_{i=1}^{I} \Pi^i\left(x^i\right) \neq \emptyset \Longleftrightarrow K = \{0\}.$$

[13] Bewley (1989) focuses on the absence of trade. His Propositions 3.1 and 5.1 then provide conditions under which a no-trade result obtains under inertia. In our setting, the same conditions are used to characterize Pareto optimality and equilibrium.

Suppose $\bigcap_{i=1}^{I} \Pi^i\left(x^i\right) \neq \emptyset$ and let $p \in \bigcap_{i=1}^{I} \Pi^i\left(x^i\right)$. For each i, let $z^i \in K^i$; then, using the definition of K^i, we have

$$\begin{aligned} 0 &< \sum_{s=1}^{S} \pi_s \left[u^i\left(z_s^i + x_s^i\right) - u^i\left(x_s^i\right)\right] \text{ for all } \pi \in \Pi^i \\ &\leq \sum_{s=1}^{S} \pi_s u^{i\prime}(x_s^i) z_s^i \text{ for all } \pi \in \Pi^i, \end{aligned}$$

where the second inequality follows from the concavity of u^i. This last inequality implies that

$$\sum_{s=1}^{S} \frac{\pi_s u^{i\prime}(x_s^i)}{\sum_t \pi_t u^{i\prime}(x_t^i)} z_s^i > 0 \text{ for all } \pi \in \Pi^i. \tag{1}$$

Since $p \in \bigcap \Pi^i(x^i)$, for each i there exists $\pi^i \in \Pi^i$ such that

$$p = \left(\frac{\pi_1^i u^{i\prime}(x_1^i)}{\sum_t \pi^i u^{i\prime}(x_t^i)}, \ldots, \frac{\pi_S^i u^{i\prime}(x_S^i)}{\sum_t \pi^i u^{i\prime}(x_t^i)}\right).$$

Since (1) holds for every $\pi \in \Pi^i$ and every i, we can substitute p in (1) and conclude that

$$\sum_{s=1}^{S} p_s z_s^i = p \cdot z^i > 0 \text{ for all } i.$$

Summing over individuals, we get

$$\sum_{i=1}^{I} p \cdot z^i > 0,$$

or

$$p \cdot \left(\sum_{i=1}^{I} z^i\right) > 0.$$

Since $p \neq 0$ this implies

$$\sum_{i=1}^{I} z^i \neq 0,$$

and thus $K = \{0\}$.

Now suppose $K = \{0\}$. We must show this implies $\bigcap_{i=1}^{I} \Pi^i\left(x^i\right) \neq \emptyset$. Let $X = \sum_{i=1}^{I}\left(K^i \cup \{0\}\right)$. Since $K = \{0\}$, $X \cap \mathsf{R}_-^S = \{0\}$. By the Separating Hyperplane Theorem, there exists $p \in \mathsf{R}^S$, with $p \neq 0$, which separates X and R_-^S. In particular, choose p so that

$$p \cdot x \geq 0 \text{ for all } x \in X.$$

Without loss of generality, take p to be normalized so that $\sum_{s=1}^{S} p_s = 1$. Note that $K_i \subset X$ for all i. Therefore

$$p \cdot x \geq 0 \text{ for all } x \in K_i, \text{ for all } i.$$

We claim that $p \in \bigcap_{i=1}^{I} \Pi^i\left(x^i\right)$. To see this, first note that for all i, $\Pi^i\left(x^i\right)$ is compact and convex (see Lemma 3 in the Appendix). Let i be given. Now if $p \notin \Pi^i(x^i)$, there exists $y \neq 0$ such that

$$p \cdot y < b < p^i \cdot y \quad \forall p^i \in \Pi^i(x^i) \text{ for some } b \in \mathsf{R}.$$

Moreover, we can take $b = 0$ without loss of generality, as if $b \neq 0$ define $\bar{y}$ by $\bar{y}_s = y_s - b$ for each s. Then $\bar{y} \neq 0$ and

$$p \cdot \bar{y} = p \cdot y - b < 0 < p^i \cdot y - b = p^i \cdot \bar{y} \quad \forall p^i \in \Pi^i(x^i).$$

Finally, there exists $\alpha > 0$ such that $\alpha\bar{y} \in K_i$ (since $\sum \pi_s^i u_s^{i\prime}(x_s^i)\bar{y}_s > 0$ for all $\pi^i \in \Pi^i$). So choose $\alpha > 0$ such that $\alpha\bar{y} \in K_i$, and note that

$$p \cdot (\alpha\bar{y}) < 0 < p^i \cdot (\alpha\bar{y}) \quad \forall p^i \in \Pi^i(x^i),$$

contradicting the definition of p. Thus $p \in \Pi^i(x^i)$. Since i was arbitrary, we conclude that $p \in \bigcap_i \Pi^i(x^i)$. ■

In the course of this proof, we have also established the useful and intuitive result concerning the set of supports of an agent's better-than set to which we alluded above: the set of "marginal beliefs" corresponding to a bundle completely characterizes the set of supports of the better-than set at that bundle.

Corollary 1 *Suppose assumptions A1-A3 hold. Let $x^i \in \mathsf{R}_{++}^S$ and*

$$B^i(x^i) = \left\{ z \in \mathsf{R}_+^S : \sum \pi_s u^i(z_s) > \sum \pi_s u^i(x_s^i) \text{ for all } \pi \in \Pi^i \right\}.$$

Then $p \in \Delta$ supports $B^i(x^i)$ at x^i if and only if $p \in \Pi^i(x^i)$. Moreover, if $p \in \Pi^i(x^i)$ then p strictly supports $B^i(x^i)$ at x^i, i.e. for every $z \in B^i(x^i)$, $p \cdot z > p \cdot x^i$.

Proof: Note that, adapting the notation of the preceding proof, $B^i(x^i) = \{x^i\} + K^i(x^i)$, where $K^i(x^i) = \{z : \sum \pi_s u^i(x_s^i + z_s) > \sum \pi_s u^i(x_s^i) \text{ for all } \pi \in \Pi^i\}$. Now the results follow by noting that the preceding proof establishes that $p \in \Delta$ supports $K^i(x^i)$ at 0 if and only if $p \in \Pi^i(x^i)$, and that each $p \in \Pi^i(x^i)$ strictly supports $K^i(x^i)$ at 0. ■

Our characterization of Pareto optimality is related to recent work on the absence of betting. Theorem 3 can be seen as a generalization of the main result in Billot, Chateauneuf, Gilboa, and Tallon (2000) (for a finite state space). Assuming the economy has no aggregate uncertainty and individuals have MEU preferences, they show that a full insurance allocation is Pareto optimal if and only if all agents share at least one prior. While distinct in general, with respect to bundles that involve no uncertainty, better-than sets for MEU preferences and incomplete preferences coincide.[14] Therefore we can apply Theorem 3 to the special case of full insurance allocations to obtain their result as follows.

[14]To see this, note that if z is a certain bundle, so that $z_s = \bar{z}$ for every s, then the better-than-z set under each model corresponds to $\{x \in \mathsf{R}_+^S : \sum_s \pi_s u(x_s) \geq u(\bar{z}) \; \forall \pi \in \Pi\}$. Thus a full insurance allocation is Pareto optimal under MEU preferences if and only if it is Pareto optimal under incomplete preferences. See also Rigotti and Shannon (2003).

Corollary 2 *Under assumptions A1-A3, there is a full insurance Pareto optimal allocation if and only if individuals have at least one common probability distribution. In this case every full insurance allocation is Pareto optimal.*

Proof: If x is a full insurance allocation, then x^i is constant in every state. Thus for each i there exists k^i such that $x^i_s = k^i$ for each state s. In this case, $\Pi^i(x^i) = \Pi^i$ for all i. Then the result follows by Theorem 3. ■

We now turn to characterizations of equilibrium allocations. Theorem 3 shows that an allocation is Pareto optimal if and only if there is a common element in the marginal belief sets of all agents at their bundles in the allocation. This leads to a characterization of equilibrium via standard Negishi-type arguments, as a Pareto optimal allocation is then an equilibrium if there is a price in this common support set at which each consumer's budget constraint is satisfied.

Theorem 4 *Under assumptions A1-A3, an interior allocation $\left(x^1, \ldots, x^I\right)$ is an equilibrium allocation if and only if*

$$\bigcap_i \Pi^i\left(x^i\right) \neq \emptyset$$

and there exists $p \in \bigcap_i \Pi^i\left(x^i\right)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i.

Proof: Let $(x^1, \ldots, x^I)$ be an interior equilibrium allocation. Then there exists a non-zero $p \in \mathsf{R}^S_+$ such that $p \cdot y > p \cdot x^i$ for every $y \in \mathsf{R}^S_+$ such that $y \succ^i x^i$. Without loss of generality, suppose p is normalized so that $\sum p_s = 1$. By Corollary 1, $p \in \Pi^i(x^i)$. Since this holds for each i, $p \in \bigcap \Pi^i(x^i)$.

Now suppose there exists $p \in \bigcap \Pi^i(x^i)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i. By Corollary 1, for each i, if $y \succ^i x^i$ then $p \cdot y > p \cdot x^i = p \cdot \omega^i$. Thus $(x^1, \ldots, x^I)$ is an equilibrium allocation supported by p. ■

As an immediate consequence, we can characterize equilibria with inertia, and then derive conditions under which no trade is the unique equilibrium with inertia.

Corollary 3 *Under assumptions A1-A3, an interior allocation $\left(x^1, \ldots, x^I\right) \neq \left(\omega^1, \ldots, \omega^I\right)$ is an equilibrium allocation with inertia if and only if for every i such that $x^i \neq \omega^i$, $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for each $\pi^i \in \Pi^i$,*

$$\bigcap_i \Pi^i\left(x^i\right) \neq \emptyset$$

and there exists $p \in \bigcap_i \Pi^i\left(x^i\right)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i.

Corollary 4 *Suppose assumptions A1-A3 hold, and in addition that for each i, u^i is strictly concave and $\Pi^i \subset$ rint Δ. Then the no-trade allocation $\left(\omega^1, \ldots, \omega^I\right)$ is the unique equilibrium allocation with inertia if and only if*

$$\bigcap_{i=1}^I \Pi^i\left(\omega^i\right) \neq \emptyset.$$

Any price $p \in \bigcap_{i=1}^I \Pi^i\left(\omega^i\right)$ supports $\left(\omega^1, \ldots, \omega^I\right)$.

These results can easily be illustrated graphically. Figure 2 depicts a no-trade equilibrium. The initial endowment is ω. Agent 1's preferences are described by the grey indifference curves; bundles preferred to ω^1 are to the northeast of these indifference curves. Agent 2's preferences are described by the black indifference curves; bundles preferred to ω^2 are to the southwest of these indifference curves. In this example, there are no allocations preferred to ω by both individuals. Hence the initial endowment is Pareto optimal and an equilibrium, and it is the unique equilibrium with inertia. A range of prices supports this as an equilibrium.

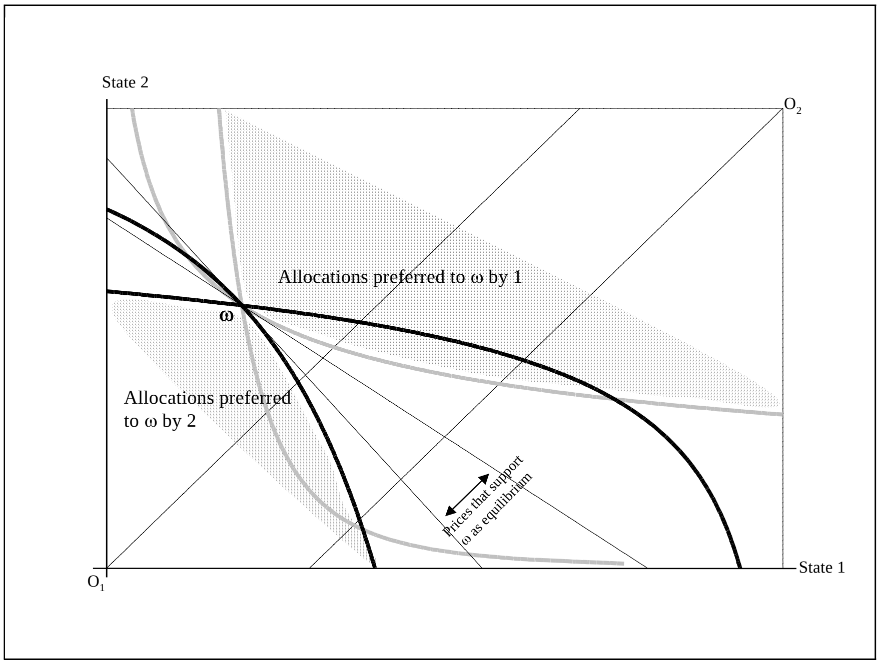

Figure 2: A no-trade equilibrium

Figure 2 helps to understand how uncertainty may reduce the set of mutually beneficial trades. In the diagram, the presence of uncertainty in each individual's beliefs is reflected by the kink at the initial endowment. More uncertainty implies a "sharper" kink since more probability distributions are used to determine the set of consumption bundles preferred to the initial endowment. Therefore, more uncertainty implies that there are fewer allocations preferred to the initial endowment by all agents. Risk-aversion also plays a role since the curvature of the utility function determines the shape of the better-than sets. Risk-aversion is reflected in an incentive for individuals to seek mutual insurance, and hence trade away from the initial endowment. The trade-off between these forces determines equilibrium.

Figure 2 also helps to highlight a relatively simple way to characterize Pareto optimal allocations. As noted in Section 2, we can think of each individual evaluating consumption bundles using a family of utilities. When an allocation is Pareto optimal, there must exist members of each individual's family of indifference curves that are tangent at that allocation. This point is made formally by the following theorem.

Theorem 5 *Under assumptions A1-A3, an interior allocation $x = (x^1, \ldots, x^I)$ is Pareto optimal if and only if for each i there exists $\pi^i \in \Pi^i$ such that x solves the social planner's problem*

$$\max_{x \in \mathsf{R}^{SI}_{++}} \quad \sum_{i=1}^{I} \lambda^i \sum_{s=1}^{S} \pi^i_s u^i(x^i_s)$$

$$s.t. \ \sum_{i=1}^{I} x^i = \sum_{i=1}^{I} \omega^i$$

for some weights $\lambda^i \geq 0$ such that $\sum_i \lambda^i = 1$.

Proof: First note that there exists $\pi^i \in \Pi^i$ for each i such that x solves the social planner's problem for some weights $\lambda^i \geq 0$ such that $\sum_i \lambda^i = 1$ if and only if x is Pareto optimal in the (risk-only) economy with unique priors $(\pi^1, \ldots, \pi^I)$. In this case there exists no allocation y such that

$$\sum_s \pi^i_s u^i(y^i_s) > \sum_s \pi^i_s u^i(x^i_s) \quad \text{for all } i = 1, \ldots, I\,,$$

which implies there exists no allocation y such that $y^i \succ^i x^i$ for each i. Thus x is also Pareto optimal in the economy with uncertainty.

To establish the converse, note that since x is Pareto optimal, $\bigcap_i \Pi^i(x^i) \neq \emptyset$ by Theorem 3. Thus there exist $\pi^i \in \Pi^i$ for each $i = 1, \ldots, I$ such that for each i, j,

$$\frac{\pi^i_s u^{i\prime}(x^i_s)}{\sum_t \pi^i_t u^{i\prime}(x^i_t)} = \frac{\pi^j_s u^{j\prime}(x^j_s)}{\sum_t \pi^j_t u^{j\prime}(x^j_t)} \text{ for all } s.$$

Then in the risk economy with priors $(\pi^1, \ldots, \pi^I)$, x is Pareto optimal. ■

The previous result can also be interpreted as a description of the relationship between Pareto optimal allocations in an economy where uncertainty is present and those in economies where there is only risk. In particular, Theorem 5 shows that an allocation is Pareto optimal in our economy if and only if it is Pareto optimal in a standard risk economy in which agents assign some unique subjective priors π^i chosen from the sets Π^i. In the next section, we expand on this theme.

4 Determinacy and Comparative Statics

In this section, we analyze the role of uncertainty in determining equilibrium outcomes. We proceed in two directions. First, we investigate how uncertainty affects the determinacy of equilibrium allocations and prices. We find robust indeterminacies for every initial endowment vector. Our results are in fact quite stark: provided there is sufficient overlap in agents' beliefs, there are a continuum of equilibrium allocations and prices, regardless of other features of agents' beliefs, initial endowments, or aggregate endowments. Our second class of results establishes that, despite such robust indeterminacies, the set of equilibria varies continuously with the amount of uncertainty agents perceive. In particular, as uncertainty goes to zero (that is, agents perceive only risk), the equilibrium correspondence converges to an equilibrium of the economy in which there is only risk.

The following is our main indeterminacy result. For this result we will use a standard Inada condition coupled with the requirement that all agents' beliefs have full support to ensure that all equilibria are interior, and hence that we can use the characterization of equilibria in Theorem 4. The addition of these assumptions also further highlights the role of uncertainty as the source of indeterminacies in our model, since these are essentially the assumptions under which generic determinacy would obtain in the absence of uncertainty. Thus here we add

(A4) for each i, $u^{i\prime}(c) \to \infty$ as $c \to 0$

(A5) for each i, $\Pi^i \subset \mathsf{R}^S_{++}$

Theorem 6 *Suppose assumptions A1-A5 hold, and* $\bigcap_i \Pi^i$ *has non-empty relative interior. Then for every initial endowment vector* $(\omega^1, \ldots, \omega^I)$*, there is a continuum of equilibrium allocations and prices.*

The idea behind this result is straightforward. First consider an economy which has sufficient uncertainty in which the belief sets of all agents admit a common probability distribution. Consider the risk-only economy defined by this common prior. An equilibrium in this economy is an equilibrium in the economy with uncertainty, but there are infinitely many other equilibria around it due to uncertainty. This fact, established in the next lemma, immediately implies our indeterminacy result.

Lemma 2 *Suppose assumptions A1-A5 hold, and* $\bigcap_i \Pi^i$ *has non-empty relative interior. Let* $\pi \in \operatorname{rint} \bigcap_i \Pi^i$ *and let* x *be an equilibrium allocation in the risk economy with common prior* π*. Let* $p \in \Delta$ *be the corresponding equilibrium price supporting* x*, so*

$$p_s = \frac{\pi_s u^{i'}(x^i_s)}{\sum_t \pi_t u^{i'}(x^i_t)} \quad \textit{for each } s.$$

Then

(i) x *is an equilibrium allocation under uncertainty and* $p \in \bigcap_i \Pi^i(x^i)$

(ii) there exists a neighborhood V *of* x *such that every neighborhood* $V' \subset V$ *of* x *contains infinitely many equilibrium allocations*

(iii) there exists a neighborhood O *of* p *such that every* $p' \in O$ *is an equilibrium price supporting some* $x' \in V$*.*

Proof: By assumptions A4 and A5, every equilibrium allocation is interior. For (i), note that there exist $\lambda^1, \ldots, \lambda^I > 0$ such that

$$u^{i\prime}(x^i_s) = \lambda^i u^{I\prime}(x^I_s) \text{ for each } i \text{ and } s.$$

Then for each $\tilde{\pi} \in \bigcap_i \Pi^i$ and for each i, j,

$$\frac{\tilde{\pi}_s u^{i\prime}(x_s^i)}{\sum \tilde{\pi}_t u^{i\prime}(x_t^i)} = \frac{\tilde{\pi}_s u^{j\prime}(x_s^j)}{\sum \tilde{\pi}_t u^{j\prime}(x_t^j)} \quad \text{for each } s.$$

Hence $\bigcap_i \Pi^i(x^i) \neq \emptyset$, and $p \in \bigcap_i \Pi^i(x^i)$. Clearly $p \cdot x^i = p \cdot \omega^i$ for each i, which implies that $(x; p)$ is an equilibrium under uncertainty.

For (ii) and (iii), first note that since $\bigcap_i \Pi^i$ has non-empty relative interior, so does $\bigcap_i \Pi^i(x^i)$. Moreover, since $\pi \in \operatorname{rint} \bigcap_i \Pi^i$, $p \in \operatorname{rint} \bigcap_i \Pi^i(x^i)$. Now by Lemma 4 in the Appendix, there exist neighborhoods V of x and O of p such that

$$O \subset \bigcap_i \Pi^i(y^i) \qquad \forall y \in V.$$

Without loss of generality, taking a smaller neighborhood if needed, we can ensure that for each $y \in V$, $y^i \succ^i \omega^i$ for each i. Then choose $y \in V$ such that $p \cdot y^i = p \cdot \omega^i$ for each i; there is a continuum of such allocations. Since $p \in \bigcap_i \Pi^i(y^i)$, y is an equilibrium allocation supported by p.

For (iii), define the correspondence $B : \Delta \to \mathsf{R}_+^{IS}$ by

$$B(q) = \{y \in \mathsf{R}_+^{IS} : q \cdot y^i = q \cdot \omega^i \ \forall i\}.$$

To establish (iii), it suffices to show that there exists a neighborhood O of p such that $B(q) \cap V \neq \emptyset$ for all $q \in O$. To that end, we claim that B is lower hemi-continuous at p. To show this, let $q \gg 0$ be arbitrary and take $q^n \to q$ and $y \in B(q)$. For each n, set

$$y^{i_n} = \frac{1}{q^{1_n}} \left(q^n \cdot \omega^i - q^n \cdot y^i\right) e^1 + y^i$$

where $e^1 = (1, 0, \ldots, 0)$. For n sufficiently large, $y^{i_n} \geq 0$. Clearly $y^n \to y$ and

$$\begin{aligned} q^n \cdot y^{i_n} &= q^n \cdot \omega^i - q^n \cdot y^i + q^n \cdot y^i \\ &= q^n \cdot \omega^i. \end{aligned}$$

Thus $y^n \in B(q^n)$ for all n sufficiently large. We conclude that B is lower hemi-continuous. Now note that $B(p) \cap V \neq \emptyset$, as $x \in B(p) \cap V$. Since V is open, the lower hemi-continuity of B implies that there exists a neighborhood O of p such that $B(q) \cap V \neq \emptyset$ for all $q \in O$. ■

This result is illustrated in Figure 3. As usual, ω is the initial endowment, agent 1's preferences are grey, while agent 2's preferences are black. The allocations x and y are both equilibria supported by the same price vector since the corresponding better-than sets do not intersect. Roughly, the kink of the better-than sets of each individual moves along the price line from x to y. Therefore, any allocation between these is also an equilibrium.

Notice that the result in Theorem 6 is independent of the precise nature of agents' beliefs, as long as their intersection has non-empty relative interior, and of the presence of aggregate

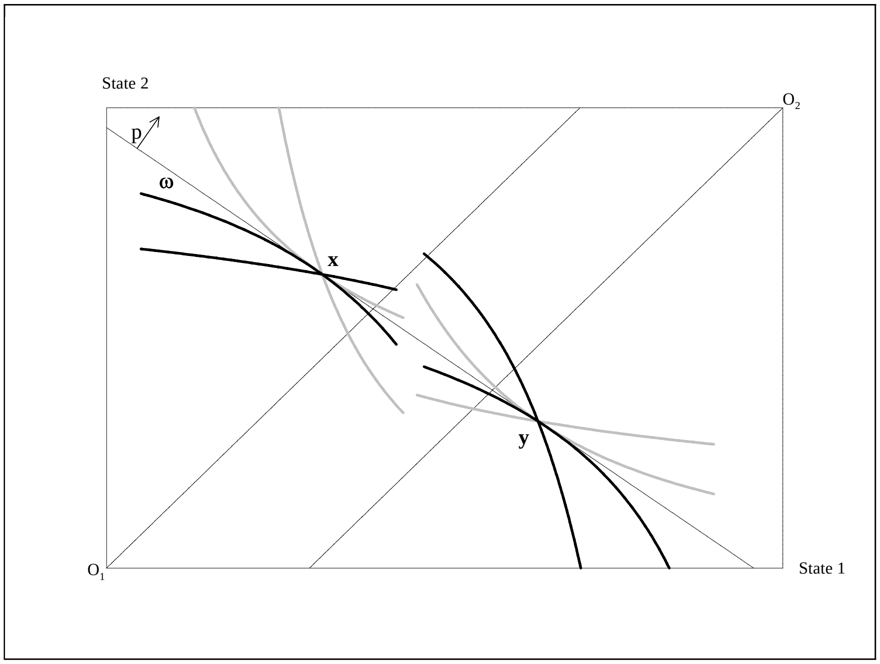

Figure 3: Multiple Equilibria

uncertainty or certainty. In this sense, uncertainty is a robust source of indeterminacies. In Section 6, we argue that this is not the case in economies where individuals have MEU or CEU preferences, where instead indeterminacies are possible only under very specific assumptions about belief sets and/or initial endowment vectors. See also Rigotti and Shannon (2003).

Next we turn to comparative statics, and analyze how the set of equilibria changes with uncertainty. Recall that by Proposition 1 we can think equivalently of varying the degree of completeness of agents' preferences or varying the size of their belief sets. With this in mind, we show first that the set of equilibrium allocations is monotone in beliefs, that is, any equilibrium of an economy with smaller belief sets, hence with more complete preferences, is an equilibrium of the economy with larger belief sets.

Theorem 7 *Suppose assumptions A1-A2 hold. Let (x,p) be an equilibrium in the economy with belief sets $(\Pi^1,\dots,\Pi^I)$. If $\Pi^i \subset \tilde{\Pi}^i$ for each i, then (x,p) is also an equilibrium in the economy with belief sets $(\tilde{\Pi}^1,\dots,\tilde{\Pi}^I)$.*

Proof: Let (x,p) be an equilibrium in the economy with belief sets $(\Pi^1,\dots,\Pi^I)$, and let $\Pi^i \subset \tilde{\Pi}^i$ for each i. Fix j. To see that (x,p) is an equilibrium in the economy with belief sets $(\tilde{\Pi}^1,\dots,\tilde{\Pi}^I)$, it suffices to note that if $\sum \pi_s u^j(y_s) > \sum \pi_s u^j(x^j_s)$ for all $\pi \in \tilde{\Pi}^j$ then $\sum \pi_s u^j(y_s) > \sum \pi_s u^j(x^j_s)$ for all $\pi \in \Pi^j$. Hence $p\cdot y > p\cdot\omega^j$ for any such bundle y, which implies that (x,p) is an equilibrium in the economy with belief sets $(\tilde{\Pi}^1,\dots,\tilde{\Pi}^I)$. ■

The next result establishes that equilibria vary continuously with uncertainty. To make this statement precise, we need some notion of distance between sets of priors. We use the Hausdorff topology on the set of all subsets of priors 2^{Δ}.

Theorem 8 *Under assumptions A1-A2, the equilibrium correspondence* $\mathcal{E} : \prod_{i=1}^{I} 2^{\Delta} \to \mathsf{R}_{+}^{IS} \times \Delta$ *is upper hemi-continuous.*

Proof: Fix $(\Pi^1, \ldots, \Pi^I)$. Let $\Pi^{i_n} \to \Pi^i$ for each i, and $(x^n, q^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ such that $x^n \to x$ and $q^n \to q$. For each n,

$$q^n \cdot x^{i_n} = q^n \cdot \omega^i \;\; \forall i, \text{ and } \; \sum x^{i_n} = \sum \omega^i.$$

So

$$q \cdot x^i = q \cdot \omega^i \;\; \forall i, \text{ and } \; \sum x^i = \sum \omega^i.$$

We must show that $(x, q) \in \mathcal{E}(\Pi^1, \ldots, \Pi^I)$. To that end, fix i and let $x \succ^i x^i$. Then for each $\pi^i \in \Pi^i$, $E_{\pi^i}[u^i(x)] > E_{\pi^i}[u^i(x^i)]$. For each $\pi^i \in \Pi^i$ there exists a neighborhood L_{π^i} of π^i and M_{π^i} of x^i such that

$$E_{\hat{\pi}^i}[u^i(x)] > E_{\hat{\pi}^i}[u^i(\hat{x}^i)]$$

for all $\hat{\pi}^i \in L_{\pi^i}$ and $\hat{x}^i \in M_{\pi^i}$. The collection $\{L_{\pi^i} : \pi^i \in \Pi^i\}$ is an open cover of Π^i, and since Π^i is compact we can find a finite subcover $\{L_1, \ldots, L_m\}$. Set $L = \cup_{k=1}^m L_k$ and $M = \cap_{k=1}^m M_k$. Then for every $\pi \in L$ and $y \in M$, we have

$$E_\pi[u^i(x)] > E_\pi[u^i(y)].$$

Now since $\Pi^{i_n} \to \Pi^i$ and $x^{i_n} \to x^i$, there exists N sufficiently large such that for $n \geq N$, $\Pi^{i_n} \subset L$ and $x^{i_n} \in M$. Thus for $n \geq N$,

$$E_{\pi^{i_n}}[u^i(x)] > E_{\pi^{i_n}}[u^i(x^{i_n})]$$

for all $\pi^{i_n} \in \Pi^{i_n}$. As $(x^n, q^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$, we conclude that

$$q^n \cdot x > q^n \cdot \omega^i \;\; \text{for all } n \geq N.$$

Thus $q \cdot x \geq q \cdot \omega^i$. This shows that for every y such that $y \succ^i x^i$, $q \cdot y \geq q \cdot \omega^i$. Now suppose, by way of contradiction, that there exists x such that $x \succ^i x^i$ and $q \cdot x = q \cdot \omega^i$. Choose $\epsilon > 0$ sufficiently small such that $y \equiv x - \epsilon\omega^i \succ^i x^i$. Then $q \cdot (x - \epsilon\omega^i) = (1 - \epsilon) q \cdot \omega^i < q \cdot \omega^i$, a contradiction. Thus if $x \succ^i x^i$, then $q \cdot x > q \cdot \omega^i$. This establishes our claim that $(x, q) \in \mathcal{E}(\Pi^1, \ldots, \Pi^I)$. ■

A similar result holds if we restrict attention to equilibria with inertia. We leave the details to the interested reader.

Theorem 8 enables us to derive several important implications. Intuitively, these are simple consequences of the trade-off between risk and uncertainty that characterizes equilibria. At one extreme, for any vector of initial endowments there is a large enough amount of uncertainty such that no-trade is the unique equilibrium allocation with inertia. At the other extreme, when uncertainty becomes smaller and incentives to insure are very strong, all equilibria converge to the equilibrium outcome with only risk. These results are formally established below.

First we show that in the presence of risk-aversion, uncertainty can always hamper trade since as it becomes sufficiently large, the no-trade allocation is the unique equilibrium with inertia.

Corollary 5 *Suppose assumptions A1-A3 hold, and u^i is strictly concave for each i. There exist belief sets $\overline{\Pi}^1, \ldots, \overline{\Pi}^I$ with $\overline{\Pi}^i \neq \Delta$ for each i such that the unique equilibrium allocation with inertia in $\mathcal{E}(\overline{\Pi}^1, \ldots, \overline{\Pi}^I)$ is $(\omega^1, \ldots, \omega^I)$. If $(\Pi^{1_n}, \ldots, \Pi^{I_n}) \to (\overline{\Pi}^1, \ldots, \overline{\Pi}^I)$ and (x^n, p^n) is an equilibrium with inertia in $\mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ for each n, then $x^n \to (\omega^1, \ldots, \omega^I)$.*

Proof: First it suffices to show that there exist belief sets $\overline{\Pi}^1, \ldots, \overline{\Pi}^I$ with $\overline{\Pi}^i \subset \text{rint } \Delta$ for each i such that $\bigcap_i \Pi^i(\omega^i) \neq \emptyset$. To that end, fix $\pi^1 \in \text{rint } \Delta$. For each $i \neq 1$, set

$$\pi^i_s = \lambda^i \frac{\pi^1_s u^{1\prime}(\omega^1_s)}{u^{i\prime}(\omega^i_s)}$$

for each $s = 1, \ldots, S$, where

$$\lambda^i = \frac{1}{\sum \frac{\pi^1_s u^{1\prime}(\omega^1_s)}{u^{i\prime}(\omega^i_s)}}.$$

Then $\pi^i \in \Delta$ and by construction,

$$\frac{\pi^i_s u^{i\prime}(\omega^i_s)}{\sum \pi^i_t u^{i\prime}(\omega^i_t)} = \frac{\pi^1_s u^{1\prime}(\omega^1_s)}{\sum \pi^1_t u^{1\prime}(\omega^1_t)}$$

for each i. For each i, choose $\overline{\Pi}^i$ such that $\pi^i \in \overline{\Pi}^i$ and $\overline{\Pi}^i \subset \text{rint } \Delta$. Then by construction, for these belief sets $\bigcap_i \Pi^i(\omega^i) \neq \emptyset$. Thus $(\omega^1, \ldots, \omega^I)$ is an equilibrium allocation, and by Lemma 1, it is the unique equilibrium allocation with inertia.

Now let $(\Pi^{1_n}, \ldots, \Pi^{I_n}) \to (\overline{\Pi}^1, \ldots, \overline{\Pi}^I)$ and (x^n, p^n) be an equilibrium with inertia in $\mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ for each n. Notice that the sequence $\{(x^n, p^n)\}$ is bounded, since feasibility ensures that for each n, $(x^n, p^n) \in [0, \omega]^I \times \Delta$, where $\omega \equiv \sum_i \omega^i$. Take any convergent subsequence $(x^{n_k}, p^{n_k}) \to (x, p)$. By Theorem 8, $(x, p) \in \mathcal{E}(\overline{\Pi}^1, \ldots, \overline{\Pi}^I)$. Fix i such that $x^i \neq \omega^i$. Then there exists $K > 0$ such that for $k \geq K$, $x^{i n_k} \neq \omega^i$. Since (x^{n_k}, p^{n_k}) is an equilibrium with inertia for each k, we conclude that

$$E_{\pi^i}[u^i(x^{i n_k})] \geq E_{\pi^i}[u^i(\omega^i)] \quad \text{for all } \pi^i \in \Pi^{i n_k}$$

and for each $k \geq K$. Letting $k \to \infty$ yields

$$E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)] \quad \text{for all } \pi^i \in \overline{\Pi}^i.$$

Thus (x, p) is an equilibrium with inertia in $\mathcal{E}(\overline{\Pi}^1, \ldots, \overline{\Pi}^I)$. By Lemma 1, $x = (\omega^1, \ldots, \omega^I)$. Since every convergent subsequence of $\{x^n\}$ must converge to $(\omega^1, \ldots, \omega^I)$, we conclude that $x^n \to (\omega^1, \ldots, \omega^I)$. ■

We now show that if individuals' beliefs are close enough to a unique prior, then the equilibria converge to the equilibria of the risk economy. This result also establishes that the equilibria in the standard risk setting are robust to adding a small degree of uncertainty.

Theorem 9 *Suppose assumptions A1-A2 hold. Let* $\pi^i \in \Delta$ *for* $i = 1, \ldots, I$, *and let* $(x,p) \in \mathcal{E}(\pi^1, \ldots, \pi^I)$. *For any sequence of belief sets* $(\Pi^{1_n}, \ldots, \Pi^{I_n}) \to (\{\pi^1\}, \ldots, \{\pi^I\})$ *such that* $\pi^i \in \Pi^{i_n}$ *for each* i *and* n:

(i) there exists a sequence $\{(x^n, p^n)\}$ *such that* $(x^n, p^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ *for each* n *and* $(x^n, p^n) \to (x, p)$.

(ii) if $\{(x^n, p^n)\}$ *is a sequence such that* $(x^n, p^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ *for each* n *and* $(x^n, p^n) \to (\hat{x}, \hat{p})$, *then* $(\hat{x}, \hat{p}) \in \mathcal{E}(\{\pi^1\}, \ldots, \{\pi^I\})$.

Proof: Part (i) follows immediately from the monotonicity of the equilibrium correspondence in belief sets established in Theorem 7, by setting $(x^n, p^n) = (x, p)$ for each n. Part (ii) follows immediately from the upper hemi-continuity of the equilibrium correspondence at $(\{\pi^1\}, \ldots, \{\pi^I\})$. ■

Finally, as the last corollary demonstrates, if the equilibrium of the risk economy happens to be unique, then all equilibria under uncertainty converge to that equilibrium as uncertainty shrinks.

Corollary 6 *Suppose assumptions A1-A2 hold, and let* $\pi^i \in \Delta$ *for each* i. *Assume that* (x,p) *is the unique equilibrium in the risk economy with priors* $(\pi^1, \ldots, \pi^I)$. *If* $(\Pi^{1_n}, \ldots, \Pi^{I_n}) \to (\{\pi^1\}, \ldots, \{\pi^I\})$ *and* $(x^n, p^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ *for each* n, *then* $(x^n, p^n) \to (x, p)$.

Proof: By Theorem 8, the equilibrium correspondence is upper hemi-continuous at $(\{\pi^1\}, \ldots, \{\pi^I\})$. Let $(\Pi^{1_n}, \ldots, \Pi^{I_n}) \to (\{\pi^1\}, \ldots, \{\pi^I\})$ and $(x^n, p^n) \in \mathcal{E}(\Pi^{1_n}, \ldots, \Pi^{I_n})$ for each n. As in the proof of Corollary 5, the sequence $\{(x^n, p^n)\}$ is bounded, since by feasibility $(x^n, p^n) \in [0, \omega]^I \times \Delta \; \forall n$. Take any convergent subsequence (x^{n_k}, p^{n_k}). Since the equilibrium correspondence is upper hemi-continuous at $(\{\pi^1\}, \ldots, \{\pi^I\})$ and (x, p) is the unique equilibrium given beliefs $(\{\pi^1\}, \ldots, \{\pi^I\})$, $(x^{n_k}, p^{n_k}) \to (x, p)$. Then as every convergent subsequence must converge to (x, p), we conclude that $(x^n, p^n) \to (x, p)$. ■

In particular, note that this corollary applies in the case in which there is no aggregate uncertainty, agents are risk averse and have a (unique) common prior, as in this case there is a unique equilibrium in the risk economy with that prior.

5 Uncertainty and Risk

In this section, we derive conditions under which endogenous market incompleteness can arise. In the previous sections, we have shown that no-trade obtains when uncertainty is ubiquitous. In that case, securities markets are extremely incomplete since no exchange takes place in equilibrium. Here, we show that when there is uncertainty only about some events, there may be equilibria in which securities contingent on these events are not traded, while securities contingent on the remaining (risky) events are traded. In this case, a more limited degree of market incompleteness is possible in equilibrium, in that risky securities are traded while uncertain securities are not.[15]

The state space contains two types of events: events that all agents perceive have a unique probability, and events that some perceive do not. To find conditions under which only risky securities are traded in equilibrium we proceed as follows. First, we define an equilibrium concept *as if* trades in securities contingent on uncertain events were not allowed. Then we give conditions under which this constrained equilibrium is an equilibrium when unrestricted trading strategies are permitted. To make this analysis precise, however, we need to introduce some new notation that allows the focus to shift to trade contingent on events rather than states.

To this end, divide the state space S into two sets, S_R and S_U. The set S_R contains all of the risky states, those to which each agent assigns a precise probability. Similarly, the set S_U contains all of the uncertain states, those to which some agents assign multiple probabilities. In parallel, the set of all possible events $E \subset S$ is divided between risky events $\mathcal{R}$ and uncertain events $\mathcal{U}$, where

$$\mathcal{R} \equiv \left\{ E \subset S: \text{ for each } i, \sum_{s \in E} \pi_s^i \text{ is unique for all } \pi^i \in \Pi^i \right\}.$$

and

$$\mathcal{U} \equiv \{E \subset S : E \notin \mathcal{R}\}$$

Notice that the union of all uncertain events is an element of $\mathcal{R}$, and so is the event given by the union of all uncertain states.

In this framework we study the incentives for trade over risky and uncertain events. Given a bundle $x \in \mathsf{R}_+^S$ and an event $E \subset S$, let $x(E) = \{x_s\}_{s \in E}$ denote the vector of consumption in the states contained in the event E. Similarly, given $\pi \in \Delta$, let $\pi(E) = \{\pi_s\}_{s \in E}$ denote the probabilities of the states contained in the event E. With slight abuse of notation, we let $u^{i\prime}(x(E))$ denote the vector of marginal utilities corresponding to $x(E)$, so $u^{i\prime}(x(E)) = \{u^{i\prime}(x_s)\}_{s \in E}$.

Suppose agents can only trade in securities contingent on risky events. Then, we can define an equilibrium subject to this restriction as follows.

Definition A feasible allocation $\left(x^1, \ldots, x^I\right)$ and a non-zero price vector $p \in \mathsf{R}_+^S$ are an *equilibrium over risky events* if

1. for each i, $x^i(E) = \omega^i(E)$ for each $E \in \mathcal{U}$,
2. for each i, if $y(E) = \omega^i(E)$ for each $E \in \mathcal{U}$ and $y \succ^i x^i$, then $p \cdot y > p \cdot \omega^i$,
3. for each i, $p \cdot x^i = p \cdot \omega^i$

[15] In a similar spirit, Mukerji and Tallon (2001) investigate the possibility that uncertainty aversion in a heterogeneous agent CEU model might lead to an endogenous breakdown in markets for some risky assets. If asset payoffs vary across states over which endowments are constant, and uncertainty is sufficiently large, every equilibrium involves no trade over these assets.

In an equilibrium over risky events, each agent must consume her endowment when an uncertain event occurs. Therefore, the individuals consider only consumption bundles that respect this restriction. One can see easily that an equilibrium over risky events is, by construction, an equilibrium with inertia.

The next theorem establishes conditions under which an equilibrium over risky events is actually an equilibrium. Intuitively, this can happen whenever individuals' initial endowments in the uncertain events are Pareto optimal contingent on those events. Thus this theorem gives conditions under which markets for trade over uncertain events do not function.

Theorem 10 *Suppose assumptions A1-A3 hold. Let (x,p) be an interior equilibrium over risky events. Then (x,p) is an equilibrium if and only if*

$$\bigcap_i \left\{ \left(\frac{\pi(U) \cdot u^{i\prime}(x^i(U))}{\sum_{E\in\mathcal{U}} \pi(E) \cdot u^{i\prime}(x^i(E))} \right)_{U\in\mathcal{U}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

Proof: If x is an equilibrium allocation over risky events, then

$$\bigcap_i \left\{ \left(\frac{\pi(R) \cdot u^{i\prime}(x^i(R))}{\sum_{E\in\mathcal{R}} \pi(E) \cdot u^{i\prime}(x^i(E))} \right)_{R\in\mathcal{R}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

By assumption,

$$\bigcap_i \left\{ \left(\frac{\pi(U) \cdot u^{i\prime}(x^i(U))}{\sum_{E\in\mathcal{U}} \pi(E) \cdot u^{i\prime}(x^i(E))} \right)_{U\in\mathcal{U}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

Thus for each i there exists $\pi^i \in \Pi^i$ such that

$$\frac{\pi^i(U) \cdot u^{i\prime}(x^i(U))}{\sum_{E\in\mathcal{U}} \pi^i(E) \cdot u^{i\prime}(x^i(E))} = \frac{\pi^1(U) \cdot u^{1\prime}(x^1(U))}{\sum_{E\in\mathcal{U}} \pi^1(E) \cdot u^{1\prime}(x^1(E))}$$

for each $U \in \mathcal{U}$. Thus for each i there exists $\lambda^i_{\mathcal{U}} > 0$ such that

$$\pi^i(U) \cdot u^{i\prime}(x^i(U)) = \lambda^i_{\mathcal{U}} \pi^1(U) \cdot u^{1\prime}(x^1(U))$$

for each $U \in \mathcal{U}$; similarly, for each i there exists $\lambda^i_{\mathcal{R}} > 0$ such that

$$\pi^i(R) \cdot u^{i\prime}(x^i(R)) = \lambda^i_{\mathcal{R}} \pi^1(R) \cdot u^{1\prime}(x^1(R))$$

for each $R \in \mathcal{R}$. Now to establish our claim it suffices to show that $\lambda^i_{\mathcal{R}} = \lambda^i_{\mathcal{U}}$ for all i. To that end, note that

$$\bigcup_{s\in S_U} \{s\} = S_U = S \setminus S_R \in \mathcal{R}$$

while for every $s \in S_U$, $\{s\} \in \mathcal{U}$. Thus for every $s \in S_U$,

$$\pi^i_s u^{i\prime}(x^i_s) = \lambda^i_{\mathcal{U}} \pi^1_s u^{1\prime}(x^1_s)$$

which implies that

$$\sum_{s \in S_U} \pi_s^i u^{i\prime}(x_s^i) = \lambda_{\mathcal{U}}^i \sum_{s \in S_U} \pi_s^1 u^{1\prime}(x_s^1)$$

or

$$\pi^i(S_U) \cdot u^{i\prime}(x^i(S_U)) = \lambda_{\mathcal{U}}^i \pi^1(S_U) \cdot u^{1\prime}(x^1(S_U)).$$

However, since $S_U \in \mathcal{R}$,

$$\pi^i(S_U) \cdot u^{i\prime}(x^i(S_U)) = \lambda_{\mathcal{R}}^i \pi^1(S_U) \cdot u^{1\prime}(x^1(S_U))$$

which implies that $\lambda_{\mathcal{R}}^i = \lambda_{\mathcal{U}}^i$ for each i. ■

This result can be viewed as a natural analogue of our equilibrium characterization results in section 3. To make this analogy more precise, define the set $\Pi^i(x^i(\mathcal{U}))$ as follows:

$$\Pi^i(x^i(\mathcal{U})) = \left\{ \left(\frac{\pi(U) \cdot u^{i\prime}(x^i(U))}{\sum_{E \in \mathcal{U}} \pi(E) \cdot u^{i\prime}(x^i(E))} \right)_{U \in \mathcal{U}} : \pi \in \Pi^i \right\}.$$

Using this notation, Theorem 10 states that an equilibrium over risky events is an equilibrium if and only if $\bigcap_i \Pi^i(x^i(\mathcal{U})) \neq \emptyset$. In sections 3 and 4, we used the corresponding set $\bigcap_i \Pi^i(x^i)$ to characterize equilibria and derive comparative statics results. Those results could easily be modified so that they apply to equilibria over risky events by using the intersection of these "conditional" belief sets $\bigcap_i \Pi^i(x^i(\mathcal{U}))$. For example, an appropriately modified version of Corollary 5 would show that if belief sets over uncertain events are large enough, then all equilibria over risky events are equilibria since the initial endowment is always conditional Pareto optimal. We leave the details of these results to the interested reader.

Theorem 10 proves particularly useful with additional information on the characteristics of the economy. For example, suppose there is no aggregate uncertainty over the uncertain states and therefore there is no aggregate uncertainty over any uncertain event. Then, mirroring the conclusion of Corollary 2, an equilibrium over risky events is an equilibrium if and only if agents have at least one prior over uncertain events in common. This result can be illustrated more precisely as follows.

Corollary 7 *Suppose assumptions A1-A3 hold. Let the set of uncertain events* $\mathcal{U}$ *be* $\{U_1, \ldots, U_m\}$. *Suppose* $\bar{R} \in \mathcal{R}$ *is a risky event such that* $\bar{R} = U_1 \cup \cdots \cup U_m$ *and that* $\bar{R}$ *contains no other proper risky subevent. Suppose there is no idiosyncratic or aggregate uncertainty over the states in* $\bar{R}$, *so for each* $s, t \in \bar{R}$, $\omega_s^i = \omega_t^i$ *for each* i. *Then an equilibrium over risky events is an equilibrium if and only if agents have one common conditional prior over the uncertain events* $U_1, \ldots, U_m$.

Proof: To establish this claim, consider an equilibrium over risky events (x, p). Since each agent's initial endowment is constant across the states in $\bar{R}$, $x^i(\bar{R})$ will be constant across states for each i as well. Now consider insurance markets for the uncertain events $U_1, \ldots, U_m$. By Theorem 10, (x, p) is an equilibrium if and only if

$$\bigcap_i \left\{ \left(\frac{\pi^i(U_k) \cdot u^{i\prime}(x^i(U_k))}{\sum_\ell \pi^i(U_\ell) \cdot u^{i\prime}(x^i(U_\ell))} \right)_{k=1,\ldots,m} : \pi^i \in \Pi^i \right\} \neq \emptyset.$$

But note that $x^i(U_k)$ is constant for each $s \in U_k \subset \bar{R}$ and for each k, thus for any k,

$$\frac{\pi^i(U_k) \cdot u^{i\prime}(x^i(U_k))}{\sum_\ell \pi^i(U_\ell) \cdot u^{i\prime}(x^i(U_\ell))} = \frac{\sum_{s \in U_k} \pi^i_s}{\sum_\ell \sum_{t \in U_\ell} \pi^i_t}.$$

Thus there is no trade on these new markets if and only if

$$\bigcap_i \left\{ \left(\frac{\sum_{s \in U_k} \pi^i_s}{\sum_\ell \sum_{t \in U_\ell} \pi^i_t} \right)_{k=1,\dots,m} : \pi^i \in \Pi^i \right\} \neq \emptyset.$$

that is, if and only if the agents share a common conditional prior over the uncertain events $U_1, \dots, U_m$. ■

A simple way to understand this result is to imagine that markets open sequentially. Initially, only markets for trades contingent on risky events are available. After all trades in these markets are completed, and the economy is in an equilibrium over risky events, the possibility to open new markets for insurance over uncertain subevents of some risky event is considered. Is there any trade in these new financial instruments? Using the previous result, we can say the new markets will succeed if and only if there is sufficient disagreement over the likelihoods of uncertain events. Otherwise, the uncertainty motive to avoid trade prevails and financial innovations fail.

6 Comparisons with Existing Literature

In this section, we compare our results about the impact of uncertainty on equilibrium with the ones obtained using two other prominent models of decision making under uncertainty: Choquet expected utility (CEU), due to Schmeidler (1989), and maxmin expected utility (MEU), due to Gilboa and Schmeidler (1989). Following Ellsberg's suggestion that the independence axiom may not be a reasonable description of individual behavior under uncertainty, both models retain completeness but relax independence. Since Bewley's model retains independence and relaxes completeness, neither CEU nor MEU is either a generalization or a special case of incomplete preferences.

The MEU and CEU models differ in the particular way in which independence is modified. CEU requires independence to hold only when comparing comonotonic alternatives, that is, alternatives that rank states in the same way. Under this assumption, Schmeidler (1989) shows that preferences are represented by their Choquet expected utility, where the expectation is taken with respect to a capacity rather than a probability measure. MEU requires independence to hold only when mixing with constant acts, and adds an uncertainty aversion axiom. Under these assumptions, Gilboa and Schmeidler (1989) show that preferences are represented by the minimum expected utility the decision maker computes using a closed, convex set of probability distributions. Under an additional similar uncertainty aversion axiom, the Choquet expected utility model becomes a special case of the MEU model. With the addition of uncertainty aversion, the CEU representation is equivalent to an MEU representation in which the agent's belief set is given by the core of the capacity (roughly, this is the set of probability distributions consistent with the capacity). Under uncertainty aversion, the capacity representing beliefs is convex, and when the capacity is convex, the core is non-empty and the Choquet expected value of a random variable with respect to the capacity is the minimum expected value over distributions in the core. Since uncertainty aversion in the Choquet setting also corresponds to convexity of the underlying preference order, this assumption is maintained in virtually every application of the CEU model, including all of those we discuss here. Thus for our purposes, we can think of CEU as a special case of MEU.

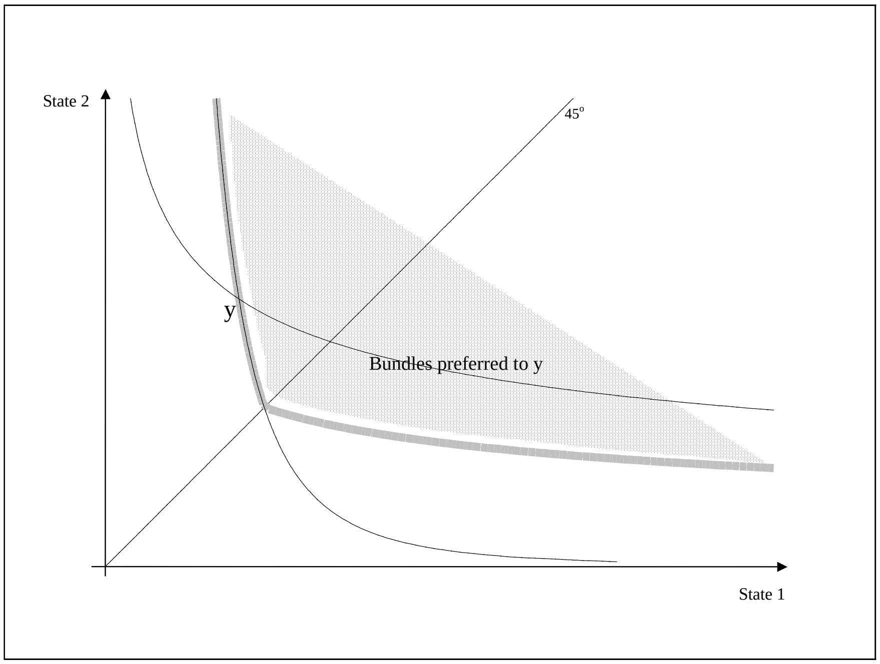

Figure 4: CEU and MEU Preferences

How do decision makers with CEU and MEU preferences compare to those with incomplete preferences? Figure 4 illustrates for the case in which there are only two states. Consider alternative y. The better-than-y set corresponding to CEU and MEU is the area above the thick grey indifference curves, while the better-than-y set under incompleteness is the area above the thin indifference curves. In particular, note that CEU and MEU preferences have a kink at every bundle on the certainty line, and nowhere else. In general, the better-than set under incomplete preferences is a subset of the better-than set under MEU.

One of the most important features of previous models of uncertainty aversion has been the possibility of indeterminacy of equilibria. This point was first raised by Dow and Werlang (1992). In a portfolio choice problem in which the agent is a Choquet expected utility maximizer with convex capacity, they show that there is an interval of (exogenously given) prices at which the agent will neither buy nor sell short a given risky asset. Epstein and Wang (1994) show that endogenously determined equilibrium prices may exhibit significant indeterminacy when agents are Choquet expected utility maximizers. They study the extension of Lucas's asset pricing model to a CEU setting with convex capacity, and show that this representative agent economy may permit a continuum of equilibrium prices.[16] This is a representative agent model, however, and equilibrium prices are simply prices supporting the representative agent's initial endowment and no trade in the assets; there are no nontrivial opportunities for risk sharing. Furthermore, equilibrium indeterminacy is not obtained even in this setting unless asset payoffs vary across states over which the agent's endowment is constant. The intuition for this can be seen in Figure 4: the representative agent's indifference curves have a kink along the certainty line and nowhere else, thus a consumption bundle varying across the states has a unique supporting price.

Whether indeterminacies can arise robustly in such economies with heterogeneous agents is even more unclear. Dana (2000) illustrates this point. She shows that in the CEU model with common convex capacity, the degree of indeterminacy is linked to the variation in aggregate endowments across states. If there is no aggregate uncertainty, any prior in the core of the agents' common capacity is an equilibrium price, and the set of equilibrium allocations is a convex polyhedron. If instead the aggregate endowment is state-revealing (i.e. it is a one-to-one mapping on the state space), then equilibria correspond to those in a standard risk economy with a fixed common prior, and hence are generically determinate.[17] Since the set of state-revealing endowment vectors is itself an open and dense set in the positive cone, generically there is no indeterminacy in either equilibrium allocations or prices in this case. In addition generically there are no observable distinctions between these models and a standard expected utility model in equilibrium. Finally, in Rigotti and Shannon (2003), we show that the generic determinacy of equilibria holds for general MEU preferences.

In contrast, when uncertainty is modeled by incomplete preferences, every initial endowment allocation gives rise to equilibrium price indeterminacy, regardless of whether there is aggregate uncertainty or not, and regardless of agents' beliefs (provided there is sufficient overlap). The heart of this distinction is the following observation. When preferences are incomplete, there is an important trade-off between risk (aversion) and uncertainty (aversion) in exchange situations. This trade-off is absent in CEU and MEU models, because in these cases agents have a strong desire for insurance stemming from their drive to equate utility *in levels* across states at the margin. In the CEU and MEU settings, uncertainty makes agents "more" risk averse.

These results can be illustrated using an Edgeworth box. Figure 5 presents the same situation illustrated in Figure 2 (which depicted a no-trade equilibrium in our setting), but assumes instead that individuals have MEU preferences. Here the initial endowment vector is ω, and the shaded area represents consumption bundles preferred to this initial endowment vector by both individuals. In this example, there are many such bundles, while instead with incomplete preferences (represented by the thin indifference curves) ω is Pareto optimal.

Figure 5 also illustrates equilibria in the CEU model with common convex capacity. With

[16] Similar results are derived by Anderson, Hansen, and Sargent (2001), Hansen, Sargent, and Tallarini (1999), and Hansen and Sargent (2001) using robust control to model uncertainty.

[17] Similarly, Chateauneuf, Dana, and Tallon (2000) show that in the CEU model with common convex capacity, Pareto optimal allocations are comonotonic. Since CEU preferences satisfy the independence axiom over comonotonic bundles, this implies that they behave (locally) like expected utility preferences for the consumption bundles relevant for equilibrium analysis.

Figure 5: Incentives to Insure with MEU Preferences

only two states, aggregate uncertainty implies immediately that the aggregate endowment is state-revealing, since it differs across the two states. This illustrates the results in Dana (2000): the equilibria correspond to equilibria in a standard risk economy with fixed priors. In contrast, if there is no aggregate uncertainty, with CEU or MEU preferences Pareto optimal allocations involve full insurance, so are on the *common* certainty line, as shown by Billot, Chateauneuf, Gilboa, and Tallon (2000). Every prior in the intersection of the cores of the agents' capacities is an equilibrium price supporting some equilibrium allocation.

7 Conclusions

We have developed a simple Arrow-Debreu economy in which Knightian uncertainty is modeled using incomplete preferences. Our model generates two intuitive features of equilibria: robust indeterminacies in both equilibrium allocations and prices and no trade for sufficient degrees of uncertainty. These results are in sharp contrast with what obtains using other approaches to Knightian uncertainty, such as Choquet expected utility and maxmin expected utility. This contrast highlights the danger in drawing inferences about equilibrium behavior from features of individual choices.

An added benefit of our model is its relative tractability. Despite the apparent complications arising from incomplete preferences, we are able to give a simple characterization of Pareto optimal allocations and equilibria using agents' sets of normalized gradients generated by their sets of priors. In a straightforward generalization of standard results, optimal allocations here are simply those at which *some* selection of priors yields equality of marginal rates of substitution across agents.

Our work suggests several natural and promising areas for further exploration. One that we have briefly discussed here is the connection between uncertainty and incomplete markets, and the extent to which Knightian uncertainty aversion can explain the endogenous failure of some insurance and asset markets. A second is to cast the model in a dynamic framework and examine the interplay between spot markets and asset markets in the face of uncertainty about fundamentals as well as future prices.

A Appendix

Proof of Theorem 1: To show that an equilibrium exists here, we will appeal to the equilibrium theorem of Gale and Mas-Colell (1975). To this end, for each i define the preference mapping $P^i : \mathsf{R}^S_+ \to 2^{\mathsf{R}^S_+}$ by

$$P^i(x) = \{y \in \mathsf{R}^S_+ : y \succ^i x\}.$$

Then P^i is irreflexive, i.e. $x \notin P^i(x)$ for every x, and has non-empty convex values by the concavity and strict monotonicity of u^i. Finally, P^i has an open graph in $\mathsf{R}^S_+ \times \mathsf{R}^S_+$. To show this, let $(x, y) \in$ graph P^i. Then

$$\sum \pi_s u^i(y_s) > \sum \pi_s u^i(x_s) \text{ for all } \pi \in \Pi^i.$$

Since u^i is continuous, for each $\pi \in \Pi^i$ there exist neighborhoods U_π about π, V_π about y and W_π about x such that

$$\sum \pi'_s u^i(y'_s) > \sum \pi'_s u^i(x'_s) \text{ for all } \pi' \in U_\pi, y' \in V_\pi, x' \in W_\pi.$$

As $\{U_\pi : \pi \in \Pi^i\}$ is an open cover of Π^i and Π^i is compact, we can find a finite subcover $\{U_{\pi^k} : k = 1, \ldots, m\}$. Let $V = \cap_k V_{\pi^k}$ and $W = \cap_k W_{\pi^k}$. Then for any $y' \in V$ and $x' \in W$,

$$\sum \pi'_s u^i(y'_s) > \sum \pi'_s u^i(x'_s) \text{ for all } \pi' \in \Pi^i \subset \cup_k U_{\pi^k}.$$

That is, $W \times V \subset$ graph P^i, which establishes that P^i has an open graph. Now by Gale and Mas-Colell's equilibrium theorem, an equilibrium exists.

That every equilibrium allocation is Pareto optimal follows from standard arguments (see Fon and Otani (1979)). ■

Proof of Theorem 2: Let $X = \{x \in \mathsf{R}^S_+ : x \leq 2\omega\}$. For each $i = 0, 1, \ldots, I$, define the correspondences V^i on $\Pi^1 \times \cdots \times \Pi^I \times \Delta \times \prod_{i=1}^I X$ as follows:
for $i = 1, \ldots, I$ let

$$V^i(\pi, p, x) = \begin{cases} \{y \in X : p \cdot y < p \cdot \omega^i\} & \text{if } p \cdot x^i > p \cdot \omega^i \\ \{y \in X : p \cdot y < p \cdot \omega^i\} \cap \{y : E_{\pi^i}[u^i(y)] > E_{\pi^i}[u^i(x^i)]\} & \text{if } p \cdot x^i \leq p \cdot \omega^i \end{cases}$$

and for $i = 0$ let

$$V^0(\pi, p, x) = \left\{ q \in \Delta : q \cdot \sum x^i > q \cdot \sum \omega^i \right\}.$$

First note that if $V^i(\pi, p, x) = \emptyset$ for each $i = 0, \ldots, I$, then (x, p) is an equilibrium in the risk economy with priors given by π.[18] To see this, note that if $V^0(\pi, p, x) = \emptyset$ then $q \cdot \sum x^i \le q \cdot \sum \omega^i$ for each $q \in \Delta$. Hence $\sum x^i \le \sum \omega^i$, i.e. x is feasible. Next, if $V^i(\pi, p, x) = \emptyset$, then $p \cdot x^i \le p \cdot \omega^i$, and thus x^i is maximal in agent i's budget set since

$$\{y \in X : p \cdot y < p \cdot \omega^i\} \cap \{y : E_{\pi^i}[u^i(y)] > E_{\pi^i}[u^i(x^i)]\} = \emptyset.$$

Now for each i define the correspondence $W^i : \Pi^1 \times \cdots \times \Pi^I \times \Delta \times \prod_{i=1}^I X \to \Pi^i$ by

$$W^i(\pi^1, \ldots, \pi^I, p, x) = \{\hat{\pi}^i \in \Pi^i : E_{\hat{\pi}^i}[u^i(x^i))] < E_{\hat{\pi}^i}[u^i(\omega^i)]\}.$$

Notice that if $V^i(\pi, p, x) = \emptyset$ for each $i = 0, 1, ..., I$ and $W^i(\pi^1, \ldots, \pi^I, p, x) = \emptyset$ for each $i = 1, \ldots, I$, then (x, p) is an equilibrium with inertia.

So define the correspondence $V : \Pi^1 \times \cdots \times \Pi^I \times \Delta \times \prod_{i=1}^I X \to \Pi^1 \times \cdots \times \Pi^I \times \Delta \times \prod_{i=1}^I X$ by $V = (W^1, \ldots, W^I, V^0, \ldots, V^I)$.

Each component of this correspondence clearly has convex values, and we claim each has an open graph. To show this, fix a point $(\pi, p, x, \pi', q, y) \in$ graph V. Then consider each component of V in turn.

First, for each i, $E_{\pi'}[u^i(x^i)] < E_{\pi'}[u^i(\omega^i)]$, so by continuity we can find neighborhoods N^i of (π, p, x) and M^i of π' such that $E_{\hat{\pi}}[u^i(\tilde{x}^i)] < E_{\hat{\pi}}[u^i(\omega^i)]$ for each $(\tilde{\pi}, \tilde{p}, \tilde{x}) \in N^i$ and $\hat{\pi} \in M^i$, i.e. such that $N^i \times M^i \subset$ graph W^i.

Next, consider V^0. Since $q \cdot \sum x^i > q \cdot \sum \omega^i$, we can find neighborhoods N^0 of (π, p, x) and M^0 of q such that $\hat{q} \cdot \sum \tilde{x}^i > \hat{q} \cdot \sum \omega^i$ for every $(\tilde{\pi}, \tilde{p}, \tilde{x}) \in N^0$ and $\hat{q} \in M^0$. Thus $N^0 \times M^0 \subset$ graph V^0.

Finally, consider V^i. If $p \cdot x^i > p \cdot \omega^i$, then $p \cdot y < p \cdot \omega^i$. In this case, clearly there exist neighborhoods L^i_1 about (π, p, x) and K^i_1 about y^i such that $L^i_1 \times K^i_1 \subset$ graph V^i. Now suppose $p \cdot x^i \le p \cdot \omega^i$. Then $p \cdot y < p \cdot \omega^i$ and $E_{\pi^i}[u^i(y^i)] > E_{\pi^i}[u^i(\omega^i)]$. Then again there exist neighborhoods L^i_2 of (π, p, x) and K^i_2 of y^i such that for every $(\tilde{\pi}, \tilde{p}, \tilde{x}) \in L^i_2$ and $\hat{y}^i \in K^i_2$, $\tilde{p} \cdot \hat{y}^i < \tilde{p} \cdot \omega^i$ and $E_{\tilde{\pi}^i}[u^i(\hat{y}^i)] > E_{\tilde{\pi}^i}[u^i(\omega^i)]$. Thus $L^i_2 \times K^i_2 \subset$ graph V^i. So set $L^i = L^i_1 \cap L^i_2$ and $K^i = K^i_1 \cap K^i_2$; we have shown that $L^i \times K^i \subset$ graph V^i.

Then by Gale and Mas-Colell's fixed point theorem (Gale and Mas-Colell (1975), p. 10), there exists $(\bar{\pi}, \bar{p}, \bar{x}) \in \Pi^1 \times \cdots \times \Pi^I \times \Delta \times \prod_{i=1}^I X$ such that

1. either $\bar{p} \in V^0(\bar{\pi}, \bar{p}, \bar{x})$ or $V^0(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$,
2. for each i, either $\bar{x}^i \in V^i(\bar{\pi}, \bar{p}, \bar{x})$ or $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$,
3. for each i, either $\bar{\pi}^i \in W^i(\bar{\pi}, \bar{p}, \bar{x})$ or $W^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$.

Now note that if $\bar{p} \cdot \bar{x}^i > \bar{p} \cdot \omega^i$, then $V^i(\bar{\pi}, \bar{p}, \bar{x}) \neq \emptyset$ and $\bar{x}^i \notin V^i(\bar{\pi}, \bar{p}, \bar{x})$ by construction. So we must have $\bar{p} \cdot \bar{x}^i \le \bar{p} \cdot \bar{\omega}^i$. Then clearly $\bar{x}^i \notin V^i(\bar{\pi}, \bar{p}, \bar{x})$, so $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$. Next, since $\bar{p} \cdot \bar{x}^i \le \bar{p} \cdot \omega^i$

[18] This is just a special case of Gale and Mas-Colell's construction with complete preferences and pure exchange.

for each i, $\bar{p} \cdot \sum \bar{x}^i \le \bar{p} \cdot \sum \omega^i$. Thus $\bar{p} \notin V^0(\bar{\pi}, \bar{p}, \bar{x})$, which implies that $V^0(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$. Thus for each $i = 0, \ldots, I$ we must have $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$, from which we conclude that $(\bar{p}, \bar{x})$ is an equilibrium in the risk economy with priors given by $\bar{\pi}$.

Using the fact that $(\bar{p}, \bar{x})$ is an equilibrium in the risk economy with priors $\bar{\pi}$, individual rationality implies that $\bar{\pi}^i \notin W^i(\bar{\pi}, \bar{p}, \bar{x})$. Thus $W^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$ for each i. From this we conclude that $(\bar{p}, \bar{x})$ is an equilibrium with inertia in our original economy. ■

Lemma 3 *Under assumptions A1-A3,* $\Pi^i : \mathsf{R}^S_{++} \to 2^{\mathsf{R}^S_+}$ *is a closed-, convex-valued, continuous correspondence.*

Proof: Fix $x \in \mathsf{R}^S_{++}$. First, we show that $\Pi^i(x)$ is convex for every x. To that end, pick a $\lambda \in [0,1]$ and $\pi, \widehat{\pi} \in \Pi^i$. Then define

$$\gamma = \frac{1}{\sum_t \pi_t u^{i\prime}(x_t)} \text{and } \widehat{\gamma} = \frac{1}{\sum_t \widehat{\pi}_t u^{i\prime}(x_t)}.$$

One can easily verify:

$$\begin{aligned}
\frac{\lambda \pi_s u^{i\prime}(x_s)}{\sum_t \pi_t u^{i\prime}(x_t)} + \frac{(1-\lambda)\,\widehat{\pi}_s u^{i\prime}(x_s)}{\sum_t \widehat{\pi}_t u^{i\prime}(x_t)} &= \lambda\gamma\pi_s u^{i\prime}(x_s) + (1-\lambda)\,\widehat{\gamma}\widehat{\pi}_s u^{i\prime}(x_s) \\
&= [\lambda\gamma + (1-\lambda)\,\widehat{\gamma}] \left(\pi_s \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} + \widehat{\pi}_s \frac{(1-\lambda)\,\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \right) u^{i\prime}(x_s) \\
&= \frac{\overline{\pi}_s u^{i\prime}(x_s)}{\sum_t \overline{\pi}_t u^{i\prime}(x_t)}
\end{aligned}$$

where

$$\overline{\pi}_s = \pi_s \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} + \widehat{\pi}_s \frac{(1-\lambda)\,\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}}$$

and

$$\begin{aligned}
\frac{1}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} &= \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \frac{1}{\gamma} + \frac{(1-\lambda)\,\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \frac{1}{\widehat{\gamma}} \\
&= \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \sum_t \pi_t u^{i\prime}(x_t) + \frac{(1-\lambda)\,\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \sum_t \widehat{\pi}_t u^{i\prime}(x_t) \\
&= \sum_t \left(\pi_t \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} + \widehat{\pi}_t \frac{(1-\lambda)\,\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\,\widehat{\gamma}} \right) u^{i\prime}(x_t) \\
&= \sum_t \overline{\pi}_t u^{i\prime}(x_t).
\end{aligned}$$

Now since Π^i is convex, $\overline{\pi} \in \Pi^i$. Thus $\sum_t \overline{\pi}_t u^{i\prime}(x_t) \in \Pi^i(x)$, which establishes that $\Pi^i(x)$ is convex.

To see that $\Pi^i(x)$ is closed, let $q^n \in \Pi^i(x)$ such that $q^n \to q$. Since $q^n \in \Pi^i(x)$, there exists $\pi^n \in \Pi^i$ such that

$$q^n_s = \frac{\pi^n_s u^{i\prime}(x_s)}{\sum \pi^n_t u^{i\prime}(x_t)}$$

Since Π^i is compact, choose a convergent subsequence $\{\pi^{n_k}\}$, converging to $\pi \in \Pi^i$. Then $q^{n_k} \to q$, and

$$q_s = \frac{\pi_s u_i'(x_s)}{\sum \pi_t u_i'(x_t)}$$

Thus $q \in \Pi^i(x)$.

Next we establish that $\Pi^i(\cdot)$ is upper hemi-continuous at x. Let $x^n \to x$ and $q^n \in \Pi^i(x^n)$ such that $q^n \to q$. Then for each n there exists $\pi^n \in \Pi^i$ such that

$$q_s^n = \frac{\pi_s^n u^{i\prime}(x_s^n)}{\sum \pi_t^n u^{i\prime}(x_t^n)}$$

As above, take $\{\pi^{n_k}\}$ to be a convergent subsequence, converging to $\pi \in \Pi^i$. Since $q^{n_k} \to q$ and $x^{n_k} \to x$, the continuity of $u^{i\prime}$ implies that

$$q_s = \frac{\pi_s u^{i\prime}(x_s)}{\sum \pi_t u^{i\prime}(x_t)}$$

Thus $q \in \Pi^i(x)$.

Finally, to see that $\Pi^i(\cdot)$ is also lower hemi-continuous at x, let $x^n \to x$ and $q \in \Pi^i(x)$. Then there exists $\pi \in \Pi^i$ such that

$$q_s = \frac{\pi_s u^{i\prime}(x_s)}{\sum \pi_t u^{i\prime}(x_t)}$$

For each n define q^n by

$$q_s^n = \frac{\pi_s u^{i\prime}(x_s^n)}{\sum \pi_t u^{i\prime}(x_t^n)}$$

Then $q^n \in \Pi^i(x^n)$ for each n and clearly $q^n \to q$. Thus $\Pi^i(\cdot)$ is lower hemi-continuous at x. As x was arbitrary, the proof is completed. ■

Lemma 4 *Suppose assumptions A1-A5 hold. If x is an equilibrium allocation for which $\bigcap_i \Pi^i(x^i)$ has non-empty relative interior, and p is an equilibrium price supporting x such that $p \in \text{rint } \bigcap_i \Pi^i(x^i)$, then there exists a neighborhood $O \subset \bigcap_i \Pi^i(x^i)$ of p and a neighborhood W of x such that for every $\tilde{x} \in W$, $O \subset \bigcap_i \Pi^i(\tilde{x}^i)$. In particular, $\bigcap_i \Pi^i(\tilde{x}^i) \neq \emptyset$ for all $\tilde{x} \in W$.*

Proof: Let x and p satisfy the hypotheses. Choose a neighborhood O of p such that $C(O) \subset \text{rint } \bigcap_i \Pi^i(x^i)$, where

$$C(O) \equiv \left\{ p \in \Delta : p_s \geq \min_{\pi \in \bigcap_i \Pi^i(x^i)} \pi_s, \ s = 1, \ldots, S \right\}$$

For each $s = 1, \ldots, S$, let

$$m_s = \min_{\pi \in \bigcap_i \Pi^i(x^i)} \pi_s$$

and let $\{e^s\}$ be the extreme points of $C(O)$, thus

$$e^s_t = \begin{cases} m_t & \text{if } t \neq s; \\ 1 - \sum_{t \neq s} m_t & \text{if } t = s. \end{cases}$$

For each s, define

$$W_s \equiv \left\{ \pi \in \text{rint} \bigcap_i \Pi^i(x^i) : \pi_t < e^s_t \;\; \text{for } t \neq s \right\}.$$

Note that W_s is non-empty and open for each s.

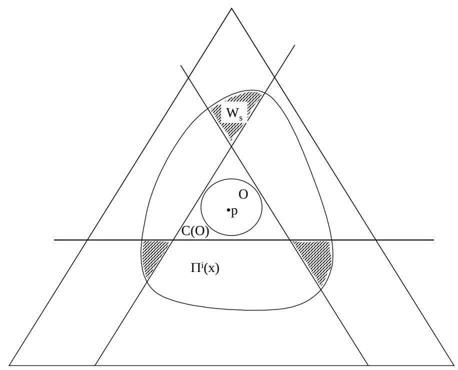

Figure 6: Construction of W_s

Now fix i. Since $\Pi^i(\cdot)$ is lower hemi-continuous, for each s there exists a neighborhood V_s of x^i such that

$$\Pi^i(x) \cap W_s \neq \emptyset$$

for all $x \in V_s$. Let $V = \cap V_s$. Then for every $x \in V$,

$$\Pi^i(x^i) \cap W_s \neq \emptyset \;\; \forall s$$

Fix $x \in V$. For each s, choose $q^s \in \Pi^i(x) \cap W_s$. Since $\Pi^i(x)$ is convex,

$$\text{con } \{q^1, \ldots, q^S\} \subset \Pi^i(x)$$

where $\text{con}\{z^1, \ldots, z^S\}$ denotes the convex hull of the set $\{z^1, \ldots, z^S\}$. But by construction, since $q^s \in W_s$ for each s,

$$O \subset C(O) \subset \text{con } \{q^1, \ldots, q^S\} \subset \Pi^i(x)$$

Thus $O \subset \Pi^i(x)$ for all $x \in V^i$.

As i was arbitrary, there exists a neighborhood V of the allocation x such that $O \subset \bigcap_i \Pi^i(y^i)$ for all $y \in V$. ■

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