The representative agent of an economy with external habit-formation and heterogeneous risk-aversion

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Abstract

For the first time in the literature, we derive an analytic expression for the representative agent of a fairly general class of economies populated by agents with “catching up with the Joneses” preferences, but with heterogeneous risk-aversion. As Chan and Kogan (2002) show numerically, the representative agent has stochastic risk-aversion that moves counter-cyclically with the state variable. However, we show that the heterogeneity of risk-aversion is unlikely to be able to explain the empirical regularities -namely the variability of the Sharpe ratio- that Campbell and Cochrane (1999) explain in a model of a representative agent with stochastic risk-aversion. We further show that many of these empirical regularities can be easily explained with a varying conditional volatility of the state variable.
1 Introduction

The consumption asset pricing literature, starting with Hansen and Singleton (1983), Mehra and Prescott (1985) and others, has tried with limited success to identify the fundamental factors that drive the level, variation and cyclical movements of asset prices and conditional asset pricing moments. This failure is attributed to the fact that aggregate consumption growth is very smooth and its covariance with asset returns is very low. Effectively, the high volatility of asset returns and the high average historical returns in excess of the risk free rate cannot be reconciled with the low consumption risk born by the assets. If agents care about consumption these two pieces of evidence imply that the unit price of risk required by investors is on average very high. Further evidence indicates that the unit price of risk also varies significantly across time. In particular, price dividend ratios exhibit high variability compared to dividends, and exhibit some forecasting power in predicting long-run stock returns, as found in Fama and French (1988), Campbell and Shiller (1988a,b) and Campbell (1991), among others. On the same note, variance decompositions of price dividend ratios have revealed that almost all variation is attributed to varying future excess returns. Based on these facts many researchers have concluded that expected excess returns are on average high and vary significantly and further that possible variations in the conditional moments of dividend growth or consumption growth are not responsible for these facts.

In parallel to the analysis of the variation of excess stock returns, a number of papers have studied stock market volatility, and they have found it to vary over the business cycle. This is shown in, for example, Bollerslev, Chou and Kroner (1992) or Ludvigson and Ng (2007). Several other papers\footnote{Like Bollerslev, Engle and Wooldridge (1988), Harvey (1989), Whitelaw (1994), Brandt and Kang (2004) and Ludvigson and Ng (2003).} have studied the contemporaneous relation between expected stock returns and conditional return volatilities and found conflicting results. The relation, however, seems to be weak but, more importantly, most of these studies indicate that the Sharpe ratio varies throughout the business cycle. In particular, it appears to increase considerably during recessions, and to fall during expansions.

It seems that at the heart of all these stylized facts is a mean-reverting and counter-cyclically varying risk premium. Furthermore, it seems to vary more than the stock market volatility, giving rise to a counter cyclically varying Sharpe ratio. In an attempt to explain this dynamics in an economy with rational agents, Campbell and Cochrane (1999) were able to explain many of the mentioned asset pricing features in a model of a representative agent with external habit preferences and counter-cyclical variation in risk aversion. Chan and Kogan (2002) argue that such a variation in the risk aversion of the representative agent can be the result of the endogenous cross-sectional redistribution of wealth in an economy with multiple agents with heterogeneous risk-aversion parameters.
We study a model similar to that of Chan and Kogan (2002), although in discrete time. We confirm that the risk-aversion of the representative agent exhibits the qualitative properties discussed in Campbell and Cochrane (1999) and Chan and Kogan (2002). However, we find that it cannot vary enough to explain the empirical regularities described before and, in particular, the variability of the Sharpe ratio. With a detailed study of the mechanics of the stationary equilibrium of our heterogeneous agents economy, we derive explicitly the time-varying risk-aversion of the representative agent. From the resulting expression, we can analyze the properties of the varying risk-aversion parameter of the economy. We find that, although the counter-cyclical pattern of Campbell and Cochrane (1999) is accurate, many more assumptions are needed in order to have more than just a marginal effect on asset price dynamics.

Chan and Kogan (2002) assume a highly persistent and slow moving external habit, as well as a particularly high level of heterogeneity in risk-aversion. Our representative agent formulation reveals that if either of these assumptions is missing, a substantially, if at all, varying Sharpe ratio does not obtain. The problem with such a slow moving state variable is that its effects will be seen over much longer periods than a business cycle, the price dividend ratios become very volatile, while it predicts significant redistribution of wealth along the business cycle. Campbell and Cochrane (1999) also assume a slowly varying state variable, namely the surplus consumption (current aggregate consumption over the external habit, given by the previous period consumption), but with the additional feature of a highly time-varying conditional variance, that also plays the role of risk-aversion. Their study is very useful because it identifies the main features that a successful asset pricing model needs to have in order to explain all the aforementioned empirical facts. However our study shows that a time-varying risk aversion fails to obtain as the result of aggregation in an economy of rational agents with standard preferences and different risk-attitudes. Further, we show that time variation in the conditional volatility of consumption growth is adequate to explain variations in the Sharpe ratio.

Even though consumption risk seems to be too low to be able to explain equity premia, there is growing evidence that variations in the conditional distribution of consumption growth might have certain significant asset pricing implications. Kandell and Stambaugh (1990) have found substantial evidence that the conditional moments of consumption growth vary along the business cycle and their variation is associated with a variation in the Sharpe ratio. Lettau, Ludvigson and Wachter (2007) using a two state regime switching model find similar evidence and in particular that an almost permanent downward shift in the conditional volatility of consumption might explain the increase in the asset pricing ratios in the late 1990’s. Many papers including Kandell and Stambaugh (1991) Cecchetti, Lam and Mark (1990, 1993) and Abel (1994) have modeled consumption growth as a Markov switching model by which the conditional moments vary through time according to a stationary discrete Markov process. We find this underlying structure very appealing for asset pricing for two reasons. First, it
appears that even small variations in the conditional moments of consumption growth are able to have a significant effect on certain asset pricing moments. Using either a conditionally heteroscedastic process or a regime-switching process for consumption growth, we find that the variation in the Sharpe ratio might be predicted solely by variations in the conditional moments, consistent with Kandell and Stambaugh’s (1990) finding. However, unlike Lettau, Ludvigson and Wachter’s (2007) we find that the variation in conditional moments have little effect on price-dividend ratios. A second reason in favor of Markov switching is that, using such a structure we can disentangle the realization of risk from its future conditional moments, a feature that might help explain certain facts. For example, we find that a model with external habit in the form of Abel (1990, 1999) 2, can predict substantial variation in the price-dividend ratio that is unrelated to the variation in the future consumption growth. At the same time, the price-dividend ratio might have some limited predictive power in forecasting future excess returns.

Heterogeneity in risk-aversion has been studied by several papers but in either more restrictive settings or with different analytical or numerical results. Dumas (1989) considers a two agent economy where one of them has logarithmic utility, and relies on numerical analysis. Wang (1996) considers also a two agent economy and concentrates on the dynamics of bond prices. Coen-Pirani (2004) focuses on the dynamics of wealth distribution among two agents with Epstein and Zin preferences. Bhamra and Uppal (2007) show that completing the market in an economy populated with heterogeneous agents might increase the stock price volatility substantially. Kogan, Makarov and Uppal (2007) show that in a two agent economy with borrowing constraints the Sharpe ratio can be high while at the same time having a low risk-free rate. In our paper we consider an arbitrary number of agents with “catching up with the Joneses” preferences where markets are dynamically complete. Using both analytical results as well as computing the exact equilibrium of several economies we find that in the absence of any frictions or incompleteness in the market the effect of heterogeneity is potentially minimal. In effect, representative agent economies can approximate well a certain family of heterogeneous agent economies.

The rest of the paper is structured as follows. In section 2 we describe the heterogeneous agent economy and solve for the competitive equilibrium. In section 3 we consider a representative agent economy that is homeomorphic in its pricing implications with the heterogeneous agent economy of section 2. We derive an expression for the stochastic risk aversion of the representative agent and analyze its properties. In section 4 using the representative agent results, we explain and analyze the effect of agent heterogeneity on asset pricing dynamics. A quantitative

2Habit formation preferences have been extensively explored in the literature in various forms. Significant contributions include Gali (1994) Ryder and Heal (1973), Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991) and Hindy, Huang and Zhu (1997).
analysis of the effects of heterogeneity and varying conditional moments of consumption growth is carried out in section 5. In section 6, we alternatively consider a four-state Markov switching model for consumption growth and compute the equilibrium. We conclude in section 7.

2 The Model

We construct an infinite horizon endowment economy in discrete time. Uncertainty is driven by some exogenous state that follows a time homogeneous Markov process. The exogenous state is perfectly observable by all agents in the economy. Financial markets are dynamically complete, in the sense that the equilibrium asset structure spans the one period uncertainty at every possible state of nature. There is a single perishable good, and agents exhibit power utility preferences with external habit, in the style of “catching up with the Joneses” of Abel (1990). We present two versions of the model: In the first version the economy is populated by a number of different types of agents who can have different coefficients of relative risk aversion; in the second version, we replace the heterogeneous agents with a representative agent with a stochastic coefficient of relative risk aversion (as in Campbell and Cochrane 1999); we then derive an expression for the stochastic risk aversion coefficient of the representative agent, that makes the two economies equivalent.

As in Chan and Kogan (2002), the catching up with the Joneses preferences are not only attractive from an economics point of view (there are some influential papers that assume this type of preferences, namely Campbell and Cochrane 1999), but they also yield a stationary equilibrium such that the wealth distribution follows a time-homogeneous Markov process. Otherwise, with standard power utility preferences, in the limit, wealth would be eventually accumulated by the less risk-averse agent.

2.1 Aggregate Uncertainty

The state variable of our model is the growth of the economy $G$. As it is customary, we model the dynamics of the logarithm of the growth process, that we denote by $g_t = \log(G_t)$. In particular, we assume that the log of the growth process satisfies the following discrete-time dynamics,

$$g_{t+1} = \mu(g_t) + \sigma(g_t)\epsilon_{t+1}, \quad t \geq 0. \quad (1)$$

where $\epsilon_t \sim i.i.d. N(0,1)$. The conditional mean is a weighted average between the realized consumption growth and the constant value $\bar{g}$,

$$\mu(g_t) = \phi_1 g_t + (1 - \phi_1)\bar{g}, \quad \forall t \geq 0,$$
where \( \phi_1 \in [0, 1] \). Similarly the conditional volatility is given by,

\[
\sigma(g_t) = \phi_2 (\bar{g} - g_t) + (1 - \phi_2) \sigma
\]

where \( \phi_2 \in [0, 1] \). The conditional volatility function is chosen so that it decreases with consumption growth. As we will see this will give us a decreasing price of risk with consumption growth. This is due to the fact that in our model the conditional price of risk, as given by the conditional Sharpe ratio, tracks closely the conditional volatility of consumption growth.

Given the above growth dynamics, the unconditional first two central moments are given by \( \bar{g} \), which for that reason we will call long-term mean, and,

\[
\sigma_g = \frac{(1 - \phi_2) \sigma}{\sqrt{1 - \phi_1^2 - \phi_2^2}}, \tag{2}
\]

respectively. \( \phi_1 \) is the first order autocorrelation of consumption growth.

### 2.2 Financial Markets

We assume that financial markets are dynamically complete in the sense that at any point in time the equilibrium asset structure locally spans the one period uncertainty. Then, to keep the notation as simple as possible and without loss of generality, we assume that there is a single productive asset that yields the single consumption good of the economy. The process \( p_t \) represents the “price” of the consumption good (or pricing kernel), so that the price of the single productive asset, that we denote by \( P^m \) (the “market portfolio”) is the present value of future dividends/endowment priced at \( p \). The one period risk free rate is derived from the price of a claim to a unit of consumption next period, denoted by \( P^f \).

### 2.3 Heterogeneous Agents

There is a set of infinitely lived agents given by \( I = \{1, \ldots, I\} \). An agent \( i \)'s preferences are represented by,

\[
U_i(c^i, x) = \mathbb{E}_0 \sum_{t \geq 0} \delta^t u^i(c^i_t, x_t), \tag{3}
\]

where \( \delta \in (0, 1) \) is the common subjective discount factor and \( c^i_t \) is the consumption of agent \( i \) at time \( t \). \( x_t \) is the external habit common to all agents. Without loss of generality, we choose the external habit to be the aggregate consumption in the previous period, \( x_t = e_{t-1} \). The
running utility is drawn from the “catching up with the Joneses” literature and is given by,

\[ u^i(c, x) = \frac{c^{1-\gamma_i} x^{\gamma_i-\rho} - 1}{1 - \gamma_i}, \quad \forall i \in \mathcal{I}. \]

\( \gamma_i \) denotes agent \( i \)'s coefficient of relative risk aversion, while \( \tau_i \), the inverse of \( \gamma_i \), is the relative risk tolerance. The parameter \( \rho \) is common to all agents and determines the relative effect that the external habit has on the marginal utility of each agent. The derivative of the marginal utility of consumption with respect to the external habit is given by,

\[ \frac{\partial u^i(c, x)}{\partial x} = (\gamma_i - \rho)c^{-\gamma_i} x^{\gamma_i-\rho-1}. \]

Since we would like to have a negative externality for all agents, we impose the restriction that \( \rho \leq \min_{i \in \mathcal{I}} \gamma_i \). With a negative externality, an increase in the level of habit will increase the value that each agent places on consumption. We also note that the smaller the habit parameter is, the bigger is the effect of the habit on the marginal utilities. Hence, a smaller \( \rho \) implies a higher variation in the marginal utilities.

### 2.4 Financial Equilibrium

Preferences and consumption growth dynamics were chosen so that the financial equilibrium is stationary in the exogenous state. This means that the consumption allocations as proportions of the aggregate endowment are functions of the state, which in the model we are considering is the realized growth \( g \) of equation (1). This also implies that both the risk-free rate and the market price-dividend ratio depend only on the state.

At any time \( t \), each agent holds a positive proportion of the aggregate wealth which is denoted by \( \theta_i^t \). Since we have complete markets, the budget constraint of each agent can be expressed as a single intertemporal budget constraint. At the initial period the intertemporal budget constraint for an agent \( i \) is given by,

\[ \mathbb{E}_0 \sum_{t \geq 0} \delta^t p_t c_i^t \leq \theta_i^0 p_0 P_0^m, \]

where \( P_0^m \) is the price of the market security at the initial period.

Every period agents choose their consumption and investment allocations, given their wealth levels and the consumption price process \( (p_s, s \geq t) \). In equilibrium, the marginal utilities of consumption of all agents are collinear and are given by,

\[ \left( \frac{c_i^t}{c_{i-1}^t} \right)^{-\gamma_i} e_{i-1}^{\rho_t} = \lambda_t p_t, \quad \forall (i, t) \in \mathcal{I} \times [0, 1, \ldots, \infty], \]
where $\tilde{\lambda}_i$ is the Lagrange multiplier of the intertemporal budget constraint of agent $i$. Denote the consumption proportion of agent $i$ at time $t$ by $\alpha_i^t = c_i^t / e_t$. Define also,

$$Z_t = \frac{p_t}{e_t^{\rho}},$$

(5)

which can be interpreted as a normalized and stationary pricing kernel. From now on we will refer to it simply as “the pricing kernel.” Rearranging (4) we get,

$$\alpha_i^t G_t = (\tilde{\lambda}_i Z_t)^{-1/\gamma_i} = \lambda_i Z_t^{-1/\gamma_i},$$

(6)

with $\lambda_i = \tilde{\lambda}_i^{-1/\gamma_i}$. Market clearing in every period implies that individual consumption proportions add up to 1. From market clearing and (6), $Z = Z(G)$ is determined from

$$\sum_{i \in I} \lambda_i Z^{-1/\gamma_i} = G.$$  

(7)

Consumption allocations $\alpha^t$ are also stationary in the exogenous state $G$ and are determined from (6) and (7). The equilibrium values of the Lagrange multipliers $\lambda_i$ depend on the initial wealth allocation $(\theta_i^0, i \in I)$. Additionally, we normalize the pricing kernel by choosing $\{\lambda_i, i \in I\}$ be such that $Z(\bar{G}) = 1$ where $\bar{G}$ is the unconditional expected growth (or average state).

A financial equilibrium is given by a pricing kernel, $\{Z(G); Z(\bar{G}) = 1\}$ and a set of consumption allocation functions or ratios of the aggregate endowment, $\{\alpha^t(G)\}$, so that (6) and (7) hold for every $G$, and the intertemporal budget constraints of all agents at the initial time are satisfied. The equilibrium pricing kernel is computed from the definition of $Z$ in (5).

At equilibrium the marginal rates of substitution between any two states are equalized across agents and are described by the following relationship,

$$\left(\frac{\alpha^t(G) G}{\alpha^t(G) \bar{G}}\right)^{-\gamma_i} = Z(G), \quad \forall G.$$  

(8)

Equation (8) will be used to derive the representative agent preferences that give an equivalent economy.

### 3 The Representative Agent Equivalent Economy

In this section we construct a representative agent economy with state-dependent risk-aversion parameter that is equivalent to the heterogeneous agent economy. An equivalent economy is one that has the same aggregate endowment process, financial market structure and exhibits
the same price kernel process. We will derive an expression for the stochastic risk-aversion parameter of the representative agent and study its properties.

3.1 Preferences and Equilibrium

As it is standard, we will re-express the previous results in terms of the log of the state variable \( g, g = \log G \); this will simplify the presentation of our results. In the representative agent economy, the representative agent optimally consumes the aggregate endowment \((e_t, t \geq 0)\) and has the following preferences,

\[
U_r(c, x) = E_0 \sum_{t \geq 0} \delta^t c_t^{1-\gamma (g_t)} x_t^{\gamma (g_t) - \rho} - 1,
\]

where the external habit \( x_t = e_{t-1} \tilde{G} \) for all \( t \geq 0 \), and \( \tilde{G} \) is the long-run average growth. The external habit of the representative agent is slightly different from the habit of the individual investors in the heterogeneous agent economy. As it will be clear later, this habit makes the computation of stochastic risk-aversion of the representative agent (with same aggregate endowment and equilibrium pricing kernel) easier. As in the heterogeneous agents economy, we require a negative externality from the habit, and for that we need the habit parameter to be always less than the risk-aversion parameter, i.e. \( \rho \leq \min_g \gamma (g) \).

Given the consumption price process \((p_t, t \geq 0)\) the representative agent maximizes the previous utility subject to the intertemporal budget constraint. The standard optimality condition is given by,

\[
\left( \frac{G_t}{G} \right)^{-\gamma (g_t)} \tilde{G}^{-\rho} = \lambda_r Z_t, \quad \forall t \geq 0. \tag{9}
\]

where \( \lambda_r \) is the Lagrange multiplier of the intertemporal budget constraint and \( Z_t \) as defined in (5). The equilibrium is trivial and since the left hand side of (9) is state dependent, so must be \( Z_t \). We denote the equilibrium pricing kernel (as opposed to any arbitrary pricing kernel) as \( \{ Z^r (g), Z^r (\bar{g}) = 1 \} \). Using (9) we derive the following,

\[
\left( \frac{G}{G} \right)^{-\gamma (g)} = Z^r (g). \tag{10}
\]

For the representative agent economy to be equivalent to the heterogeneous agent economy, it suffices to have the same equilibrium pricing kernel processes in the two economies. Hence,
we require that $Z(g) = Z^r(g)$ for all $g$. Using (10) we get,

$$\left( \frac{G}{\bar{G}} \right)^{-\gamma(g)} = Z(g), \quad \forall g.$$  \hspace{1cm} (11)

In accord with our previous notation, we make $z = \log Z$. Given condition (11), the representative agent economy is equivalent to the heterogeneous agent economy if and only if the stochastic risk aversion coefficient of the representative agent is given by,

$$\gamma(g) = \begin{cases} 
-\frac{z(g)}{\bar{g} - \tilde{g}}, & G(g) \neq \bar{G} \\
\frac{1}{\bar{g}}, & G(g) = \bar{G}
\end{cases},$$  \hspace{1cm} (12)

where $\tilde{g}$ is a measure of the average risk-aversion in the economy during “normal” times, and will be defined later on. We will see that $\tilde{g}$ is chosen so that the above definition of $\gamma(\cdot)$ gives a continuous function.

### 3.2 Characterization of $\gamma(\cdot)$

In equilibrium, the agents must consume the totality of the endowment process. Since the proportion of the aggregate endowment each agent consumes is given by (6) it is convenient to interpret that equilibrium consumption share as a probability. Therefore, we say that the probability distribution of agent types (characterized by their risk aversion parameters ) for any given growth $g$ is,

$$\mathbb{P}_I(i|g) = \exp(-g)\lambda_i Z(g)^{-\frac{1}{\tau}}, \quad \forall i \in I, g.$$  \hspace{1cm} (13)

Note that by (4) and the definition of $\lambda_i$, in equilibrium the probability measure of an agent $i$ is its consumption allocation $\alpha^i(g)$. Note also that this expression characterizes the distribution of the total endowment for a set of $\lambda_i$’s. Each $\lambda_i$ will depend on the distribution of wealth, which we discuss later. Let $\tau$ denote the reciprocal of the coefficient of relative risk aversion which, also known as the coefficient of relative risk tolerance. Then define the average risk aversion across agents for any growth $g$ according to,

$$\mu_I(\gamma|g) = \mu_I(\tau|g)^{-1},$$  \hspace{1cm} (14)

or

$$\mu_I(\tau|g) = \mathbb{E}_I[\tau_i|g].$$  \hspace{1cm} (15)
\( \tilde{\gamma} \) therefore denotes the average risk aversion at the average growth, \( \mu(T|\bar{g}) \), with \( \bar{\tau} = 1/\tilde{\gamma} \). The standard deviation of risk-tolerance for any \( g \) is given by,

\[
\sigma_T(\tau|g) = \sqrt{V_T(\bar{\tau}|g)}.
\] (16)

For the average growth, this is denoted by \( \sigma_\tau = \sigma_T(\tau|\bar{g}) \).

In order to examine the properties of the function \( \gamma(\cdot) \), we first need to study the properties of the pricing kernel \( z(\cdot) \). From equation (7) we can establish the following,

\[
\lim_{g \to +\infty} z(g) = -\infty, \quad (z1)
\]
\[
\lim_{g \to -\infty} z(g) = +\infty, \quad (z2)
\]
\[
z'(g) = -\mu_T(\gamma|g), \quad (z3)
\]
\[
z''(g) = \frac{\sigma_T(\tau|g)^2}{\mu_T(\tau|g)^3}. \quad (z4)
\]

The prime and double prime denote the first and second derivative. Therefore, \( z \) is monotonically decreasing and convex. It is interesting to note that the convexity of the function \( z \) is increasing with the level of heterogeneity in the economy as it is given by the variance of the coefficients of risk tolerance across agents. In the special case of homogeneity in the economy \( z(g) \propto -\gamma g \) and has a constant first derivative. Furthermore, from the fact that

\[
\lim_{g \to +\infty} \frac{P_T(i|g)}{P_T(j|g)} = 0
\]

when \( \gamma_i > \gamma_j \), we have that \( \sigma_T(\tau|g) \) tends to zero as \( g \) tends to infinity and also,

\[
\lim_{g \to +\infty} z'(g) = -\min_{i \in I} \gamma_i, \quad (z5)
\]
\[
\lim_{g \to +\infty} z''(g) = 0. \quad (z6)
\]

Similarly we have that

\[
\lim_{g \to -\infty} z'(g) = -\max_{i \in I} \gamma_i, \quad (z7)
\]
\[
\lim_{g \to -\infty} z''(g) = 0. \quad (z8)
\]

Let \( \Delta g = g - \bar{g} \), then the coefficient of risk aversion of the representative agent is given by \( \gamma(g) = -z(g)/\Delta g \) for all \( g \neq \bar{g} \) and equal to \( \tilde{\gamma} \) when growth is equal to that of the steady state.
In order for $\gamma(\cdot)$ to be continuous we need that,

$$\lim_{g \to \bar{g}} \gamma(g) = -z'(\bar{g}) = \bar{\gamma},$$

which is true from property (z3) and a simple application of l'Hôpital's rule. Similarly, from properties (z1) and (z2) we have that,

$$\lim_{g \to +\infty} \gamma(g) = \min_{i \in I} \gamma_i,$$

$$\lim_{g \to -\infty} \gamma(g) = \max_{i \in I} \gamma_i.$$

Figure 1 shows a graph of function $z$. The graph also shows that $\gamma(g)$ for some given $g$ is the absolute value of the slope of the line segment connecting the points $z(g)$ and $z(\bar{g})$. We can also observe, due to the convexity of function $z$, that the risk-aversion of the representative agent is always less than the average risk aversion in the economy as defined by $\mu_I(\gamma|g)$ when $g < \bar{g}$, and greater when $g > \bar{g}$. In addition, the first derivative of the risk-aversion function w.r.t. $g$ is negative since,

$$\gamma'(g) = -\frac{1}{\Delta g} [\gamma(g) - \mu_I(\gamma|g)].$$

From the expression of the first derivative we can also deduce that the first derivative tends to zero from below when $g$ tends to either plus or minus infinity. This is because the entire wealth in the economy tends to be held by the least or more risk-averse agent in the economy.
respectively. Mathematically we have that,
\[
\lim_{g \to \pm \infty} \gamma'(g) = 0.
\]

Finally, the second derivative of \( \gamma(\cdot) \) is given by,
\[
\gamma''(g) = \frac{1}{\Delta g} \left[ -2\gamma'(g) - z''(g) \right].
\]

The second derivative is negative below some given value and positive otherwise. As \( g \) tends to plus infinity it tends to zero from above and when growth tends to minus infinity then the second derivative tends to zero from below. Figure 2 shows the graph of \( \gamma(\cdot) \).

![Figure 2: Graph of function \( \gamma(g) \).](image)

### 3.3 Computation of \( \gamma(\cdot) \)

In order to compute \( (\gamma(g)) \) we need to first specify the entire distribution of heterogeneous agents and then solve for the equilibrium. However, we would like to derive an expression for \( \gamma(g) \) using only the dynamics of the growth rate and the mean and dispersion of the distribution of agents at the steady state. Under a normality assumption we can derive the risk aversion of the representative agent in closed form, otherwise we can approximate it closely using Taylor expansion.
Using the probability measure (13) at the steady state we can express (7) in the following way;

\[ \mathbb{E}_\mathcal{I} \left[ e^{-\tau_i z(g)} \left| \bar{g} \right. \right] = 1. \]  

(17)

If \( \tau_i \) is normally distributed at the steady state then we have that,

\[ -\bar{\tau} z(g) + \frac{1}{2} \sigma^2 \bar{\tau} z(g)^2 - \Delta g = 0. \]

which is a quadratic function in \( z(g) \). With some rearrangement of terms we express the above as a quadratic equation in \( -\bar{\tau} z(g) / \Delta g \) which has the following positive solution,

\[ h_0(g) = \sqrt{1 + 2(\sigma \bar{\gamma} \Delta g - \frac{1}{\Delta g (\sigma \bar{\gamma})^2}} \], \quad \forall g \geq \bar{g} - \frac{1}{2(\sigma \bar{\gamma})^2}, \; g \neq \bar{g}. \]  

(18)

The positive solution was chosen because of the properties of \( \gamma(g) \) that we have derived earlier. The domain over which the above expression can be applied is restrictive since the expression in the square root needs to be positive. In order to be able to extend such an expression over a bigger domain we need to compute \( z(g) \) in terms of the distribution of agents at a smaller \( g \). However, if the distribution of the risk-tolerance of agents is normal at the steady state then the distribution is skewed in any other state. In particular for \( g < \bar{g} \) the distribution is positively skewed and negatively skewed for growth levels above the average growth. For any growth level \( g_0 \) we have that,

\[ \mathbb{E}_\mathcal{I} \left[ e^{-\tau_i[z(g)-z(g_0)]-(g-g_0)} \left| g_0 \right. \right] = 1. \]

We use the Taylor expansion of \( e^x \) around zero\(^3\) and omit the third and higher order terms. After some rearrangement of terms we get,

\[ (g - g_0 - 2) + 2(1 - g + g_0) h(g - g_0 | g_0) + (g - g_0) (\eta(g_0) + 1) h(g - g_0 | g_0)^2 \approx 0, \]

where the error term is of third order in the deviations of \( g \) from growth \( g_0 \). We have used the following definitions,

\[ h(g - g_0 | g_0) = -\frac{z(g) - z(g_0)}{g - g_0} \mu_\mathcal{I} | g_0. \]

\(^3\)The Taylor expansion up to the second order term is,

\[ e^x = 1 + x + \frac{1}{2} x^2 + O(x^3). \]
and

$$\eta(g) = \frac{\sigma_I(\tau|g)^2}{\mu_I(\tau|g)^2}. \quad (19)$$

When we assume exact equality we get a quadratic equation in \(h(g - g_0|g_0)\) which has the following positive solution,

$$h_1(\Delta|g_0) = \frac{\Delta - 1 + \sqrt{1 + \eta(g_0)\Delta(2 - \Delta)}}{\Delta (\eta(g_0) + 1)}, \quad \forall \Delta \in D(g_0), \quad (20)$$

where

$$D(g) = \left[1 - \sqrt{1 + \eta(g)}^{-1}, 0\right].$$

Note that \(h_1(\Delta|\bar{g}) \simeq h_0(\Delta)\). For the first two derivatives of \(h_1\) we have that \(h'_1(\Delta|g) < 0\) and \(h''_1(\Delta|g) > 0\) for all \(\Delta \in D(g)\) and for all \(g \in G\).

Using functions \(h_0\) and \(h_1\) we can now define the stochastic risk aversion function of the representative agent. We use \(h_0\) because we assume that the distribution of the risk-tolerances across agents is normally distributed at the steady state. Alternatively we can use only function \(h_1\) for the entire domain. Define therefore,

$$\tilde{\gamma}(g) = \begin{cases} \gamma h_0(g) & \text{if } g \geq g_0 \text{ and } g \neq \bar{g}, \\ \bar{\gamma} & \text{if } g = \bar{g}, \\ \tilde{\gamma} h_0(g_0) + h_1(g - g_0|g_0)\mu_I(\gamma|g_0)\frac{g - g_0}{\Delta g} & \text{when } g - g_0 \in D(g_0). \end{cases} \quad (21)$$

Note that \(\tilde{\gamma}(\cdot)\) is a continuous function\(^4\). \(g_0\) must lie between the minimum growth level for which function \(h_0\) can be used and the average growth level,

$$\bar{g} < g_0 < \bar{g} - \frac{1}{2\bar{\eta}}.$$

where \(\bar{\eta} = \eta(\bar{g})\). In order to compute \(\tilde{\gamma}(g)\) for \(g < g_0\) we need to compute the first two moments

\(^4\)The definition of \(\tilde{\gamma}(\cdot)\) as given by (21) gives us a continuous function in \(g\) since we have that,

$$\lim_{\Delta g \to 0} h_0(\Delta g) = 1,$$

and also,

$$\lim_{g \to g_0} h_1(g - g_0|g_0) = 1.$$
of the distribution of agent risk-tolerances. Using $h_0$ we have that,

$$z(g) = \frac{1 - \sqrt{1 + 2\eta \Delta g}}{\eta}, \quad \text{for } g \geq g_0 \text{ and } g \neq \bar{g}. \quad (22)$$

Then from the properties $(z3)$ and $(z4)$ and the first two derivatives of $z(g)$ as expressed in (22), we get,

$$\mu_I(\gamma|g) = \frac{\bar{\gamma}}{\sqrt{1 + 2\eta \Delta g}}, \quad (23)$$

$$\eta(g) = \frac{\eta}{1 + 2\eta \Delta g}. \quad (24)$$

Naturally we see that the average risk aversion in the economy decreases with growth. Surprisingly, though, we obtain that the volatility stays constant as from (23) and (24) we derive that $\sigma_I(\tau|g) = \sigma_\tau$. This result as well as the behavior of these functions when growth tends to infinity come from the assumption of normality. However, when the risk-aversion of the agents is bounded above and below, properties $(z5)$ and $(z6)$ show that volatility tends to zero while the mean tends to the minimum risk-aversion coefficient in the economy. Despite these discrepancies (23) and (24) are considered to be good descriptions of the behavior of these quantities at the growth levels that we consider.

Under the normality assumption we have that $\tilde{\gamma}(g) = \gamma(g)$ for $g \geq g_0$ while for lower levels of growth the error between the approximated and the exact risk aversion function is of third order in the deviations of the log aggregate consumption growth from $g_0$;

$$\gamma(g) = \tilde{\gamma}(g) + O\left((g - g_0)^3\right).$$

The level of this approximation has turned out to be extremely good in practice. For the economies we have solved for, the maximum percentage error is negligible. The error also depends on the level of heterogeneity in the economy. When there is no heterogeneity in the economy, i.e. when $\sigma_\tau = 0$, then for all $g \in G$ we have $\tilde{\gamma}(g) = \gamma(g) = \bar{\gamma}$.

### 3.4 The variation of $\gamma(\cdot)$

Function $h(\Delta|g)$, which is expressed by $h_0$ and $h_1$, is the ratio of the risk-aversion coefficient of the representative agent when the growth deviates from some $g$ by $\Delta$, to the risk-aversion coefficient at $g$. In particular,

$$h(\Delta|g) = \frac{\gamma(g + \Delta)}{\gamma(g)}.$$
It has been constructed in this way so that we can study the variation in the risk-aversion coefficient of the representative agent without assuming an average risk aversion at the steady state in the economy. However, we do need to assume some values for $\sqrt{\bar{\eta}} = \sigma_{\tau \bar{\gamma}}$.

Figure (3) plots function $h$ for three different values of $\sigma_{\tau \bar{\gamma}}$, 0, 1 and 2. Consider for example a case where the average risk-aversion is 10 and therefore a standard deviation for the risk-tolerance up to 0.2. Note that the risk-tolerance is a number between 0 and 1 and therefore a standard deviation of 0.2 is possibly unrealistically high. We plot the function for a range of values for $\Delta$ from $-10\%$ to $+10\%$. The first thing we note is that the possible variation in the coefficient of risk-aversion for the representative agent is small. For example when the ratio of the standard deviation of the risk tolerance to the average risk tolerance is one, the increase in the risk aversion coefficient goes only up to 5%. For a ratio of two this increase goes up to 40% which is still not a particular high number. This is a first indication that the effect of risk-aversion heterogeneity on asset prices is potentially minimal.

In an economy with rational investors the risk-attitude of the representative agent can be time-varying due to two reasons. The first one, which is what drives the risk-aversion coefficient in this model, is the evolution of the cross-sectional wealth distribution. Unless the variation in the state vector that determines the cross sectional wealth distribution is very high, the wealth reallocation across time cannot potentially have a quantitatively very important effect on asset prices. In figure (3) we plot function $h$ for deviations of the state variable up to 10%. In our model the state variable is the aggregate consumption growth which has an annual standard deviation of around 2 to 3%. The second possible reason for time-variation in the risk-
aversion coefficient of the representative agent is time-variation in the individual risk-aversion coefficients. In order to have a high variation in the risk-preferences of the representative agent then we would also need the individual preferences to be moving together and towards a specific direction, i.e. decreasing with consumption growth in our model. Otherwise the distribution of wealth across the various risk-aversion levels will not be varying enough to justify a high variation in the risk-aversion of the representative agent. In particular the entire distribution needs to be moving up and down with the state variable.

We also notice that the effect of heterogeneity is non-symmetrical for negative and positive deviations from the steady state. Due to the fact that the second derivative of the \( h \) function is positive, the increase in the risk aversion coefficient of the representative agent when the deviations are negative, is greater than the decrease for positive deviations.

4 Asset Pricing Dynamics

In this section we examine the asset pricing implications of the model described so far for a number of financial variables that have been at the center of the asset pricing literature. In particular, the equity premium, the market price of risk as given by the Sharpe ratio, the volatility of the market returns and the price dividend ratio of the market. Following the asset pricing literature we assume that the market security is the claim to the aggregate endowment.

4.1 The Stochastic Discount Factor

The fundamental price of an asset is the expected value of the discounted future dividends. Let \( M_{t+1} \) denote the one-period stochastic discount factor between periods \( t \) and \( t + 1 \). Since \( p_t \) denotes the price of a unit of consumption in period \( t \), the stochastic discount factor is given by,

\[
M_{t+1} = \frac{\delta p_{t+1}}{p_t}, \quad \forall t > 0.
\]

In equilibrium, the price of consumption is given by \( p_t = e^{-\alpha_t}Z(g_t) \). Substituting in the expression for the stochastic discount factor, we get that it is a function of the current and future state of nature, i.e. \( M_{t+1} = M(g_t, g_{t+1}) \), given by,

\[
M(g, g') = \delta \frac{Z(g')}{Z(g)} G^{-\rho}, \quad \forall (g, g').
\]  

(25)

In terms of the risk aversion of the representative agent,

\[
M_{t+1} = \delta \left( \frac{G_{t+1}}{G} \right)^{-\gamma(g_{t+1})} \left( \frac{G}{G_t} \right)^{-\gamma(g_t)} G_t^{-\rho}.
\]
Let $m_t = \log(M_t)$, and recall that $\Delta g = g - \bar{g}$, then,

$$m_{t+1} = \log(\delta) - \gamma(g_{t+1})\Delta g_{t+1} + \gamma(g_t)\Delta g_t - \rho g_t.$$  \hfill (26)

When agents are homogeneous $\gamma_t$ is constant and equal to the common risk aversion parameter $\gamma$, and standard preferences are obtained when $\rho = \gamma$.

As we have already discussed, the model of Campbell and Cochrane (1999) has often been perceived as a model of a representative agent with countercyclical variation in risk aversion. Chan and Kogan (2002) argue that such a countercyclical variation in the risk-aversion in the economy can be produced endogenously in an economy populated with agents heterogeneous in their risk-aversion parameters. We show that result in our derivation of a representative agent economy equivalent in its pricing implications to a heterogeneous agent economy. In particular, the risk-aversion of the representative agent is negatively related to the growth in the economy, through the fact that $\gamma'(g) < 0$. However, as we showed in section 3.5, the possible variation of the parameter of risk-aversion of the representative agent is relatively modest for realistic parameter values and it seems difficult to argue that it will be able to explain the variation of the risk-premia, price of risk, and other empirical facts.

In addition, the state of an economy is not only determined by growth but mostly by the potentials of the economy. For example an economy when recovering from contraction might have the same growth as during another period when the economy is close to reaching its peak. Clearly, the economy is at different states at those two time periods and the risk-premia can differ only due to the difference in the conditional distributions of future growth. In fact we will see that quantitatively it seems much more plausible that variations in the conditional probabilities are responsible for variations in the risk-premia rather than variations in the risk-aversion in the economy.

One way to try to examine our claim is to try to relate the conditional distribution of the stochastic discount factor of our model, as given in (26), to the one of Campbell and Cochrane. In their model the logarithm of the stochastic discount factor is given by,

$$m_{t+1}^{cc} = \log(\delta) - \gamma g + \gamma(1 - \varphi)\Delta s^{cc}_t - \gamma [1 + \lambda(s^{cc}_t)] \Delta g_{t+1},$$  \hfill (27)

where $\Delta g_{t+1} \sim N(0, \sigma^2)$ and $\sigma$ is the long-run standard deviation of consumption growth. $s^{cc}$ is the logarithm of the consumption surplus ratio, relative to an external habit, and follows a mean-reverting process. It is assumed to move along with the business cycle. A low surplus ratio occurs when the economy is at a trough. $\lambda(s^{cc})$ is a sensitivity function that affects the conditional volatility of the state of the economy. It moves countercyclically with the surplus ratio and it is responsible for many of the asset pricing predictions in their model.
\( \Delta s^c_t = s^c_t - \bar{s}^c \) where \( \bar{s}^c \) is the steady state of their economy. Comparing (26) with (27) we can observe many differences. Firstly, we see that in Campbell and Cochrane the conditional distribution of \( m_t \) is normal whereas in our model it is not since the consumption growth interacts with a varying risk-aversion function. In fact if the conditional distribution of growth is normal then the conditional distribution of the logarithm of the stochastic discount factor is positively skewed. This is due to the fact that the risk-aversion coefficient of the representative agent decreases with growth.

The most important differences, however, lie in the variations of the states and the conditional volatilities of the log stochastic discount factor. In Campbell and Cochrane the deviations from the steady state can be much bigger than the deviations in growth that we consider. In fact \( \Delta s^c \) ranges from at least \(-400\%\) up to \(+50\%\) while the deviations in consumption growth cannot be expected to exceed an absolute value of 10\% either way. The main issue is whether we believe that there is such variation in the endogenously determined cross-sectional distribution of wealth. I think it is hard to accept that economic agents are faced with so large macroeconomic risks that their different risk-attitudes would lead to big changes in the cross-sectional distribution of wealth.

The second important difference is the conditional volatility of \( m_{t+1} \). In Campbell and Cochrane this is given by \( \gamma \sigma [1 + \lambda (s^c_t)] \) and the sensitivity function ranges from 0 to 50. The sensitivity function firstly introduces variability into the conditional volatility and secondly it increases considerably when the surplus ratio decreases. If we accept the interpretation of stochastic risk-aversion this implies that when the economy is at a trough of a business cycle the risk-aversion increases up to fiftyfold. However, we have seen that in a heterogeneous agent economy like ours, we cannot expect an increase of more than 50\%. The variation in the conditional volatility of the stochastic discount factor in our economy of heterogeneous agents comes from the convexity of the stationary part of the pricing kernel and the variation in the conditional moments of growth. The conditional variance of \( M \) is given by,

\[
\mathbb{V} [m(g, g')|g] = \mathbb{V} [z(g')|g] = \mathbb{V} [-\gamma(g')\Delta g'|g],
\]

where \( g' \) is next period’s value. \( \mathbb{V}(\cdot|g) \) denotes the conditional variance when the current state of the economy is \( g \). The first thing we note is that the conditional volatility of the stochastic discount factor varies if the conditional distribution of \( g \) varies, regardless of whether there is heterogeneity in the economy or not. Chan and Kogan (2002) study a special case where the conditional variance of \( g \) is constant while the conditional mean varies. In their case homogeneity does not produce varying conditional volatility because \( z(g) = -\gamma \Delta g \), and therefore the conditional volatility of \( z(g) \) is just \( \gamma \sigma \) where \( \sigma \) is the constant conditional volatility of consumption growth. Once heterogeneity is introduced \( z(g) \) becomes a convex function of growth.
and therefore its conditional volatility depends on the location of the conditional distribution. To see this let us return back to our economy and approximate the conditional volatility of $m$. Using a second order Taylor expansion of $z(g)$ around the conditional mean of growth\textsuperscript{5}, the conditional variance is approximately given by,

$$\nabla \left[ m(g, g') | g \right] \simeq \frac{\mu_2}{2} (\gamma | \mu(g) |) \sigma (g)^2 \left[ 1 + \frac{1}{2} \eta (\mu(g))^2 \sigma (g)^2 \right],$$

where $\mu (g)$ and $\sigma (g)$ are the conditional mean and volatility of consumption growth. The above expression is derived from properties $(z3)$ and $(z4)$, as well as the definition of $\eta$ as given in (19). From (23) and (24) we get,

$$\sqrt{\nabla \left[ m(g, g') | g \right]} \simeq \gamma \sigma (g) \frac{\sqrt{(1 + 2\eta \Delta \mu (g))^2 + 0.5 \eta^2 \sigma (g)^2}}{(1 + 2 \eta \Delta \mu (g))^{3/2}}.$$ 

We see therefore that when there is homogeneity in the economy $\bar{\eta}$ is zero and the conditional volatility of $m$ becomes simply $\gamma \sigma (g)$. If further, the conditional volatility of consumption growth is constant then the conditional volatility of $m$ is constant as well. When there is heterogeneity in the economy then we also need at least the conditional mean of consumption growth to be varying. However, in order to get significant variation in the conditional volatility of the pricing kernel we need both significant variation in the conditional mean as well as a high level of heterogeneity. Let,

$$\kappa (g) = \frac{\sqrt{(1 + 2\eta \Delta \mu (g))^2 + 0.5 \eta^2 \sigma (g)^2}}{(1 + 2 \eta \Delta \mu (g))^{3/2}},$$

denote the amplification factor of the conditional volatility of $m$ due to risk-preference heterogeneity in the economy. Figure 4 plots $\kappa (g)$ for three different values of $\sqrt{\bar{\eta}}$. We see that unless there is an unrealistically high level of heterogeneity as well as great variation in the conditional mean of consumption growth, the variation in $\kappa (g)$ is minimal. For example, for the high heterogeneity level of $\sqrt{\bar{\eta}} = 1$ and for a high variation of $\mu (g)$ from $-5\%$ to $+5\%$, the amplification factor ranges from 1.06 to 0.96. Therefore, the variability in the conditional variance of the pricing kernel coming from the heterogeneity of risk-attitudes is so small that

\textsuperscript{5}The Taylor expansion of $z(g')$ around the conditional mean $\mu (g)$ is given by,

$$z(g') = z(\mu (g)) + z' (\mu (g))(g' - \mu (g)) + \frac{1}{2} z'' (\mu (g))(g' - \mu (g))^2 + \frac{1}{6} z''' (\mu (g))(g' - \mu (g))^3 + O((g' - \mu (g))^4).$$

The conditional first two moments of $z$ are therefore given by

$$\mathbb{E} \left[ z(g') \right] = z(\mu (g)) + \frac{1}{2} z''(\mu (g)) \sigma (g)^2 + \mathbb{E} \left[ O((g' - \mu (g))^4) \right],$$

$$\nabla \left[ z(g') \right] = z' (\mu (g))^2 \sigma (g)^2 + \frac{1}{2} z'' (\mu (g))^2 \sigma (g)^4 + \nabla \left[ O((g' - \mu (g))^3) \right].$$
not only it can never be expected to play the role of \( [1 + \lambda(s^{cc})] \), but also it is almost negligible and unimportant to be considered as a major factor for the variation in the conditional moments of risky asset returns. The sensitivity factor of Campbell and Cochrane most probably proxies for a varying conditional volatility of the state variable. This said, however, it is hard to imagine a macroeconomic variable that would exhibit such radical variation. Fortunately, as we will show for certain asset pricing features like the variation of the Sharpe ratio, only a small variation is needed while it has almost no effect on the variability of the price-dividend ratio.

4.2 The Risk-Free Rate

The price of a risk free bond that pays a unit of consumption the next period from a given state \( s \), denoted by \( P^f(g) \), is simply given by the conditional expectation of the one-period stochastic discount factor. The continuously compounded risk-free rate is the negative logarithm of the bond price, and after using the expression for \( m \), it is given by,

\[
    r^f(g) = - \log(\delta) + z(g) + pg - \log \mathbb{E} \left[ e^{z(g')} | g \right], \quad \forall s \in S, (29)
\]

where \( g' \) is the next period’s growth rate. For a discrete set \( S \) the risk-free rate for every state \( s \) is computed as in (29) using the transition probabilities and the equilibrium values of \( z \).

Using a second order Taylor expansion and the expressions for \( \mu_T(\gamma | \mu(g)) \) and \( \eta(\mu(g)) \) we get

Figure 4: Graph of function \( \kappa(g) \) for three different values of \( \bar{\eta} \), 0.5, 1 and 2. Volatility of consumption growth \( \sigma \) was chosen to be 4%.
the following approximate risk-free rate,

\[ r_f(g) \approx -\log(\delta) + z(g) - z(\mu(g)) + \rho g - \frac{1}{2} \left( z''(\mu(g)) + \bar{\gamma}^2 \kappa(g)^2 \right) \sigma(g)^2, \]

where \( \kappa(g) \) is as defined before and, as we see in Figure 4, it is larger than one when growth is below the average, and lower than one when growth is above the average. Therefore, precautionary savings, given by the last term of the previous expression, are higher than in the homogeneous agents case for growth levels below average and lower than in the homogenous agents case otherwise. However, the effect is minimal since \( \kappa(g) \) does not vary much.

When there is homogeneity in the economy the risk-free rate gets the following usual form as as in Abel (1990, 1999),

\[ r_f(g) = -\log(\delta) - \bar{\gamma} g + \bar{\gamma} \mu(g) + \rho g - \frac{1}{2} \bar{\gamma}^2 \sigma(g)^2. \]

Note that in the case of homogeneity the approximation of the risk-free rate becomes exact since \( z''(\cdot) \) becomes zero and \( \kappa(g) \) becomes one.

The habit parameter \( \rho \) can potentially have a big effect on the dynamics of the risk-free rate. We recall from our discussion at the end of section 2.3 that the value of \( \rho \) needs to be less or equal to the minimum possible risk-aversion parameter, so that habit formation will imply a negative externality: The smaller it is, the larger is the effect on marginal utility from consumption. Therefore, we see that as the difference between \( \rho \) and the average risk-aversion increases the risk-free rate decreases; in addition, since part of the volatility of the interest rate comes from \( (\bar{\gamma} - \rho) g \), the larger the difference, the higher the volatility. As the habit becomes stronger then a high growth level will lead agents to invest more in order to increase their future consumption. This is turn decreases interest rates. Therefore, in principle, the variation of the risk free rate can be much higher than what is observed in the market. However, a pro-cyclical variation in the conditional mean of future consumption growth will decrease this variation. For example, if the conditional volatility is constant and if the conditional mean is given by,

\[ \mu(g) = \frac{\bar{\gamma} - \rho}{\bar{\gamma}} g \]

the risk-free rate in the homogeneous economy would be constant and only slightly varying in the heterogeneous agent economy. A strong enough habit however, as it will be needed to produce a substantially varying price-dividend ratio, would require a very persistent consumption growth. A small counter-cyclical variation in consumption volatility on the other hand would require a significantly less persistent state variable.
4.3 The Stock Price

The price $P_t$ of an infinitely lived asset is equal to the expected discounted future dividends and satisfies the usual Euler equation,

$$P_t = \mathbb{E}_t [M_{t+1} (P_{t+1} + D_{t+1})],$$  \hspace{1cm} (30)

where $D_t$ is the dividend of the asset in period $t$. In our pure exchange economy, the dividend of the single risky security (the “market portfolio,” whose price we denote by $P^m$) is given by the aggregate endowment, which grows at a rate $G$, the state variable. We denote the price dividend ratio of the risky security as $\hat{P}^m$, $\hat{P}^m = P^m / D = P^m / e$. After straightforward manipulation of (30) and using (25),

$$\hat{P}^m(g) = \mathbb{E}\left[M(g, g')G \left(1 + \hat{P}^m(g')\right)\middle| g\right], \hspace{1cm} \forall(g, g').$$  \hspace{1cm} (31)

When the set of states is discrete and finite, the price dividend ratio function is easily computed once $M(g, g')$ is available. We write equation (31) for all states in matrix format and after some manipulation we can compute the vector of price dividend ratios in terms of the transition matrix, the stochastic discount factor matrix and the growth vector.\(^6\) When the set of states in continuous then we first discretize it and then follow the same procedure.

From (31) we can represent the logarithm of the price-dividend ratio in the following form,

$$\log \hat{P}^m(g) = \log \delta - \rho g - z(g) + \log \mathbb{E}\left[Z(G')G' \left(\hat{P}^m(g') + 1\right)\middle| g\right].$$

If the realized consumption growth $g$ does not depend in a substantial way in the current state,\(^7\) and taking into account $(z3)$, the partial derivative of the log price-dividend ratio w.r.t. $g$ is

\(^6\)The Euler equation for some state $g$ can be written in matrix notation as follows;

$$G'Z\hat{P}(g) = \delta[\pi(g) \circ G'] \left[Z \circ \hat{P} + \pi(g) \circ [Z \circ G]\right],$$

where $\circ$ denotes the element wise vector or matrix multiplication and $Z$, $\hat{P}$ and $G$ denote the vectors for all states. $\pi(g)$ denotes the vector of transition probabilities from state $g$. $X'$ denotes the transpose of some vector $X$. Then we can write the Euler equation for all states as,

$$\text{diag}(G') \left[Z \circ P\right] = \delta[\Pi \circ (G' \otimes 1)] + \delta\Pi [Z \circ G],$$

$\text{diag}(G')$ denotes the diagonal matrix where the $g$ diagonal element is $G'$. $\otimes$ denotes the Kronecker product and $1$ denotes the identity vector. Therefore we have that,

$$Z \circ P = \delta N\Pi [Z \circ G],$$

where

$$N = \left[\text{diag}(G') - \delta[\Pi \circ (G' \otimes 1)]\right]^{-1}.$$  \hspace{1cm} (32)

\(^7\)In our numerical exercises we have verified that such is the case.
given by,
\[
\frac{\partial \log \hat{P}_m(g)}{\partial g} = \mu_I(\gamma|g) - \rho > 0.
\]
The partial derivative is positive due to the fact that the habit parameter is less or equal to the minimum possible risk-aversion parameter. We see that agent heterogeneity adds some more variability to the price-dividend ratio since the difference between the slopes of the heterogeneous and homogeneous economies, given by \( \mu_I(\gamma|g) - \bar{\gamma} \), is positive for \( \Delta g > 0 \) and negative otherwise. However, the additional variation is again expected to be minimal since the variation of the average risk aversion in the heterogeneous agent economy cannot be expected to be significant.

The return on the market is also a function of the current and future state and is given in terms of the price dividend ratios in the following way,
\[
R_m(g, g') = G'' \hat{P}_m(g') + 1 - \hat{P}_m(g'), \quad \forall (g, g').
\]  
(32)

Let \( r_m \) denote the logarithm of the market return. Therefore we have that,
\[
r_m(g, g') = g' + \log \left( 1 + \hat{P}_m(g') \right) - \log \hat{P}_m(g).
\]

From this equation we can conclude that the conditional volatility of market returns is approximately equal to the sum of the conditional volatilities of consumption growth and log price-dividend ratio. In order to get a feeling for how volatile the price-dividend ratio is, we approximate the log price-dividend ratio in some state \( g' \) using (31) and a second order Taylor expansion of \( z \) around the conditional expected consumption growth \( \mu(g) \);
\[
\log \hat{P}_m(g') \simeq \log \delta - \rho \mu(g) - z(\mu(g)) + (\mu_I(\gamma|\mu(g)) - \rho)(g' - \mu(g)) - \frac{\mu_I(\gamma|\mu(g))\eta(\mu(g))}{2}(g' - \mu(g))^2 + \log \mathbb{E} \left[ Z(G''G'' \hat{P}_m(g'' + 1) \bigg| g' \right]
\]
where \( G'' \) denotes growth in two periods ahead. First, we see that the price-dividend ratio is constant if there is no heterogeneity in the economy, the habit parameter is equal to the risk-aversion parameter and the conditional probabilities do not vary.\(^8\) The conditional variance of \( \log \hat{P}_m(g') \) is almost equal to the sum of the variances coming from agent heterogeneity, habit and variation in conditional probabilities. The variance coming from the agent heterogeneity is almost negligible since it is equal to \( \mu_I(\gamma|\mu(g))^2\eta(\mu(g))^2\sigma(g)^4 \). Similarly, as we explained before, the last term in the previous expression does not depend heavily on the

\(^8\)When there is no heterogeneity \( \mu_I(\gamma|\mu(g)) = \bar{\gamma}, \eta(\mu(g)) = 0 \) and \( z(\mu(g)) = -\bar{\gamma}\Delta \mu(g) \). When there is no habit then \( \rho = \bar{\gamma} \).
current state. On the other hand, the conditional variance coming from habit, which is equal to \((\mu_\gamma|\mu(g)) - \alpha)^2\sigma(g)^2\), may potentially be significant if the difference between the habit and the risk-aversion parameter is big. These variances are amplified if the conditional probabilities vary as well, through the variance of the log of the conditional expectation. Hence, the variation in the conditional volatility of returns depends heavily on the variation of the conditional moments of consumption growth and especially of the conditional volatility, while the risk-aversion heterogeneity adds little to this variation.

4.4 The Price of Risk and the Equity Premium

We have already discussed that the empirical literature has found significant evidence that the price of risk varies, and furthermore that it varies along with the business cycle, most probably with a counter-cyclical pattern. The level of variation or whether the conditional excess return is positively or negatively related to the conditional return volatility is debateable.

The price of risk is a key variable in the economy because it summarizes the potentials of the economy and is independent of the fundamentals of the economy in the current state. In this section we show that in order to have a variable price of risk is crucial the existence of variation in the conditional probabilities of the state of the economy. The important point we want to make is that the price of risk at a given point in time is independent of the wealth distribution across agents. It is affected by the agent heterogeneity only indirectly through the varying conditional distribution of the state. The current state affects the price of risk only to the extend that it determines the conditional distribution.

The price of risk is measured by the Sharpe ratio. The conditional Sharpe ratio for a given state \(s\) is given by,

\[
SR(g) = \frac{\mathbb{E}[R^e(g,g')|g]}{\sqrt{\mathbb{V}[R^m(g,g')|g]}},
\]

(33)

where \(R^e(g,g') = R^m(g,g') - R^f(g)\) is the excess market return over the risk-free rate. The first thing we have to establish is that in an economy with rational agents and stationary equilibrium, the conditional Sharpe ratio varies if the conditional probabilities of the state
variable vary. We multiply the numerator and denominator in (33) with \( \hat{P}^m(g) \) to get,

\[
SR(g) = \frac{E \left[ G' \left( \hat{P}^m(g') + 1 \right) \mid g \right] - \hat{P}^m(g)}{\sqrt{V \left[ G' \left( \hat{P}^m(g') + 1 \right) \mid g \right]}},
\]

and note that,

\[
\hat{P}^m(g) = \frac{G^0 Z(g) \hat{P}^m(g)}{G^0 Z(g) P^f(g)} = \frac{E \left[ Z(g') G' \left( \hat{P}^m(g') + 1 \right) \mid g \right]}{E \left[ Z(g') \mid g \right]}.
\]

We have obtained a representation of the conditional Sharpe ratio, in some state \( g \), in terms of conditional expectations of fixed functions of only the future state \( g' \). Therefore, unless the conditional probabilities \( \pi(g, g') \) vary, the conditional price of risk does not vary across states either.

In Chan and Kogan (2002) the state is conditionally normally distributed and the conditional mean of the state varies while the conditional volatility does not. In their model, the Sharpe ratio only varies when there is heterogeneity in the economy. Here, we will see that this is a special case. First, we have to express the conditional Sharpe ratio in terms of the state variable. For that purpose, using (25), we reexpress the Euler equation in the following way,

\[
\frac{1}{\delta} Z(g) G^{\rho} = E \left[ Z(g') R(g, g') \mid g \right],
\]

in order to obtain,

\[
0 = E \left[ Z(g') R^e(g, g') \mid g \right],
\]

The above equation then implies that,

\[
SR(g) = -\varphi_g \left( Z(g'), R(g, g') \right) \frac{\sqrt{V \left[ Z(g') \mid g \right]}}{E \left[ Z(g') \mid g \right]}, \tag{34}
\]

where \( \varphi_g(\cdot, \cdot) \) denotes conditional correlation. Let also,

\[
\overline{SR}(g) = \frac{\sqrt{V \left[ Z(g') \mid g \right]}}{E \left[ Z(g') \mid g \right]},
\]

This result is general and it not only applies when the state is a single exogenous variable. It applies when the equilibrium is time-homogeneous Markovian in terms of a vector of variables. The state vector may include both exogenous as well as endogenous variables. The only requirement is that the stochastic discount factor \( M_{t+1} \) is additively or multiplicatively separable in a fixed function of \( \mathcal{F}_t \) measurable variables and a fixed function of \( \mathcal{F}_{t+1} \) measurable variables of the state vector. \( \mathcal{F}_t \) is the conditioning information set at time \( t \).
denote the maximum possible Sharpe ratio in the economy in state $g$. We have already shown that in the case of homogeneous agents, $Z(g') = e^{\gamma \Delta g'}$ and therefore it is log-normally distributed if consumption growth is also log-normally distributed. Therefore, the maximum Sharpe ratio in some state $g$ is given by,

$$
\text{SR}(g) = \sqrt{\frac{\mathbb{E}[e^{2z(g')}|g] + \mathbb{E}[e^{z(g')}|g]^2}{\mathbb{E}[e^{z(g')}|g]}} = \sqrt{\exp(\mathbb{V}[z(g')|g]) - 1},
$$

(35)

In the case of homogeneous agents we also have that $\kappa(g)$ is equal to one in all states and therefore the maximum Sharpe ratio is constant as long as the conditional volatility is too. The conditional mean does not enter the relation due to the log-normality assumption and hence the result of Chan and Kogan (2002).

In the case of heterogeneous agents relation (35) holds only approximately since we’ve seen that $z$’s conditional distribution is positively skewed. However, this approximate relation can still give us the extend to which a heterogeneous agent economy can give us a variable Sharpe ratio. As we have already seen $\kappa(g)$ varies only slightly unless we assume an unrealistically high level of heterogeneity along with a very high variation in the conditional mean of consumption growth. Otherwise the additional variability that risk-aversion heterogeneity incorporates into the Sharpe ratio is potentially minimal. In Chan and Kogan’s model the conditional Sharpe ratio moves counter-cyclically. This is due to two reasons; (i) the fact that $\kappa(g)$ is negatively related to the conditional mean of the state variable and (ii) due to the assumption that the conditional mean is monotonically increasing in the current level of the state variable while the conditional volatility is constant. If the latter does not hold then the conditional Sharpe ratio will behave differently since the effect of heterogeneity is only through the variation in the conditional mean as well as the variation in the conditional volatility.

The approximate relation (35) has one other important implication. In this rational agent economy with heterogeneous risk-aversion parameters, the conditional Sharpe ratio changes from time to time either due to the heterogeneity in the economy and changes in the conditional moments of consumption growth, or only due to changes in the conditional volatility. Whether the Sharpe ratio moves counter or pro-cyclically depends on how the conditional moments vary along with the business cycle. If there is no heterogeneity in the economy then the conditional Sharpe ratio moves with the conditional volatility of $g$. If there is heterogeneity then we need to see more closely the effect of $\kappa(g)$. The partial derivatives of $\kappa(g)$ with respect to the

\footnote{In our numerical exercises we verify that $\varphi_g(\cdot, \cdot)$ is close to $-1$ and does not vary in a significant way, so our results about the volatility of the maximum Sharpe ratio are robust to the actual Sharpe ratio.}
conditional mean and standard deviation of consumption growth are given by,

\[
\frac{\partial \kappa(g)}{\partial \mu(g)} = -\frac{\bar{\eta}}{\kappa(g)} \left[ \frac{1}{(1 + 2\bar{\eta} \Delta \mu(g))^2} + \frac{1.5\bar{\eta}^2 \sigma(g)^2}{(1 + 2\bar{\eta} \Delta \mu(g))^4} \right],
\]

and

\[
\frac{\partial \kappa(g)}{\partial \sigma(g)} = \frac{\bar{\eta}^2 \sigma(g)}{2\kappa(g) (1 + 2\bar{\eta} \Delta \mu(g))^3},
\]

respectively. Clearly we see that \( \kappa(g) \) decreases with the conditional mean and increases with the conditional variance. Therefore, if for example during recessions the conditional mean decreases while the conditional volatility increases and the opposite happens during expansions, then all effects have the same direction and the conditional Sharpe ratio moves counter-cyclically. If however the conditional moments move differently then the Sharpe ratio might move either way and it might not behave monotonically along the business cycle.

The equity premium as given by the conditional expected excess return of the market over the risk-free rate in some state \( s \) is given by,

\[
\mathbb{E} \left[ R^e(g, g') | g \right] = -\rho_s \left( R^e(g, g'), Z(g') \right) \sqrt{\mathbb{V} \left[ R^e(g, g') | g \right]}. 
\]

Since we do not expect the conditional correlation to vary significantly across states we determine that the variation in the equity premium will only be coming from the variation of the conditional maximum Sharpe ratio and the conditional return volatility. We have already seen that both of these conditional quantities will follow the behavior of the conditional volatility of consumption and they are little affected by the level of heterogeneity in the economy. Hence, if the conditional volatility of consumption varies counter-cyclically then so will the conditional equity premium.

5 Quantitative Analysis

In this part of the paper we will try to quantify the relative importance of the risk-aversion heterogeneity in the economy and the varying conditional moments. In particular, we are interested in the variation of quantities like the price of risk, the equity premium, the price dividend ratio as well the conditional volatility of stock returns. We have seen that theoretically even though agent heterogeneity may have some role in the variation of these quantities, it is the variation of the conditional moments of the state of the economy that have the larger effect.

We examine economies with different parameters for the consumption growth process. Parameter \( \phi_1 \) controls the persistence in the conditional mean of the process while \( \phi_2 \) makes the
conditional volatility varying and is related to the persistence in the conditional volatility when \( \phi_1 \) is positive. For example after a high realized growth the conditional mean increases and the conditional volatility decreases. In turn higher growth realizations are expected which implies lower expected conditional volatility for next period.

In these economies we will vary the level of risk-aversion heterogeneity by varying the risk-tolerance standard deviation across agents at the steady state. The average risk-tolerance will always be 1/3 and we will consider one homogeneous economy and two heterogeneous economies with risk-tolerance standard deviations of 0.05 and 0.1 respectively. Figure 5 plots the distribution across agent risk-aversion parameters for the two heterogeneous cases. As we see the level of heterogeneity for \( \sigma_r = 0.1 \) is quite high.

The parameters were chosen in these exercises in order to constitute the comparison between the different cases easy, while having asset pricing quantities not very far away from the observed ones. No attempt was made to fit quantities like the average risk-free rate or average premium. For example the volatility of consumption growth was chosen to be around 4% which is higher than what is observed but at the same time lower than the average growth of aggregate dividends. This in effect predicts lower risk-premia, higher risk-free rate volatilities and lower Sharpe ratios.

\( \phi_1 \) was chosen to be 0.6 which is relatively high in terms of the first-order autocorrelation observed in the data of growth in the consumption of non-durable goods and services. This
choice decreases considerably the risk-free rate volatility, increases slightly the stock return volatility as well as the price-dividend ratio volatility. For examining the effect of the variability of the conditional consumption growth volatility we selected three different values for \( \phi_2 \), 0, 0.05 and 0.1. We will see that even such a small persistence in the conditional volatility has a large impact on the variability of certain conditional asset pricing moments. The long-term mean for consumption growth, \( \bar{g} \), was chosen to be 2% while \( \sigma \) was set to 4%. Figure 6 shows the unconditional distribution of consumption growth for the different values of \( \phi_2 \). We see that the distribution has high volatility relative to the growth process that we observe. One reason for this was to amplify the effect of agent heterogeneity and see if in fact this kind of heterogeneity has a significant effect on the asset pricing quantities we examine. In this context it can be loosely regarded as the state variable that determines the cross sectional wealth distribution. For example market incompleteness or frictions in the economy might prevent perfect risk sharing in which case the wealth distribution will be more volatile than what our frictionless market predicts.

We start by examining the behavior of the price of risk. Figure 7 plots the conditional Sharpe ratios across states, for different combinations of \( \sigma_\tau \) and \( \phi_2 \). Panels A and C show the conditional Sharpe ratio and the conditional maximum Sharpe ratio for the three different levels of heterogeneity. As we see the price of risk becomes counter-cyclical in consumption growth, once we introduce heterogeneity. The variation of the Sharpe ratio increases with the level of heterogeneity but only marginally. For example when \( \sigma_\tau = 0.1 \) the maximum Sharpe ratio ranges from 0.1215 to 0.1195 when growth ranges within 3 standard deviations, left and right

![Unconditional Distributions of consumption growth for different values of \( \phi_2 \). The long run mean is \( \bar{g} = 2\% \) and \( \sigma = 4\% \).](image)

Figure 6: Unconditional distributions of consumption growth for different values of \( \phi_2 \). The long run mean is \( \bar{g} = 2\% \) and \( \sigma = 4\% \).
of the steady state. For these two graphs conditional volatility is constant. Whereas, as conditional volatility starts varying the conditional Sharpe ratio becomes highly variable. In panels B and D we plot the conditional Sharpe ratio for the three selected values of $\phi_2$ and when the level of heterogeneity is at $\sigma_\tau = 0.05$. The variation of the Sharpe ratio clearly increases with the persistence of the conditional volatility and when $\phi_2 = 0.1$ the $\overline{SR}$ ranges from around 0.15 to around 0.7.

Figure (8) panels A and B show the variation of the risk-free rates across states when we vary again the level of heterogeneity as well as the persistence in the conditional volatility. We observe that none of the two factors is able to significantly impact the behavior of the risk-free rate. The level of heterogeneity has virtually no effect while increasing $\phi_2$ increases the variability of the risk-free rate slightly. The equilibrium risk-free rate as determined by the inter-temporal substitution, in our model is dominated by the habit through the $\alpha$ parameter rather than the elasticity of intertemporal substitution. When the habit parameter closes on the average risk-aversion then the variation in the EIS becomes more significant. The impact of the volatility persistence is through the precautionary savings. When growth is low and conditional volatility becomes lower than average, then the precautionary savings decrease leading to an increase in the risk-free rate. The opposite obviously happens when growth is higher than average.

Panels C and D of figure (8) plot the conditional equity premium across the various levels of consumption growth. Firstly, we see that the equity premium moves counter-cyclically with growth for all cases considered. For the case of agent homogeneity and no variation in the conditional volatility, the equity premium variation is driven by the variation of the conditional return volatility that we will see next. Panel C indicates that risk-aversion heterogeneity is able to explain some counter-cyclical variation in the equity premia but once more the effect is quite insignificant. Comparing the two extreme cases the variation goes from around (1.85%, 2.22%) in the homogeneous case to around (1.82%, 2.27%) in the case of $\sigma_\tau = 0.1$. Whereas in panel D we observe that the variation in the conditional volatility is able to produce a highly variable equity premium. In the case of $\phi_2 = 0.1$ the equity premium ranges from around 0.5% to around 3.4%. Note that the conditional volatility only ranges from around 4.8% when growth is at its lowest, to around 2.2% when growth is at its highest.

A very similar behavior to that of the equity premium is observed for the behavior of the stock return conditional volatilities. Panels A and B of figure 9 plot the variation in the conditional volatilities across states and across the different cases we consider. Once more the impact of the risk-aversion heterogeneity is almost negligible while variation in the conditional volatility can significantly impact the counter-cyclical behavior of the stock return volatility. The same however, cannot be said for the log price-dividend ratio. In panels C and D we observe that
none of these factors can significantly affect the variation of the price-dividend ratio across states. This asset pricing quantity is mostly determined by the level of habit as well as the persistence in the conditional mean of consumption growth. Naturally, if the conditional mean is more persistent then the price-dividend ratio is even higher when growth is high and even lower for low growth realizations. As for habit, when its level increases through smaller values of $\alpha$, the incentive for intertemporal substitution varies more across states. This in turn increases the price-dividend ratio but proportionally to the level of growth.

In table 1 we provide several unconditional moments of asset returns for all different cases. We show the first two moments of the risk-free rate, the average equity premium, the long run stock return volatility, the first two moments of the Sharpe ratio and those of the price-dividend ratio. The first we note is that the agent heterogeneity has almost no quantitative effect on these quantities, even in the case where the conditional volatility is more varying. The average equity premium shows a consistent increase with the level of heterogeneity and the risk-free rate a consistent decreases. However the variation once more is negligible. Looking at the variation in $\phi^2$ the most notable is the increase in the volatility of the Sharpe ratio. It increases from almost zero to around 0.013 while it doubles when $\phi^2$ increases from 0.05 to 0.1. Further we note a decrease in the average equity premium which seems to be partly coming from a decrease in the return volatility. However, the decrease in the equity premium is relatively bigger since the average Sharpe ratio decreases as well.

Concluding this exercise once again we have to emphasize that in a complete market setup having heterogeneous agents in their risk-aversion parameter does not help to explain either the variation or the level of the asset pricing quantities like the price of risk, equity premia or conditional volatilities. Unless the aggregate risks that the agents in the economy face are unrealistically high along with an unrealistically high level of heterogeneity, the representative agent assumption is able to approximate well the asset pricing behavior of a heterogeneous agent economy. As for the variation in the conditional moments of the aggregate uncertainty we have to stress that it seems to be able to provide a good explanation for the variation in the conditional asset pricing moments. This variation however, does not necessarily provide any explanation as to the level of these asset pricing quantities.

6 Regime Switching Along the Business Cycle

We have discussed briefly at the introduction that a Markov-switching process for consumption growth has certain interesting asset pricing implications. These implications are mainly due to the disentanglement of the conditional distribution of consumption growth from the realized consumption growth. So far we have assumed a conditionally heteroscedastic process for the logarithm of consumption growth with the conditional moments depending on the
Table 1: Unconditional asset pricing moments for various combinations of risk-aversion heterogeneity and persistence in the conditional volatility of consumption growth. The other parameters used are, $\bar{\gamma} = 3$, $\bar{g} = 2\%$, $\sigma = 4\%$, $\phi_1 = 0.6$, $\alpha = 0.5$ and $\delta = 0.96$. $\mathbb{E}(\cdot)$ denotes unconditional mean and $\sigma(\cdot)$ denotes unconditional standard deviation.

<table>
<thead>
<tr>
<th>$\phi_2 = 0$</th>
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<th>$\phi_2 = 0.1$</th>
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<td>$\sigma_\tau = 0$</td>
<td>$\sigma_\tau = 0.05$</td>
<td>$\sigma_\tau = 0.1$</td>
</tr>
<tr>
<td>$\sigma(R^f)$ (%)</td>
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<td>3.6580</td>
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<tr>
<td>$\mathbb{E}(R^e)$ (%)</td>
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<td>2.0379</td>
</tr>
<tr>
<td>$\sigma(R^m)$ (%)</td>
<td>17.6733</td>
<td>17.6664</td>
</tr>
<tr>
<td>$\mathbb{E}(SR)$</td>
<td>0.1179</td>
<td>0.1180</td>
</tr>
<tr>
<td>$\sigma(SR)$</td>
<td>0.0001</td>
<td>0.0003</td>
</tr>
<tr>
<td>$\log \mathbb{E}(\hat{P}^m)$</td>
<td>3.5158</td>
<td>3.5158</td>
</tr>
<tr>
<td>$\log \sigma(\hat{P}^m)$</td>
<td>1.6624</td>
<td>1.6621</td>
</tr>
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</table>
realized growth. This is because the asset prices are influenced both from the realized consumption growth, through the cross-sectional allocation of wealth and the marginal utility of consumption, and the potentials of the economy as given by the conditional distribution.

At a given point in time the realization of the uncertainty facing an economy does not necessarily determine the future distribution of the uncertainty. Regime switching seems to be a necessary ingredient in modeling asset prices because it seems that the state of the economy as it moves through business cycles, moves slower than the quarterly or yearly realizations of the uncertainty. In this paper we are examining two distinct factors of the asset pricing formation process. The first one is the wealth distribution across agents with different risk appetites and the second one is the conditional distribution of the one period risk. The second factor need not be the actual conditional distribution but what the economy perceives it to be. The realization of the uncertainty at a given point in time is what vastly determines the cross sectional distribution of wealth, along with the aggregate wealth, while the state of the economy is what determines the future distribution of the uncertainty. It seems natural therefore to disentangle these two factors in modeling asset prices. Further, we have identified a key asset pricing quantity, namely the conditional Sharpe ratio, that only depends on the conditional distribution of the uncertainty while it is independent of the current total wealth as well as the wealth distribution. Even though the conditional price of risk is not directly observable, this feature of it constitutes any attempts of measuring its evolution very important. This kind of evidence will point directly to the process of the conditional distribution of the aggregate uncertainty or in fact the variation of the economy’s beliefs about the economy wide uncertainty. In this section we will also find out that the price-dividend ratio as opposed to the Sharpe ratio, is virtually unaffected by the potentials of the economy but it is vastly determined by the current realization of the uncertainty. Unlike the Sharpe ratio, the information content of the price-dividend ratio is limited in predicting future excess returns and dividend growth as shown empirically in Campbell (1991) and Cochrane (2001).

The state of the economy is given by the pair \((y, g)\) where \(y\) is a Markov chain that determines the conditional moments of consumption growth. In particular, the consumption growth process is given by,

\[
g_{t+1} = \mu(y_t) + \sigma(y_t)\epsilon_{t+1}, \quad \forall t \geq 0,
\]

where \(\epsilon_t \sim i.i.d. N(0,1)\). The regime variable \(y\) takes values in a discrete set \(\mathcal{Y}\) according to a transition matrix \(Q\). Let \(q\) denote the stationary distribution so that,

\[
q = qQ.
\]

The regime variable \(y\) is independent of \(\epsilon\). The unconditional mean of log consumption growth
Table 2: Regime switching process parameters.

<table>
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</thead>
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<td>Expansion</td>
<td>Peak</td>
<td>Contraction</td>
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<td></td>
<td>0.0300</td>
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<td>$\sigma(y)$</td>
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<td>0.0400</td>
<td>0.0300</td>
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</tr>
<tr>
<td>Transition Probabilities</td>
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<td>0.6000</td>
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</tr>
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<td>4</td>
<td>0.2000</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
</tbody>
</table>

is given by,

$$
\bar{g} = \sum_{y \in Y} q(y) \mu(y),
$$

and the unconditional volatility is given by

$$
\sigma_g = \sqrt{\sum_{y \in Y} q(y)(\mu(y) - \bar{g})^2 + \sum_{y \in Y} q(y)\sigma(y)^2}.
$$

We interpret the different regimes as the different stages of a business cycle starting from a trough, going through expansion, a peak and ending at the last stage of contraction. The parameters of the process were arbitrarily chosen. Table 2 shows these parameters. The first row shows the conditional mean of consumption growth during each respective regime and the second row shows the conditional volatilities. The trough is a period of high mean and high volatility while the peak is a period of low mean and low volatility. The expansion and the contraction periods have the same conditional volatility while during expansion we assume a higher mean. We further assume that the economy spends the same amount of time in peaks and in troughs and also equal times in contractions and expansions. The unconditional distribution of $g$ is shown in figure 10 along with the average time spent in the four stages of the business cycle.

Since the effect of agent heterogeneity in risk-aversion is almost unnoticeable we present the equilibrium results only of the economy for which $\sigma_\tau = 0.05$. Figure 12 panel A shows the log price-dividend ratio for the four different regimes across the different growth levels. The variation across the different regimes is almost non-existing. The price dividend ratio is driven by the habit and note that is almost linear with a slope of $\bar{\gamma} - \alpha$. The different regimes cause only a small parallel shift. The same is almost true for the risk-free rate as shown in panel C of the same figure. In contrast the equity premium, shown in panel B, and the conditional stock
return volatilities, shown in panel D, exhibit high variation both across growth levels as well as across regimes.

From these results it is evident that the price-dividend ratio in such an economy would be unable to predict future expected dividend or consumption growth. Further, it’s variability would naturally be found to be almost entirely due to variations in the future expected excess returns due to the fact that significant part of the variation in the risk-premium is due variations in the consumption growth. For the same reason a possible predictive power of the price-dividend ratio especially in longer horizons could be found.

Finally, figure 11 shows the conditional Sharpe ratio over the business cycle. Since it only depends on the conditional distribution within each regime the conditional Sharpe ratio is constant. Further, we see that it behaves counter-cyclically mostly due to the fact that the conditional volatility of consumption growth varies counter-cyclically. From our theoretical analysis we have seen that the conditional mean does not have a significant effect and for this reason the Sharpe ratio during contraction and expansion are almost the same. The small difference is only due to the indirect effect of agent heterogeneity.

7 Conclusion

At the heart of many asset pricing phenomena seems to lie a varying risk-premium that moves counter-cyclically with the business cycle. In an efficient market with rational agents this could be attributed either to counter-cyclical variation in the risk-aversion of the economy or a cyclical variation in the joint distribution of asset returns with consumption growth. Campbell and Cochrane (1999) are able to explain many of the asset pricing facts in a model with external habit that leads to a counter-cyclical variation in risk-aversion. However, in their attempt to match most of the asset pricing moments, including the average equity premium, require the risk-aversion to have an unrealistically high variation. Chan and Kogan (2002) show that the counter cyclical pattern is obtained in an economy with heterogeneous risk-aversion and argue that this heterogeneity can produce a varying Sharpe ratio. We consider a similar model and derive explicitly the risk aversion of the economy and find that such variation is unlikely to be produced in a realistic economy. Moreover, if we forget the level of the equity premium many other observed asset pricing features like the varying price-dividend ratio and the varying price of risk can be predicted in an economy with habit in the form of Abel (1990) and varying conditional moments of consumption growth.

Even though the consumption risk of asset prices does not seem to be high enough to justify such a high equity premium, it does possess certain features that are able to explain several other asset pricing facts. Markov switching models of consumption growth seem to be a natural way
of modeling macroeconomic uncertainty since the varying conditional distribution can account for the variation of the price of risk. As for explaining its level one more ingredient is needed which could be associated with the underlying structure of macroeconomic risks. Asset prices are formed on expectations or beliefs about these risks and naturally the sentiment of the investors, the way they update these beliefs or the uncertainty they have about the conditional distribution might be able to explain why investors require such high compensation for every unit of ex-post risk they take.
A The function $Z(g)$

For a distribution of agents \( \{ (\lambda_i, \gamma_i), i \in I \} \) where \( \lambda_i > 0 \) and \( \gamma_i > 1 \) for all \( i \in I \), function \( Z \) is determined by the equation,

\[
\sum_{i \in I} \lambda_i Z(g)^{-\frac{1}{\gamma_i}} = \exp(g), \quad \forall g \in \mathbb{R}.
\]

Let \( Z'(g) \) denote the partial derivative w.r.t. \( g \). Differentiating the above equation we get,

\[
-Z''(g) \frac{Z(g)}{Z'(g)} \sum_{i \in I} \lambda_i Z(g)^{-\frac{1}{\gamma_i}} = \exp(g).
\]

Define

\[
a_1(g) = \exp(-g) \sum_{i \in I} \lambda_i Z(g)^{-\frac{1}{\gamma_i}}.
\]

Since \( \gamma_i > 1 \) for all \( i \in I \) we have \( a_1(g) \in (0, 1) \) for all \( g \in \mathbb{R} \). Hence, we have that,

\[
Z'(g) = -\frac{Z(g)}{a_1(g)}, \quad \forall g \in \mathbb{R}.
\]

Differentiating one more time we get,

\[
-Z''(g) a_1(g) \exp(g) + \left( \frac{Z'(g)}{Z(g)} \right)^2 \sum_{i \in I} \lambda_i \left( 1 + \frac{1}{\gamma_i} \right) Z(g)^{-\frac{1}{\gamma_i}} = \exp(g).
\]

Define also,

\[
a_2(g) = -\exp(-g)Z'(g) \sum_{i \in I} \lambda_i \gamma_i Z(g)^{-\frac{1}{\gamma_i} - 1},
\]

and note that \( a_2(g) \in (0, 1) \) for all \( g \in \mathbb{R} \). Substituting in \( a_1(g) \) and \( a_2(g) \) and after rearranging we get,

\[
Z''(g) = \frac{1 + a_2(g) - a_1(g)}{a_1(g)^2} Z(g), \quad \forall g \in \mathbb{R}.
\]
The above expression implies that $Z''(g) > 0$ for all $g$. Let $z(g) = \log(Z(g))$. Then the first two derivatives of $z(g)$ w.r.t. $g$ are given by,

$$z'(g) = -\frac{1}{a_1(g)},$$

$$z''(g) = \frac{a_2(g) - a_1(g)}{a_1(g)^2},$$

for all $g \in \mathbb{R}$. The first derivative is clearly negative, while the second derivative is positive iff $a_2(g) > a_1(g)$. In order to show that this is true we re-express $a_2(g)$ as follows,

$$a_2(g) = \frac{\exp(-g) \sum_{i \in I} \lambda_i Z(g)^{-\frac{1}{\gamma_i}}}{a_1(g)},$$

and require that,

$$\exp(-g) \sum_{i \in I} \lambda_i Z(g)^{-\frac{1}{\gamma_i}} > a_1(g)^2.$$

To make exposition easier define a probability measure over the set of agents, for a given $g$ as follows,

$$P_I(i|g) = \exp(-g)\lambda_i Z(g)^{-\frac{1}{\gamma_i}}.$$

Then the requirement becomes,

$$E_I[\gamma_i^{-2} | g] > E_I[\gamma_i^{-1} | g]^2,$$

which is true from Jensen’s inequality. Let

$$\mu_I(\tau|g) = E_I[\tau_i | g],$$

$$\sigma_I(\tau|g)^2 = \mathbb{V}_I[\tau_i | g],$$

denote the mean and the variance of the risk-tolerance of agents when realized growth is $g$. Then we have that,

$$a_1(g) = \mu_I(\tau|g),$$

$$a_2(g) = \frac{\sigma_I(\tau|g)^2 + \mu_I(\tau|g)^2}{\mu_I(\tau|g)},$$
and therefore the first two derivatives of $z$ are given by,

$$z'(g) = -\mu_I(\tau|g)^{-1},$$

$$z''(g) = \frac{\sigma_I(\tau|g)^2}{\mu_I(\tau|g)^3}. $$
B The function $h_1(\Delta)$

Let $\eta = (\sigma_x \epsilon)^2$, then $h$ is given by,

$$h_1(\Delta) = \frac{\Delta - 1 + \sqrt{1 + 2\eta \Delta - \eta \Delta^2}}{\Delta(1 + \eta)},$$

where we need that,

$$1 + 2\eta \Delta - \eta \Delta^2 \geq 0.$$ 

which in turn requires that,

$$\Delta \in [1 - \sqrt{1 + \eta^{-1}}, 1 + \sqrt{1 + \eta^{-1}}].$$

Let the domain be given by,

$$D = [1 - \sqrt{1 + \eta^{-1}}, 0) \cup (0, 1].$$

The first derivative of $h$ is given by,

$$h'_1(\Delta) = \frac{1}{(1 + \eta)\Delta^2} \left[ 1 + \frac{\eta \Delta(1 - \Delta)}{\sqrt{1 + 2\eta \Delta - \eta \Delta^2}} - \sqrt{1 + 2\eta \Delta - \eta \Delta^2} \right],$$

which is negative for $\Delta \in D$. It suffices to show that the expression in square brackets is negative. This holds since,

$$\sqrt{1 + 2\eta \Delta - \eta \Delta^2} > 1 + \frac{\eta \Delta(1 - \Delta)}{\sqrt{1 + 2\eta \Delta - \eta \Delta^2}},$$

$$1 + 2\eta \Delta - \eta \Delta^2 > \sqrt{1 + 2\eta \Delta - \eta \Delta^2} + \eta \Delta(1 - \Delta),$$

$$1 + \eta \Delta > \sqrt{1 + 2\eta \Delta - \eta \Delta^2},$$

$$1 + 2\eta \Delta + \eta^2 \Delta^2 > 1 + 2\eta \Delta - \eta \Delta^2.$$ 

The second derivative of $h$ is given by,

$$h''_1(\Delta) = \frac{1}{\Delta^3 A(\Delta) \sqrt{A(\Delta)}} \left[ \left( A(\Delta) - \sqrt{A(\Delta)} \right)^2 - (\eta \Delta(1 - \Delta))^2 \right],$$

where $A(\Delta) = 1 + 2\eta \Delta - \eta \Delta^2$. For $\Delta \in D$ the second derivative is always positive. It is enough to show that the expression in square brackets is positive when $\Delta$ is positive and negative when $\Delta$ is negative. The expression in square brackets is a difference in squares and can be expressed
as,
\[
\left(1 + \eta \Delta - \sqrt{A(\Delta)}\right) \left(1 + 3\eta \Delta - 2\eta \Delta^2 - \sqrt{A(\Delta)}\right).
\]

The first parenthesis is always positive since \(A(\Delta) < (1 + c\Delta)^2\). The second parenthesis is positive when \(\Delta > 0\) since \(A(\Delta) > 1\), \(\sqrt{A(\Delta)} < A(\Delta)\) and therefore
\[
1 + 3\eta \Delta - 2\eta \Delta^2 - \sqrt{A(\Delta)} > 1 + 3\eta \Delta - 2\eta \Delta^2 - A(\Delta) = \eta \Delta(1 - \Delta) > 0.
\]

Similarly when \(\Delta < 0\),
\[
1 + 3\eta \Delta - 2\eta \Delta^2 - \sqrt{A(\Delta)} < 1 + 3\eta \Delta - 2\eta \Delta^2 - A(\Delta) = \eta \Delta(1 - \Delta) < 0.
\]
References


Figure 7: Conditional Sharpe Ratio and conditional maximum Sharpe ratio across states, for different combinations of $\phi_2$ and $\sigma_\tau$. 
Figure 8: Risk-free rate and conditional equity premium across states, for different combinations of $\phi_2$ and $\sigma_\tau$. 
Figure 9: Conditional stock return volatility and log of price-dividend ratio across states, for different combinations of $\phi_2$ and $\sigma_\tau$. 

A. Stock Return Volatility (vary $\sigma(\tau)$), ($\phi_1=0.6$, $\phi_2=0$, $\alpha=0.5$, $\delta=0.96$)
g (%)

B. Stock Return Volatility (vary $\phi_2$), ($\sigma(\tau)=0.05$, $\phi_1=0.6$, $\alpha=0.5$, $\delta=0.96$)
g (%)

C. Log Price/Div. Ratio (vary $\sigma(\tau)$), ($\phi_1=0.6$, $\phi_2=0$, $\alpha=0.5$, $\delta=0.96$)

D. Log Price/Div. Ratio (vary $\phi_2$), ($\sigma(\tau)=0.05$, $\phi_1=0.6$, $\alpha=0.5$, $\delta=0.96$)

log(PD)
Figure 10: Unconditional distribution of consumption growth and average time spent in the four regimes.

Figure 11: Conditional Sharpe ratios across the four regimes.
Figure 12: Log price-dividend ratio, conditional equity premium, conditional stock return volatility and risk-free rate across different growth levels and regimes. Each line corresponds to one regime.