Abstract

We present a parsimonious and tractable general equilibrium model featuring a continuum of overlapping generations, as in Blanchard (1985). In addition, we assume that agents have standard utilities exhibiting constant relative risk aversion and can be born with differing risk aversions and endowments. Equilibrium asset prices are determined as if the economy was populated by a single representative agent with time-varying risk aversion that follows a stationary process. For this reason, our model is observationally similar to the model of Campbell and Cochrane (1999), and is therefore successful at addressing a number of stylized facts about asset prices. Importantly, the time variation in the risk aversion of the representative agent arises endogenously as a result of aggregating standard life cycle consumption and portfolio choice problems.
1 Introduction

A significant body of research over the last two decades has focused on uncovering the link between variation in asset prices and fundamental macroeconomic risks. This has proven to be a challenging task. The baseline textbook consumption-smoothing model predicts too low equity premia, too high risk-free rates, and asset prices that are substantially less volatile than in the data, to name just a few of the widely documented failures. This has led many researchers to become pessimistic about the potential of rational consumption-based pricing models to explain observed asset valuations.

In this paper we argue that many of the failures of the standard consumption-based asset-pricing paradigm are linked to some of the simplifying assumptions behind it. Once we extend the standard textbook model of consumption smoothing in several simple and realistic dimensions, we find that the model can account reasonably well for many of the perceived failures of the consumption-based asset pricing model.

In particular, we take the following four main departures from the textbook model: a) Instead of assuming infinitely-lived agents, we acknowledge the fact that lives are finite and generations may not be altruistically linked through gifts or bequests; b) agents age, and their ability to produce declines with age; c) agents need not have the same endowments; some agents are workers, whereas some other agents run their own firms, and d) (aggregate) consumption and (aggregate) dividends are not equal — in particular, dividends are more volatile than consumption, but the two quantities are cointegrated over the long run.

All four extensions appear realistic and plausible. Furthermore, despite the vast diversity in the population that is introduced by the continuous arrival and departure of agents, their aging, and the differences in their preferences and endowments, we are able to obtain a fairly tractable model that addresses several asset-pricing puzzles in a transparent fashion.

The model is an extension of the perpetual youth model of Blanchard (1985). As in Blanchard (1985), agents arrive and die according to independent Poisson processes with constant intensity. In contrast to Blanchard (1985), however, firms are faced with a stochastic productivity process following a random walk. Hence, in contrast to Blanchard (1985), our
model is stochastic, so that we can analyze equity premia.

An advantage of assuming overlapping generations, as opposed to a single representative agent, is that the model can address the risk-free-rate puzzle. Since agents are faced with declining labor income over their life cycles, there is a constant pressure to save when they are young. This increases savings and reduces the real rate. This simple and intuitive mechanism is absent in models with infinitely-lived agents because of the absence of life-cycle motivations for savings.

A further departure from the standard model is that agents can have different risk aversions, and can derive their incomes from different sources. Less risk averse agents are more exposed to aggregate risk. Because of this, their relative importance in the wealth distribution increases (declines) in response to positive (negative) news. Thus, during good times, they are relatively relatively more important in the economy and Sharpe ratios reflect their low risk-aversion levels. By contrast, in bad times, the economic importance of the less risk averse agents declines, and risky assets need to be held by the more risk averse agents, making Sharpe ratios reflect the high risk aversion of the latter group. Hence, even though each agent has constant relative risk aversion, assets are priced as if there existed a representative agent with time-varying risk aversion. Importantly, because of the birth and death of agents, aggregate risk aversion, interest rates, valuation ratios, etc., follow stationary processes. This stationarity can easily fail in models where agents are infinitely lived, making it hard to compare these models to the data.

Another important feature of the model is the low volatility of the risk-less rates. This is driven by the fact that agents have different endowments and hence saving behaviors. In the model, the less risk averse agents are also the entrepreneurs. Assuming that entrepreneurial income is on average more ephemeral, the less risk averse/entrepreneurial agents have a stronger propensity to save in order to smooth consumption intertemporally. On the other hand, the more risk averse/non-entrepreneurial agents have stronger precautionary motives for saving. As a consequence, even if the relative importance of the two types of agents

\footnote{See e.g. Dumas (1989) and Wang (1996)}
changes in response to productivity shocks, aggregate savings (and the equilibrium interest rate) are unaffected. Hence, as in the Campbell and Cochrane (1999) model, almost all variation in discount rates is driven by changes in excess returns.

Finally, by introducing an explicit labor-leisure choice and a production function with non-constant dividend and labor shares, we can reproduce the fact that dividends are more volatile than consumption, even though they are cointegrated over the long run. The higher volatility of dividends compared to consumption, along with the countercyclical variation in discount rates due to changing risk aversion, makes the volatility of the stock market high, which helps us obtain a reasonably high equity premium. Furthermore, because of the time-varying price of risk, the model can produce substantial predictability of excess returns.

Last, but not least, the model is quite tractable and easy to compute and analyze. Technically, the paper exploits the agents’ ability to trade dynamically in order to derive the dynamics of the cross-sectional consumption distribution, an object that is easier to analyze than the wealth distribution.\(^2\) Utilizing this insight, we are able to fully characterize the equilibrium in terms of a system of ordinary differential equations, which can be solved almost instantaneously by a modern computer. Moreover, there is no need for approximate solution concepts, whose accuracy is hard to assess.

The paper is related to various strands of the literature. There exists a vast literature on asset pricing that explains some of the stylized asset-pricing facts by utilizing habit formation. Constantinides (1990) and Abel (1990) were early contributions in this literature. Campbell and Cochrane (1999), in a highly influential paper, pursued the idea of external habit formation further. They succeeded in engineering a utility function exhibiting external habit formation that addresses several asset-pricing puzzles simultaneously.

In the model that we propose, the state variable that governs time variation in asset prices resembles in many ways the “surplus” ratio of Campbell and Cochrane (1999). Hence, we are able to obtain a model that is observationally similar to Campbell and Cochrane (1999), but

\(^2\)See also Basak and Cuoco (1998), who use a similar idea in the context of a model with limited-participation.
whose economic mechanisms and justification are substantially more standard. Therefore, besides being useful for understanding asset prices our model can also be used as a workhorse for performing welfare exercises, policy experiments, etc., inside a framework where asset prices are reasonably well matched, and the standard welfare theorems are not invalidated by substantial externalities.

Additionally, in our model dividends and consumption are different, yet co-integrated, so that the model can provide a laboratory to investigate the net present values of consumption, dividends, and labor income as separate quantities. These distinctions have attracted the attention of the recent asset-pricing literature.\(^3\)

We also relate to Chan and Kogan (2002). Chan and Kogan (2002) present an interesting and insightful approach to obtaining a stationary wealth distribution in the presence of heterogeneity, based on habit-formation preferences. An advantage of their approach is that it allows for a continuum of risk aversions. However, it also produces substantial variability in interest rates, so that a substantial fraction of the equity premium is due to a pure term premium.

Several papers utilize variations in the cross-sectional wealth distribution due to some incompleteness to obtain implications for asset prices. This literature is vast and we do not attempt to summarize it. The papers that relate more closely to ours include Basak and Cuoco (1998), Guvenen (2005), Storesletten, Telmer, and Yaron (2007), and Gomes and Michaelides (2007). The first two of these papers assume infinitely lived agents, and the presence of limited participation allows the time variation of the wealth distribution to \(^3\)See, for instance, the work of Lettau and Ludvigson (2005), and Lustig and Van Nieuwerburgh (2007). We further relate to Santos and Veronesi (2006) and Menzly, Santos, and Veronesi (2004), since both these papers produce a dividend process that is not identical to consumption in the short run, but is cointegrated over the long run. The important difference is that in our paper this share process arises endogenously and jointly with the time variation in discount rates. We do not have to exogenously assume a structure for the joint dynamics of dividends and discount rates. Hence our approach complements Santos and Veronesi (2006) and Menzly, Santos, and Veronesi (2004) and lets the economic mechanisms of the model dictate this crucially important choice.
affect returns. A common implication of models with infinitely lived agents is that wealth eventually concentrates in the hands of agents who participate in markets. Even though ours is not a model of limited stock market participation, the presence of differing risk aversions has observationally similar implications. More importantly, the assumption of overlapping generations implies that all agents start and end life with zero wealth, so that the equity premium will not asymptotically reflect only the risk aversion of one group. Furthermore, an improvement over Guvenen (2005) is that our interest-rate volatility is very low and our consumption process is practically unpredictable. Storesletten, Telmer, and Yaron (2007) and Gomes and Michaelides (2007) study overlapping generations models and introduce frictions. Storesletten, Telmer, and Yaron (2007) study changes in the cross sectional variation of consumption shocks as Constantinides and Duffie (1996). Gomes and Michaelides (2007) analyze a rich setup (costly participation, heterogeneity in both preferences and income, etc.) and focus on understanding individual portfolio holdings in general equilibrium. However, both frameworks impose effectively an exogenous stock market volatility, by adopting a setup that is closer in spirit to Cox, Ingersoll, and Ross (1985) than to Lucas (1978). In such a framework the quantity rather than the price of capital absorbs all economic shocks, since Tobin’s q equals 1. Hence, any volatility in stock valuations results from exogenous assumptions on stochastic depreciation, and the volatility of aggregate output is driven to a large extent by changes in the capital stock, rather than total factor productivity. As Storesletten, Telmer, and Yaron (2007) admit, “solving the analogous endowment economy is substantially more difficult”. The reason is that, in an endowment economy of the Lucas (1978) kind, volatility in asset prices is endogenous. Because volatility is both challenging and central for many other moments (such as the equity premium, the predictability of returns etc.), we believe that our framework allows us to address a broader set of asset-pricing puzzles compared to previous literature.

There is a vast literature on overlapping-generations models. We do not attempt to

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4Guvenen (2005) sidesteps this problem by assuming a non-growing economy. Because of overlapping generations we can allow growth and still obtain a stationary wealth distribution.
summarize this literature. A partial listing of interesting applications of OLG frameworks to asset pricing include Abel (2003), Constantinides, Donaldson, and Mehra (2002), and Heaton and Lucas (2000). Most models in the OLG tradition share the feature that the minimal time periods of the model correspond to decades. The advantage of using a Blanchard (1985) framework is that the model produces implications for any time interval of interest. Given that most empirical regressions are run with monthly, quarterly or yearly data, this makes it easier to relate the model to the empirical asset-pricing literature.

Finally, there is a large literature on entrepreneurship and inequality in general equilibrium, which we do not attempt to summarize here. This literature has a fundamentally different focus than this paper. The goal of this literature is to understand decisions of entrepreneurs and their implications for inequality. Typically, these papers abstract from aggregate uncertainty so as to preserve tractability. Because of our focus on asset pricing, we cannot do the same. Instead, we abstract from many realistic aspects of entrepreneurship and focus on the asset pricing implications of the model. For our purposes, the most important aspect of entrepreneurship is the fact that entrepreneurs save more than non-entrepreneurs, a fact that is consistent with the data and reproduced by the model.

Section 2 describes the model. Section 3 presents the solution of the model. Section 4 presents a qualitative discussion and section 5 contains quantitative implications. Section 6 contains a brief discussion of the model’s implications beyond asset pricing. Section 7 concludes. All proofs are contained in the appendix.

2 Model

2.1 Agents’ Preferences and Endowments

There is a continuum of agents whose mass we normalize to 1. Existing agents face a constant hazard rate of death $\pi > 0$ throughout their lives. Furthermore newly born agents also arrive at a rate of $\pi$ per unit of time, so that the population remains constant. These demographic assumptions are identical to Blanchard (1985) and are key for the tractability of all the
aggregation results.

As is standard in the literature, we furthermore assume that agents have constant relative risk aversion. A key departure from prevailing representative-agent approaches is that we explicitly allow for the possibility that agents have heterogeneous preferences. The most parsimonious way to introduce heterogeneity is to follow Dumas (1989) and Wang (1996) and assume the presence of two types of agents, which we label as “type-A” and “type-B” agents.

Letting \( s \) denote the time of an agent’s birth and \( \tau \) the (stochastic) time of her death, type-A agents have mass \( \nu \) and preferences of the form

\[
E_s \int_s^{\tau} e^{-\rho_A(t-s)} \left( \frac{c^A_{t,s}}{1 - \gamma_A} \right) dt.
\]

(1)

\( \rho_A > 0 \) is the subjective discount rate and \( \gamma_A > 0 \) the relative risk aversion of type-A agents. Throughout we follow the notational convention that \( c^i_{t_1,t_2} \) denotes the time \( t_1 \) consumption by an agent of type \( i \in \{A,B\} \) who was born at time \( t_2 \leq t_1 \). By integrating over the (exponential) distribution of the time of death in (1), the agent’s problem amounts to maximizing\(^5\)

\[
E_s \int_s^{\infty} e^{-(\rho_A+\pi)(t-s)} \left( \frac{c^A_{t,s}}{1 - \gamma_A} \right) dt.
\]

(2)

Type-A agents come into life endowed with units of non-tradeable (human) capital, which they supply inelastically. We postpone a more detailed discussion of a type-A agent’s endowment until the next section, when we describe the production technology.

The second type of agents (type-B agents) have mass \( 1 - \nu \). They arrive in life with an endowment of hours, which they can supply in markets in exchange for wage income. We follow Blanchard (1985) and assume that the agents’ endowments of hours decline exponentially over the life-cycle at the rate \( \chi \). Blanchard (1985) argues that this simple assumption captures the idea that agents retire, so that their income over the life cycle is downward-

\(^5\)This is a standard property of the Blanchard-Yaari model. See Blanchard (1985).
Type-$B$ agents maximize an objective of the form

$$E_s \int_s^\infty e^{-(\rho^B + \pi)(t-s)} \left( \left( e^B \right)^\psi \left( \frac{\pi + \chi}{\pi} \frac{1}{1-v} e^{-\chi(t-s)} - h^B_{t,s} \right)^{1-\psi} \right)^{1-\gamma^B} \frac{dt}{(1-\gamma^B)}.$$  \hfill (3)

$\rho^B > 0$ is the subjective discount rate and $\gamma^B > 0$ the relative risk aversion of type-$B$ agents. The constant $\psi \in (0,1]$ controls how much type-$B$ agents care about leisure, and $h^B_{t,s}$ refers to the time-$t$ hours that are supplied by an agent who was born at time $s$. \(\frac{\pi + \chi}{\pi} \frac{1}{1-v} e^{-\chi(t-s)}\) gives the time $t$ endowment of hours for an agent who was born at time $s$. The normalizing constant \(\frac{\pi + \chi}{\pi} \frac{1}{1-v}\) that multiplies a type-$B$ agent’s endowment implies that the aggregate endowment of hours in the economy is unity, since

$$\left(1-v\right) \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \frac{\pi + \chi}{\pi} \frac{1}{1-v} e^{-\chi(t-s)} ds = 1.$$  \hfill (4)

Note that Equation (4) accounts for the mass of type-$B$ agents $(1-v)$ and their age distribution by the term $\pi e^{-\pi(t-s)}$ inside the integral.

Throughout, we shall assume that $\gamma^A \leq \gamma^B$, so that type-$B$ agents are at least as risk averse as type-$A$ agents.

### 2.2 Technology

Firms come in two varieties: publicly traded and privately held. We start by describing the first kind and then turn to the second kind.

The publicly traded firms are competitive and identical to each other, and own a total (physical) capital stock equal to $K_{t}^{\text{pub}}$. Assuming constant returns to scale, we can therefore
proceed as if there was a single “representative” firm that owns $K^\text{pub}_t$ units of the (physical) capital stock. This firm produces output $Y^\text{pub}_t$ by utilizing a technology of the form

$$Y^\text{pub}_t = Z_t K^\text{pub}_t f \left( \frac{H^\text{pub}_t}{K^\text{pub}_t} \right),$$

where $Z_t$ follows a geometric Brownian motion

$$\frac{dZ_t}{Z_t} = \mu_Z dt + \sigma_Z dB_t$$

for two positive constants $\mu_Z$ and $\sigma_Z$. $H^\text{pub}_t$ denotes the aggregate hours worked at the representative public firm. The constant-returns-to-scale assumption is captured by the fact that increasing both capital and hours in (5) by a proportionality factor $\varphi$, will result in a proportional increase in output by $\varphi$.

The function $f(\cdot)$ in (5) is an increasing and concave function of the hours-to-capital ratio $\eta_t \equiv H^\text{pub}_t/K^\text{pub}_t$. In particular, we assume that $f(\eta_t)$ solves the following ordinary differential equation:

$$f'(\eta) = \frac{\alpha(\eta) f(\eta)}{\eta}, \quad f(0) = 0 \quad (6)$$

and that $\alpha(\eta)$ is a continuous function satisfying

$$\alpha(\eta) \in (0, 1), \quad \alpha'(\eta) \leq 0. \quad (7)$$

First, note that $f'(\eta) > 0$, given the above assumptions. Second, differentiating both sides of (6) and using (7), it follows that $f''(\eta) < 0$. Furthermore, in the special case where $\alpha(\eta)$ is constant and equal to $\alpha$, the resulting solution to (6) is the familiar Cobb-Douglas production function $f(\eta) = \eta^\alpha$. When $\alpha(\eta)$ is chosen as $\alpha(\eta) = \frac{(1-b)\eta^{-\nu}}{(1-b)\eta^{-\nu}+b}$ for some $\nu > 0$ and some $0 < b < 1$, then $f(\eta)$ specializes to the CES production function. As in Lucas (1978), we assume that $K^\text{pub}_t = K^\text{pub}$ remains constant throughout, i.e., the (physical) capital stock does not depreciate. Finally, publicly traded firms are fully equity financed with the total supply of their equity normalized to 1.

We now turn to privately held firms. We think of these as entrepreneurial firms whose existence at time $t$ depends on some crucial skill of an agent who is alive at time $t$. In
particular, we assume that type-A agents are entrepreneurial. Once a type-A agent is born, an entrepreneurial firm is born, and when a type-A agent dies, her firm perishes as well. In the baseline model, we simply assume that type A agents are simultaneously entrepreneurial and have lower risk aversion than type B agents. We revisit this issue in section 6.2, and show how to obtain the link between entrepreneurship and lower risk aversion in an extended model with risky person-specific entrepreneurial skills.

Given these simplifying assumptions, it is possible to think of the present setup as a situation where type-A agents arrive in life endowed with units of (entrepreneurial) “capital” stock $K_{s,s}^{\text{priv}}$. Following a popular approach in the literature\footnote{See, e.g., Quadrini (2000).}, we think of entrepreneurial capital as a productive factor in a broad sense, i.e., as an idea or managerial talent, that can be used to produce output once combined with labor. Purely for parsimony, we assume that entrepreneurial capital gives entrepreneurs access to a technology that produces output (once combined with labor) according to a production technology that is similar to (5), i.e.,\footnote{One possible interpretation of equation (8) is that entrepreneurs can run their firms more efficiently than public firms, and hence can use their skill to economize on physical capital. In that interpretation physical and entrepreneurial capital are (perfect) substitutes.}

$$Y_{t,s}^{\text{priv}} = Z_t K_{t,s}^{\text{priv}} \frac{h_{t,s}^A}{K_{t,s}^{\text{priv}}},$$

where $h_{t,s}^A$ are the time-$t$ hours that are demanded by the firm of a type-A agent who was born at time $s$. To capture the idea that the ongoing arrival of new entrepreneurial firms may make the ideas and skills of previous entrepreneurs partially obsolete, we allow the capital stock of entrepreneurs to depreciate with their age at the rate $\delta \geq 0$, so that $K_{t,s}^{\text{priv}} = e^{-\delta(t-s)} K_{s,s}^{\text{priv}}$.

Importantly, type A agents cannot trade the units of the (human) capital stock that they are endowed with. For them, the output $Y_t^{\text{priv}}$ (net of the wages that they pay to the workers that they hire) corresponds to their non-traded income. All generations of type-A agents come endowed with the same (human) capital stock, and we will simplify notation by writing $K_{s,s}^{\text{priv}} = K^{\text{priv}}$.

Finally, as a matter of normalization, we assume that $K^{\text{pub}} \in (0, 1)$ and that $K^{\text{priv}} = \ldots$
\( \frac{\delta}{\pi + \delta} (1 - K_{\text{pub}}) \). Given these assumptions, the total capital stock \( K_t \) in the economy is constant and given by

\[
K_t = uK_{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi + \delta)(t-s)} ds + K_{\text{pub}} = uK_{\text{priv}} \frac{\pi}{\pi + \delta} + K_{\text{pub}} = 1. \tag{9}
\]

The integral in equation (9) adds up the capital stock that belongs to all surviving generations of type-A agents, taking into account the mass \( u \) of these agents and their age distribution. Given that the capital stock adds up to 1, \( K_{\text{pub}} \) can be interpreted as the fraction of the capital stock that belongs to the publicly traded firm.

### 2.3 Budget Constraints

A type-B agent who supplies \( h_{t,s}^{B} \) hours of labor at time \( t \) earns a labor income of \( w_t h_{t,s}^{B} \), where \( w_t \) is the prevailing wage at time \( t \). Similarly, a type-A agent receives dividend from her firm that are equal to \( Y_{t,s}^{\text{priv}} - w_t h_{t,s}^{A} \). Hence, the time-\( t \) income of an agent of type \( i = \{A, B\} \) who was born at time \( s \) is given by

\[
y_{t,s}^{i} = \begin{cases} 
Y_{t,s}^{\text{priv}} - w_t h_{t,s}^{A}, & \text{if } i = A; \\
w_t h_{t,s}^{B}, & \text{if } i = B. 
\end{cases} \tag{10}
\]

Both agents can also trade in a risk-less bond, a stock, and competitively priced annuities as in Blanchard (1985). The rate of return on bonds is given by \( r_t \). The stock is a claim that delivers a dividend flow given by \( D_t \equiv Y_{t,s}^{\text{pub}} - w_t H_{t,s}^{\text{pub}} \). It is reasonable to conjecture that the stock-price process follows a diffusion:

\[
dS_t = (\mu_t S_t - D_t) dt + \sigma_t S_t dB_t \tag{11}
\]

for some processes \( \mu_t \) and \( \sigma_t \). The processes for \( w_t, r_t, \mu_t, \) and \( \sigma_t \) will be jointly determined later so that markets clear. Finally, competition amongst competitive life insurers will drive the annuity income per dollar anuitized toward the actuarially fair flow of \( \pi \) per unit of time.\(^9\)

\(^9\)See Blanchard (1985) for details.
For future reference, it is convenient to define the stochastic discount factor process as
\[
\frac{d\xi_t}{\xi_t} \equiv -r_t dt - \kappa_t dB_t,
\]
(12)
where \(\kappa_t\) is the Sharpe ratio in the market defined as \(\kappa_t = \frac{\mu_t - r_t}{\sigma_t}\). Given the assumptions, the financial wealth \(W^j_{t,s}\) of agent \(i = \{A, B\}\) evolves as
\[
dW^i_{t,s} = (r_t W^i_{t,s} + \pi W^i_{t,s} - c^i_{t,s} + \theta^i_{t,s}(\mu_t - r_t) + y^i_{t,s}) dt + \theta^i_{t,s} \sigma_t dB_t,
\]
(13)
where \(\theta^i_{t,s}\) denotes the dollar investment in publicly traded stocks. Equation (13) is a standard dynamic budget constraint. The term \(\pi W^i_{t,s}\) captures the fact that the agent has no bequest motive and hence will choose to annuitize her entire wealth\(^\text{10}\). Letting \(W_t\) denote aggregate wealth, insurance companies collect \(\pi W_t\) per unit of time from agents who die, and hence can deliver payments equal to \(\pi W_t\) to the survivors. As a result of that, Blanchard (1985) shows that insurance companies break even at each point in time and the competitive annuity price is identical to the actuarially fair price. For the rest of the paper, we will be concerned with clearing the remaining markets.

### 2.4 Markets and Equilibrium

There are four markets that must clear in equilibrium: 1) the labor market; 2) the current-consumption-good market; 3) the bond market, where agents trade a zero net supply bond, and 4) the stock market, where agents trade claims to the dividend of the publicly traded company.

The definition of equilibrium is standard:

**Definition 1** An equilibrium is defined as a set of progressively measurable processes \(\{c^i_{t,s}, \theta^i_{t,s}, h^i_{t,s}\}\) for \(i \in \{A, B\}\) and a set of progressively measurable processes for the rate of return in the

\(^\text{10}\)To be more specific, annuities work as follows in this context. The agent signs an instantaneous contract that delivers competitive insurers a fraction \(\eta_{t,s}\) of her wealth upon death in exchange for an income of \(\hat{\pi}\eta_{t,s}W_{t,s}\) while the agent is alive. Since the agent has no bequest motives, and annuities are fairly priced (i.e., \(\hat{\pi} = \pi\)), \(\eta_{t,s} = 1\). For details see Blanchard (1985).
bond market \( (r_t) \), wages \( (w_t) \) and an appropriate stock market process of the form (11) with progressively measurable coefficients \( \mu_t, \sigma_t \) such that:

1. Given the process for \( \{r_t, w_t, \mu_t, \sigma_t\} \), for all \( s \) and \( t \) with \( t \geq s \), the processes \( \{c^i_{t,s}, \theta^i_{t,s}, h^i_{t,s}\} \) for \( i = A, B \) maximize (2) (objective [3] respectively) subject to (13), the initial condition \( W^i_{t,t} = 0 \) and the transversality condition \( E_s \lim_{t \to \infty} e^{-\pi t} \xi_t W^i_{t,s} = 0 \) for all \( s \).

2. Given \( w_t \), public firms\(^{11}\) choose hours so as to maximize profits:

\[
H^\text{pub}_t = \arg \max_{H^\text{pub}_t} D_t
\]  

(14)

3. Given \( c^i_{t,s}, h^i_{t,s}, \theta^i_{t,s} \) for \( i \in \{A, B\} \), markets for goods and labor clear, i.e.,

\[
(1 - v) \int_{-\infty}^{t} \pi e^{-\pi (t-s)} h^B_{t,s} ds = H^\text{pub}_t + v \int_{-\infty}^{t} \pi e^{-\pi (t-s)} h^A_{t,s} ds
\]

(15)

\[
\int_{-\infty}^{t} \pi e^{-\pi (t-s)} (uc^A_{t,s} + (1 - v) c^B_{t,s}) ds = Y^\text{pub}_t + v \int_{-\infty}^{t} \pi e^{-\pi (t-s)} Y^\text{priv}_t ds
\]

(16)

and markets for stocks and bonds clear, as well:

\[
\int_{-\infty}^{t} \pi e^{-\pi (t-s)} (v \theta^A_{t,s} + (1 - v) \theta^B_{t,s}) ds = S_t
\]

(17)

\[
\int_{-\infty}^{t} \pi e^{-\pi (t-s)} (v (W^A_{t,s} - \theta^A_{t,s}) + (1 - v) (W^B_{t,s} - \theta^A_{t,s})) ds = 0.
\]

(18)

Equation (15) states that the aggregate hours supplied by type-B agents alive at time \( t \) equal the total hours demanded by publicly traded and privately held firms. Equations (16), (17), and (18) capture the analogous requirements for the goods market, the stock market, and the bond market.

\(^{11}\)Note that it is sufficient to require that public firms maximize profits. The requirement that private firms should also maximize profits is implied by the first part of the definition. The reason is that type-A agents choose the hours demanded by their firms in an expected-utility maximizing way. As is shown in the next section, this is equivalent to maximizing the flow of profits at each state and date.
3 Solution

In this section we construct an equilibrium. We start by letting \( Y_t = Y_t^{\text{pub}} + \int_{-\infty}^{t} \pi e^{-\pi(t-s)} Y_t^{\text{priv}} \, ds \) denote the aggregate output in the economy, and \( X_t \) denote the consumption share of type-\( A \) agents, namely

\[
X_t \equiv \frac{v \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c_{t,s}^A \, ds}{Y_t}.
\]  

Since the consumptions of both types of agents are non-negative, the goods-market clearing condition (16) implies that \( X_t \in [0, 1] \). In the remainder of this section we construct an equilibrium with the following properties: a) \( X_t \) is a Markov process b) \( r_t, \mu_t, \sigma_t, \kappa_t \) are functions of \( X_t \) exclusively, and \( w_t \) has the form \( w_t = Z_t \omega(X_t) \) for an appropriate function \( \omega \) that we determine explicitly. In practical terms, this implies that a single variable, namely \( X_t \) is sufficient to characterize the equilibrium interest rate, expected stock market returns and volatility, despite the heterogeneity created by overlapping generations and differences in preferences and endowments.

3.1 Labor Markets

To establish the claims above, we start by examining the labor market. Labor supply by type-\( B \) agents is determined by the familiar condition that the marginal rate of substitution between leisure and consumption should be equal to the real wage. Letting \( u(c_{t,s}^B, h_{t,s}^B) \) denote the utility of a type-\( B \) agent, we obtain

\[
- \frac{u_h}{u_c} (c_{t,s}^B, h_{t,s}^B) = w_t.
\]  

Using the functional-form of \( u(c_{t,s}^B, h_{t,s}^B) \) and (20) gives the following relationship between hours, consumption, and wages:

\[
h_{t,s}^B = \frac{1}{1 - v} \pi \frac{\pi}{\pi} e^{-\chi(t-s)} - \frac{(1 - \psi) c_{t,s}^B}{\psi w_t} \quad \text{for } i \in \{A, B\}.
\]
Letting $H_t$ denote the aggregate hours supplied in the economy and using (21) along with (19) gives
\[ H_t = \int_{-\infty}^{t} \pi e^{-\pi(t-s)} (1-v) h_{t,s} ds = 1 - \frac{Y_t}{w_t} \frac{(1 - \psi)}{\psi} (1 - X_t). \] (22)
This expression gives the aggregate labor-supply relation implied by the model. To clear the labor market, we turn attention to the aggregate labor demand. Whether privately held or publicly traded, a firm can maximize its profits state by state by simply setting the marginal product of labor equal to the real wage. For both publicly traded and privately held firms this leads to the first order conditions
\[ Z_t f'(\frac{H_{t}^{\text{pub}}}{K_{t}^{\text{pub}}}) = Z_t f'(\frac{h_{t,s}^{A}}{K_{t,s}^{A}}) = w_t. \] (23)
An implication of (23) is that all firms in the economy will have the same hours-to-capital ratio, which in turn will be equal to the aggregate hours-to-capital ratio. Letting that ratio be denoted by $\eta_{t}^{\text{aggr}}$, it follows that in equilibrium $\eta_{t}^{\text{aggr}} = \frac{H_{t}^{\text{pub}}}{K_{t}^{\text{pub}}} = \frac{h_{t,s}^{A}}{K_{t,s}^{A}} = \frac{H_t}{K_t} = H_t$ since $K_t = 1$. Using these observations to compute aggregate output gives
\[ Y_t = Y_{t}^{\text{pub}} + v \int_{-\infty}^{t} \pi e^{-\pi(t-s)} Y_{t,s}^{\text{priv}} ds = \]
\[ = Z_t K_{t}^{\text{pub}} f \left( \frac{H_{t}^{\text{pub}}}{K_{t}^{\text{pub}}} \right) + v Z_t K_{t}^{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} f \left( \frac{h_{t,s}^{A}}{K_{t,s}^{A}} \right) ds \]
\[ = Z_t f(H_t) \left[ K_{t}^{\text{pub}} + v K_{t}^{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} ds \right] = Z_t f(H_t) K_t = Z_t f(H_t). \] (24)

Using $Y_t = Z_t f(H_t)$ together with (6), (5), and (23) and noting that $f'(\frac{H_{t}^{\text{pub}}}{K_{t}^{\text{pub}}}) = f'(H_t)$ leads to
\[ \frac{Y_t}{w_t} = \frac{H_t}{\alpha(H_t)}. \] (25)
Using (25) inside (22) results in
\[ H_t = 1 - \frac{H_t}{\alpha(H_t)} \frac{(1 - \psi)}{\psi} (1 - X_t). \] (26)
Given a value of $X_t$, equation (26) determines the equilibrium quantity of hours implied by the model. We shall therefore write $H_t = H(X_t)$ to denote this dependence on $X_t$.
Furthermore, equation (23) implies that the equilibrium wage can be written in the form 
\[ w_t = Z_t f'(H(X_t)) \]. It will be useful to define \( \omega(X_t) \equiv f'(H(X_t)) \), so that the resulting equilibrium wage can be expressed as \( w_t = Z_t \omega(X_t) \) as we conjectured at the beginning of this section.

### 3.2 Intertemporal Consumption Allocations

To study agents’ intertemporal consumption decisions, we start by assuming that there exists a stochastic discount factor \( \xi_t \). Given that agents can trade dynamically in stocks and bonds and there is a single source of uncertainty, the results in Karatzas and Shreve (1998), Chapter 3, imply that agent A’s optimal consumption choices satisfy the intertemporal first order condition

\[
e^{-\pi A}(t-s) \left( \frac{c^A_{t,s}}{c^A_{s,s}} \right)^{-\gamma A} = e^{-\pi (t-s)} \frac{\xi_t}{\xi_s}.
\]  

Equation (27) captures the intertemporal aspect of agent A’s problem. Roughly speaking, it states that the marginal benefit of an additional unit of consumption in a given state (as measured by the marginal utility of consumption) should equal the “cost” of a unit of consumption in that state. In turn this “cost” is measured by the product of the stochastic discount factor and the probability that the consumer will live until time \( t \) (namely \( e^{-\pi (t-s)} \)).

Agent B’s intertemporal first order condition is given by

\[
e^{-\pi B}(t-s) \left( \frac{\pi + \chi}{\pi} \frac{1}{1 - \psi} e^{-\chi (t-s)} - h^B_{t,s} \right)^{1-\gamma B} \left( 1-\psi \right) \left( \frac{c^B_{t,s}}{c^B_{s,s}} \right)^{\psi (1-\gamma B)-1} = e^{-\pi (t-s)} \frac{\xi_t}{\xi_s}.
\]  

Equation (28) is similar to equation (27), modified by the fact that the marginal utility of consumption depends also on leisure when \( \psi < 1 \). Re-arranging (28) leads to

\[
\frac{c^B_{t,s}}{c^B_{s,s}} = e^{-\frac{\rho B}{1-\psi(1-\gamma B)}(t-s)} \left( \frac{\pi + \chi}{\pi} \frac{1}{1 - \psi} e^{-\chi (t-s)} - h^B_{t,s} \right)^{\frac{(1-\psi)(\gamma B-1)}{1-\psi(1-\gamma B)}} \left( \frac{\xi_t}{\xi_s} \right)^{\frac{1}{1-\psi(1-\gamma B)}}.
\]  

We observe that, since (21) has to hold at all dates and states, it implies the following
relation between consumption, hours, and wages between two different points in time:

$$\frac{\pi + \chi}{\pi} \frac{1}{1 - \upsilon} e^{-\chi(t-s)} - h_{t,s}^B = \frac{c_{t,s}^B w_s}{c_{s,s}^B w_t},$$

Combining (29) with (30) and rearranging leads to

$$\frac{c_{t,s}^B}{c_{s,s}^B} = e^{-\frac{\rho B}{\gamma B}(t-s)} \left( \frac{w_t}{w_s} \right)^{\frac{1}{\gamma B} - 1} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\gamma B}}. \quad (31)$$

Similarly, rearranging (27) leads to

$$\frac{c_{t,s}^A}{c_{s,s}^A} = e^{-\frac{\rho A}{\gamma A}(t-s)} \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\gamma A}}. \quad (32)$$

Given $c_{s,s}^A, c_{s,s}^B$ equations (31) and (32) give the entire stochastic process of any agent’s consumption profile as a function of wages and the stochastic discount factor. In order to determine the initial consumption ($c_{t,s}^i$) for $i \in \{A, B\}$ we use the definition of income $y_{t,s}^i$ in equation (10) along with the inter-temporal budget constraint to obtain

$$E_s \int_s^\infty e^{-\pi(t-s)} c_{t,s}^i \left( \frac{\xi_t}{\xi_s} \right) dt = E_s \int_s^\infty e^{-\pi(t-s)} y_{t,s}^i \left( \frac{\xi_t}{\xi_s} \right) dt. \quad (33)$$

This states that the consumer’s net present value of consumption over the life cycle should equal the net present value of her non-traded income (since she is born with zero financial wealth).

Letting $\Phi_{s}^A$ denote the time $s$ net present value of non-traded income for a newly-born type-A agent, substituting (10) into (33), using (8), (6) and (23) along with $K_{t,s}^{priv} = K^{priv} e^{-\delta(t-s)}$ and recalling that $K_t = 1$ give

$$\Phi_{s}^A \equiv E_s \int_s^\infty e^{-\pi(t-s)} y_{t,s}^i \left( \frac{\xi_t}{\xi_s} \right) dt = K^{priv} E_s \int_s^\infty e^{-(\pi+\delta)(t-s)} Y_t (1 - \alpha(H_t)) \left( \frac{\xi_t}{\xi_s} \right) dt. \quad (34)$$

Similarly, for agents of type B using (10), and (21) inside the right hand side of (33) and rearranging gives

$$E_s \int_s^\infty e^{-\pi(t-s)} c_{t,s}^B \left( \frac{\xi_t}{\xi_s} \right) dt = \psi \frac{1}{1 - v} \frac{\pi + \chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} w_t \left( \frac{\xi_t}{\xi_s} \right) dt. \quad (35)$$
In light of this equation, it is convenient to define $\Phi^B_s$ as the human capital of a type-B agent:

$$
\Phi^B_s \equiv \psi \frac{1}{1 - \nu} \pi + \chi \int_s^\infty e^{-(\pi + \chi)(t-s)} w_t \left( \frac{\xi_t}{\xi_s} \right) dt.
$$

An implication of our conjecture that $X_t$ is a Markov process and that $r_t$ and $\kappa_t$ are functions of $X_t$ is the following Lemma.

**Lemma 1** Assuming that $X_t$ is a Markov process and that $r_t$ and $\kappa_t$ are functions of $X_t$, there exist four appropriate functions $\phi^i(X_t)$ and $\beta^i(X_t)$ for $i \in \{A, B\}$, such that $\Phi^i_s = \phi^i(X_s) Y_s$ and $c^i_{s,s} = \beta^i(X_s) Y_s$.

The next sub-section derives the dynamics of $X_t, r_t, \kappa_t$ and confirms the conjecture that $X_t$ is Markov, along with the conjecture that $r_t, \kappa_t$ are functions of $X_t$.

### 3.3 Dynamics of the Stochastic-Discount Factor and the Consumption Share

We start by determining the processes for the drift $\mu_X(t)$ and diffusion coefficients $\sigma_X(t)$ of

$$
dX_t = \mu_X dt + \sigma_X dB_t.
$$

To simplify notation we will use the short-hand notation $(\mu_X, \sigma_X)$ instead of $(\mu_X(t), \sigma_X(t))$. To determine the dynamics of $X_t$, we start by defining the function $g(X_t)$ as

$$
g(X_t) \equiv \frac{Y_t}{Z_t} = f(H(X_t)).
$$

Using (24), (38), and Ito’s Lemma implies that $dY_t/Y_t = \mu_Y dt + \sigma_Y dB_t$, where

$$
\mu_Y = \mu_Z + \frac{g'}{g} (\mu_X + \sigma_X \sigma_Z) + \frac{\sigma_X^2 g''}{2} \left( \frac{Z_t}{Y_t} \right),
$$

$$
\sigma_Y = \sigma_Z + \frac{g'}{g} \sigma_X.
$$

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Combining (32), Lemma 1, the definition of \( g \) in equation (38), and the definition of \( X_t \) in equation (19) leads to

\[
X_t Y_t = v \int_{-\infty}^{t} \pi e^{-\left(\frac{\rho A - \gamma A}{\gamma A}\right) (t-s)} \beta_s^A Z_s g_s \left( \xi_t \xi_s \right)^{-\frac{1}{\gamma A}} ds, \tag{41}
\]

where we have used the shorthand notation \( g_s = g(X_s), \beta_s^A = \beta^A(X_s) \). Applying Ito’s Lemma on both sides of (41), using (12), and matching the resulting diffusion coefficients on the left- and the right-hand sides implies that

\[
\frac{\sigma_X}{X_t} + \sigma_Y = \frac{\kappa_t}{\gamma A}. \tag{42}
\]

Similarly, matching the drift coefficients on both sides gives

\[
\mu_X + \mu_Y X_t + \sigma_Y \sigma_X = X_t \left[ \frac{r_t - \rho A}{\gamma A} + \frac{\kappa_t^2 \gamma A + 1}{2 (\gamma A)^2} - \pi \right] + v \pi \beta_t^A. \tag{43}
\]

To solve for \( \mu_X \) and \( \sigma_X \) from equations (42) and (43) we need to obtain expressions for \( r_t \) and \( \kappa_t \), which we do by using the goods-market clearing condition (16). Specifically, combining (16) with (31) gives

\[
Y_t = Z_t g(X_t)
\]

\[
= \int_{-\infty}^{t} \pi e^{-\left(\frac{\rho B - \gamma B}{\gamma B}\right) (t-s)} v \beta_s^B Z_s g_s \left( \xi_t \xi_s \right)^{-\frac{1}{\gamma B}} ds + \int_{-\infty}^{t} \pi e^{-\left(\frac{\rho B - \gamma B}{\gamma B}\right) (t-s)} (1 - v) \beta_s^B Z_s g_s \left( \frac{Z_t \omega_t}{Z_s \omega_s} \right) \left( \frac{1 - \psi}{\gamma B} \right) \left( \frac{1 - \psi}{\gamma B} \right) \left( \xi_t \xi_s \right)^{-\frac{1}{\gamma B}} ds. \tag{44}
\]

Once again, applying Ito’s Lemma to both sides of (44) and matching diffusion terms on the left- and the right-hand sides yield

\[
\sigma_Y = X_t \frac{\kappa_t}{\gamma A} + (1 - X_t) \left[ \frac{\kappa_t}{\gamma B} + \frac{(1 - \psi)}{(1 - \psi)} \left( \frac{\omega' \omega}{\omega' \omega} \right) \sigma_X + \sigma_Z \right]. \tag{45}
\]

Similarly, by matching drift coefficients we obtain

\[
\mu_Y = \sum_{i \in \{A, B\}} v^i \pi \beta_i^i + X_t \left[ \frac{r_t - \rho A}{\gamma A} + \frac{\kappa_t^2 \gamma A + 1}{2 (\gamma A)^2} - \pi \right] + (1 - X_t) \left[ \mathcal{D} \left( \frac{1 - \psi}{\gamma B} \right) - \left( \pi + \frac{\rho B}{\gamma B} \right) \right], \tag{46}
\]

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where we have used the shorthand notation \( v^A = v, v^B = 1 - v \), and

\[
\mathcal{D} (w^\xi) \equiv a \left( \mu_Z + \frac{\sigma_Z^2}{2} (a - 1) + \frac{\omega'}{\omega} (\mu_X + a \sigma_x \sigma_X - \kappa_x b \sigma_X) - b \kappa_x \sigma_Z \right) + \frac{\sigma_Z^2}{2} \left( a (a - 1) \left( \frac{\omega'}{\omega} \right)^2 + a \frac{\omega''}{\omega} \right) - b \left( r_t + \frac{\kappa_x^2}{2} (1 - b) \right). \tag{47}
\]

Fixing a value of \( X_t \), equations (42) and (45) form a linear system in \( \sigma_X \) and \( \kappa_t \) that can be solved explicitly. This yields \( \sigma_X \) and \( \kappa_t \) as functions of \( X_t \). Having obtained \( \sigma_X \) and \( \kappa_t \), equations (43) and (46) also form a linear system in \( r_t \) and \( \mu_X \) that can be solved explicitly, yielding \( r_t \) and \( \mu_X \) as functions of \( X_t \), as conjectured at the beginning of the section. Since both \( \mu_X \) and \( \sigma_X \) are functions of \( X_t \), the consumption-share process \( X_t \) is a Markov process, verifying the guess at the beginning of the section.

The last step in the construction of the equilibrium stochastic discount factor is the explicit determination of the functions \( \phi^i (X_t) \) and \( \beta^i (X_t) \). The following Lemma shows how to obtain these functions.

**Lemma 2** Let \( \sigma_X(X_t), \kappa(X_t), \mu_X(X_t), \) and \( r(X_t) \) denote the solution to the linear system of equations (42), (43), (45), (46). Then the function \( \phi^A(X_t) \) and \( \phi^B(X_t) \) are the solutions to the differential equations

\[
0 = \frac{\sigma_X^2}{2} (\phi^A)'' + (\phi^A)' \left( \mu_X + \sigma_X (\sigma_Y - \kappa) \right) + \phi^A (\mu_Y - r - \sigma_Y \kappa - \pi - \delta) + K^{\text{priv}} (1 - \alpha (H_t)) \tag{48}
\]

\[
0 = \frac{\sigma_X^2}{2} (\phi^B)'' + (\phi^B)' \left( \mu_X + \sigma_X (\sigma_Y - \kappa) \right) + \phi^B (\mu_Y - r - \sigma_Y \kappa - \pi - \chi) + \psi \frac{1}{1 - v} \frac{\pi + \chi \omega (X_t)}{\pi g(X_t)}, \tag{49}
\]

where we have used the simpler notation \( \sigma_x, \mu_x, r, \kappa, \) rather than \( \sigma_X(X_t), \mu_X(X_t) \), etc. Finally, the functions \( \beta^i, i \in \{A, B\} \), are given as \( \beta^i (X_t) = \phi^i (X_t) / \zeta (X_t) \), where \( \zeta^A (X_t) \) and \( \zeta^B (X_t) \) solve the differential equations

\[
-1 = \frac{\sigma_X^2}{2} (\zeta^A)'' + (\zeta^A)' \left( \mu_X - \sigma_X \frac{\gamma^A - 1}{\gamma^A} \kappa \right) - \zeta^A \left( \pi + r + \frac{\rho^A - r}{\gamma^A} + \frac{\gamma^A - 1}{(\gamma^A)^2} \frac{\kappa^2}{2} \right), \tag{50}
\]

\[
-1 = \frac{\sigma_X^2}{2} (\zeta^B)'' + (\zeta^B)' \left( \mu_X + \sigma_X \frac{(1 - \psi) (\gamma^B - 1)}{\gamma^B} \right) \left( \sigma_Z + \frac{\omega' (X_t) \omega (X_t) - \sigma_Z \gamma^B - 1}{\omega (X_t) \sigma_X} \kappa \right) + \zeta^B \left[ \mathcal{D} \left( \frac{(1 - \psi) \gamma^B - 1}{\xi_t \gamma^B} \frac{1}{\xi_t} \right) - \left( \pi + \frac{\rho^B}{\gamma^B} \right) \right]. \tag{51}
\]

\[\Xi_{\omega, r, \kappa}^A \Xi_{\omega, r, \kappa}^B \Xi_{\omega, r, \kappa}^C \Xi_{\omega, r, \kappa}^D \Xi_{\omega, r, \kappa}^E \Xi_{\omega, r, \kappa}^F \Xi_{\omega, r, \kappa}^G \Xi_{\omega, r, \kappa}^H \Xi_{\omega, r, \kappa}^I \Xi_{\omega, r, \kappa}^J \Xi_{\omega, r, \kappa}^K \Xi_{\omega, r, \kappa}^L \Xi_{\omega, r, \kappa}^M \Xi_{\omega, r, \kappa}^N \Xi_{\omega, r, \kappa}^O \Xi_{\omega, r, \kappa}^P \Xi_{\omega, r, \kappa}^Q \Xi_{\omega, r, \kappa}^R \Xi_{\omega, r, \kappa}^S \Xi_{\omega, r, \kappa}^T \Xi_{\omega, r, \kappa}^U \Xi_{\omega, r, \kappa}^V \Xi_{\omega, r, \kappa}^W \Xi_{\omega, r, \kappa}^X \Xi_{\omega, r, \kappa}^Y \Xi_{\omega, r, \kappa}^Z \]
From the solution $\zeta^i(X_t), \phi^i(X_t)$ to the differential equations above we obtain $\beta^i(X_t) = \phi^i(X_t)/\zeta^i(X_t)$, and then solve for $r_t, \kappa_t, \mu_X$ and $\sigma_X$ as functions of $X_t$. As we illustrate in section 4.1, in some special cases it is even possible to obtain closed-form expressions for some of these quantities. Before doing this, we determine the value of the stock market.

### 3.4 Stock Price

Given $\xi_t$, it is possible to define the stock market value as follows

$$ S_t \equiv E_t \int_t^\infty \left( \frac{\xi_u}{\xi_t} \right) D_u du. \quad (52) $$

We assume throughout that $S_t < \infty$. To verify that the constructed allocation forms an equilibrium, it remains to verify conditions (17) and (18). Adding up these two equations, and using Walras’ law it suffices to verify that the aggregate financial wealth is equal to the stock market value

$$ v \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^A ds + (1 - v) \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^B ds = S_t \quad (53) $$

The next Lemma asserts that equation (53) holds.

**Lemma 3** Let $S_t$ be defined as (52). Then equation (53) holds.

It is also possible to give a simple expression for $S_t$ in terms of the functions $\zeta^i, \phi^i$.

**Lemma 4** The stock market value is given as

$$ S_t = Y_t \left[ \zeta^A(X_t) X_t + \frac{\zeta^B(X_t)}{\psi} (1 - X_t) - v \phi^A(X_t) \frac{\pi}{\pi + \delta} - (1 - v) \frac{\phi^B(X_t)}{\psi} \frac{\pi}{\pi + \chi} \right] \quad (54) $$

From (54) we obtain the price-dividend ratio as

$$ p(X_t) = \frac{S_t}{D_t} = \frac{S_t}{Y_t K^{pub} [1 - \alpha (H(X_t))]}.$$

Finally, applying Ito’s lemma to (54) together with the definitions of $p(X_t), D_t, Y_t$ gives the stock-market volatility and the stock market expected return as

$$ \sigma_t = \sigma(X_t) = \sigma_Z + \left( \frac{p'}{p} - \frac{\alpha' H'}{1 - \alpha} + \frac{g'}{g} \right) \sigma_X \quad (55) $$

$$ \mu_t = \mu(X_t) = r_t + \kappa_t \sigma_t. \quad (56) $$
4 Qualitative Features of the Model

Before proceeding with an analysis of the quantitative implications of the model, it is instructive to start by examining some special cases that clarify the channels behind the model.

4.1 Overlapping Generations and the Risk-Free Rate Puzzle

We start our analysis with the special case $\gamma^A = \gamma^B = \gamma$, $\psi = 1$, and $\rho^A = \rho^B = \rho$, so that agents of type $A$ and agents of type $B$ have identical preferences, but different endowments. Then we obtain the following result.

**Lemma 5** An equilibrium is characterized by the following properties: 1) $H_t = 1$, and $\sigma_X = 0$, so that the evolution of $X_t$ is deterministic. 2) $\kappa_t$ is constant and given by $\kappa_t = \kappa = \gamma\sigma_Z$, 3) In steady state, the interest rate $r$ is constant and is given by the solution to the equation

$$0 = \frac{\pi + r + \frac{\rho - r}{\gamma} + \frac{\sigma_Z^2}{2} (\gamma - 1)}{r + \gamma\sigma_Z^2 + \pi + \delta - \mu_Z} (1 - K^{pub}) (\pi + \delta) (1 - \alpha(1)) + \frac{\pi + r + \frac{\rho - r}{\gamma} + \frac{\sigma_Z^2}{2} (\gamma - 1)}{r + \gamma\sigma_Z^2 + \pi + \chi - \mu_Z} (\pi + \chi) \alpha(1)
+ \frac{r - \rho}{\gamma} + \frac{\sigma_Z^2}{2} (\gamma + 1) - (\pi + \mu_Z).$$

(57)

Figure 1 plots the solution to (57) and illustrates the effects of changing $\chi$ and $\delta$ on the equilibrium interest rate for various degrees of risk aversion. There are two conclusions that follow from Figure 1. First, the interest rate declines as either $\chi$ or $\delta$ increases. Second, this decline in the interest rate can be substantive. The graph reconfirms (in a stochastic environment) the observations originally made by Blanchard (1985) (in a deterministic setting): If agents face a downward sloping profile of earnings over the life cycle (say, because of retirement), interest rates are substantially lower in equilibrium, even for high levels of $\gamma$.

This is intuitive. In the standard infinite-horizon representative-agent framework, there is little need to save for the future, as the representative agent is entitled to the aggregate endowment both presently and in the future. In contrast, in an OLG framework an agent with income constituting a decreasing fraction of the aggregate endowment over her life cycle will need to save for her retirement. This effect increases savings and hence reduces the equilibrium interest rate.
Figure 1: Equilibrium interest rates for different levels of $\chi$, $\delta$, and $\gamma = \gamma^A = \gamma^B$. The rest of the parameters are $\rho = 0.01$, $\pi = 0.01$, $\mu = 0.0172$, $\sigma = 0.041$, $\alpha(1) = 0.75$, and $K^{\text{pub}} = 0.4$. The line denoted by $\pi = \chi = \delta = 0$ corresponds to the infinite-horizon representative-agent model.

Even though the decline of labor income over the life cycle can help explain the low real rates that are observed in reality, a model with identical agents produces a constant price-to-dividend ratio and hence cannot explain why the stock market is more volatile than dividends, which in turn are more volatile than consumption. Next, we utilize the heterogeneity of
preferences to introduce variation in discount rates and the price-to-dividend ratio.

4.2 Heterogenous Agents

4.2.1 Sharpe Ratio

In what follows we keep the same assumptions as in section 4.1, except that we allow for the possibility that $\gamma^A < \gamma^B$. This special case is particularly attractive for comparison purposes, because now $X_t$ is a time-varying stochastic process with a non-degenerate stationary distribution. However, since $\psi = 1$, equation (26) continues to imply that $H_t = 1$. Since hours are constant, this implies that the aggregate output $Y_t$ satisfies $dY_t/Y_t = dZ_t/Z_t$, so that the aggregate endowment follows a geometric Brownian motion, as is commonly assumed in the literature. Furthermore, since hours are not time varying, both $g(X_t) = f(H(X_t))$ and $\omega(X_t) = f'(H(X_t))$ are constant functions, and so is the fraction of output that accrues to labor, i.e., $\alpha(H(X_t)) = \alpha(1)$.

We start our analysis by defining $\Gamma(X_t)$ as the weighted harmonic average of agents’ risk aversions:

$$\Gamma(X_t) \equiv \frac{1}{\frac{X_t}{\gamma^A} + \frac{(1-X_t)}{\gamma^B}}.$$  \hspace{1cm} (58)

Using this definition and the fact that hours are constant, equation (45) simplifies to

$$\kappa_t = \Gamma(X_t) \sigma_Z.$$  \hspace{1cm} (59)

Since $\Gamma'(X_t) < 0$, it follows that $\kappa_t$ is a declining function of $X_t$. Furthermore, equations (42) and (59) lead to the following expression for $\sigma_X$:

$$\frac{\sigma_X}{X_t} = \sigma_Z \left( \frac{\Gamma(X_t)}{\gamma^A} - 1 \right).$$  \hspace{1cm} (60)

Since $X_t \in [0,1]$, equation (58) implies that $\Gamma(X_t) > \gamma^A$, so that $\sigma_X \geq 0$. Hence, the state variable $X_t$ increases in response to positive innovations to the exogenous productivity process $Z_t$ and hence to positive news about the aggregate endowment $Y_t$. Since $\kappa_t$ is declining in $X_t$, this implies that the Sharpe ratio in the economy is countercyclical.
This property of the model is a first illustration of the consequences of aggregation: Less risk-averse agents (type-$A$ agents) have portfolios that expose them more to aggregate productivity risks. As a result, their wealth increases more than the wealth of more risk-averse agents (type-$B$ agents) in response to positive economic news, and so does their relative importance in the economy, which is captured by $X_t$. Furthermore, by equation (59), the Sharpe ratio is proportional to the (harmonic) weighted average of the risk aversions of the two agents, where the weights are given by $X_t$ and $1 - X_t$. Accordingly, the Sharpe ratio declines in response to good news.

Interpreting $\Gamma(X_t)$ as the risk aversion of the “representative agent” shows the analogy to Campbell and Cochrane (1999). Even though in our framework each agent has constant relative risk aversion, equation (59) shows that assets are priced as if there exists a single representative agent with countercyclical risk aversion. Importantly, the interaction of heterogeneity with overlapping generations ensures that $X_t$ is stationary, a property that may fail in economies with heterogeneous but infinitely lived agents.\footnote{The reason is intuitive. Given a positive risk premium, and the fact that less risk averse agents hold more stock implies that the expected growth rate of the less risk averse agents’ wealth is higher than that of the more risk averse. Over sufficiently long horizons, this leads to the “extinction” of the more risk averse agents.} Stationarity of $X_t$ implies stationarity of valuation ratios, interest rates etc., making it possible to calibrate the model to the data.

4.2.2 Interest Rate

One of the most challenging facts for asset pricing models is that interest rates do not seem to vary as much as risk premia.\footnote{For instance, the original habit-formation models (see e.g. Abel (1990), Constantinides (1990)) were successful at reproducing high equity premia, but at the cost of fairly volatile interest rates.} One of the main innovations of the Campbell and Cochrane (1999) model was to engineer a consumption-based stochastic discount factor where precautionary savings motives exactly cancel variations in the intertemporal smoothing motive, so that interest rates remain constant.
In the present model, it is possible to maintain interest-rate volatility at low levels, but through a different channel. To gain intuition on this issue and simplify the formulae, we make the same parametric assumptions as in section 4.2.1. Substituting (59) into (46) and solving for \( r_t \) gives

\[
r_t = \rho + \Gamma(X_t) \left[ \mu_Z + \pi \left( 1 - v \beta^A(X_t) - (1 - v)\beta^B(X_t) \right) \right] - \frac{\sigma^2_Z}{2} \Gamma^3(X_t) \Delta(X_t),
\]

where \( \Delta(X_t) \) is defined as

\[
\Delta(X_t) \equiv \left\{ \frac{X_t}{\gamma^A} \left( \frac{\gamma^A + 1}{\gamma^A} \right) + \frac{1 - X_t}{\gamma^B} \left( \frac{\gamma^B + 1}{\gamma^B} \right) \right\} = \frac{1}{\Gamma(X_t)} + \frac{X_t}{\gamma^2_A} + \frac{(1 - X_t)}{\gamma^2_B}.
\]

Notice that, for large values of \( \gamma^A \) and \( \gamma^B \), \( \Gamma(X_t)\Delta(X_t) \) is approximately equal to 1. Hence (61) is approximated by

\[
r_t \approx \rho + \Gamma(X_t)\mu_Z - \frac{\sigma^2_Z}{2} \Gamma^2(X_t) + \Gamma(X_t) \left[ \pi \left( 1 -v\beta^A(X_t) - (1 - v)\beta^B(X_t) \right) \right].
\]

Equation (62) is similar to the expressions that are typically obtained in infinite horizon representative agent models. The first term on the right hand side of (62) is the subjective discount rate, the second term is related to the intertemporal smoothing motive, and the third term is related to the precautionary-savings motive of consumers. The major difference between the infinite horizon representative agent model and the present OLG model is the last term, and specifically the term inside the square brackets. Intuitively, this term adjusts for the fact that aggregate consumption growth and the consumption growth of the surviving agents (who matter for asset pricing) are not identical. On the one hand, agents that are alive at date \( t \) perish at the rate \( \pi \), and this increases the expected growth rate of consumption to the survivors. On the other hand, new agents also arrive at the rate \( \pi \), and claim a share \( v\beta^A(X_t) + (1 - v)\beta^B(X_t) \) of output \( Y_t \) as their consumption. Adding up the two effects gives the term inside square brackets in equation (62).

Equation (62) is useful in showing how the model can produce low levels of interest-rate volatility. Collecting terms inside (62) gives

\[
r_t \approx \rho + \Gamma(X_t) \left[ \mu_Z - \frac{\sigma^2_Z}{2} \Gamma(X_t) + \pi \left( 1 - v\beta^A(X_t) - (1 - v)\beta^B(X_t) \right) \right].
\]
Equation (62) shows that in order to obtain interest rates that are low and non-volatile, there are two conditions that need to be satisfied. First, in order to obtain low interest rates, the term inside square brackets in equation (63) needs to be relatively small. Otherwise, the “representative” risk aversion $\Gamma(X_t)$ that premultiplies this term (and that typically is high, in order to account for the Sharpe ratio in the data) makes the interest rate implausibly large. Second, in order to obtain roughly constant interest rates, the term inside the square brackets in equation (63) needs to be increasing in $X_t$, since the term $\Gamma(X_t)$ that pre-multiplies the expression inside square brackets is declining in $X_t$.

The first property typically obtains with a declining labor income over the life cycle, as we discussed in section 4.1. The second property requires that $v\beta_A(X_t) + (1-v)\beta_B(X_t)$ is increasing in $X_t$, to counteract the fact that $\frac{\sigma^2}{2}\Gamma(X_t)$ is declining in $X_t$.

Intuitively phrased, variations in $X_t$ have two effects on the term inside square brackets in equation (62). On the one hand, they change the risk aversion of the representative agent, and hence the extent of precautionary savings. On the other hand, they affect the survivors’ consumption growth rate, by determining the fraction of aggregate consumption that accrues to newly born agents of each type, namely $\beta_i(X_t)$. If an increase (decrease) in precautionary savings is counterbalanced by an increase (decrease) in survivors’ consumption growth rates, then aggregate savings and interest rates will be roughly unaffected by changes in the representative agent’s risk aversion. Hence, changes in $X_t$ will affect mostly the Sharpe ratio and not the interest rate, as in the Campbell and Cochrane (1999) model.

In the next section we study circumstances under which $v\beta_A(X_t) + (1-v)\beta_B(X_t)$ is increasing in $X_t$.

### 4.3 Dividends and Labor

Since the two types of agents are endowed with different income streams, a first step towards examining $v\beta_A(X_t) + (1-v)\beta_B(X_t)$ involves the properties of dividends and labor income in the model.

To obtain non-trivial variation in hours and hence in the labor and the dividend share of
output we need to assume $\psi < 1$. Applying the implicit function theorem to (26) shows that

$$\frac{1}{H_t} \frac{dH_t}{dX_t} = \frac{\left(1 - \frac{1}{\psi}\right)}{\alpha'(H_t)(1 - H_t)} - \frac{\alpha(H_t)}{H_t}. \tag{64}$$

Since $\alpha'(H_t) \leq 0$, $\alpha(H_t) \geq 0$ and $H_t \leq 1$, the denominator on the right hand side of (64) is negative. If $\psi < 1$, then the numerator is also negative and hours are procyclical, i.e., an increasing function of $X_t$. The existence of stationary variation in hours implies that the model endogenously produces some cyclical variation in output alongside the variation caused by shocks to the (random walk) productivity process $Z_t$. Using the definition of $g$ in equation (38), the fact that $H_t$ is an increasing function of $X_t$ also implies that output is increasing in $X_t$. Mathematically, $g'(X_t) > 0$.

Since we are interested in the asset-pricing implications of the model, we do not focus on these effects. Instead, we calibrate the model so as to ensure that hours supplied are roughly constant and as a result consumption is approximately a random walk. To achieve this, we choose $\psi$ close to 1. The resulting small and procyclical variation in hours, in conjunction with our assumptions on the production function (equations [6] and [7]), results in a countercyclical labor share $\alpha(X_t)$ and hence a procyclical dividend share $1 - \alpha(X_t)$ of output. More specifically, the definition of dividends for either publicly traded or privately held firms, together with (6), implies that the volatility of (log) dividends $\sigma_D$ for either type of firm is given by

$$\sigma_D \equiv \sigma_Y - \frac{\alpha'H'}{1 - \alpha(H(X_t))} \sigma_X. \tag{65}$$

Given the assumption $\psi < 1$, all three terms in the above expression are positive, since $\alpha' \leq 0$, $\sigma_X > 0$, and $H' > 0$ by equation (64). This means that dividends are more volatile than output (and hence consumption).\footnote{The property $\sigma_Y < \sigma_D$ follows from (40).} However, over longer horizons, (log) aggregate dividends and (log) output are cointegrated, since $(1 - \alpha(X_t))$ is stationary.

Another observation concerning (64) is that the extra volatility in dividends is driven by variation in $X_t$. (For instance if $\sigma_X$ were 0, then $\sigma_D = \sigma_Z$). This dependence of dividends on
variation in $X_t$ translates into an increased sensitivity of the value of type A’s endowment $\phi^i(X_t)$ to variation in $X_t$, since type A agents arrive in life with a dividend income stream. Since $\beta^i(X_t) = \phi^i(X_t)/\zeta^i(X_t)$, in calibrated versions of the model this increased sensitivity of $\phi^A$ to variation in $X_t$ is sufficiently strong to ensure that $v\beta^A(X_t) + (1 - v)\beta^B(X_t)$ is increasing in $X_t$. This is particularly the case when the entrepreneurial and less risk averse type-A agents save and accumulate more assets over their lifetimes, so that it is mostly variations in $\beta^A(X_t)$ that drive variations in $v\beta^A(X_t) + (1 - v)\beta^B(X_t)$.

To conclude our discussion of dividends and labor, we also note that the assumption of a countercyclical labor share together with a countercyclical Sharpe Ratio makes the model consistent with the three observations about labor-income growth reported in Lustig and Van Nieuwerburgh (2007). First, dividend growth and labor-income growth are negatively correlated in our framework.\(^{15}\) Second, anticipated labor-income growth is positively correlated with “current” shocks to the productivity process.\(^{16}\) Third, periods of high expected return coincide with periods of low anticipated income growth.\(^{17}\)

5 Quantitative Results

5.1 Parameter Choice and Calibration

To calibrate the model we need to choose eleven parameters along with a functional form for $\alpha(H_t) = \alpha(H(X_t))$.

The parameters that we use for the calibration are given in Table 1. The parameters $\mu_Z$

\(^{15}\)Hence, when $\alpha(X_t)$ is above (below) its stationary mean, it can be expected to decline (increase). Therefore dividends can be expected to increase (decline) as a fraction of the aggregate endowment, while labor income can be expected to decline (increase).

\(^{16}\)Shocks to the productivity shock $Z_t$ increase $X_t$ (by equation [60]) and hence make $\alpha(X_t)$ decline since $\alpha'(X_t) \leq 0$. Because $\alpha(X_t)$ can be expected to mean revert after such a shock, anticipated labor-income growth is positively correlated with “current” shocks to the productivity process.

\(^{17}\)Periods of high expected returns occur when $X_t$ is below its stationary mean. Therefore $\alpha(X_t)$ is above its stationary mean and can be expected to mean revert.
and $\sigma_Z$ are chosen so as to match the mean growth rate and the volatility of consumption growth respectively. As in Chan and Kogan (2002), we choose the volatility of instantaneous consumption to be higher than in yearly discrete-time data, in order to account for the effects of time aggregation.

The parameter $\pi$ is chosen so that the median agent dies at age 69. As we have already discussed in section 2.1, $\chi$ can be interpreted as the arrival rate of a health shock that eliminates an agent’s ability to work. By letting $\chi = 0.0054$, a type B agent has on average the ability to work for $\frac{\pi}{\pi + \chi} = 65\%$ of her lifetime. Admittedly, the stylized assumptions of the model make it hard to calibrate $\pi$ and $\chi$. In real life neither $\pi$ nor $\chi$ are constant. Nevertheless, given the tractability of aggregation that is allowed by these assumptions, we believe that our choices for $\pi$ and $\chi$ are reasonable quantitatively.

The model assumes that a type-A agent arrives with an idea that is modeled as a unit of entrepreneurial capital stock. The parameter $\delta$ controls the depreciation of such entrepreneurial capital - a parameter that is clearly hard to calibrate. We use a common choice for depreciation by setting $\delta = 0.08$. This number is consistent with the fact that entrepreneurial businesses seem to have a short life-span in the data. Moreover, this assumption allows us to reproduce and study the general equilibrium consequences of the empirical finding that entrepreneurial saving is higher than non-entrepreneurial saving.

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18 See in particular footnote 6.
19 See Gentry and Hubbard (2002).
20 See in particular Gentry and Hubbard (2002).
The parameter $\nu$ controls the fraction of the population that is comprised of the less risk averse / entrepreneurial agents. We set that number to 10%, which is close to the fraction of entrepreneurs in the population as reported in Gentry and Hubbard (2002).\textsuperscript{21} $K^{\text{pub}}$ controls the fraction of the capital stock that is owned by publicly traded corporations. To choose a value of $K^{\text{pub}}$, we use annual NIPA data since 1929. In NIPA data the ratio of “income from assets” to “proprietors’ income” is roughly one to one. Interpreting proprietors’ income as proceeds of entrepreneurial equity, this suggests a value of $K^{\text{pub}}$ close to one half. However, to allow for the possibility that some entrepreneurial businesses are incorporated we choose $K^{\text{pub}} = 0.4$.

The parameters that pertain to agent preferences are given in the second column of Table 1. We set type-$A$ agents’ risk aversion to 4 and type-$B$ agents’ risk aversion to 12, with two motivations in mind. First, we want to match the average Sharpe ratio in the data and, second, we want to match the evidence in Vissing-Jorgensen, Malloy, and Moskowitz (2007), who argue that the “long-run” covariance between consumption and returns for wealthier, stock-holding households is up to 4 times larger than the equivalent covariance for the rest of the households. One can show that in our setup type-$A$ agents have a covariance between long run consumption growth and returns that is $\gamma^B/\gamma^A = 3$ times higher than the equivalent quantity for type-$B$ agents, well within the reasonable range of values supported by the data.

For parsimony, we choose $\rho^A = \rho^B = 0.001$. The low value of the discount rate is motivated by the fact that in our OLG framework the “effective” discount rate is $\rho + \pi = 0.011$.

Finally, the parameter $\psi$ that controls type-$B$ agents’ disutility of work is intentionally chosen very close to one. As we have discussed, equation (26) implies that when $\psi \approx 1$, hours do not vary considerably, and hence the predictable components of consumption growth become negligible. In particular, by combining values of $\psi \approx 1$ with a steeply declining $\alpha(H(X_t))$ we can ensure that the volatility in $X_t$ affects almost exclusively the dividend share.

\textsuperscript{21}Gentry and Hubbard (2002) report numbers between 8.7% and 11.5% depending on the definition used to classify entrepreneurs.
of output, and not the predictable components of consumption. To have enough flexibility to obtain these properties, we parameterize $\alpha(H(X_t))$ as

$$\alpha(H(X_t)) = (\beta_1 - \beta_2) \mathcal{N}(\beta_3(X - \beta_4)) + \beta_2,$$

where $\mathcal{N}$ is the cumulative normal distribution function and $\beta_1, \beta_2, \beta_3,$ and $\beta_4$ are constants that we can choose to match certain properties of the data. Equation (66) implies that $\alpha \in (\beta_1, \beta_2)$ for any value of $X_t$, so that $\beta_1$ and $\beta_2$ control the range of $\alpha$. The constants $\beta_3$ and $\beta_4$ control the steepness of the function and the point at which it achieves its maximum slope (in absolute value).

Our choices of $\beta_1, \beta_2, \beta_3,$ and $\beta_4$ control the production function of the economy, and hence the dividend and labor shares of output. We choose these numbers so as to approximately match (i) the average dividend and labor share of national income and (ii) the year-to-year volatility of the dividend share. In computing this dividend share we include both proprietors' income and income from assets.\footnote{However, we excluded rental income of persons. The model has no role for land, and the asset pricing focus of the paper implies that one should match as closely as possible the fraction of income that is paid out to entrepreneurs and holders of corporate claims in the economy. Including rental income of land as part of the income that accrues to shareholders raises the “dividend share” from 24.4 percent to 29.7. As a robustness check, we also calibrated the model to this higher number, and obtained similar results.}

### 5.2 Unconditional Moments

Table 2 compares the model’s performance with some key moments in the data. As Chan and Kogan (2002), we calibrated the model to the long-sample data reported in Campbell and Cochrane (1999). The model’s performance is not as good as in Campbell and Cochrane (1999), but it does explain a significant fraction of some asset-pricing facts. Most moments are within a reasonable distance from their empirical counterparts. The main moment that is underpredicted by the model is the volatility of equity.

There are two remarks about these results.
Table 2: Unconditional annual moments of the data. With the exception of the fourth, the eighth, and the ninth row, all data are from the long sample of Campbell and Cochrane (1999). The volatility of the interest rate is from Chan and Kogan (2002) and it refers to the volatility of the ex-post real rate. Hence, it overstates the volatility of the ex-ante riskless rate, because it doesn’t account for inflation surprises. The eighth and ninth row are from NIPA data provided by the Bureau of Economic Analysis, spanning the years 1929-2005. We add proprietors’ income and income from assets and divide by national income in computing the “dividend share” in the data. Simulation data are based on 10,000 simulated years of data for publicly traded companies, using a time increment of one month. Data are time-aggregated to yield yearly data.

First, the model does not assume any financial leverage. We have made this assumption in order to study how much dividend volatility can be generated by the procyclicality of the dividend share. As Table 2 shows, the resulting dividend volatility is 8.4%, which is about 70% of the dividend volatility in the data. Hence, it should not be surprising that the model can only replicate 60% of the volatility of asset prices and 2/3 of the equity premium. What is more important for our purposes is that the volatility of prices is higher than the volatility of dividends, which in turn is substantially higher than the volatility of consumption. As
we discuss in the next section, this implies that the variability of anticipated discount rate changes is larger than the variability of anticipated dividend growth, a fact with important implications for the predictability of excess returns and dividends.

Second, the stationary standard deviation of the interest rate is about 20 basis points in the model, i.e., the interest rate is practically constant. This implies that the variability in discount rates is almost exclusively driven by changes in excess returns, not interest rates. We note here that the model fit could be improved further by making assumptions on subjective discount rates that would increase the volatility of the interest rate, thus raising the volatility of the price/dividend ratio, and hence the equity premium and volatility. However, we have chosen to not make such assumptions in order to illustrate the model’s ability to drive fluctuations in asset prices without relying on volatile interest rates.

The main conclusion is that the model explains a significant fraction of the unconditional asset-pricing moments commonly studied, despite the usage of standard expected-utility specifications and without relying on excessive interest-rate volatility.

5.3 Conditional Moments

Figure 2 gives a depiction of the instantaneous Sharpe ratio, risk-free rate, conditional volatility, and equity premium as functions of \( X_t \). The range of values of \( X_t \) corresponds to \( \pm 1 \) (stationary) standard deviations around its stationary mean. Figure 2 shows that the range of values for the conditional equity premium is substantially larger than the equivalent range for the riskless rate. Hence, it confirms that most of the variation in discount rates is related to variations of the equity premium, not the interest rate, consistent with the data.

Figure 3 addresses another feature of the model that is consistent with the data and presents a challenge for many models: the joint presence of a procyclical dividend share and a procyclical price-to-dividend ratio. Figure 3 presents the dividend share in the economy.

\[ \text{We have explored versions of the model where } \rho^A < \rho^B. \text{ This assumption leads to a countercyclical interest rate, and amplifies the countercyclicality of discount rates, raising thus the volatility of prices and hence of the equity premium.} \]
Figure 2: The four panels depict the Sharpe ratio, the interest rate, the stock volatility and the equity premium respectively as a function of the consumption share of type A agents (less risk averse agents), which is denoted as $X$. The range of $X$ values corresponds approximately to ±1 standard deviations around the stationary mean of $X$.

and the P/D ratio as a function of $X_t$. Note that both the dividend share and P/D are increasing in $X_t$. This picture implies that there is time variation in anticipated changes to discount rates, and that this time variation is sufficiently strong to offset the time variation in anticipated dividend growth rates.

To see this point most clearly, consider figure 4. To simplify matters, assume that consumption is a random walk in logs, which is a close approximation to our model. The figure shows the effects of a 1 percent increase in current consumption. Since this effect is perma-
Figure 3: The top panel of the figure presents the dividend share of output as a function of the consumption share of type-A agents (less risk averse agents), which is denoted as $X$. The bottom panel presents the price-to-dividend ratio as a function of $X_t$.

...ent, it shifts permanently the path of consumption at all future dates. Furthermore, since the dividend share is procyclical, dividends rise by $k > 1$ in the short run. However, because of co-integration, we also know that in the long run dividends will rise by 1 percent. These simple observations imply that in response to a positive shock to aggregate consumption, the anticipated growth rate of dividends will decline instead of increase, as is illustrated by the declining line in figure 4. If discount rates were constant, this would imply that the price-to-
Figure 4: The implications of co-integration between dividends and consumption.

The dividend ratio would decline in response to good news. Alternatively put, in response to a one percent increase in consumption, prices would rise by a percentage strictly smaller than $k$ and, hence, dividends would be more volatile than asset prices.

In our setup, however, discount rates are not constant: Instead, they decline in response to positive shocks, as explained in section 4.2.1. In the calibrated version of the model, the decline in the anticipated dividend growth is smaller than the decline in discount rates, and thus the price-to-dividend ratio is procyclical, as in the data. This is consistent with the evidence in Lettau and Ludvigson (2005), who point out that the comovement of discount rates with the anticipated growth rate in dividends can help account for the observed inability
Table 3: Campbell-Shiller Variance Decompositions. The data are from the long sample of Campbell and Cochrane (1999). Simulation data are based on 10,000 simulated years of data for publicly traded companies. Time increments are monthly and are time-aggregated to yearly data.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividends</td>
<td>-10</td>
<td>-35</td>
</tr>
<tr>
<td>Returns</td>
<td>101</td>
<td>128</td>
</tr>
</tbody>
</table>

Additionally, the model can help explain the finding that the price-dividend ratio typically predicts dividend growth with the wrong sign. (See e.g. Cochrane (2005), p. 392). Table 3 shows a direct implication of figure 4. Since high P/D ratios are associated with high levels of the dividend share, mean reversion implies that high P/D ratios anticipate a moderation in dividend growth. Table 3 shows how this can help explain the puzzling finding that Campbell-Shiller type decompositions in the data assign a negative fraction of the variation in the P/D ratio to changes in dividend growth.

Table 4 gives yet a different perspective on these effects by showing the strong predictive ability of the P/D ratio for excess returns. The model overpredicts the absolute value of the coefficients in the predictive regressions for excess returns. This is partly driven by the fact that the model underpredicts the volatility of the (log) P/D ratio. This is to be expected, because of the offsetting effects of dividend growth on the time variation in discount rates. The $R^2$ of the regression, which is less affected by this issue, has the right order of magnitude when compared with the data.

As an additional “goodness of fit” test, we also estimated the autocorrelation of the P/D ratio, and compare it to the data. To account for well known finite-sample biases, we adopted the same design setup as in Table 4, and estimated this autocorrelation in repeated 100-year long simulated samples. We obtained a median autocorrelation coefficient of 0.94.\textsuperscript{24} In the data, Campbell and Cochrane (1999) report an autocorrelation of 0.87 based on CRSP

\textsuperscript{24}The 95\% coverage interval for this parameter was [0.79, 1.01].
<table>
<thead>
<tr>
<th>Horizon (Years)</th>
<th>Coefficient</th>
<th>$R^2$</th>
<th>Coefficient</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
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<td>0.23</td>
<td>-1.37</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 4: Long Horizon Regressions of excess returns on the log P/D ratio. To account for the well documented finite sample biases that are driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 100-year long samples, where the initial condition for $X_0$ for each of these simulation paths is drawn from the stationary distribution of $X_t$. For each of these 100-year long simulated samples, we run predictive regressions of the form $R_{t-t+h} = \alpha + \beta \log \left( \frac{P_t}{D_t} \right)$, where $h$ is the horizon for returns in years. We report the median values for the coefficient $\beta$ and the $R^2$ of this regression.

data and an autocorrelation of 0.78 based on the S&P sample. Therefore, even though the autocorrelation of the P/D ratio in the model is somewhat higher than in the data, it seems to be roughly consistent with the empirical magnitude of persistence.

6 Discussion and Extensions

6.1 The Dynamics of Cross-Sectional Inequality

The focus of the present model is on asset pricing. Nevertheless, it is interesting to check that the asset-pricing predictions of the model are not driven by counterfactual dynamics of cross-sectional consumption and wealth inequality.

As a first pass, the model’s key mechanism, namely that the wealthier (type $A$) agents become comparatively richer when the stock market performs well and poorer agents become
comparatively poorer when the stock market performs badly, is consistent with the data.\textsuperscript{25} However, does the model need “too much” high-frequency variation in cross-sectional inequality to explain returns? This question is a common concern in all asset-pricing models that feature heterogeneity, since high-frequency changes in inequality are small.

Figure 5 depicts the stationary distribution of the share of consumption of type-\(A\) agents \((X_t)\) in the top panel and the diffusion coefficient \(\sigma_X\) in the bottom panel. Intuitively, the bottom panel depicts how much one can expect the consumption share of type-\(A\) agents to change from year to year. The bottom panel of Figure 5 shows that the consumption share of type-\(A\) agents changes by about \(\pm1\) percent from year to year. In simulations we find that this translates into a yearly change in the Gini coefficient of consumption inequality of about 0.68 percent. This is consistent with the numbers given in Cutler and Katz (1992), based on repeated CES samples in the seventies and eighties.\textsuperscript{26}

To understand how small year-to-year changes in inequality can have substantial effects on asset prices it is important to examine the persistence of \(X_t\). Figure 5 depicts the stationary distribution of \(X_t\). There are two observations from this figure. First, type-\(A\) agents account for about 37\% of aggregate consumption, even though they are just 10\% of the population. Second, the stationary distribution of \(X_t\) is substantially more “spread out” than the year-

\textsuperscript{25}Wolff (1992) provides some direct evidence to that effect. With data that go back to the twenties he shows that the wealth distribution becomes more uneven in response to positive excess stock market returns, even after controlling for changes in the income distribution. Furthermore, Vissing-Jorgensen, Malloy, and Moskowitz (2007) present evidence that the consumption share of shareholders is useful in predicting subsequent excess returns.

\textsuperscript{26}Cutler and Katz (1992) report the Gini coefficient for consumption inequality for the years 1960, 1972, 1980, 1984, and 1988. Computing the differences in Gini coefficients between those years, dividing by the square root of the time distance between the years (to account for heteroskedasticity in the observations) and then computing the standard deviation gives 0.70 percent. As a check that this number has the right order of magnitude, we also fitted an AR(1) to yearly post-war CPS Gini coefficient data on income inequality, since yearly consumption inequality data are not available for such a long period of time. We isolated the residuals and estimated their standard deviation to obtain 0.56 percent. Even though this number is for income rather than consumption inequality, it confirms that the variability of inequality that is produced by the model is reasonable.
to-year increments to $X_t$. This illustrates that the state variable $X_t$ is highly persistent. This is consistent with the data, where cross-sectional inequality behaves almost like a random walk.\footnote{\textsuperscript{27}The Gini coefficient of \textit{income inequality} in the CPS data behaves almost like a random walk (autocorrelation coefficient 0.996). This is approximately equal to the autocorrelation of $X_t$ in the model, which is about 0.99. Admittedly, this comparison between the model and the data is not exact, because the data refer to income inequality, whereas $X_t$ is about consumption inequality. However, as Cutler and Katz (1992) argue, consumption and income inequality behave similarly over longer horizons.}

We conclude by noting that, despite the model’s main focus on asset pricing, it makes two additional predictions about inequality that are consistent with the data: a) Inequality increases over the life cycle as documented by Deaton and Paxson (1994), and b) Type-A agents (entrepreneurs) have higher saving rates than type-B agents (non-entrepreneurs), as has been documented by Gentry and Hubbard (2002). Specifically, Deaton and Paxson (1994) impute a Gini coefficient for consumption inequality of 0.29 for 25-year olds and 0.38
for 55-year olds. In our model the equivalent Gini coefficients are about 0.20 for 25-year olds and about 0.24 for 55-year olds. What drives this increase in consumption inequality over the life cycle is that type-A agents both save more and bear more aggregate risk than type-B agents. The latter effect is clearly due to their lower risk aversion and it implies that type-A agents’ wealth increases faster than that of type-B agents’, due to the positive equity premium. The higher savings owe to the combination of an income profile that decays faster for type-A than for type-B agents (δ > χ) and an intertemporal elasticity of substitution\(^{28}\) that is higher for type-A than for type-B agents \(1/\gamma^A > 1/\gamma^B\).

Clearly, the model is too simple to facilitate a thorough study of consumption and income inequality. Such a theory would require the introduction of time-varying idiosyncratic shocks. However, as this section has shown, the model’s prediction for inequality are qualitatively consistent with the data. We believe that the introduction of time-varying idiosyncratic shocks would further strengthen the mechanisms of the paper. For instance, in our parsimonious model, we can account for differences in savings rates between type-A agents and type-B agents by assuming that the income of entrepreneurs is more ephemeral than the income of non-entrepreneurs. It is reasonable to conjecture that a more complex model with idiosyncratic shocks to entrepreneurs would further amplify their incentives to save for precautionary reasons. However, such a model would be substantially less tractable.

### 6.2 The Link Between Low Risk Aversion and Entrepreneurship

Our baseline model simply assumes that agents with low risk aversion are entrepreneurs and agents with high risk aversion are workers, so that there is no occupational choice. Here we sketch how to extend the model in a simple way to allow for occupational choice without changing any of the asset pricing implications of the model.

Specifically, assume that at time \(s\) (i.e., the time of their birth) agents can choose whether to become entrepreneurs or workers. To simplify matters, assume furthermore

\(^{28}\)For expected utility preferences, the intertemporal elasticity of substitution is simply the inverse of risk aversion.
that this choice is irreversible. Also, extend the model to allow for an individual-specific entrepreneurial skill $\Theta^i$, so that an entrepreneur’s capital stock is given by $\Theta^i e^{-\delta(t-s)} K^\text{priv}_{s,s}$. The shock $\Theta^i$ is drawn from a log-normal distribution with mean 1 and variance $\Sigma$. Importantly, this random variable becomes known at time $s^+$, after the agent made her choice to become an entrepreneur. Finally, suppose that becoming an entrepreneur is associated with a (multiplicative) utility benefit of $(U^E)^{(1-\gamma^i)}$, where $U^E > 1$. That is, the ratio of an agent’s felicity function assuming that she chooses to become an entrepreneur to the same agent’s felicity if she chooses to become a worker is given by $(U^E)^{(1-\gamma^i)}$ for $i \in A, B$. A motivation for this assumption is that agents (of both types) don’t like to work for others, and enjoy the independence of entrepreneurship.

In such a modified setup it is possible to show that there always exist values of $\Sigma$ and $U^E > 1$, such that type-$A$ agents always choose to become entrepreneurs and type-$B$ agents always choose to become workers, given the equilibrium prices of section 3. The intuition is that the uncertainty introduced by $\Theta^i$ deters more risk averse agents from entrepreneurship, but not type-$A$ agents.

Even though this simple extension makes the link between risk aversion and entrepreneurship endogenous, it does not affect any of the asset pricing implications of the model. The reason is simple and intuitive. Because of homogeneity, the consumption of entrepreneur $i$ scales proportionately with $\Theta^i$, which is not time varying. Since $\Theta^i$ has a mean of 1, the consumption share that accrues to entrepreneurs (i.e. type-$A$ agents) collectively is not be affected by $\Theta^i$ shocks. Therefore, the equilibrium prices of Section 3 still clear markets.

7 Conclusion

In this paper we have presented a model that addresses a number of stylized facts about asset prices. The model combines three key ingredients: a) Agents are finitely lived, b) they can be heterogeneous in their preferences and endowments and c) consumption and dividends

\footnote{To keep the paper within a manageable size, we make the proof of this statement available upon request}
These natural assumptions help explain simultaneously several asset-pricing phenomena: 

a) Risk-less rates are low, since life-cycle motivations enhance agents’ incentive to save.

b) The Sharpe ratio is volatile, since variations in the wealth distribution determine the relative importance of agents with differing risk aversion. This composition effect makes our model resemble an economy populated by a representative agent with time-varying and countercyclical risk aversion.

c) Since dividends are procyclical and more volatile than consumption, and discount rates vary countercyclically, stock market prices are volatile and the equity premium is reasonably high.

d) Most of the variation in discount rates is due to changes in equity premia, not interest rates.

e) The price-to-dividend ratio predicts excess returns.

f) Even though dividends are predictable, the time variation in expected dividend growth is offset by changes in the stochastic discount factor, making the P/D ratio procyclical.

g) Dividends are more volatile than consumption in the short run, but the two quantities are cointegrated over the long run.

h) Consumption is practically a random walk.

These facts are consistent with the data. Moreover, calibrated versions of the model produce a satisfactory, albeit not perfect, quantitative fit.

Accordingly, we believe that the broad conclusion of the model is that overlapping generations along with preference heterogeneity can go a long way towards explaining prevailing asset-pricing puzzles. Observationally, our framework resembles a model of exogenous habit formation of the type proposed by Campbell and Cochrane (1999). However, the economic mechanisms of the models differ fundamentally.

Furthermore, our model allows us to draw a distinction between a claim to consumption and a claim to dividends in a framework where the joint dynamics of dividends and consumption are modeled realistically. This allows us to use our framework as a laboratory in order to propose an explanation for the findings of a recent empirical literature that exploited this distinction.30

Last but not least, an important advantage of the model is its analytic tractability. It

30See e.g. Lettau and Ludvigson (2005).
provides us with a simple way of reproducing some key asset-pricing facts in a framework that can be used in various applications. For instance, the model could be expanded to investigate the effect of demographic shocks (such as a baby boom) on asset prices within a model that reproduces key asset-pricing facts. The conventional utilities that we use also facilitate policy experiments, such as the effects of a switch from pay-as-you-go to a fully funded system. We leave such extensions and applications for future research.
\textbf{A Appendix}

\textbf{Proof of Lemma 1.} Re-write $\Phi_s^A$ as $\Phi_s^A = \phi^A Y_s$, where
\begin{equation}
\phi^A = K^{\text{priv}} E_s \int_s^\infty e^{-(\pi+\delta)(t-s)} \left( \frac{Y_t}{Y_s} \right) (1 - \alpha(H_t)) \left( \frac{\xi_t}{\xi_s} \right) dt.
\end{equation}

Now note that (24) implies that $Y_t = \frac{Z_t f(H_t)}{f(H_s)} = \frac{f(H_t)}{f(H_s)} e^{\left(\mu_Z - \frac{\sigma_Z^2}{2}\right)(t-s) + \sigma_Z (B_t-B_s)}$. By (26) $H_t$ is a function of $X_t$, while the exponential $e^{\left(\mu_Z - \frac{\sigma_Z^2}{2}\right)(t-s) + \sigma_Z (B_t-B_s)}$ does not depend on $Y_t$. Similarly, because $r_t$ and $\kappa_t$ are functions of $X_t$ only, $\xi^t_s$ does not depend on $Y_t$. Finally $(1 - \alpha(H_t))$ is independent of $Y_t$, since $H_t$ is a function of $X_t$. Therefore, $\phi^A$ is independent of $Y_t$. Since $\phi^A$ is an expected integral of elements that depend on future values of $X_t$ and $X_t$ is Markovian (by assumption), it follows that $\phi^A$ is a function of $X_s$.

A similar argument shows that $\Phi_s^B$ can be written as $\phi^B Y_s$, where
\begin{equation}
\phi^B = \phi^{\pi + \chi \pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} \frac{w_t}{Y_t} \left( \frac{\xi_t}{\xi_s} \right) dt.
\end{equation}

Noting that $w_t = Z_t f'(H_t)$ and $Y_t = Z_t f(H_t)$ shows that $w_t/Y_t$ is a function of $H_t$ and hence of $X_t$ only. Therefore, the same arguments as the ones we gave above show that $\phi^B$ is a function of $X_t$ only. To show that $c^i_{s,s} = \beta^i (X_s) Y_s$ note that
\begin{equation}
E_s \int_s^\infty e^{-\pi(t-s)} c^i_{t,s} \left( \frac{\xi_t}{\xi_s} \right) dt = c^i_{s,s} \left[ E_s \int_s^\infty e^{-\pi(t-s)} \frac{c^i_{t,s}}{c^s_{s,s}} \left( \frac{\xi_t}{\xi_s} \right) dt \right].
\end{equation}

Equations (32) and (31) imply that the term inside square brackets depends exclusively on $X_t$. Therefore equations (33) and (35) together with the fact that $\Phi_s^i = \phi^i (X_s) Y_s$ imply that $c^i_{s,s} = \beta^i (X_s) Y_s$. \end{proof} 

\textbf{Proof of Lemma 2.} For any $T < s$, equation (67) implies that
\begin{equation}
e^{-(\pi+\delta)s} K^{\text{priv}} Y_s \xi_s + \int_T^s e^{-(\pi+\delta)t} Y_t (1 - \alpha(H_t)) \xi_t dt = E_s \int_T^\infty e^{-(\pi+\delta)t} Y_t (1 - \alpha(H_t)) \xi_t dt.
\end{equation}

The right hand side of this expression is a martingale, since it is a conditional expectation. Accordingly, applying Ito’s Lemma to the left hand side and setting the drift coefficient equal to zero implies (48). Applying a similar argument to (68) leads to (49). To obtain the functions $\beta^i$, define
\begin{equation}
\zeta^i = E_s \int_s^\infty e^{-\pi(t-s)} \frac{c^i_{t,s} \xi_t}{c^s_{s,s} \xi_s} dt.
\end{equation}
Next use (69) along with equation (33), and \( \Phi^i = \phi^i Y_s \) to obtain \( \beta^i(X_t) = \frac{\phi^i(X_t)}{\zeta(X_t)} \). Furthermore, equations (32) and (31) imply

\[
\zeta^A = E_s \int_s^\infty e^{-\left(\pi + \frac{1}{\gamma^A}\right)(t-s)} \left( \frac{\xi_t}{\xi_s} \right)^{1-\frac{1}{\gamma^A}} dt.
\]

\[
\zeta^B = E_s \int_s^\infty e^{-\left(\pi + \frac{1}{\gamma^B}\right)(t-s)} \left( \frac{w_t}{w_s} \right) \frac{(1-\psi)(\xi^B)}{\xi_s}^{1-\frac{1}{\gamma^B}} dt.
\]

By applying a similar argument to the one given for \( \phi^A, \phi^B \) one arrives at (50), (51).

**Proof of Lemma 3.** Applying Ito’s lemma to compute \( d \left( e^{-\pi s\zeta_s}W^i_s \right) \) and integrating leads to

\[
W^i_{t,s} = E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left( c^i_{u,s} - y^i_{u,s} \right) du
\]

(71)

We next observe that generations that are born after \( t \) neither consume, nor supply hours, nor own any wealth at time \( t \). This means that for any \( i \in \{A, B\}, W^i_{t,s} = 0, c^i_{t,s} = 0, y^i_{t,s} = 0 \) as long as \( s > t \). Combining this observation with (71) and using the short-hand notation \( \nu^A = \nu, \nu^B = 1 - \nu \), we obtain

\[
\sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \nu^i W^i_{t,s} ds = \sum_{i \in \{A, B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(t-s)} \nu^i W^i_{t,s} ds
\]

\[
= \sum_{i \in \{A, B\}} \pi e^{-\pi(t-s)} \nu^i \left( E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left( c^i_{u,s} - y^i_{u,s} \right) du \right) ds
\]

\[
= E_t \int_t^\infty \frac{\xi_u}{\xi_t} \left( \sum_{i \in \{A, B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(u-s)} \nu^i \left( c^i_{u,s} - y^i_{u,s} \right) ds \right) du
\]

\[
= E_t \int_t^\infty \frac{\xi_u}{\xi_t} D_u du,
\]

where the last line follows from the equilibrium condition (16), (15), (10) and the definition of \( D_t \).

\[\blacksquare\]
Proof of Lemma 4. As we have shown in Lemma 3,

\[
S_t = \sum_{i \in \{A, B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} u^i W_{t,s}^i ds
\]

(72)

\[
= \sum_{i \in \{A, B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} u^i \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds
\]

(73)

\[
- \sum_{i \in \{A, B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} u^i \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,i} du \right] ds.
\]

using the short-hand notation \(u^A = u, u^B = 1 - u\). We can compute the first term in (73) as

\[
\sum_{i \in \{A, B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} u^i \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds
\]

\[
= \sum_{i \in \{A, B\}} u^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c_{t,s}^i \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds.
\]

Now note that equation (31) and (32) implies that \(c_{u,s}^i/c_{t,s}^i\) for \(i \in \{A, B\}\) is independent of \(s\); i.e., \(c_{u,s}^i/c_{t,s}^i = c_{t,t}^i/c_{t,s}^i\). Using this observation together with (70) leads to

\[
\sum_{i \in \{A, B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} u^i \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds =
\]

\[
= \sum_{i \in \{A, B\}} u^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c_{t,s}^i (X_t) ds = Y_t \left[ \zeta^A (X_t) X_t + \zeta^B (X_t) (1 - X_t) \right].
\]

(74)

Similarly we can compute the second term in (73) separately for each agent. By using (34) and \(\Phi^i(X_t) = \phi^i(X_t) Y_t\), we obtain for agent \(A\)

\[
v \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,i}^A du \right] ds =
\]

\[
= K^{\text{priv}} v \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} e^{-\delta(u-s)} Y_u (1 - \alpha (H_u)) du \right] ds
\]

\[
= K^{\text{priv}} v \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} e^{-\delta(u-t)} Y_u (1 - \alpha (H_u)) du \right] ds
\]

\[
= Y_t \phi^A (X_t) \left[ v \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} ds \right] = Y_t \left[ \phi^A (X_t) \frac{v \pi}{\pi + \delta} \right].
\]

(75)

---

31To see this, fix a time of birth \(s\), apply equation (31) at two different points in time, say \(u\) and \(t\), and then derive \(c_{u,s}^i/c_{t,s}^i\) which is independent of \(s\).
Similarly, by using (21) we obtain for agent B
\[
(1 - v) \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \xi_u \frac{\xi_t}{\xi_t} Y_{u,s} du \right] ds
\]
\[
= (1 - v) \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \xi_u \left( \frac{\pi + X - \chi(u-s)}{\pi} e^{-\chi(u-s)} w_u - \frac{1 - \psi}{\psi} \zeta_u \right) du \right] ds
\]
\[
= (1 - v) \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \xi_u \left( \frac{\pi + X - \chi(u-s)}{\pi} \right) w_u du \right] ds \tag{76}
\]
\[- Y_t \frac{(1 - \psi)}{\psi} \zeta^B (X_t) (1 - X_t).
\]

Using (36), the first term in (76) can be further rewritten as
\[
(1 - v) \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[ E_t \int_{t}^{\infty} e^{-\pi(u-t)} \xi_u \left( \frac{\pi + X - \chi(u-s)}{\pi} \right) w_u du \right] ds
\]
\[
= (1 - v) \int_{-\infty}^{t} e^{-(\pi+\chi)(t-s)} \left[ E_t \int_{t}^{\infty} e^{-(\pi+\chi)(u-t)} \xi_u (\pi + \chi) w_u du \right] ds
\]
\[
= (1 - v) Y_t \left[ \frac{\phi^B (X_t)}{\psi} \pi \left( \int_{-\infty}^{t} e^{-(\pi+\chi)(t-s)} ds \right) \right] = Y_t \frac{\pi (1 - v)}{\pi + \chi} \frac{\phi^B (X_t)}{\psi} \tag{77}
\]

Combining (77) with (76), (74), and (75), we arrive at (54). ■

**Proof of Lemma 5.** Solving for $\sigma_X$ and $\kappa_t$ from equations (42) and (45) gives $\sigma_X = 0$ and $\kappa_t = \gamma \sigma_Z$. Therefore, $X_t$ is deterministic. Moreover, equation (26) implies that $H_t = 1$ for all $X_t$. By (24) it follows that $\sigma_Y = \sigma_z, \mu_Y = \mu_Z$. Furthermore, since hours do not vary with $X_t$, it also follows that $g', g'', \omega, \omega''$ are all zero. Using these observations inside equations (43) and (46) gives
\[
\mu_X + \mu_Z X_t = X_t \left[ \frac{r_t - \rho}{\gamma} + \frac{\kappa^2}{2} \frac{\gamma + 1}{\gamma^2} - \pi \right] + v \pi \beta_t^A \tag{78}
\]
\[
\mu_Z = \sum_{i \in \{A,B\}} v^i \pi \beta_t^i + \frac{r_t - \rho}{\gamma} + \frac{\kappa^2}{2} \frac{\gamma + 1}{\gamma^2} - \pi. \tag{79}
\]

Given knowledge of $\beta_t^i = \beta^i (X_t)$, the above two equations form a system in $r_t, \mu_X$. We will be interested in determining the steady state value of $X_t$ and $r$. Since $H_t = 1$, equation (6) implies that $\omega (H_t (X_t)) \equiv \frac{f'(1)}{f(1)} = \frac{\alpha(1)}{1} = \alpha(1)$. Also, in steady state $\mu_X = 0$, and hence equations (48) and (49) imply
\[
\phi^A = \frac{K^{\text{priv}} (1 - \alpha(1))}{r + \sigma_Z \kappa + \pi + \delta - \mu_Z}, \quad \phi^B = \frac{\frac{1}{1 - v} \frac{\pi + \chi}{\pi} \alpha(1)}{r + \sigma_Z \kappa + \pi + \chi - \mu_Z} \tag{80}
\]
\[
\zeta^A = \frac{1}{\pi + r + \frac{\rho - \rho}{\gamma} + \frac{\kappa^2}{2}}, \quad \zeta^B = \frac{1}{\pi + r + \frac{\rho - \rho}{\gamma} + \frac{\kappa^2}{2}}. \tag{81}
\]

Substituting $\kappa = \gamma \sigma_Z$ and $K^{\text{priv}} = \frac{1}{v} \frac{\pi + \delta}{\pi} (1 - K^{\text{pub}})$ inside (80) and (81), using $\beta^i (X_t) = \phi^i (X_t) / \zeta^i (X_t)$, and then substituting into (79) gives (57). ■
References


