A Bayesian Approach to Real Options: The Case of Distinguishing Between Temporary and Permanent Shocks*

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Abstract

This paper studies the optimal timing of investment in the presence of uncertainty about both future and past shocks. Specifically, we allow for both temporary and permanent shocks to cash flows and assume that the firm is unable to distinguish between them. As a result, the evolving uncertainty is driven by Bayesian updating, or learning. We solve for the optimal investment rule and show that the implied investment behavior differs significantly from that predicted by standard real options models. For example, unlike the standard real options implications, investment may occur at a time of stable or decreasing cash flows, respond sluggishly to positive cash flow shocks, and critically depend on the maturity structure of the project cash flows.

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1 Introduction

During the past two decades the real options approach to valuation of irreversible investment opportunities has become part of the mainstream literature in financial economics. The central idea is that the opportunity to invest is equivalent to an American call option on the underlying investment project. As a consequence, the problem of optimal investment timing is analogous to the optimal exercise decision for an American option. Applications of the real options approach are now numerous.[1] McDonald and Siegel (1986) provided the (now standard) setting, that was later extended to account for a time-to-build feature (Majd and Pindyck (1987)) and strategic interactions among several option holders (Grenadier (1996, 2002), Lambrecht and Perraudin (2003), Novy-Marx (2007)). Real options modeling is used to study specific industries such as natural resources (Brennan and Schwartz (1985)) or real estate (Titman (1985) and Williams (1991)). Recent developments include incorporating agency conflicts (Grenadier and Wang (2005)) and behavioral preferences (Grenadier and Wang (2007), Nishimura and Ozaki (2007)) into the standard setting.

One feature which is common to virtually all real options models is that the underlying uncertainty is about future shocks only. Since the future value of the asset is uncertain, there is an important opportunity cost of investing today: the value of the asset might go up so that tomorrow will be an even better time to invest. This opportunity cost is often referred to in the literature as the "option to wait." The implied investment strategy in this setting is to wait until the value of the asset reaches some upper threshold value and then invest.

In this paper we propose a very different kind of real options problem. While uncertainty about future shocks is important, it may not be the most important, let alone unique uncertainty faced by a firm. Indeed, another critical source of uncertainty is uncertainty about *past* shocks. This is the case when the firm observes the past shock, but fails to identify its exact properties. As time passes, the firm updates its beliefs about the past shock. Therefore, unlike the standard models, the evolving uncertainty is driven by Bayesian updating, or learning. This gives rise to a trade-off between investing now and waiting to learn.

Specifically, this paper focuses on the case where uncertainty about past shocks comes from the firm's inability to distinguish between temporary and permanent shocks to the cash flow process. To gain intuition underlying the trade-off between investing now and learning, consider a positive shock to cash flows. In the standard setting the firm's strategy would be rather simple: the firm should invest if and only if the shock is high enough so that the value of the cash flow process exceeds some threshold level. However, when the firm does not know if the shock is temporary or permanent, it may want to wait in order

[1] The early literature is well summarized in Dixit and Pindyck (1994).

to learn more about the identity of the shock. Indeed, if the firm waits until tomorrow and the value of the cash flow process is still at the high level, then the past shock is more likely to represent a positive fundamental change. Similarly, if the cash flow process decreases, then there is a greater chance that the past positive shock was simply a result of temporary non-fundamental fluctuations.

Our argument is based on two building blocks that distinguish this model from the real options literature. The first building block is the presence of both temporary and permanent shocks of cash flows. While virtually all real options models focus solely on permanent shocks, the presence of temporary shocks is a natural feature of many real-world economic environments. The focus on only permanent shocks is clearly a simplification that permits one to model the cash flow process as a geometric Brownian motion. However, as Gorbenko and Strebulaev (2008) show, assuming a geometric Brownian motion cash flow process leads to a number of undesirable empirical properties. For example, it implies that the volatilities of the cash flow and asset growth are equal, while empirically volatilities of cash flow growth are much higher than volatilities of asset value growth.

The second building block is the inability of the firm to distinguish between permanent and temporary shocks. This is an especially important feature of investments in natural resources, where it is often unclear when a change in the commodity price represents a fundamental or ephemeral shift. Indeed, as the World Bank's Global Economic Prospects annual report for the year 2000 states:

> "Most important, distinguishing between temporary and permanent shocks to commodity prices can be extraordinary difficult. The swings in commodity prices can be too large and uncertain to ascertain their causes and nature. The degree of uncertainty about duration of a price shock varies. For example, market participants could see that the sharp jump in coffee prices caused by the Brazilian frost of 1994 was likely to be reversed, assuming a return to more normal weather. By contrast, most analysts assumed that the high oil prices during the mid-1970s and early 1980s would last indefinitely."[2]

We show that this novel feature has a number of very important consequences for the optimal investment policy. First, we show that accounting for uncertainty about past shocks may lead to a failure of the "record-setting news principle" stating that the value of the cash flow process at the time of exercise is the highest in the whole history.[3] As intuition suggests, in the Bayesian setting investment may occur even when the value of the cash flow process does not change (or even decreases), simply because the firm becomes sufficiently sure that the past shocks are permanent. Therefore, the paper

[2] World Bank, Global Economic Prospects and Developing Countries 2000, p. 110.

[3] See Boyarchenko (2004) for a discussion of the record-setting news principle, a term credited to Maxwell Stinchcombe.

provides a rationale for observing investment in stable periods and the sluggish responses of investment to shocks.

Second, we find that once we account for uncertainty about past shocks, the optimal investment strategy is no longer expressed as a single boundary in the value of the cash flow process. Indeed, in our setting the investment boundary is a function of the firm's beliefs about the past shocks. Specifically, we find that, other things being equal, the firm is unlikely to invest when many shocks occurred in the recent past. Intuitively, the most recent shocks are the most uncertain ones. Thus, if many shocks occurred recently, the value of waiting to learn is very high.

Third, we find that, unlike in the standard setting, in the context of valuations that are driven by Bayesian updating about past shocks, the timing of cash flows is very important. Specifically, we show that the lesser the "front-loadedness" of the project's cash flows, the later the investment will take place. Intuitively, the importance of the nature of past shocks decreases in the front-loadedness of the project cash flows. For example, consider the investment project that consists of buying an asset at some fixed price and selling it at a market price. This is the case when the project is extremely front-loaded. Clearly, since the firm gets all cash flows from the project immediately at the chosen time, the nature of the past shocks has no effect on the firm's optimal investment strategy. On the other hand, when the firm's project is an investment in the development of an oil well, cash flows are relatively back-loaded: the project is unlikely to yield cash flows in the first several years. In this case learning if a past increase in oil prices was a fundamental shift or simply a temporary fluctuation has important option value for the firm.

These results hold in virtually all versions of our model. To develop the main intuition we start with the simplest setting in which the cash flow process is subject to only one shock which can be either permanent or temporary. We completely characterize the optimal investment strategy in this case and solve for the corresponding investment option value. In later sections we extend the basic model by allowing for multiplicity of potential shocks, positive drift in the cash flow process and Brownian uncertainty. Interestingly, while additional complications prevent us from computing the investment option values in closed form, optimal investment strategies can be characterized and in some cases computed explicitly.

Several papers deserve mentioning as being the most closely related to our model. Gorbenko and Strebulaev (2008) incorporate temporary shocks into a contingent claims framework of capital structure and show that the presence of temporary shocks provides a potential explanation of several puzzles in corporate finance. Their paper differs from ours in two important respects. First, and most importantly, our argument heavily relies on the inability of the firm to distinguish between temporary and permanent shocks, while Gorbenko and Strebulaev (2008) do not have any learning in their model. Second, while we consider firms' investment decisions, their focus is on financing policy. Our paper is also related to Decamps et al. (2005) who solve for the optimal investment timing when the firm is uncertain about the drift of the state process. There are two major differences between their paper and ours. First, while in our paper the firm learns about the nature of past shocks to the state process, in Decamps et al. (2005) the firm learns about the parameters of the state process itself. Second, in Decamps et al. (2005) the timing and degree of uncertainty is pre-specified, while in our model uncertainty is proportional to past cash flow shocks whose timing is also uncertain to the firm. Another related paper is Miao and Wang (2007) who solve for the optimal entrepreneur's decision to exit a business in the presence of idiosyncratic nondiversifiable risk. Specifically, Miao and Wang (2007) consider an entrepreneur with incomplete information about his entrepreneurial abilities who chooses between continuing entrepreneurial activity and taking a safe job. While the focus of their paper is not on investment decisions but on the impact of idiosyncratic risk, their model of Bayesian learning is very similar to ours.[4] Finally, our paper also shares the learning feature with Lambrecht and Perraudin (2003) who study competition between two firms for a single investment opportunity when information about investment costs is private. Because of this, as time goes by, and the competitor has not invested yet, each firm updates its belief about the competitor's investment costs upward.[5]

The remainder of the paper is organized as follows. Section 2 considers the basic setting of irreversible investment in a Bayesian framework. Section 3 provides a multi-shock extension of the same model. Section 4 incorporates positive drift and Brownian uncertainty into the model of Section 2. Section 5 discusses the importance of differences in cash flow timing for optimal investment policy. Finally, Section 6 concludes.

2 A Simple Model

In this section we consider the simplest setting for incorporating the essential features of investment timing into a Bayesian framework. Subsequent sections will provide extensions of this base case model.

2.1 The Investment Option

Consider a standard real option framework in which a firm contemplates irreversible investment. By paying the investment cost I, the firm obtains the perpetual cash flow $X(t)$. The firm is free to invest in the project at any time it so chooses.

[4] See also other papers on optimal experimentation that deal with problems of optimal control under learning (e.g., Jovanovic (1979) Bolton and Harris (1999), Keller and Rady (1999), Moscarini and Smith (2001))

[5] Our paper is also related to Moore and Schaller (2002), who extend the neoclassical q theory of investment by allowing for permanent and temporary shocks to interest rates.

The cash flow process is relatively simple. At all times prior to the arrival of the shock, the cash flow stream is fixed at X. Upon the arrival of a shock it jumps to $X(1+\varphi)$, with $\varphi > 0$. The shock may be either temporary or permanent. With a permanent shock the cash flow remains at the higher level $X(1+\varphi)$ forever after, but with a temporary shock the cash flow eventually reverts to the level X at some (random) point in the future. Importantly, the firm is unable to distinguish between permanent and temporary shocks, but uses Bayesian updating to assess their relative likelihoods. We assume that the arrival and reversal (of the temporary shock) follow independent Poisson jump processes. Specifically, we assume that the permanent jump arrives with intensity λ_1, the temporary jump arrives with intensity λ_2, and an existing temporary shock reverses with intensity λ_3.

To summarize, the cash flow of the project prior to any shock is fixed at X. Upon the stochastic arrival of a shock (either permanent or temporary), the cash flow process jumps to the level $X(1+\varphi)$. If the shock is permanent, it remains at this level; if it is temporary, then at a stochastic reversal time the cash flow reverts back to the level X forever after.

2.2 The Bayesian Learning Process

Before and at the moment of the arrival of a shock, the firm assesses the prior probability of a shock being temporary as $p_0 = \lambda_2/(\lambda_1+\lambda_2)$. However, after a shock occurs, the firm continuously updates its assessment of this probability. Obviously, once the shock reverses, the firm updates its belief that the shock is temporary to probability one.

Consider any moment t after the arrival of the shock at t_0. Let $p(t)$ denote the conditional probability that the shock is temporary, for any $t > t_0$. If the shock is temporary, over a short period of time dt it reverts back with probability $\lambda_3 dt$. Thus, conditional on the shock having a temporary nature, the probability that there are no jumps over dt equals $1-\lambda_3 dt$. If the shock is permanent, it never reverts back. Thus, conditional on the shock having a permanent nature, the probability that there are no jumps over dt equals 1. Using Bayes rule, the posterior probability $p(t+dt)$ is given by

$$p(t+dt) = \frac{p(t)\left(1-\lambda_3 dt\right)}{p(t)\left(1-\lambda_3 dt\right)+1-p(t)}, \tag{1}$$

where $p(t_0)=p_0$. At the moment the shock reverses, $p(t)$ immediately increases to one. Note that this is analogous to the updating process in Miao and Wang (2007) in their problem of learning about the quality of their entrepreneurial business.

We can rewrite (1) as

$$\frac{p(t+dt)-p(t)}{dt} = \frac{-\lambda_3 p(t)\left(1-p(t)\right)}{p(t)\left(1-\lambda_3 dt\right)+1-p(t)}. \tag{2}$$

Taking the limit as $dt \to 0$,

$$\frac{dp(t)}{dt} = -\lambda_3 p(t)(1 - p(t)). \tag{3}$$

The solution to this differential equation is:

$$p(t) = \frac{\lambda_2}{\lambda_1 e^{\lambda_3 (t-t_0)} + \lambda_2}. \tag{4}$$

The dynamics of $p(t)$ has two interesting properties. First, $p(t)$ decreases over time. Intuitively, the longer a shock persists, the more likely it is to be permanent, and hence, the less likely it is to be temporary. In the limit as $t \to \infty$, $p(t)$ converges to 0, meaning that if the shock persists over infinite time, the firm learns that it is permanent. Second, the speed of learning is proportional to λ_3. In other words, if temporary shocks are more short-term, then the firm updates its beliefs faster than if they are more long-term.

2.3 Valuing the Option While the Shock Persists

In this subsection we look at the most important part of the valuation model: choosing the optimal time to invest in a Bayesian framework. Importantly, the option exercise problem represents a fundamental trade-off between the current benefits of investing to receive the cash flow, and the benefits of waiting to learn more about the identity of the shock before investing.

Let us begin by calculating some simple values. If the firm knew for sure that the shock was permanent, then the value of investing would be $\frac{X(1+\varphi)}{r} - I$, which is simply the present value of the perpetual cash flow $X(1+\varphi)$ minus the cost of investment. Conversely, if the firm knew for sure that the shock was temporary, then the value of investing would equal $\frac{X(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I$, which is the net present value of receiving a flow of $X(1+\varphi)$ until the shock is reversed and X thereafter. To ensure that a non-trivial solution exists for the problem, we make the following assumption on the values of parameters:

Assumption 1. *The initial value of the cash flow process X satisfies*

$$\frac{rI}{1+\varphi} < X < \frac{r + \lambda_3 p_0}{1 + \varphi + \frac{\lambda_3}{r} p_0} I.$$

Assumption 1 puts lower and upper bounds on X. Economically, the lower bound means that if the shock is known to be permanent, the net present value of investing into the project is positive, $\frac{X(1+\varphi)}{r} - I > 0$, or else there would never be any investment in this Bayesian setting. An upper bound, as we show later in this section, is equivalent to an assumption that learning has positive value. It guarantees that the solution to the investment timing problem is non-trivial. In addition, this assumption implies that if the shock is known to be temporary, then it is not optimal to invest: $\frac{X(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I < 0$.

Let $G\left[p(t)\right]$ denote the value of the option to invest, while a shock persists, and where $p(t)$ is the current value of the belief process. We assume risk neutrality, where r is the riskless rate of interest. Therefore, over the range of $p(t)$ at which the option is not exercised, $G\left[p(t)\right]$ must satisfy the equilibrium differential equation[6]:

$$\left(r + p\lambda_3\right) G = -G_p\lambda_3 p(1-p) + p\lambda_3 H\left(X\right), \tag{5}$$

where $H(X)$ is the value of the option to invest conditional on the cash flow being reversed to the level X forever after:

$$H\left(X\right) = \begin{cases} \frac{X}{r} - I & if\ X > rI, \\ 0 & otherwise. \end{cases} \tag{6}$$

Note that since $X < \frac{r+\lambda_3 p_0}{1+\varphi+p_0\lambda_3/r} I < rI$, $H\left(X\right) = 0$. That is, if the option is not exercised when a temporary shock reverses, it will never be exercised. Thus, we can rewrite (5) as:

$$\left(r + p\lambda_3\right) G = -G_p\lambda_3 p(1-p). \tag{7}$$

The general solution to (7) is:

$$G\left[p(t)\right] = C \cdot \left(1 - p(t)\right)\left(\frac{1}{p(t)} - 1\right)^{\frac{r}{\lambda_3}}, \tag{8}$$

where C is a constant determined by appropriate boundary conditions.

The option will be exercised when the conditional probability of the shock being temporary decreases to a lower threshold. Intuitively, it is optimal for the firm to invest only when it becomes sufficiently sure that the project will yield high cash flows for a long period of time. Let $\bar{p}$ denote the trigger at which the option is exercised. The exercise trigger $\bar{p}$ and constant C are jointly determined by the following boundary conditions:

$$G(\bar{p}) = \left(1-\bar{p}\right)\frac{X(1+\varphi)}{r} + \bar{p}\frac{X(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I, \tag{9}$$

$$G_p\left(\bar{p}\right) = -\frac{X(1+\varphi)}{r} + \frac{X(1+\varphi+\lambda_3/r)}{r+\lambda_3}. \tag{10}$$

The first equation is the value-matching condition. It reflects the fact that upon exercise, the firm receives the conditional expected value of receiving $X(1+\varphi)$ forever if the shock is permanent, or of receiving $X(1+\varphi)$ until the shock is reversed and then X thereafter if

[6] See Dixit and Pindyck (1994), Chapter 3, for a derivation of the equilibrium differential equation for Poisson processes.

the shock is temporary. The second equation is the smooth-pasting condition.[7] It ensures that the trigger $\bar{p}$ maximizes the value of the investment option.

Combining (8) with (9) and (10) yields the optimal investment threshold $\bar{p}$ and the constant C:

$$\bar{p} = \frac{X(1+\varphi) - rI}{\lambda_3 \left(I - \frac{X}{r}\right)}, \tag{11}$$

$$C = \frac{1}{1-\bar{p}} \left(\frac{1}{\bar{p}} - 1\right)^{-\frac{r}{\lambda_3}} \left[(1-\bar{p}) \frac{X(1+\varphi)}{r} + \bar{p} \frac{X(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I\right]. \tag{12}$$

Now we can see that the upper bound on X from Assumption 1 is equivalent to assuming that $\bar{p} < p_0$. This restriction seems entirely reasonable since it ensures that there is some benefit to learning. Combined with the lower bound on X, this restriction guarantees that $\bar{p} \in (0, p_0)$. Thus, the investment trigger is reachable in finite (and positive) time.

Fig. 1 plots the option value, $G(p)$, as a function of the firm's beliefs. As we can see, the option value smoothly pastes to the project's net payoff at the investment trigger $\bar{p}$. For each value of p above $\bar{p}$, the firm has a valuable option to learn measured by the distance between $G(p)$ and the project's net payoff.

2.4 Valuing the Option Prior to the Arrival of a Shock

In this base-case model, the option will never be exercised prior to the arrival of the shock.[8] This is because the value of exercising prior to the arrival of the shock is dominated by the value of waiting until the shock arrives and then exercising immediately (which itself is dominated by waiting an additional period of time in order to learn). To see this, note that the difference between exercising prior to the arrival of the shock and exercising upon the arrival of the shock is equal to the value of getting the cash flow X until a shock occurs minus the cost savings between paying I now and waiting until a shock arrives. This difference is equal to $\frac{X}{r+\lambda_1+\lambda_2} - \left(1 - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}\right) I$, or $\frac{X-rI}{r+\lambda_1+\lambda_2}$, which is ensured to be negative by Assumption 1.

Let F denote the initial value of the option, before a shock occurs. This value must satisfy

$$(r + \lambda_1 + \lambda_2) F = (\lambda_1 + \lambda_2) G(p_0). \tag{13}$$

Solving for F:

$$F = \frac{\lambda_1 + \lambda_2}{r + \lambda_1 + \lambda_2} C (1 - p_0) \left(\frac{1}{p_0} - 1\right)^{\frac{r}{\lambda_3}}, \tag{14}$$

[7] This condition is also known as the high-contact condition (see Krylov (1980) and Dumas (1991) for a discussion).

[8] This result will not hold in several generalized versions of the model that follow.

where the constant C is given in (12).

We can now fully summarize the optimal exercise strategy in the following proposition:

Proposition 1. *The optimal exercise strategy for the model set out in Section 2, and subject to the parameter restrictions in Assumption 1, is as follows. The firm should not exercise prior to the arrival of a shock. Upon the arrival of the shock, the firm should exercise the first moment that the posterior probability that the shock is temporary, $p(t)$, decreases to the trigger $\bar{p}$ outlined in Eq. (11).*

2.5 Discussion

While the basic model outlined in this section will later be generalized in several directions, it highlights the key notion of Bayesian learning in a real options context. Perhaps the best way to gain intuition on the optimal investment policy is to re-write the expression for the optimal trigger $\bar{p}$ outlined in Eq. (11) as:

$$X(1+\varphi) = \bar{p}\lambda_3\left(I - \frac{X}{r}\right) + rI. \tag{15}$$

The intuition behind expression (15) is the trade-off between investing now versus investing a moment later if the past shock persists, where the value of waiting is explicitly an "option to learn." If the firm invests now it gets the benefit of the cash flow $X(1+\varphi)$ over the next instant. This is the term on the left hand side of the equal sign. If the firm waits a moment and invests if the past shock persists, it faces a small chance of the shock reversing, in which case the expected value gained by not investing is equal to $\bar{p}\lambda_3\left(I - \frac{X}{r}\right)$. Importantly, by waiting that additional moment, it gains the opportunity to forgo investment should the past shock prove to be temporary. The second term on the right hand side of the equal sign, rI, is the savings from delaying the investment cost by an instant. At the optimal Bayesian trigger, $\bar{p}$, these two sides are exactly equal, and the firm is indifferent between investing now or waiting a moment to learn.

Fig. 2 plots a simulated sample paths of the cash flow process, $X(t)$, and the firm's belief process, $p(t)$. Before the arrival of the shock, the investment is suboptimal since the project does not generate enough cash flows. When the shock arrives, the cash flow process jumps from X to $X(1+\varphi)$ and the NPV of the project becomes positive. Nevertheless, the firm finds investment suboptimal because of the valuable option to learn more about the nature of the past shock. As time goes by and the shock does not revert back, the firm updates its beliefs downwards. When the firm becomes sufficiently sure that the past shock is permanent, it invests. In the example in Fig. 2 this happens more than 1.5 years after the arrival of the shock.

While standard real options models imply that investment will be triggered when future shocks push the cash flow level up to an upper threshold, this model implies that investment is triggered when uncertainty regarding the nature of past shocks falls to a lower threshold. As we also discuss in following sections, this effect can lead to the failure of the record-setting news principle that holds for a large class of real options models (Boyarchenko (2004)) in which investment is seen to occur only at points in which cash flows rise to their historical maximum. In this Bayesian setting in which uncertainty about past shocks is critical, the firm may invest even when the cash flow process is stable or declining simply because it has become more certain about the permanent nature of the past shocks. Empirically, we certainly see examples of firms (and industries) investing in markets where cash flows are stable, or even declining. As detailed in Grenadier (1996), during the late 1970's and early 1980's several U.S. cities saw explosive growth in office building development in the face of rapidly increasing office vacancy rates. Specifically, consider the cases of the Denver and Houston office markets. Over the thirty-year period from 1960 through 1990, over half of all office construction was completed in a four-year interval: 1982-1985. Since office space takes an estimated average length of time between the initiation and completion of construction of 2.5 years, this investment was likely initiated over the period from 1979-1983. Throughout this period, office vacancies in these two cities were around 30%, considerably above previous levels. Notably, these two cities (as well as most of the cities experiencing unprecedented office growth during this period) were oil-patch cities where developers likely concluded (incorrectly) that high oil prices in the late 1970's and early 1980's would last indefinitely, as discussed in the quote provided in the Introduction.

Another interesting feature of the optimal exercise policy is that it implies a sluggish response of investment to shocks. Intuitively, a forward-looking firm is interested not only in the current value of the cash flow process, but also in the likelihood that past positive shocks were caused by temporary fluctuations. As a consequence, when there is substantial uncertainty about past shocks, firms may prefer to wait in order to learn more information about the nature of the past shock. Due to this waiting period, investment may respond quite slowly to shocks. A similar result was obtained by Moore and Schaller (2002) in the context of the neoclassical q theory of investment and interest rate shocks.

3 A Model with an Unlimited Number of Shocks

In this section, we generalize the model by allowing for an unlimited number of shocks.[9] This is an important extension, since it shows that the intuition established in the simple model also holds in more general environments.

Specifically, suppose that at any point in time the firm can have any number $n = 0, 1, ...$

[9]The authors have also solved the model for any finite number of potential shocks. The results are very similar to those presented for the case of a countably infinite number of shocks, but with some additional notational burdens.

of shocks outstanding. As before, the reversal of each outstanding temporary shock occurs with intensity λ_3 independently of all other processes in the economy. In addition, at each time a new permanent shock occurs with intensity λ_1, and a new temporary shock occurs with intensity λ_2.

Given our assumptions, at each time t the state of the economy can be described by a vector of variables $(X(t), p(t))$, where $X(t)$ is the current value of the cash flow process and $p(t) = (p_1(t), p_2(t), \ ...)'$ is the infinitely dimensional vector of the firm's beliefs. Specifically, $p_k(t)$ is the probability at time t that there are k outstanding temporary jumps. Obviously, the firm's belief at time t that there are no outstanding temporary jumps equals $1 - \iota' p(t)$, where ι is the infinitely dimensional vector of ones. The initial state can be described by $(X(0), p(0)) = (X(0), \mathbf{0})$, and at any time there exists $\hat{k}$ such that for all $k > \hat{k}$, $p_k(t) = 0$. Intuitively, $\hat{k}$ corresponds to the total number of outstanding shocks. Note that $X(t)$ can only equal values of $X(0)(1+\varphi)^k$, where $k = 0, 1, ...$

3.1 The Bayesian Learning Process

Before proceeding with solving for the optimal investment policy, we generalize the Bayesian learning process outlined in Section 2.2 to the case of an unlimited number of shocks. Consider moments t and $t + dt$ for any t and infinitesimal positive dt. Depending on the history between t and $t + dt$, there are three cases to be analyzed:

1. no new shocks occur or outstanding shocks reverse between t and $t + dt$;
2. a new shock occurs;
3. an outstanding shock reverses.[10]

If no new shocks or reversions occur between t and $t + dt$, as in the first case above, the posterior probability $p_k(t + dt)$ of having k outstanding temporary shocks at time $t + dt$, can be calculated as

$$p_k(t+dt) = \frac{p_k(t)(1 - (\lambda_1 + \lambda_2)dt - k\lambda_3 dt)}{1 - (\lambda_1 + \lambda_2)dt - \lambda_3 \sum_{i=1}^{\infty} p_i(t)\, i dt}, \tag{16}$$

for $k = 1, 2, ...$ Eq. (16) is a direct application of the Bayes rule. The numerator is the joint probability of having k temporary outstanding jumps and observing no jumps or reversions between t and $t + dt$. The denominator is the sum over $i = 0, 1, ...$ of joint probabilities of having i temporary outstanding jumps and observing no jumps and reversions between t and $t + dt$.

[10] Note that the probability of observing more than one new shock or reversion between t and $t+dt$ has the order $(dt)^2$. Since dt is infinitesimal, we can ignore these cases. The same is true for the likelihood of both a new shock and a reversal occurring at the same instant.

We can rewrite (16) as

$$\frac{p_k(t+dt)-p_k(t)}{dt} = -\frac{\lambda_3 p_k(t)\left(k-\sum_{i=1}^{\infty} p_i(t)\, i\right)}{1-(\lambda_1+\lambda_2)\, dt - \lambda_3 \sum_{i=1}^{\infty} p_i(t)\, i dt}. \tag{17}$$

Taking the limit as $dt \to 0$,

$$\frac{dp_k(t)}{dt} = -\lambda_3 p_k(t)\left(k - \sum_{i=1}^{\infty} p_i(t)\, i\right). \tag{18}$$

The dynamics of $p_k(t)$ are very intuitive. First, as in the one jump case, the speed of learning is proportional to λ_3. Second, $p_k(t)$ increases in time when $k < \sum_{i=1}^{\infty} p_i(t)\, i$, and decreases in time, otherwise. Intuitively, when $k < \sum_{i=1}^{\infty} p_i(t)\, i$, the likelihood of reversion conditional on having k outstanding temporary jumps is lower than the unconditional likelihood of reversion. Therefore, when the firm does not observe a jump or reversion, it updates its beliefs $p_k(t)$ upward. The opposite is true when $k > \sum_{i=1}^{\infty} p_i(t)\, i$.

Now, consider the second case. If a new shock occurs between t and $t+dt$, then the updated beliefs equal

$$p_k(t+dt) \equiv \hat{p}_k(p(t)) = \begin{cases} p_{k-1}(t)\, p_0 + p_k(t)(1-p_0) & \text{for } k = 2, 3, ..., \\ (1-\iota' p(t))\, p_0 + p_1(t)(1-p_0) & \text{for } k = 1. \end{cases} \tag{19}$$

The intuition behind (19) is relatively simple. When a new shock occurs, it can be either permanent or temporary. After the shock, there can be k temporary shocks outstanding either if there were $k-1$ temporary shocks and the new shock is temporary or if there were k temporary shocks and the new shock is permanent. The probability of the first event is $p_{k-1}(t)\, p_0$, while the probability is the second event is $p_k(t)(1-p_0)$. Combining them yields (19).

Finally, considering the third case, if an outstanding shock reverses between t and $t+dt$, by Bayes rule the firm's updated beliefs equal

$$p_k(t+dt) \equiv \tilde{p}_k(p(t)) = \frac{p_{k+1}(t)(k+1)}{\sum_{i=1}^{\infty} p_i(t)\, i} \text{ for } k = 1, 2, ... \tag{20}$$

When a shock reverses, the firm learns that it was temporary. Hence, there are k outstanding temporary jumps after the reversal if and only if there were $k+1$ outstanding temporary jumps before that. The joint probability of having $k+1$ temporary jumps at time t and observing a reversion between t and $t+dt$ equals $p_{k+1}(k+1)\lambda_3 dt$. The probability of observing a reversion between t and $t+dt$ conditional only on being at time t equals $\sum_{i=1}^{\infty} p_i(t)\, i\lambda_3 dt$. Dividing the former probability by the latter yields (20).

Equations (18)-(20) fully characterize the dynamics of the firm's beliefs.

3.2 Optimal Investment

Given the dynamics of the firm's beliefs described by equations (18)-(20), we proceed with solving for the optimal time to invest in this Bayesian framework. The optimal investment problem represents a trade-off between the benefits of receiving immediate cash flows, and the benefits of waiting both to learn more about the state of the economy and to have a chance at investing at a higher X.

Let $S(X,p)$ denote the value of the underlying project. It is the expected discounted value of cash flows that the firm gets if it immediately exercises the investment option. Using the standard arguments, $S(X,p)$ must satisfy:

$$\begin{aligned}(r+\lambda_1+\lambda_2+\lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i)\, S(X,p) &= -\lambda_3 \textstyle\sum_{i=1}^{\infty} \frac{\partial S}{\partial p_i} p_i \left(i - \sum_{j=1}^{\infty} p_j j\right) \\ &+ (\lambda_1+\lambda_2)\, S\left(X(1+\varphi), \hat{p}(p)\right) + \lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i S\left(\frac{X}{1+\varphi}, \tilde{p}(p)\right) + X.\end{aligned} \tag{21}$$

The solution can be written as

$$S(X,p) = a_0 X + \sum_{i=1}^{\infty} p_i (a_i - a_0) X, \tag{22}$$

where constants $a_0, a_1, ...$ are defined in the appendix.

Let $G(X,p)$ denote the value of the investment option. Using standard arguments, prior to exercise $G(X,p)$ must satisfy:

$$\begin{aligned}(r+\lambda_1+\lambda_2+\lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i)\, G(X,p) &= -\lambda_3 \textstyle\sum_{i=1}^{\infty} \frac{\partial G}{\partial p_i} p_i \left(i - \sum_{j=1}^{\infty} p_j j\right) \\ &+ (\lambda_1+\lambda_2)\, G\left(X(1+\varphi), \hat{p}(p)\right) + \lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i G\left(\frac{X}{1+\varphi}, \tilde{p}(p)\right).\end{aligned} \tag{23}$$

The optimal investment decision can be described by a trigger function $\bar{X}(p)$. Eq. (23) is solved subject to the following value-matching and smooth-pasting conditions:

$$\begin{aligned}G\left(\bar{X},p\right) &= S\left(\bar{X},p\right) - I, \\ \lambda_3 \textstyle\sum_{i=1}^{\infty} \left(\frac{\partial G\left(\bar{X},p\right)}{\partial p_i} - \frac{\partial S\left(\bar{X},p\right)}{\partial p_i}\right) p_i \left(i - \sum_{j=1}^{\infty} p_j j\right) &= 0.\end{aligned} \tag{24}$$

These boundary condition are analogous to those from the previous section. The first equation is the value-matching condition. It guarantees that at the moment of investment the value of the investment option equals the net present value of the project. The second equation is the smooth-pasting condition. It states that at the moment of exercise the derivative of the investment option value with respect to time equals the derivative of the underlying asset value with respect to time.

Combining (21), (23) and (24) gives us:

$$\begin{aligned}\bar{X} - rI = (\lambda_1+\lambda_2)&\left[G\left(\bar{X}(1+\varphi), \hat{p}(p)\right) + I - S\left(\bar{X}(1+\varphi), \hat{p}(p)\right)\right] \\ &+\lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i \left[G\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right)\right].\end{aligned} \tag{25}$$

When the state of the economy (X, p) is very close to the trigger $\bar{X}(p)$, it is clear that the arrival of new positive shock will result in immediate investment. The intuition is simple. If the firm is indifferent between investing and waiting today, an upward jump in X will certainly induce the firm to invest immediately. This implies:

$$G\left(\bar{X}(1+\varphi), \hat{p}(p)\right) = S\left(\bar{X}(1+\varphi), \hat{p}(p)\right) - I. \tag{26}$$

Therefore,

$$\bar{X} = \lambda_3 \sum_{i=1}^{\infty} p_i i \left[G\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right)\right] + rI. \tag{27}$$

Given the similarity between (27) and (15), it becomes clear how the multi-shock case generalizes the single-shock case of the previous section. The intuition behind expression (27) is the trade-off between investing now versus investing a moment later if all of the past shocks persist, where the value of waiting is explicitly an "option to learn." If the firm invests now it gets the benefit of the cash flow $\bar{X}$ over the next instant. This is the term on the left hand side of the equal sign. If the firm waits a moment and invests only if all of the past shocks persist, it faces a small chance of one of the shocks reversing, in which case the expected value gained by waiting is equal to first term on the right hand side. Specifically, $G\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}}{1+\varphi}, \tilde{p}(p)\right)$ is the value gained by not investing should an existing shock reverse over the next instant. The second term on the right hand side of the equal sign, rI, is the savings from delaying the investment cost by an instant. At the optimal trigger, $\bar{X}(p)$, these two sides are exactly equal, and the firm is indifferent between investing now or a moment later.

We can now summarize the solution to the optimal investment timing problem in this multi-shock setting.

Proposition 2. *The optimal investment strategy for the model outlined in this section is to invest when $X(t)$ exceeds $\bar{X}(p(t))$ for the first time.*

3.3 Discussion

By allowing for multiple shocks, the model of this section illuminates the nature of the Bayesian investment problem beyond that of the simpler one-shock model of the previous section. Consider the contrast between the structure of the standard real option investment strategy and that of this model. The traditional real option dictum is to invest when the level of cash flows rises to a fixed upper trigger. Importantly, this trigger is a static function of the underlying parameters of the problem. In contrast, the implied investment strategy of this problem is explicitly a dynamic one. Specifically, the trigger itself, $\bar{X}(p)$, is a function of the evolution of past shocks as well as the evolving uncertainty with regard to the determination of their underlying nature. For any given historical sample path of shock arrivals, there exists a distinct prescribed investment policy.

Fig. 3 plots a simulated path of the cash flow process $X(t)$ along with the corresponding investment trigger $\bar{X}(p(t))$. In this example, each shock increases the value of the cash flows by 5%. A new permanent shock occurs, on average, every two years ($\lambda_1 = 0.5$). A new temporary shock occurs, on average, every year ($\lambda_2 = 1$) and takes six months to revert ($\lambda_3 = 2$). The dynamics of the cash flow process is very straightforward. In the 5-year period there are four arrivals of new shocks, corresponding to upward jumps in $X(t)$. Two out of the four shocks revert corresponding to downward jumps in $X(t)$. The dynamics of the investment boundary is more interesting. Prior to the arrival of the first shock, there is no uncertainty about past shocks. Since there is no Bayesian updating at this time, the trigger is constant at $\bar{X}(\mathbf{0})$. After the first shock arrives, the trigger jumps up to $\bar{X}(p_0, 0, ...)$ as the firm becomes unsure if the outstanding shock is permanent or temporary. As time goes by and the cash flow process does not revert back, the first shock is more likely to be permanent. As a result, the firm's beliefs become more optimistic, and the investment trigger decreases. This intuition underlies the whole dynamics of the trigger in Fig. 3. An arrival of a new shock leads to an upward jump in the trigger due to an increase in uncertainty. When the cash flow process is stable, the trigger goes down as the firm updates its beliefs about the past shocks. When an outstanding shock reverses, the firm learns for sure that one of the outstanding shocks is temporary. Hence, reversal leads to a downward jump in the trigger. The investment occurs when the $X(t)$ exceeds the investment trigger $\bar{X}(p(t))$ for the first time. In the example in Fig. 3, this happens at $t = 2.7$ when the firm becomes sufficiently sure that the outstanding shock is permanent.

This example illustrates some of the interesting properties of investment behavior in the Bayesian setting. First, investment responds very sluggishly to innovations in the cash flow process. In the example in Fig. 3 the firm exercises the investment option at $t = 2.7$, while the last positive shock occurred almost 2 years before that. The intuition behind this finding is the same as in the one-jump model. When uncertainty about past shocks is important, the firm has a valuable option to learn more about past shocks. Specifically, if the firm waits after a shock occurs and the value of the cash flow process does not revert back, then the past shock is more likely to be permanent. The more the firm waits, the more "optimistic" beliefs it has about the past shocks. Because of that, eve though the value of the cash flow process is the same at $t = 2.7$ and $t = 1$, the firm believes that the investment opportunity is substantially more attractive at $t = 2.7$.

Second, investment is related to the past realized volatility of cash flows. Specifically, the model predicts that, other things being equal, the firm is less likely to invest after the recent arrival of several new shocks. Even though dependence of the investment strategy of a forward-looking firm on past cash flow volatility might seem irrational, it is perfectly rational in the Bayesian framework. Intuitively, if many new shocks occurred in the recent past, uncertainty about their nature is very high. Because of that, the firm's option to wait in order to reduce uncertainty is very valuable. As a result, the firm is better off postponing the investment opportunity to the future.

In addition, since the investment boundary evolves over time, it can be the case that investment occurs when $X(t)$ is not at its historical highest level. Thus, the record-setting news principle that holds for a large class of real options models (Boyarchenko (2004)) is violated in the Bayesian case. Consider the following scenario that illustrates this potential occurrence. Suppose that a market experienced a series of positive shocks during the past. While this would have been considered good news, firms would have waited a period of time prior to investment in order to take advantage of multiple "options to learn" about the nature of these shocks. Now, suppose that during this learning period several of these shocks reversed themselves, proving that they were indeed temporary. At such a point, the level of $X(t)$ would be strictly below its historic high. If a sufficient period of time goes by without these remaining shocks reversing, the firm will gain enough confidence in their permanent nature to justify investment. This scenario is exactly what happened in the example in Fig. 3. As we can see, after the arrival of the second shock the value of the cash flow process was even higher than at the exercise time. Nevertheless, the firm invested later at a time of lower cash flows simply because it became sufficiently sure that all outstanding shocks were permanent.

4 Interaction Between Bayesian and Brownian Uncertainties

In the models of Sections 2 and 3, the cash flow process $X(t)$ is a pure jump process. Since most traditional real options models are based on Brownian uncertainty, it is interesting to investigate how the option to wait in order to learn about the past shocks is affected by the addition of Brownian uncertainty. To do this, we generalize the model by allowing $X(t)$ to follow a combined Poisson and geometric Brownian motion process. We find that the addition of Brownian uncertainty does not undo the effect that Bayesian learning has on the optimal exercise rule. In fact, the impact of Brownian uncertainty is additive, in that we essentially have an investment rule that mirrors the previously derived Bayesian trigger, but with an additional convexity adjustment due to Brownian uncertainty. Investors now have two potentially valuable options to wait: an option to learn about past shocks, and an option to wait for positive growth in future cash flows.

Note that this generalization makes the mathematics of the solution considerably more complicated. For tractability, we return to the assumption of a single jump.

4.1 Optimal Investment While the Shock Persists

As before, consider the situation when there is an outstanding jump which can be either permanent or temporary. In addition to being subject to the jump process described in Section 2, the cash flow process $X(t)$ follows the geometric Brownian motion. Specifically, after the shock occurs $X(t)$ evolves as

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB_t - \mathbf{1}_{temp}\frac{\varphi X(t)}{1+\varphi}dN_t, \tag{28}$$

where $\mathbf{1}_{temp}$ is an indicator function of a temporary outstanding jump, α is the instantaneous conditional expected percentage change in X per unit time, σ is the instantaneous conditional standard deviation per unit time, B is a standard Wiener process, and N is a reversion process which equals 1 after reversion occurs and 0 before. As before, reversion occurs with intensity λ_3. We assume that $\alpha < r$, to ensure finite values. As in Section 2, the conditional probability that an existing shock is temporary follows the equation of motion:

$$\frac{dp(t)}{dt} = -\lambda_3 p(t)(1-p(t)). \tag{29}$$

Similarly to Section 2, we begin by calculating some simple values. Suppose that the current value of the cash flow process is X. If the firm knows for sure that a shock is permanent, then the value of investing immediately would be $\frac{1}{r-\alpha}X - I$. If the firm knows for sure that the shock is temporary, then the value of investing immediately would equal $\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}X - I$.

Let $G(X,p)$ denote the value of the option to invest while the shock persists, where X and p are the current values of the cash flow and the belief processes, respectively. Analogous to equation (5), in the range of (X,p) at which the option is not exercised, $G(X,p)$ must satisfy the equilibrium partial differential equation:

$$(r + p\lambda_3)\,G = \alpha X G_X + \frac{\sigma^2}{2}X^2 G_{XX} - \lambda_3 p(1-p)G_p + p\lambda_3 H\left(\frac{X}{1+\varphi}\right), \tag{30}$$

where $H(X)$ is the value of the option when no more jumps can occur. After the shock reverts back, the problem becomes equivalent to the standard real options problem that has been solved many times in the literature (e.g., Dixit and Pindyck (1994)). The solution for the option when no more jumps can occur is

$$H(X) = \begin{cases} \left(\frac{X}{X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right) & if\ X < X^*, \\ \frac{X}{r-\alpha} - I & otherwise, \end{cases} \tag{31}$$

where is β is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta -$

$r = 0$:

$$\beta = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1, \tag{32}$$

and X^* is the critical value at which it is optimal to invest:

$$X^* = \frac{(r-\alpha)\,\beta}{\beta - 1} I. \tag{33}$$

Let $\bar{X}(p)$ denote the exercise trigger function. Conjecture that it is optimal for the firm to invest prior to $X(t)$ reaching $X^*(1+\varphi)$, and we shall later confirm this. Equation (30) is solved subject to the following value-matching and smooth-pasting conditions:

$$G\left(\bar{X}, p\right) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right]\bar{X} - I, \tag{34}$$

$$G_X(\bar{X}, p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right], \tag{35}$$

$$G_p(\bar{X}, p) = \left[-\frac{1}{r-\alpha} + \frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right]\bar{X}. \tag{36}$$

Intuitively, the value-matching condition (34) captures the fact that at the time of investment the value of the investment option equals the expected payoff from immediate investment, while the smooth-pasting conditions (35) - (36) guarantee that the trigger function is chosen optimally.

Evaluating (30) at $\bar{X}(p)$, plugging in the boundary conditions (34) - (36) and simplifying, provides the following expression for the optimal trigger function $\bar{X}(p)$:

$$\begin{aligned} \bar{X}(p) = p\lambda_3 &\left[\left(\tfrac{\bar{X}(p)}{(1+\varphi)X^*}\right)^{\beta}\left(\tfrac{X^*}{r-\alpha} - I\right) - \left(\tfrac{\bar{X}(p)}{(1+\varphi)(r-\alpha)} - I\right)\right] \\ &+ rI + \tfrac{\sigma^2}{2}\bar{X}(p)^2\, G_{XX}\left(\bar{X}(p), p\right). \end{aligned} \tag{37}$$

In comparing the expression for $\bar{X}(p)$ to the trigger functions without Brownian uncertainty obtained in the previous sections, we find that we now have the same general form of the trigger, plus a new convexity term, $\frac{\sigma^2}{2}\bar{X}(p)^2\, G_{XX}\left(\bar{X}(p), p\right)$. Here, the trigger equals the sum of the value of the option to learn to see if the shock reverts and a convexity term that represents the traditional option to wait in the real options literature.[11] In this sense, Brownian uncertainty is additive to Bayesian uncertainty. In other words, the addition of Brownian uncertainty to the model increases the investment trigger by a further component due to the option value of waiting for the evolution of Brownian uncertainty over future cash flows.

It is relatively straightforward to demonstrate that, while the shock persists, the firm chooses to exercise at a trigger that is below $(1+\varphi)X^*$, as conjectured earlier. Consider the problem above, where the shock persists. Now, let us modify the problem in the following way. Suppose that while the firm waits, the jump cannot revert back, and after the firm invests, it reverts back in the same way as in the original problem. Since for any (X,p) this modification does not affect the value of immediate investment, but increases the value of waiting relative to the initial problem, the corresponding investment trigger $\bar{X}_{\text{mod}}(p)$ is higher than $\bar{X}(p)$ for any $p \in (0, p_0]$. Since the jump cannot revert back before investment, the firm does not update its beliefs in the modified problem. As a result, the investment trigger in the modified problem can be explicitly computed:

$$\bar{X}(p) < \bar{X}_{\text{mod}}(p) = \frac{\beta}{\beta-1} \frac{r-\alpha}{1 - p\frac{\lambda_3 \varphi}{(r-\alpha+\lambda_3)(1+\varphi)}} I < (1+\varphi) X^*. \tag{38}$$

This implies that indeed, it is optimal for the firm to invest at a trigger that is below $(1+\varphi)X^*$.

By a similar argument to that of the preceding paragraph, it is clear that the trigger function $\bar{X}(p)$ is increasing in p. For any two values of p at exercise, the expected payoff from exercise is higher for the case of the lower value of p. Thus for lower values of p, exercise will occur earlier due to the higher expected payoff. Given this monotonicity, we can invert the function and also express the exercise trigger by the function $\bar{p}(X)$. In this formulation, the firm's investment strategy can be characterized in the following way. At any time t, given the current value of the cash flow process $X(t)$, the firm compares its beliefs $p(t)$ with the boundary level $\bar{p}(X(t))$ and invests at the first instant when $p(t)$ becomes lower than $\bar{p}(X(t))$.

While the exercise trigger $\bar{p}(X)$ is characterized by (37), it is not solvable in closed-form, since the value function $G(X,p)$ itself is not available in closed-form. However, for the special case in which $\sigma = 0$, $\alpha \geq 0$, the closed-form solution for the trigger is

$$\bar{p}(X)\,|_{\sigma=0} = \frac{X - rI}{\lambda_3 \left[I + \left(\frac{X}{rI(1+\varphi)} \right)^{\frac{r}{\alpha}} \frac{\alpha I}{r-\alpha} - \frac{X}{(1+\varphi)(r-\alpha)} \right]}. \tag{39}$$

[11] In order to ensure optimality of the exercise trigger, $\bar{X}(p)$, $G(X,p)$ must be convex at $\bar{X}(p)$. To see this, note that for a given p, $G(X,p) > h(p)X - I$ for all $X < \bar{X}(p)$, where $h(p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\phi)}}{r-\alpha+\lambda_3} \right]$. From the value-matching condition, at $\bar{X}(p)$, $G(\bar{X}(p),p) = h(p)\bar{X}(p) - I$, and from the smooth-pasting condition $G_X(\bar{X}(p),p) = h(p)$. Thus, at $\bar{X}(p)$, it must be the case that $G_{XX}(\bar{X}(p),p) > 0$.

The corresponding value of the investment option equals

$$G(X,p)|_{\sigma=0} = p\frac{\alpha I}{r-\alpha}\left(\frac{X}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}} + (1-p)X^{\frac{r}{\alpha}}\Gamma\left(X\left(\frac{1}{p}-1\right)^{-\frac{\alpha}{\lambda_3}}\right), \tag{40}$$

where

$$\begin{aligned}\Gamma(y) = \left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y \Bigg(& \tfrac{\lambda_1}{r-\alpha} e^{(\alpha-r)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)} + \lambda_2 \tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{(r+\lambda_3-\alpha)(1+\varphi)} e^{(\alpha-r-\lambda_3)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)}\Bigg) \\ & - \left(\tfrac{y}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}} \tfrac{\alpha I}{r-\alpha}\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{r}{\lambda_3}} \lambda_2 e^{-\lambda_3 t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)},\end{aligned} \tag{41}$$

where $t^*(z)$ is a function defined implicitly by

$$\frac{\lambda_2\lambda_3}{\lambda_1 e^{\lambda_3 t^*} + \lambda_2} = \frac{ze^{\alpha t^*} - rI}{\left(\frac{z}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}} \frac{\alpha I e^{rt^*}}{r-\alpha} + I - \frac{ze^{\alpha t^*}}{(r-\alpha)(1+\varphi)}}. \tag{42}$$

Notice that when $\alpha = 0$, investment does not occur when the jump reverts. Therefore, in this case, the trigger (39) coincides with (11)[12].

The quantitative effects of the addition of Brownian uncertainty to the model are illustrated by Fig. 4 which shows the investment trigger function $\bar{X}(p)$ for different values of the volatility parameter.[13] Both Bayesian and Brownian uncertainties lead to a significant increase in the investment trigger. If there were no Brownian or Bayesian uncertainty and all shocks were permanent, the trigger would equal 1. Now, consider the case in which we add only Bayesian uncertainty, corresponding to the bottom curve in which $\sigma = 0$. We shall consider trigger values calculated at $p = 2/3$, equaling the value of p_0 for our parameter specification. Here we find that the impact of pure Bayesian uncertainty increases the investment threshold by 12.6%, which is labeled as the "Bayesian effect" in Fig. 4. Now, consider the additional impact of Brownian uncertainty. The middle and top curves correspond to the cases of $\sigma = 0.05$ and $\sigma = 0.10$, respectively. The addition of Brownian uncertainty leads to an *additional* increase in the threshold by 5.6% for the case of $\sigma = .05$ and by 20% for $\sigma = .10$, which are labeled as the "Brownian effects" in Fig. 4.[14] As discussed above, the addition of Brownian uncertainty does not

[12] Note that since in (11) X denotes the level of $X(t)$ before the positive jump occurred, we need to use $\bar{p}(X(1+\varphi))$ to ensure equivalence.

[13] To compute the trigger functions we used a variation of the least-squares method developed by Longstaff and Schwartz (2003). Since $p(T) \to 0$ as $T \to \infty$ and $\bar{X}(0) = X^*$, we can approximate $\bar{X}(p(T))$ for some large T by $\bar{X}(0)$. After that, we use least squares to estimate the second derivative of the conditional expected payoff from continuation. This estimate and (37) are used to compute $\bar{X}(p(t-\Delta))$ for some small Δ given $\bar{X}(p(s))$ for all $s \in [t, T]$.

[14] Note that this implies that the total instantaneous volatility of the cash flow process is higher than σ, since it also includes volatility due to the jumps.

undo the effect that Bayesian learning has on the optimal exercise rule. Indeed, the shape of the trigger function $\bar{X}(p)$ does not change much as the Brownian volatility parameter σ increases. For any σ, $\bar{X}(p)$ is an increasing and concave function of p, and the value of the option to learn is significant.

4.2 Optimal Investment Prior to the Arrival of a Shock

To complete the solution of the model, consider the value of the investment option before the jump occurs. In this case, the only underlying uncertainty concerns the future path of $X(t)$. Specifically, $X(t)$ evolves as

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB_t + \varphi X(t)\,dM_t, \tag{43}$$

where M is a process which equals 1 after a shock occurs and 0 before that. As before, the intensity of M is $\lambda_1 + \lambda_2$: a permanent jump occurs with intensity λ_1, while a temporary jump occurs with intensity λ_2.

Denote the option value by $F(X)$, where X is the current value of the state variable. Prior to the investment, $F(X)$ solves

$$(r + \lambda_1 + \lambda_2) F(X) = \alpha X F'(X) + \frac{1}{2}\sigma^2 X^2 F''(X) + (\lambda_1 + \lambda_2) G(X(1+\varphi), p_0), \tag{44}$$

where $G(X,p)$ is the value of the investment option when the shock persists. If the investment option is exercised prior to the arrival of a shock, the firm receives:

$$\frac{X}{r-\alpha} + \frac{\varphi X}{r - \alpha + \lambda_1 + \lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right) - I. \tag{45}$$

The first term of (45) is the discounted cash flows that the firm gets if the jump never occurs, while the last term is the investment cost that the firm needs to incur to launch the project. The two other terms correspond to the additional cash flows the firm gets from the shocks. If the shock is permanent, the firm gets additional expected cash flows of $\frac{\varphi X e^{\alpha\tau}}{(r-\alpha)}$ at the time of the shock τ. If the shock turn out to be temporary, the additional expected cash flows at the time of the shock τ are $\frac{\varphi X e^{\alpha\tau}}{(r-\alpha+\lambda_3)}$. Integrating over τ yields the second and third terms of (45).

Let $\hat{X}$ denote the optimal investment trigger. Conjecture that it is strictly optimal to wait when $X(t)$ is below $\bar{X}(p_0)/(1+\varphi)$, that is, $\hat{X} \geq \bar{X}(p_0)/(1+\varphi)$. We confirm this conjecture below. Then, the investment option value $F(X)$ can be divided into two parts $F_L(X)$ and $F_H(X)$, corresponding to the lower and the higher regions, respectively. By Itô's lemma, $F_L(X)$ and $F_H(X)$ satisfy the following differential equations:

- in the region $X < \bar{X}(p_0)/(1+\varphi)$,

$$(r+\lambda_1+\lambda_2) F_L(X) = \alpha X F_L'(X) + \frac{1}{2}\sigma^2 X^2 F_L''(X) + (\lambda_1+\lambda_2) G(X(1+\varphi), p_0); \tag{46}$$

- in the region $\bar{X}(p_0)/(1+\varphi) < X < \hat{X}$,

$$\begin{aligned}(r+\lambda_1+\lambda_2) F_H(X) &= \alpha X F_H'(X) + \tfrac{1}{2}\sigma^2 X^2 F_H''(X) \\ &+ \left(\lambda_1 \tfrac{1+\varphi}{r-\alpha} + \lambda_2 \tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{r-\alpha+\lambda_3}\right) X - (\lambda_1+\lambda_2) I.\end{aligned} \tag{47}$$

The differential equations (46) and (47) differ due to the implied investment behavior at the moment of the arrival of a shock. In the lower region the arrival of a shock does not induce immediate investment, while in the higher region the arrival of a shock implies immediate investment.

Differential equations (46) and (47) are solved subject to the following boundary conditions:

$$F_H\left(\hat{X}\right) = \frac{\hat{X}}{r-\alpha} + \frac{\varphi\hat{X}}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, \tag{48}$$

$$F_H'\left(\hat{X}\right) = \frac{1}{r-\alpha} + \frac{\varphi}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right), \tag{49}$$

$$\lim_{X\uparrow\bar{X}(p_0)/(1+\varphi)} F_L(X) = \lim_{X\downarrow\bar{X}(p_0)/(1+\varphi)} F_H(X), \tag{50}$$

$$\lim_{X\uparrow\bar{X}(p_0)/(1+\varphi)} F_L'(X) = \lim_{X\downarrow\bar{X}(p_0)/(1+\varphi)} F_H'(X), \tag{51}$$

$$\lim_{X\to 0} F_L(X) = 0. \tag{52}$$

As before, the value-matching condition (48) imposes equality at the exercise point between the value of the option and the expected payoff from immediate investment, while the smooth-pasting condition (49) ensures that the exercise point is chosen optimally. Conditions (50) and (51) guarantee that the value of the investment option is continuous and smooth. Finally, (52) is a boundary condition that reflects the fact that $X = 0$ is the absorbing barrier for the cash flow process.

In the appendix we show that combining equations (46)-(47) with the boundary conditions (48)-(52) yields the following investment trigger $\hat{X}$:

$$\begin{aligned}\hat{X} &= \tfrac{\gamma_1}{\gamma_1-1}\tfrac{(r-\alpha+\lambda_1+\lambda_2)}{r+\lambda_1+\lambda_2}\left[rI + \left(\tfrac{\bar{X}(p_0)}{(1+\varphi)\hat{X}}\right)^{-\gamma_2}(\lambda_1+\lambda_2) I\right] \\ &+ \left(\tfrac{\bar{X}(p_0)}{(1+\varphi)\hat{X}}\right)^{-\gamma_2}\left[-\left(\tfrac{\lambda_1}{r-\alpha} + \tfrac{\lambda_2+\frac{\lambda_2\lambda_3}{(1+\varphi)(r-\alpha)}}{r-\alpha+\lambda_3}\right)\bar{X}(p_0) + \tfrac{2(\lambda_1+\lambda_2)(r-\alpha+\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-1)}\Gamma_2\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\right],\end{aligned} \tag{53}$$

where γ_1 and γ_2 are the positive and negative roots the fundamental quadratic equation $\frac{1}{2}\sigma^2\gamma(\gamma-1)+\alpha\gamma-r-\lambda_1-\lambda_2=0$,

$$\gamma_1=\frac{1}{\sigma^2}\left[-\left(\alpha-\frac{\sigma^2}{2}\right)+\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2\left(r+\lambda_1+\lambda_2\right)\sigma^2}\right]>\beta>1, \tag{54}$$

$$\gamma_2=\frac{1}{\sigma^2}\left[-\left(\alpha-\frac{\sigma^2}{2}\right)-\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2\left(r+\lambda_1+\lambda_2\right)\sigma^2}\right]<0, \tag{55}$$

and

$$\Gamma_2(X)\equiv X^{\gamma_2}\int\frac{G\left(X\left(1+\varphi\right),p_0\right)}{X^{\gamma_2+1}}dX. \tag{56}$$

The corresponding value of the investment option $F(X)$ is shown in the appendix.

To complete the solution it remains to demonstrate that for any X below $\bar{X}(p_0)/(1+\varphi)$, it is strictly optimal for the firm to postpone the investment opportunity. This can be done using an argument similar to the one in the previous subsection. By contradiction, suppose that it is optimal to invest at some trigger X' below $\bar{X}(p_0)/(1+\varphi)$. Let us modify the problem in the following way. Suppose that the upward jump occurs immediately, that is, $\lambda_1+\lambda_1=+\infty$ with p_0 being unchanged. Notice that for each sample path the project in the modified problem yields the same cash flows as the project in the original problem with the difference that the range of extra cash flows generated by the upward shock occurs earlier. Hence, investment in the modified problem occurs earlier than in the original problem. In particular, since it was optimal to invest at X' in the original problem, it is optimal at X' in the modified problem. Since in the modified problem the jump occurs immediately, the value of the investment option is $G(X(1+\varphi),p_0)$. However, from the previous subsection we know that it is strictly optimal to wait for all $X(1+\varphi)<\bar{X}(p_0)$. Therefore, it cannot be optimal to invest at any X'. This implies that indeed, it is strictly optimal to wait for any X below $\bar{X}(p_0)/(1+\varphi)$. Therefore, the optimal investment policy is characterized by the critical value (53) at which it is optimal to invest.

We can now fully summarize the optimal investment strategy in the following proposition:

Proposition 3. *The optimal investment strategy for the model with both Bayesian and Brownian uncertainty is:*

1. *If the shock does not occur until $X(t)$ reaches $\hat{X}$, then it is optimal for the firm to invest when $X(t)=\hat{X}$;*

2. *If the shock occurs before the point when $X(t)$ reaches $\hat{X}$, and at the time of the shock $X(t)$ is above $\bar{X}(p_0)/(1+\varphi)$, then it is optimal for the firm to invest immediately after the shock;*

3. *If the shock occurs before $X(t)$ reaches $\hat{X}$, and at the time of the shock $X(t)$ is below $\bar{X}(p_0)/(1+\varphi)$, then it is optimal for the firm to invest at the first time when $X(t) = \bar{X}(p(t))$;*

4. *If the shock occurs before $X(t)$ reaches $\hat{X}$, at the time of the shock $X(t)$ is below $\bar{X}(p_0)/(1+\varphi)$, and it reverts back before $X(t)$ reaches $\bar{X}(p(t))$, then it is optimal to invest when $X(t) = X^*$.*

4.3 Discussion

Proposition 3 demonstrates that there are four fundamentally different scenarios for the firm's investment in the model with both Bayesian and Brownian uncertainty. Under the first scenario, the cash flow process $X(t)$ increases up to $\hat{X}$ before the jump occurs. Prior to the arrival of the shock there is no learning, so the investment trigger is constant over time. One simulated sample path that satisfies this scenario is shown in the upper left corner of Fig. 5.

Under the other three possible scenarios, the shock occurs before the cash flow process reaches $\hat{X}$, so the firm does not make investment prior to the arrival of the shock. After the arrival of the shock, the investment trigger is a function of the firm's beliefs about the past shock given by (37). Thus, as time passes and the shock persists, the firm learns more about the nature of the past shock. When the firm observes that the shock does not revert back, it updates (lowers) its assessment of the probability that the past shock was temporary. Since immediate investment is more attractive when the past shock was permanent, the investment trigger $\bar{X}(p(t))$ decreases over time.

Under the second scenario, the value of the cash flow process immediately after the shock overshoots $\bar{X}(p_0)$. In this case, the investment occurs immediately after the shock arrives. A simulated sample path satisfying this scenario is shown in the upper right corner of Fig. 5.

In the third scenario, the cash flow process $X(t)$ reaches the investment trigger $\bar{X}(p(t))$ prior to any potential reversion of the shock. A simulated sample path illustrating this scenario is shown in the lower left corner of Fig. 5. Notice that this graph provides an illustration of several interesting properties of investment in a Bayesian setting. First, the graph illustrates the sluggish response of investment to shocks. Even though the cash flow process exhibits the largest increases between times 0.4 and 1.0, the firm invests much later at a time when the cash flow process does not exhibit such large increases. Intuitively, in our setting the firm values not only high cash flows, but also confidence that current cash flows will persist into the future. Hence, investment responds sluggishly to the past shock because of the value of waiting to learn. Second, the graph illustrates the violation of the record-setting news principle. While the cash flow process peaks at the level around 1.33 soon after the shock, the firm invests later at a considerably lower level of the cash flow process (around 1.23).

Finally, under the fourth scenario the past temporary shock reverts back before the firm invests. If this happens, the problem becomes standard. After the reversal, the investment trigger is constant over time at the level X^*. The firm invests at the first time when the cash flow process $X(t)$ reaches X^*. A simulated sample path that describes the fourth scenario is shown in the lower right corner of Fig. 5.

5 The Implications of Cash Flow Timing

In the standard real options literature, there is a simple equivalence between options that pay off in cash flows and those that pay off with an identical lump sum value. For example, Chapter 5 in Dixit and Pindyck (1994) considers the optimal exercise rule for options that pay off with a lump sum value of $V(t)$. Then, in Chapter 6, they perform a similar analysis for options that pay off with a perpetuity cash flow of $P(t)$, with identical present value to the lump sum value $V(t)$. They show that the optimal exercise rules are identical.

However, in the context of valuations that are driven by the possibility of both temporary and permanent shocks, the timing of cash flows can be quite important. The greater the "front-loadedness" of the option payoff, the less important is the assessment of the relative likelihood that a shock is temporary or permanent. In this section we consider a simple parameterization of the front-loadedness of the option payoff, ranging from payoffs that are equivalent to a one-time lump sum to payoffs that are equivalent to perpetual cash flows.

Consider the model of Section 2, but with one alteration. Assume now that if an option is exercised at time τ, it provides a stream of payments $\left(1+\frac{k}{r}\right)e^{-k(t-\tau)}X(t)$, $t \geq \tau$. Parameter $k \in [0,+\infty)$ captures the degree of front-loadedness of the project. Projects with low values of k are relatively back-loaded: much of their cash flows are generated long after the exercise time. High values of k mean that the project is relatively front-loaded, with most cash flows coming relatively close to the exercise time. The particular parameterization was chosen so as to make the present value of cash flows from the immediate exercise of the project in the no-shock case independent of k: $\int_{\tau}^{\infty} X(\tau)\left(1+\frac{k}{r}\right)e^{-k(t-\tau)}e^{-r(t-\tau)}dt = \frac{X(\tau)}{r}$. Of course, other reasonable parameterizations are possible.

This specification of cash flows captures two cases widely used in the real options literature. First, when $k = 0$, the model reduces to the one studied in Section 2. In this case, the project pays a perpetual flow of $X(t)$ upon exercise. Second, if $k \to \infty$, payments from the project converge to a one time lumpy payment of $\frac{X(\tau)}{r}$ at the time of exercise τ.

Similar to Assumption 1 of Section 2, we now make Assumption 2 to ensure that the project has a potentially positive net present value and that there is positive value to learning:

Assumption 2. *The initial value of the cash flow process X satisfies*

$$\frac{rI}{1+\varphi} < X < \frac{r + \lambda_3 p_0}{1 + \varphi + \frac{\lambda_3}{r} p_0 + \frac{\lambda_3}{r} p_0 \frac{k\varphi}{(r+\lambda_3+k)}} I.$$

Compared to Assumption 1, Assumption 2 puts the same lower bound and a more restrictive upper bound. As previously, these bounds guarantee that the solution to the investment timing problem is non-trivial.

As in Section 2.3, let the value of the option while the shock persists be denoted by $G\left[p(t)\right]$. Over the range of $p(t)$ at which the option is not exercised, the standard argument implies

$$(r + p\lambda_3)\, G = -G_p \lambda_3 p(1-p). \tag{57}$$

This equation has the general solution

$$G\left[p(t)\right] = C_1 (1 - p(t)) \left(\frac{1}{p(t)} - 1\right)^{\frac{r}{\lambda_3}}, \tag{58}$$

where C_1 is a constant. Note this general solution coincides with the one obtained in Section 2, equation (8). However, because of the more general cash flow timing assumption, the boundary conditions are now different.

Because the payoff from the project if the current shock is temporary is affected by the parameter k, the value-matching condition at the exercise trigger $\bar{p}_k$ is now:

$$G(\bar{p}_k) = (1 - \bar{p}_k) \frac{X(1+\varphi)}{r} + \bar{p}_k \frac{X\left(1 + \varphi + \frac{\lambda_3}{r} + \frac{k(1+\varphi)}{r}\right)}{r + \lambda_3 + k} - I. \tag{59}$$

Note that although the present value of the firm's cash flow in the case of a permanent jump does not depend on k, in the case of a temporary jump it depends positively on k. Intuitively, a more front-loaded project allows the firm to capture more of the temporarily high cash flows than does a more back-loaded project.

As in Section 2, the exercise trigger is chosen to maximize the value of the option (or equivalently, to satisfy the smooth-pasting condition), giving the resulting optimal trigger value:

$$\bar{p}_k = \frac{X(1+\varphi) - rI}{\lambda_3\left(I - \frac{X}{r} - \frac{Xk\varphi}{r(r+\lambda_3+k)}\right)}. \tag{60}$$

Note, when $k = 0$, $\bar{p}_0$ gives us the same trigger ($\bar{p}$) we obtained in Section 2.

Again, given that there is value to learning, it is straightforward to show that the option will never be exercised prior to the arrival of the shock. Thus, the optimal investment rule is indeed for the firm to invest at the first moment that the posterior probability $p(t)$ falls to the trigger $\bar{p}_k$, and never if the trigger is not reached.

Consider how the parameter of front-loadedness affects the trigger value:

$$\frac{\partial \bar{p}_k}{\partial k} = \frac{r\varphi(r+\lambda_3)}{\lambda_3} \frac{X(1+\varphi) - Ir}{\left[X\left(r+\lambda_3+k(1+\varphi)\right) - Ir\left(r+\lambda_3+k\right)\right]^2} X > 0. \tag{61}$$

We therefore find that the greater the front-loadedness, the earlier the option is exercised. In contrast, learning has greater value for projects whose payoffs arrive further in the future.

6 Conclusion

This paper proposes a novel kind of real options problem in which uncertainty about both past and future shocks is important. Specifically, we assume that when the firm observes a shock, it is unable to identify whether it is permanent or temporary. As a consequence, unlike the standard models, the evolving uncertainty is driven by Bayesian updating, or learning. This leads to a conflict between two opposing forces: the desire of the firm to take advantage of the option to learn, and the desire to invest early to capture current cash flows.

We solve for the optimal investment rule in this framework, and show that it implies an investment behavior which differs significantly from that predicted by prior models. Specifically, we find three new results. First, the “record-setting news principle” may not holds in the Bayesian setting, and investment might occur at a time of stable or even decreasing cash flows. Second, investment respond sluggishly to positive cash flow shocks. Finally, investment behavior is affected not only by the net present value of the project, but also by the maturity structure of its cash flows.

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Appendix

Derivation of $S(X,p)$.

Conjecture that equation (21) is solved by (22) for some constants a_0,a_1,... . Plugging (22) into (21), we get

$$\begin{array}{c}\left(r+\lambda_1+\lambda_2+\lambda_3\sum_i ip_i\right)a_0+\left(r+\lambda_1+\lambda_2\right)\sum_i\left(a_i-a_0\right)p_i=\\ =\left(\lambda_1+\lambda_2\right)\left(1+\varphi\right)\left(p_0a_1+\left(1-p_0\right)a_0+\sum_i\left[p_0\left(a_{i+1}-a_1\right)+\left(1-p_0\right)\left(a_i-a_0\right)\right]p_i\right)\\ +\frac{\lambda_3}{1+\varphi}\sum_i a_{i-1}ip_i-\lambda_3\sum_i\left(a_i-a_0\right)ip_i+1.\end{array} \tag{62}$$

This equation must holds for any p. This happens if and only if coefficients with 1, p_1,p_2,... on the left hand and right hand side are equal. Matching the coefficients, we get

$$a_k=\begin{cases}\frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1}a_1+\frac{1}{r+\lambda_2-\varphi\lambda_1} \text{ for } k=0,\\ \frac{1}{r+\lambda_2-\varphi\lambda_1+\lambda_3 k}+\frac{\lambda_3 k}{(1+\varphi)(r+\lambda_2-\varphi\lambda_1+\lambda_3 k)}a_{k-1}+\frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1+\lambda_3 k}a_{k+1} \text{ for } k=1,2,...\end{cases} \tag{63}$$

Hence, coefficients a_0,a_1,... are defined as solutions to this recurrence relation subject to the boundary condition $\lim_{k\to\infty} a_k=0$.

Derivation of the investment trigger $\hat{X}$ **and the investment option value** $F(X)$ **for the model with Brownian uncertainty.**

The general solutions to Eq. (46) and (47) are given by

$$F_L(X)=C_1X^{\gamma_1}+C_2X^{\gamma_2}+\frac{2\left(\lambda_1+\lambda_2\right)}{\sigma^2\left(\gamma_1-\gamma_2\right)}\left(\Gamma_2(X)-\Gamma_1(X)\right), \tag{64}$$

$$F_H(X)=A_1X^{\gamma_1}+A_2X^{\gamma_2}+\frac{\left(\lambda_1+\lambda_2\right)\left(\left(1+\varphi\right)\left(r-\alpha\right)+\lambda_3\right)+\lambda_1\lambda_3\varphi}{\left(r-\alpha+\lambda_1+\lambda_2\right)\left(r-\alpha\right)\left(r-\alpha+\lambda_3\right)}X-\frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I, \tag{65}$$

where $\Gamma_2(X)$ is given by (56) and

$$\Gamma_1(X)=X^{\gamma_1}\int\frac{G\left(X\left(1+\varphi\right),p_0\right)}{X^{\gamma_1+1}}dX. \tag{66}$$

We have five boundary conditions (48)-(52) to determine four unknown constants (A_1, A_2, C_1 and C_2) and the investment trigger $\hat{X}$. The fifth boundary condition implies that

$C_2 = 0$. The other four boundary conditions give the following system of equations:

$$
\begin{array}{c}
A_1 \hat{X}^{\gamma_1} + A_2 \hat{X}^{\gamma_2} = \frac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \frac{rI}{r+\lambda_1+\lambda_2} \\
\gamma_1 A_1 \hat{X}^{\gamma_1} + \gamma_2 A_2 \hat{X}^{\gamma_2} = \frac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} \\
A_1 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + A_2 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2} + \frac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \frac{\bar{X}(p_0)}{1+\varphi} - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I \\
= C_1 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\Gamma_2\left(\frac{\bar{X}(p_0)}{1+\varphi}\right) - \Gamma_1\left(\frac{\bar{X}(p_0)}{1+\varphi}\right)\right) \\
\gamma_1 A_1 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \gamma_2 A_2 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2} + \frac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \frac{\bar{X}(p_0)}{1+\varphi} \\
= \gamma_1 C_1 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\gamma_2\Gamma_2\left(\frac{\bar{X}(p_0)}{1+\varphi}\right) - \gamma_1\Gamma_1\left(\frac{\bar{X}(p_0)}{1+\varphi}\right)\right),
\end{array}
\tag{67}
$$

Combining the last two equations, we get

$$
\begin{array}{c}
A_2 = \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{-\gamma_2} \left[\frac{1-\gamma_1}{\gamma_1-\gamma_2} \frac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \frac{\bar{X}(p_0)}{1+\varphi}\right. \\
\left. + \frac{\gamma_1}{\gamma_1-\gamma_2} \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \Gamma_2\left(\frac{\bar{X}(p_0)}{1+\varphi}\right)\right].
\end{array}
\tag{68}
$$

Combining the first two equations, we get

$$
\hat{X} = \frac{\gamma_1}{\gamma_1 - 1} \frac{(r-\alpha+\lambda_1+\lambda_2)\, rI}{r+\lambda_1+\lambda_2} + \frac{\gamma_1-\gamma_2}{\gamma_1-1} A_2 \hat{X}^{\gamma_2}. \tag{69}
$$

Plugging (68) into (69) yields the expression for the investment trigger (53).

The corresponding value of the investment opportunity is

$$
F(X) = \begin{cases}
C_1 X^{\gamma_1} + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\Gamma_2(X) - \Gamma_1(X)\right), & X \le \frac{\bar{X}(p_0)}{1+\varphi} \\
A_1 X^{\gamma_1} + A_2 X^{\gamma_2} + \frac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} X - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I, & \frac{\bar{X}(p_0)}{1+\varphi} \le X \le \hat{X} \\
\frac{X}{r-\alpha} + \frac{\varphi X}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, & X \ge \hat{X},
\end{cases}
\tag{70}
$$

where A_2 is given by (68), and A_1 and C_1 satisfy

$$
A_1 = \hat{X}^{-\gamma_1} \left(\frac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \frac{rI}{r+\lambda_1+\lambda_2} - A_2 \hat{X}^{\gamma_2}\right), \tag{71}
$$

$$
\begin{array}{c}
C_1 = A_1 + A_2 \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2-\gamma_1} + \left(\frac{\bar{X}(p_0)}{1+\varphi}\right)^{-\gamma_1} \left(\frac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \frac{\bar{X}(p_0)}{1+\varphi}\right. \\
\left. - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\Gamma_1\left(\frac{\bar{X}(p_0)}{1+\varphi}\right) - \Gamma_2\left(\frac{\bar{X}(p_0)}{1+\varphi}\right)\right)\right).
\end{array}
\tag{72}
$$

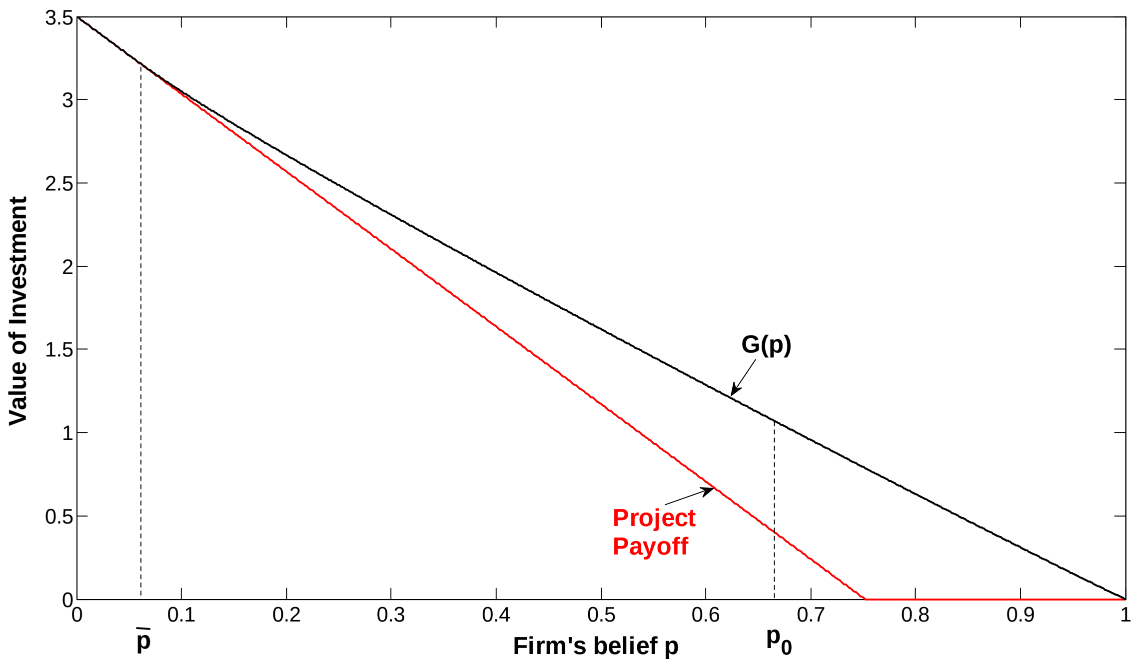

Figure 1. Value of Investment Opportunity. The graph plots the value of the investment opportunity (the black solid line) and the value of immediate investment (the red solid line) as functions of the firm's belief that the outstanding shock is temporary. The vertical distance between the lines represents the value of the option to learn. The investment occurs when the firm's belief falls to a lower threshold, $\bar{p}$. The parameter values are r=0.04, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X=0.95.

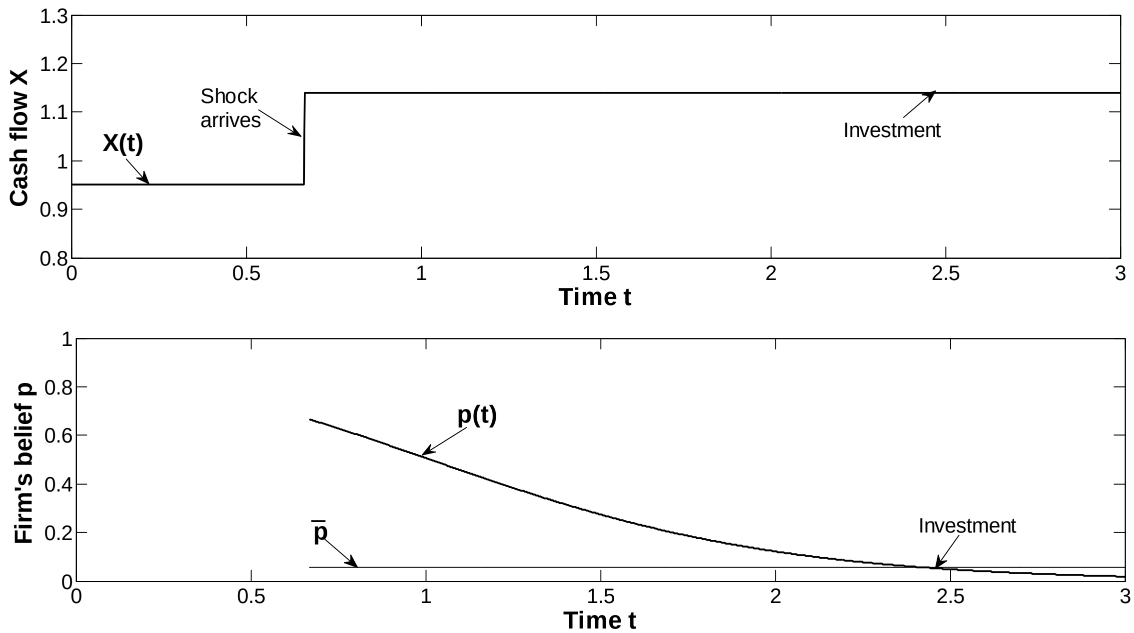

Figure 2. Simulation of the firm's investment strategy. The top graph shows the simulated sample path of the cash flow process, X(t). An upward step corresponds to the arrival of a shock. The bottom graph shows the dynamics of the firm's belief process as well as the investment threshold $\bar{p}$. The investment occurs when the firm's belief process falls to $\bar{p}$. The parameter values are r=0.04, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X=0.95.

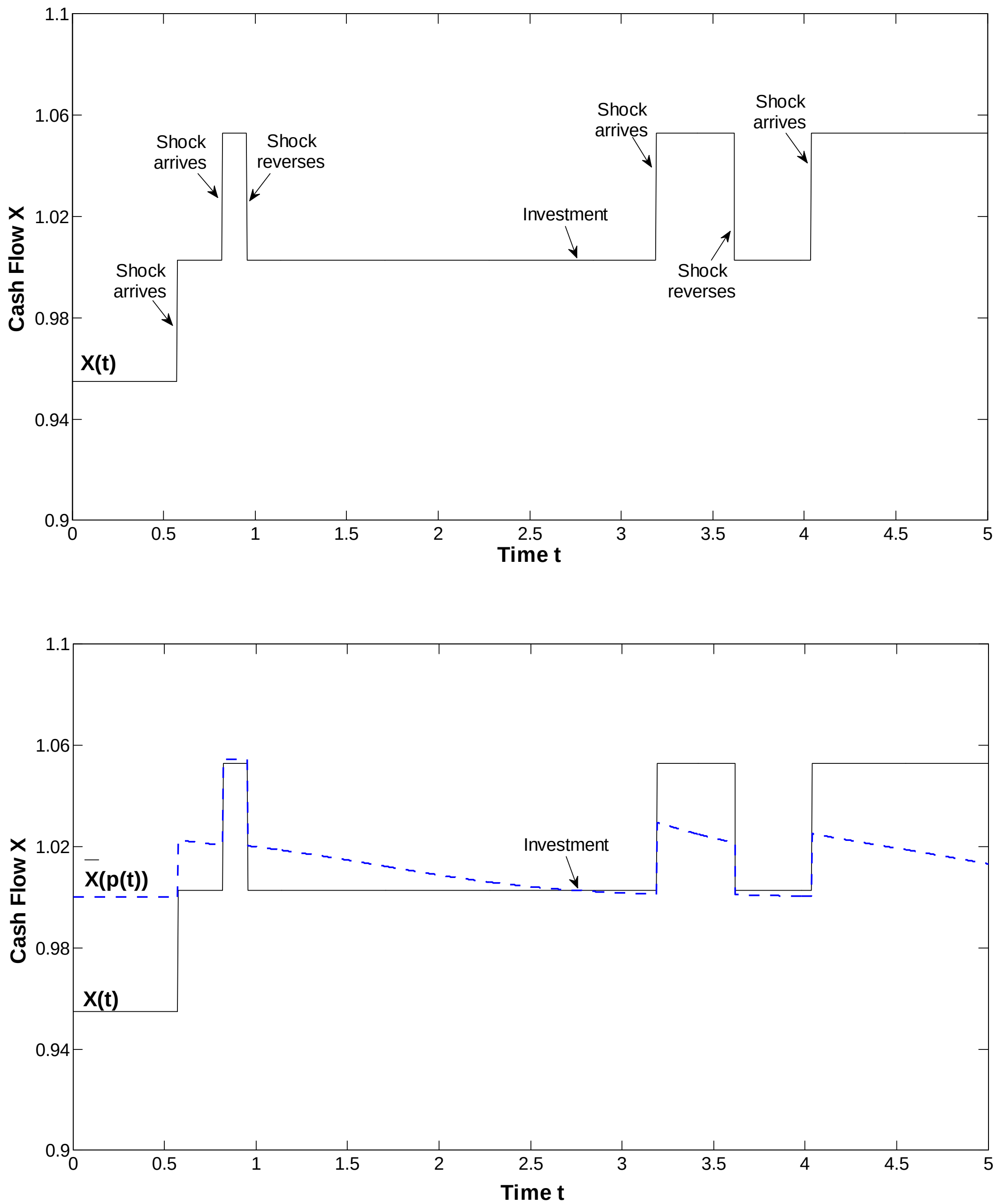

Figure 3. Simulation of the firm's investment strategy. The top graph shows the simulated sample path of the cash flow process, X(t). Each of the four upward steps corresponds to the arrival of a new shock. Similarly, each of the two downward steps corresponds to the reversal of an outstanding shock. The bottom graph adds the investment trigger $\bar{X}(p(t))$ to the simulated sample path from the top graph. The optimal investment strategy is to invest at the first time when X(t) reaches the investment trigger. The parameter values are r=0.04, φ=0.05, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X(0)=0.955.

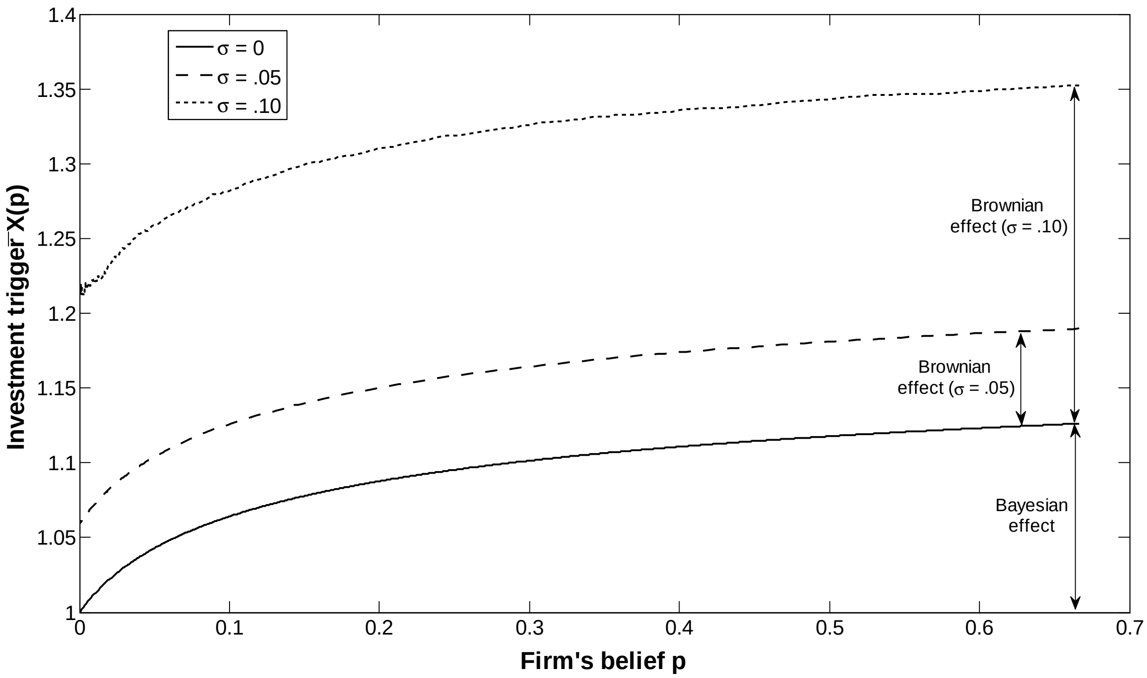

Figure 4. Investment trigger functions for different values of the volatility parameter. The graph plots the investment trigger function $\bar{X}(p)$ for different values of the volatility parameter σ. The bottom curve corresponds to the case of pure Bayesian uncertainty (σ=0). The middle and the top curves correspond to the cases of both Bayesian and Brownian uncertainties (σ=0.05 and σ=0.10, respectively). As a result, the change of the trigger along each curve is due to the impact of Bayesian uncertainty, while the upward shift of the whole trigger function is due to the impact of Brownian uncertainty. The parameter values are α=0.02, r=0.04, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, and I=25.

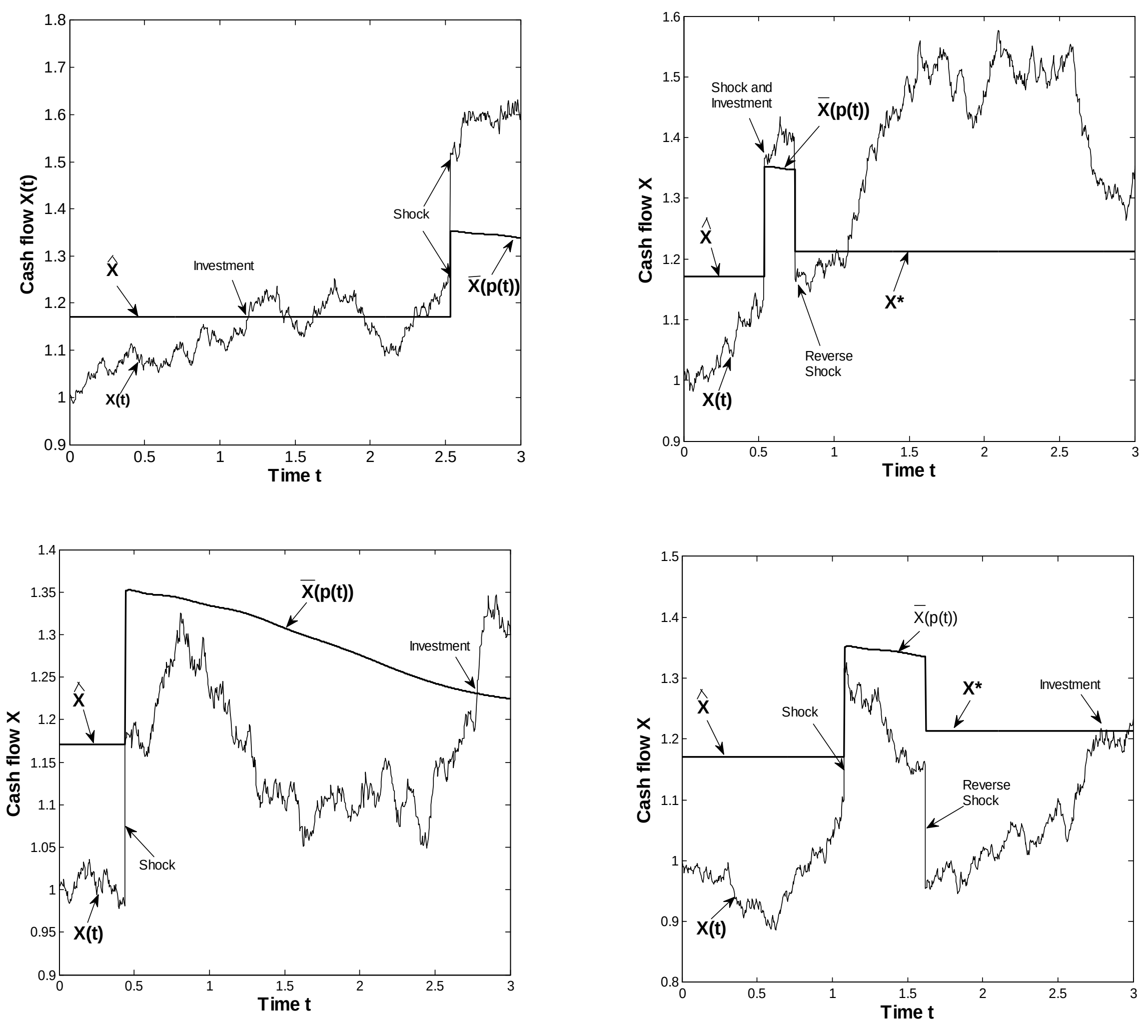

Figure 5. Simulations of the firm's investment strategies. The figure shows the simulated paths of the cash flow process, X(t), (plotted in thin lines) and the corresponding investment triggers (plotted in bold lines) for four different scenarios specified in Proposition 3. The optimal exercise strategy is to invest at the first time when X(t) reaches the investment trigger for the first time. The parameter values are α=0.02, r=0.04, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X(0)=1.