

**Supplement to "The Shape of Incomplete Preferences"
Mathematical programs for the example in section 5**

The original basis vectors for \mathcal{B}^{**} in this example are as follows:

$$\begin{aligned}
 \mathbf{B}_1 &= \mathbf{E}_1(\mathbf{H}_2 - \mathbf{H}_0), & \mathbf{B}_2 &= \mathbf{E}_2(\mathbf{H}_2 - \mathbf{H}_0), & \mathbf{B}_3 &= \mathbf{E}_3(\mathbf{H}_2 - \mathbf{H}_0), \\
 \mathbf{B}_4 &= \mathbf{E}_1(\mathbf{H}_1 - \mathbf{H}_2), & \mathbf{B}_5 &= \mathbf{E}_2(\mathbf{H}_1 - \mathbf{H}_2), & \mathbf{B}_6 &= \mathbf{E}_6(\mathbf{H}_1 - \mathbf{H}_2), \\
 \mathbf{B}_7 &= \mathbf{H}_{\mathbf{E}_1} - \mathbf{H}_{0.1}, & \mathbf{B}_8 &= \mathbf{H}_{\mathbf{E}_2} - \mathbf{H}_{0.1}, & \mathbf{B}_9 &= \mathbf{H}_{\mathbf{E}_3} - \mathbf{H}_{0.1}, \\
 \mathbf{B}_{10} &= \mathbf{X} - \mathbf{H}_{0.5} \\
 \mathbf{B}_{11} &= \mathbf{X} - \mathbf{H}_{0.5} + \mathbf{E}_1(\mathbf{H}_2 - \mathbf{H}_{0.9}) \\
 \mathbf{B}_{12} &= \mathbf{X} - \mathbf{H}_{0.5} + \mathbf{E}_2(\mathbf{H}_{0.1} - \mathbf{H}_2)
 \end{aligned}$$

Note that $\mathbf{B}_1, \dots, \mathbf{B}_6$ are the contribution of axioms A4 and A6 together; $\mathbf{B}_7, \mathbf{B}_8$, and \mathbf{B}_9 are the differences between LHS and RHS of (5.1) for $\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$; and $\mathbf{B}_{10}, \mathbf{B}_{11}$, and \mathbf{B}_{12} are the differences between LHS and RHS of (5.2)–(5.4), after dropping the multiplicative constants $\frac{1}{2}$ and cancelling the common \mathbf{Z} terms.

Axioms A1–A4 establish that the cone \mathcal{B}^{**} representing this preference relation minimally consists of the convex hull of the rays whose directions are $\mathbf{B}_1, \dots, \mathbf{B}_{12}$. The question is whether axiom A6 enlarges this cone. To prove that it does not, we can solve a series of linear programs to show that there are no non-negative linear combinations of $\mathbf{B}_1, \dots, \mathbf{B}_{12}$ which are of the form $\mathbf{E}\mathbf{B}$, where \mathbf{E} is an event and \mathbf{B} is constant, and which are nontrivial in the sense that $[\mathbf{B}]_{\min} < 0$. (Recall that $[\mathbf{B}]_{\min}$ is defined as the minimum possible state-dependent expected utility assignable to \mathbf{B} . If this is not negative, then \mathbf{B} is simply a linear combination of $\mathbf{B}_1, \dots, \mathbf{B}_6$, as is $\mathbf{E}'\mathbf{B}$ for any other event \mathbf{E}' , so the application of A6 to this \mathbf{B} does not enlarge the cone.)

If \mathbf{B} is constant, then $[\mathbf{B}]_{\min}$ is the minimum of $B(s, 1)$ and $B(s, 1) + B(s, 2)$ for any event s . To formulate this problem as a linear program, we must separately minimize each of these two quantities over all non-negative $\alpha_1, \dots, \alpha_{12}$ such that

$$\sum_i \alpha_i \mathbf{B}_i = \mathbf{E}\mathbf{B},$$

$$B(1, c) = B(2, c) = B(3, c), \quad c = 0, 1, 2,$$

for every possible conditioning event \mathbf{E} . There are seven distinct events that are subsets of S in this case, namely $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, $\{\mathbf{E}_1, \mathbf{E}_2\}$, $\{\mathbf{E}_1, \mathbf{E}_3\}$, $\{\mathbf{E}_2, \mathbf{E}_3\}$, $\{\mathbf{E}_1\}$, $\{\mathbf{E}_2\}$, and $\{\mathbf{E}_3\}$. Altogether, we must solve 14 linear programs (2 objective functions \times 7 conditioning events) to see whether any has a negative optimal objective value. As it happens, none does. Hence, the cone \mathcal{B}^{**} is merely the set of non-negative linear combinations of $\mathbf{B}_1, \dots, \mathbf{B}_{12}$. To establish the lower expected utility for \mathbf{X} that can be deduced from A1–A6, we can solve the primal inference problem, namely find the maximum value of u for which

$$\sum_i \alpha_i \mathbf{B}_i = \mathbf{X} - \mathbf{H}_u$$

has a solution in non-negative $\alpha_1, \dots, \alpha_{12}$. This maximum turns out to be 0.5.

Alternatively, we can find the lower expected utility for \mathbf{X} by solving the dual problem, which is:

$$\min_v U_v(\mathbf{X})$$

subject to

$$U_v(\mathbf{B}_i) \geq 0, \quad i = 1, \dots, 12,$$

$$v \geq 0, \quad \sum_s v(s, 1) = 1$$

This yields, as before, a value of 0.5 for the lower expected utility of \mathbf{X} given A1–A6.

To determine the lower expected utility of \mathbf{X} among all *probability/utility pairs* agreeing with the basis, which would be the lower expected utility given A1–A7, we can solve the same problem with the additional nonlinear constraints:

$$v(1, 2)/v(1, 1) = v(2, 2)/v(2, 1) = v(3, 2)/v(3, 1).$$

Non-negativity of the denominators is ensured by the lower-probability constraints. This nonconvex program has a global optimum of 0.564314, which is achieved at $p(1) = 0.41$, $p(2) = 0.1$, $p(3) = 0.49$, $u(2) = 0.743137$. There is also a local minimum of 0.565306 achieved at $p(1) = 0.1$, $p(2) = 0.44444\dots$, $p(3) = 0.45555\dots$, $u(2) = 0.246939$. These programs were formulated and solved using Microsoft Excel; copies of the worksheet file are available from the author.