Dynamic Information Asymmetry, Financing, and Investment Decisions

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Abstract

We extend the classical information asymmetry model of Myers and Majluf (1984) to the full dynamic environment. Firms possess an option to delay their investment projects and the market can learn the quality of firms over time by observing the cash flows generated by assets in place. In the dynamic equilibrium, unlike in the static one, firms optimally wait in the inaction region. A higher quality firm invests only if the market's belief reaches an optimal upper threshold that results in the trade-off between waiting to reduce between-firm transfer and the lost time value of money. A lower quality firm invests also if the market's belief reaches a lower threshold, at which it is indifferent between waiting to mimic a high quality firm and investing now. Unlike in the static game, all investment projects are eventually undertaken, but delay can still cause welfare loss.

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1 Introduction

A classical paper of Myers and Majluf (1984) starts by stating the following problem: "Consider a firm that has assets in place and also a valuable real investment opportunity. However, it has to issue common shares to raise part or all of the cash required to undertake the investment project. If it does not launch the project promptly, the opportunity will evaporate. There are no taxes, transaction costs or other capital market imperfections."

In reality, however, most investment opportunities do not "evaporate" if not undertaken immediately. A firm usually has an option to delay the investment-issuance decision, if the market conditions are unfavorable. The real options literature shows that the option to wait has a significant value and should be taken into account in investment decisions (e.g., McDonald and Siegel (1985), Dixit and Pindyck (1994)). In this paper, we extend the classical static problem of Myers and Majluf (1984) to a fully dynamic environment, in which firms can choose the timing of their projects and the market conditions change over time as the market learns about firm quality. The static equilibrium falls apart in the dynamic economy and, although many of its features continue to apply, the dynamic equilibrium that we study reveals many novel and important characteristics that are inherently dynamic.

One of the main findings of Myers and Majluf (1984) is that adverse selection can cause a financial market breakdown. The market cannot ascertain the quality of assets in place and will mix "bad" and "good" firms in pricing their equity. A firm with existing assets of higher quality may be unable to obtain equity financing at an agreeable price, even though it is commonly known that investment project has positive net present value. Then, the only type of firms that are able to undertake the real investment opportunity would be a firm with assets of lower quality. Effectively, by issuing equity a high quality firm transfers some of its value to a low quality firm, and if this transfer is sufficiently high, it prefers to forego investment. Such an equilibrium, however, falls apart, if firms can wait before investing. Indeed, in a dynamic setting, the moment all the low type firms invest, the market should realize that only high quality firms remain and thus should offer equity financing to these firms at an attractive valuation. Anticipating this, the low type firms will not invest in the first place and prefer to wait a little to mimic the high quality firms.

In the dynamic environment, the market observes cash flows generated by existing assets and learns over time about the quality of the firm. A high quality firm thus faces the following trade-off: by waiting long enough, the market will eventually learn the true high quality that will minimize any transfer from high to low quality firms. The cost of waiting, however, is in the lost time value of the investment project. Thus, we should expect a high quality firm always to invest at the higher market assessment than in the static game, but still willing to pool with a low quality firm that results in a (lower) transfer. A low quality firm thus also faces a trade-off: by waiting it hopes to be taken for a high quality firm when that firm invests, but it also loses the time value of money.

These considerations give rise to a strategic dynamic equilibrium that is quite different from the static one. Unlike in the static equilibrium, in the dynamic setting investment eventually is always taken by any firm. However, there is an initial period of inaction, which can be quite extended, when no investment takes place. As the market's belief about the firm quality becomes sufficiently optimistic, both types invest. Intuitively, this upper belief threshold is optimally chosen by a high quality firm. As the market's belief, on the other hand, turns sufficiently pessimistic, a low quality firm starts investing. The lower threshold is effectively chosen by a low quality firm, so that it is indifferent between continuing to wait and investing now. Interestingly, a low quality firm invests probabilistically at the lower threshold. In a way, it tries to play a cat-and-mouse game with the market. If the firm does not invest at the lower threshold, the market does not lower its assessment of the firm. This behavior, resulting in the so-called reflective barrier, contrasts with the static game, in which a low quality firm always invests and the market's belief becomes certainty if it realizes that a high quality firm would not have invested in such a situation.

We analyze many properties of this dynamic equilibrium by building a continuous-time signaling model. One of the interesting questions is whether waiting improves social welfare, measured as the combined value all the firms. High quality firms always welcome an opportunity to wait, and the attitude of low quality firms is more ambiguous, because they can lose the valuable pooling opportunity. A welfare trade-off is that investment in the dynamic economy always happens but it can be delayed compared to the static environment. We show that in many inactions regions introducing delay leads to lower social welfare, because the time value of money lost in waiting dominates investment eventuality.

This paper relates to a burgeoning literature on dynamic signaling. For example, Grenadier and Malenko (2010) also model costly dynamic signaling in the real options context, but signaling in their model is essentially static, because there is no uncertainty about the cash flow process and the market can learn only by observing investment outcomes. The equilibrium in our model is closely related to two recent important papers, Gul and Pesendorfer (2012) and Daley and Green (2012). Gul and Pesendorfer (2012) consider a dynamic model of political competition, in which the parties provide information to tilt the balance to their advantage. Because voter's preferences are perfectly correlated with one of the parties, their equilibrium has only one lower reflecting barrier. The optimal strategy for a high quality party is degenerate – never stop providing information. Daley and Green (2012) extend a classical lemon's problem of Akerlof (1970) to a fully dynamic setting and derive a two threshold equilibrium, on which the equilibrium in our model is closely based. In a way, our paper can be viewed to have the same relation to the paper of Myers and Majluf (1984) as the paper of Daley and Green (2012) to Akerlof (1970).

By allowing firms to postpone investment decisions, we follow the rich tradition of the real options literature that started with the seminal studies of Brennan and Schwartz (1985) and McDonald and Siegel (1985). The basic result of the real options literature is that timing flexibility leads firms to delay optimally the exercise of investment options. We build on this insight to study how flexibility affects investment and financing decisions in the presence of dynamic information asymmetry. The inaction region is also a feature of many real option models (see Stokey (2009)), although the reason for inaction in that literature is the presence of fixed costs, not adverse selection.

The paper is organized as follows. We present the static game in the next section. Section 3 develops the dynamic model and then derives and discusses the equilibrium. Section 4 explores the economic properties of the dynamic equilibrium, and Section 5 concludes. All the proofs are provided in the Appendix.

2 Model Setup and Static Equilibrium

In this section we introduce the model setup and consider the classical case of static investment and financing decisions that will serve as a useful benchmark later on. Importantly, as we will frequently comment, the results that we present are generic, but for the ease of exposition we choose a familiar but nevertheless a specific investment problem. Consider an all-equity firm with assets in places and a growth option. Managers' interests are perfectly aligned with the current shareholders, an assumption that Myers and Majluf (1984) and a number of later papers discuss at length. Equivalently, we can assume that the manager is an entrepreneur who owns the firm and naturally maximizes her shareholder value. The familiar economic problems that the firm faces are that it does not have sufficient internal resources to finance the NPV-positive growth option and that the market does not have the same quality of information about the assets in place as the manager does. The firm is unable to spin off the growth option. This classical problem is equivalent to the one faced by the firm in Myers and Majluf (1984). The purpose of this section is to formulate and solve this problem in a way that will enable us to fully explore dynamic issues later on.

2.1 Model Setup

The firm belongs to one of two types θ, $\theta \in \{H, L\}$. The type is private information of the firm. All other parameters are common knowledge. The assets in place produce in expectation free cash flow μ_θ per unit of time, where $\mu_H > \mu_L > 0$. The cumulative cash flows of type-θ firm at time t, X_t^θ, follows:

$$dX_t^\theta = \mu_\theta dt + \sigma dB_t, \tag{1}$$

where $B = (B_t, \mathcal{F}_t^B)_{t \geq 0}$ is a standard Brownian motion endowed with a natural filtration defined on a canonical probability space (Ω, $\mathcal{F}$, $\mathbf{Q}$), and μ_θ and σ are constants.

In addition to its ownership of the assets in place, the firm has the growth option that consists of a monopoly access to a new investment technology. At the time of investment, the firm pays a one-time cost of I, and the expected free cash flow increases from μ_θ to $\mu_\theta + k$, where k is a positive constant. The net present value of the investment opportunity is positive, $\frac{k}{r} > I$, where r is the constant risk-free rate. Consistent with the original Myers and Majluf setup, cash flows from the new project do not depend on the type of the assets in place. Although we analyze this setup for the sake of simplicity and clarity, our conclusions qualitatively hold for generic joint distributions of cash flows from assets in place and the growth option, such as when the two are perfectly correlated with each other.

Competitive outside investors (that we will call "the market") do not observe θ but have a prior, p_0, at time 0 that the firm is of type H, and $1 - p_0$ that the firm is of type L where $p_0 \in [0, 1]$. The type θ of the firm is independent of B. For simplicity, we assume that all agents are risk-neutral.

Finally, to make the firm's problem non-trivial, we assume that the cost I must be entirely financed by outside investors. Alternatively, we can interpret the cost I as the net funds that the firm must raise above and beyond its internal resources. As in the original Myers and Majluf model, we concentrate here on the case when only external equity financing is feasible, a case that turns out to have rich implications in a dynamic setting.

Before exploring a full dynamic version in Section 3, we first consider the static investment decision of the firm, where the investment opportunity is a take-it-or-leave-it offer at date 0. Our model then is equivalent to the one discussed by Myers and Majluf (1984). Although the solution to their model is well known, we still derive the signaling equilibrium in detail to benchmark our discussion of the dynamic model later on.

2.2 Static Decisions: Timing

The signaling game has two players, the firm and the market. The timing of the static game is as follows:

1. Nature draws the type θ of the firm from the set of $\Theta = \{H, L\}$ according to the prior probability distribution $(p_0, 1 - p_0)$.

2. The firm observes θ and the market does not. The firm's strategy $\pi(\theta)$ is a probability distribution over the set of feasible actions $a \in A = \{0, 1\}$. In this simple game, the only action is either to invest ($a = 1$) or to pass the investment opportunity ($a = 0$) at time 0. Effectively, $\pi(\theta)$ is the probability of the investment by type θ. The decision to invest is equivalent to an equity offering by the firm of size I. If the firm decides not to invest at time 0, the game is over.

3. The market observes the action a chosen by the firm. If $a = 1$, the market updates its belief that the firm is of type H to q_0, competitively prices the value of the firm, and demands a fraction λ of the firm's total equity in return for investment I.

4. If the firm issues equity and invests, the expected payoff to the entrepreneur, E, is:

$$E_\theta(a,\lambda) = \begin{cases} \frac{\mu_\theta}{r}, & \text{if } a = 0 \\ (1-\lambda)\frac{\mu_\theta + k}{r}, & \text{if } a = 1 \end{cases} \tag{2}$$

   and the payoff to the market, S, is:

$$S_\theta(a,\lambda) = \begin{cases} 0, & \text{if } a = 0 \\ \lambda\frac{\mu_\theta + k}{r} - I, & \text{if } a = 1. \end{cases} \tag{3}$$

   Because the market does not observe θ, we let $S(a,\lambda)$ be the expected payoff to the market, given a and λ. That is,

$$S(a,\lambda) = \mathbb{E}[S_\theta(a,\lambda)|a,\lambda]. \tag{4}$$

Without loss of generality, we impose the following two tie-breaking rules in the static model. First, if the firm is indifferent between, on one hand, investing and issuing equity, and, on the other hand, not investing, then the firm issues equity and invests. Second, if the market is indifferent between buying the firm's equity and not buying, then the market buys.

2.3 Static Equilibrium

We are now ready to define the Perfect Bayesian equilibrium of this static signalling game.[1]

Definition 1 *A Perfect Bayesian equilibrium of the static game consists of a pair of strategies $\pi^*(\theta)$ of the firm and λ^* of the market*[2]*, and a belief q_0 of the market such that:*

1. *Conditional on observing an action $a = 1$, the market has a belief $q_0 \in [0,1]$ that it is facing a high type firm (respectively, $1 - q_0$ that it is facing a low type firm).*

2. *Given the market's belief q_0, the market's valuation strategy λ^* satisfies the participation constraint:*

$$S(1,\lambda^*) = q_0 S_H(1,\lambda^*) + (1-q_0)S_L(1,\lambda^*) \geq I. \tag{5}$$

[1] A note on terminology. In game-theoretic applications, these games are often called dynamic games of incomplete information to distinguish them from the games, such as auctions, in which the equilibrium concept is Bayesian-Nash equilibrium (see e.g. Gibbons (1992)). "Dynamic" there refers to players moving sequentially. We use "dynamic" to denote that actions can be taken at different moments in time.

[2] This is without loss of generality since all parties are risk neutral and this is the terminal node of the game, thus, playing a mixed strategy ζ is pay-off equivalent to playing $\bar{\lambda} = \int \lambda \, d\zeta(\lambda)$.

3. *For each type θ, the strategy $\pi^*(\theta)$ maximizes the value of the entrepreneur, given the market's strategy λ^*:*

$$\pi^*(\theta) \in \arg\max_{\pi\in[0,1]} \left[\pi E_\theta(1,\lambda^*) + (1-\pi)E_\theta(0,\lambda^*)\right]. \tag{6}$$

4. *If there is a type θ that issues equity with positive probability, then the posterior belief of the market satisfies the Bayes rule:*

$$q_0 = \frac{p_0\pi^*(H)}{p_0\pi^*(H) + (1-p_0)\pi^*(L)}. \tag{7}$$

Since we assume perfectly competitive markets, the participation constraint (5) is satisfied with equality and becomes a break-even constraint. To gauge the evolution of the market's belief, note that in the case of pure strategies posterior (7) becomes:

$$q_0 = \begin{cases} p_0, & \text{if } \pi^*(H) = 1, \pi^*(L) = 1, \\ 1, & \text{if } \pi^*(H) = 1, \pi^*(L) = 0, \\ 0, & \text{if } \pi^*(H) = 0, \pi^*(L) = 1. \end{cases} \tag{8}$$

In other words, if only one type invests, the market immediately separates between the two types and updates its belief accordingly, and if both types invest with probability one, the market has no further information to modify its prior.

We now derive the optimal choice of the type-θ firm given the posterior belief $(q_0, 1-q_0)$. Since (5) is satisfied with equality in equilibrium we can write the break-even constraint of the market as:

$$I = \lambda\left(q_0\frac{\mu_H + k}{r} + (1-q_0)\frac{\mu_L + k}{r}\right). \tag{9}$$

The right hand-side of (9) is the value of outside investors' equity. The market thus demands a fraction λ of the total firm value, such that:

$$\lambda = \frac{Ir}{q_0(\mu_H + k) + (1-q_0)(\mu_L + k)} \tag{10}$$

Intuitively, the L type strictly prefers investing; the worst scenario for the L type is to invest in the positive-NPV project when everybody knows the type identity, and the best scenarios is to pool with the H type. Thus:

$$E_L(1,\lambda) > E_L(0,\lambda) \tag{11}$$

for any belief q_0. For an H type firm an investment decision results in a trade-off between the benefits of the profitable investment opportunity and the costs of subsidizing an L type firm. An H type firm strictly prefers investing (thus pooling) if and only if:

$$E_H(1,\lambda) > E_H(0,\lambda). \tag{12}$$

By Equation (10), (12) holds if and only if:

$$q_0 \geq p^* \equiv \frac{Ir}{k}\frac{\mu_H + k}{\mu_H - \mu_L} - \frac{\mu_L + k}{\mu_H - \mu_L}. \tag{13}$$

It is easy to check that $p^* < 1$. Intuitively, the H type always prefers investing in the positive-NPV project when beliefs are sufficiently high. To rule out the uninteresting case when the H type always finds it profitable to invest, we assume that $p^* > 0$, which implies from (13) that:

$$\frac{k}{r} < \frac{\mu_H + k}{\mu_L + k} I. \tag{14}$$

The next lemma shows all possible Perfect Bayesian equilibria in the static model.

Lemma 1 (Static Equilibrium) *There are at most two Perfect Bayesian equilibria in the static model with initial belief of p_0.*

1. **Separating Equilibrium**. *H type does not invest and L type invests. The posterior belief is updated to $q_0 = 0$, and the market demands a fraction $\lambda_S = Ir/(\mu_L + k)$ of the firm.*
2. **Pooling Equilibrium**. *If $p_0 \geq p^*$, where p^* is given by (13), then both H and L types invest. The posterior belief stays at $q_0 = p_0$, and the market demands a fraction λ_P of the firm, where λ_P is given in (10).*

When the fraction of high type firms in the population is sufficiently high, i.e. if $p_0 > p^*$, Lemma 1 posits the existence of two equilibria. However, the separating equilibrium is economically unreasonable. It implies that though the average quality of the pool is high, the market scares off all the high quality firms and finances only the low quality firms. It is straightforward to show that it is clearly inefficient. In both equilibria, the market always breaks even. In the pooling equilibrium, however, an L type entrepreneur is better off by selling a smaller fraction of the firm, and an H type entrepreneur is better off by being able to invest in a positive-NPV growth option. Thus, the pooling equilibrium, when it exists, is socially more efficient than the separating one. These results are summarized in the following proposition:

Proposition 1 *For any p_0 there exists a unique Pareto dominant Perfect Bayesian equilibrium:*

1 **Separating equilibrium.** *If $p_0 < p^*$, where p^* is given by (13), an H type firm does not invest and an L type firm invests. The market's belief is updated to $q_0 = 0$, and offers to buy a fraction $\lambda_L = Ir/(\mu_L + k)$ of the firm for I.*

Figure 1: Static investment and financing decisions

2 **Pooling equilibrium.** *If $p_0 \geq p^*$, then both H and L types invest. The market's belief stays at $q_0 = p_0$, and the market offers to buy a fraction λ_P of the firm for I, where λ_P is given in (10).*

Figure 1 shows a celebrated result of Myers and Majluf (1984), in the mold of Akerlof (1970), that asymmetric information can lead to the partial failure of the market ability in providing financing for the positive NPV projects. The economic thrust of the mechanism is that a decision to invest and issue equity can be interpreted by the market as a signal of low quality about the assets in place, which is immediately reflected in the share price. Low share prices deter a high type firm from investing in the first place, supporting the separating equilibrium.

2.4 Discussion of the Static Equilibrium

The degree of inefficiency in the static model depends on the threshold value p^*. The following table summarizes the comparative statics of the threshold, the equilibrium fractions of the firm that outside investors demand, as well as the entrepreneur's value with respect to all the model parameters.

Table 1: Comparative Statics of the Static Model

	Sign of change in variable for an increase in:						
Variable	μ_H	μ_L	I	k	r	p_0	σ
p^*	+	−	+	−	+	0	0
λ_L	0	−	+	−	+	0	0
λ_P	−	−	+	−	+	−	0
$E_L(1, \lambda_L) - \frac{\mu_L}{r}$	0	0	−	+	−	0	0
$E_L(1, \lambda_P) - \frac{\mu_L}{r}$	+	−	−	+	−	+	0
$E_H(1, \lambda_P) - \frac{\mu_H}{r}$	−	+	−	+	−	+	0

The cut-off p^* is increasing in the high type expected cash flow, μ_H, and decreasing in the low type expected cash flow, μ_L, because a larger difference between the profitability of two types implies a larger underpricing of an H type firm and makes it less willing to pool. For a similar reason, a higher cost I, a lower benefit k, or a higher discount rate r reduces the marginal value of the growth option and makes it less appealing to an H type firm. As the NPV of the growth option decreases, the market also demands a higher fraction of the firm in any equilibrium. On the other hand, as expected cash flows from either assets in place or the growth option increase, the market break-even constraint is satisfied for a lower fraction of the firm.

The final three lines of Table 1 show the comparative statics of the entrepreneurs's surplus, which is defined as the additional value the firm gets from exercising its growth option. For an L type firm, we consider both the pooling and separating equilibria. All results are intuitive: the surplus is higher for a larger growth option benefit k and the initial belief of the market p_0, and a smaller discount rate r and the cost I. Note, importantly, that the volatility of cash flows has no impact on any variables in the model. In the static model this is obviously driven by the risk-neutrality of agents. However, volatility plays a major role in the fully dynamic environment, as we shall see in the next section.

3 Dynamic Investment and Financing

In this section, we extend the static framework of Section 2 to a fully fledged dynamic environment. There are two critical changes that the dynamic case brings about. First, the firm becomes flexible in deciding the timing of its investment: the investment option does not "evaporate" if not taken at date 0. Second, the market is learning the true quality of the firm over time even in the absence of investment announcements. In postponing the investment, a high type firm therefore weighs the cost of losing cash flows by not exercising the growth option immediately and the benefit of gaining from market's belief update over time. A low type firm, on the other hand, faces the trade-off between investing immediately, which instantaneously increases cash flows at the cost of revealing its true type, and waiting for a high type to invest, which gives it an opportunity to pool. These intricate trade-offs result in changing significantly the equilibria of the model.

3.1 Timing of the Dynamic Model

The firm can decide to invest at any time t, $t \geq 0$. The total amount required for investment must be raised at the same time.[3] The market's belief that it is facing a high (low) type firm at any time t, conditional on no equity issuance up to time t, is given by p_t (respectively, $1 - p_t$), $0 \leq p_t \leq 1$. If the firm issues equity at time t, the market updates its belief to q_t and $1 - q_t$ for the high and low types, respectively. All players' decisions at time t are conditioned on the history of cash flows and market

[3] Allowing for several stages of fund raising would not change our results, because a high type firm finds it optimal to raise the full amount in one stage and a low type firm would be forced to mimic.

beliefs available up to time t or, equivalently, are adapted to the filtration $\mathcal{G}_t$, generated by the family of random variables $\{X_s, p_s, q_s;\ s \leq t\}$.

The timing of the dynamic game is as follows:

1. Nature draws the type θ of the firm from the set of $\Theta = \{H, L\}$, according to a probability distribution given by the prior $(p_0, 1-p_0)$ independently from B.
2. The firm observes θ and the market does not. The firm's action is defined by a process $\pi_t(\theta)$ which is a cumulative probability that a type-θ firm has announced investment and approached the market to raise capital before time t.[4] We allow the firm not to invest at all, i.e. $\mathbf{P}\left(\lim_{t\to\infty} \pi_t(\theta) = 1\right)$ could be less than 1.

   As an example, consider a case when an H type firm chooses the following "pure" strategy: invest with probability 1 as soon as the market's belief p_t crosses an upper threshold $\bar{p}$. The corresponding $\pi(H)$ is:

$$\pi_t(H) = \begin{cases} 0, & \text{if } \sup_{s\leq t} p_s < \bar{p}; \\ 1, & \text{if } \sup_{s\leq t} p_s \geq \bar{p}. \end{cases} \tag{15}$$

   The general definition of $\pi(\theta)$ allows firms to randomize decisions at any threshold.
3. The market observes only the cash flows of the firm, X_s, $0 \leq s \leq t$. Therefore, the beliefs p_t and q_t are conditioned only on the history $\mathcal{G}_t$.
4. If the firm invests at time t, the market updates its belief to q_t, conditional on the observed game history $\mathcal{G}_t$. It then prices the value of the firm and offers equity financing I in return for a fraction λ_t of the firm's equity.

If the firm raises equity and invests at time t, the payoff to the entrepreneur is:

$$E_\theta(q,t) = (1-\lambda(q_t))\frac{\mu_\theta + k}{r}, \tag{16}$$

and the payoff to the market is given by:

$$S_\theta(q,t) = \lambda(q_t)\frac{\mu_\theta + k}{r} - I. \tag{17}$$

As the market does not observe θ, $S(q_t,t)$ is the expected payoff to the market, given $t, \mathcal{G}_t$, and λ:

$$S(q,t) = q_t S_H(q,t) + (1-q_t) S_L(q,t). \tag{18}$$

[4] More rigorously, $\pi(\theta)$ is a cádlág (right continuous with existing left limits) non-decreasing process that satisfies $0 \leq \pi_t(\theta) \leq 1$. It allows for both flow and jumps in the firm's distribution of investment decisions.

3.2 Equilibrium of the Dynamic Model

We start by defining a Perfect Bayesian equilibrium in the dynamic model.

Definition 2 *A Perfect Bayesian equilibrium of the dynamic game consists of a pair of strategies $\pi^*(\theta)$ of the firm and λ^* of the market, and market beliefs (p^*, q^*) of the market, such that:*

1. *Conditional on the firm's investment at time t, the market's valuation strategy λ^* satisfies the participation constraint:*
$$S(q^*, t) \geq I. \tag{19}$$

2. *For each type θ, the strategy $\pi^*(\theta)$ maximizes the value of the entrepreneur, given the market's strategy λ^*:*
$$\pi^*(\theta) \in \arg\max_{\pi} \mathbf{E}\left[\int_0^{\infty} \left(\int_0^t e^{-ru} dX_u + e^{-rt} E_\theta(q^*, t)\right) d\pi_t\right]. \tag{20}$$

3. *Market's beliefs p_t and q_t are consistent with $\pi^*(\theta)$, i.e., they satisfy the Bayes rule along the equilibrium path:*
$$\frac{p_t}{1-p_t} = \frac{p_0}{1-p_0} \cdot \frac{\varphi_t^H(X_t)}{\varphi_t^L(X_t)} \cdot \frac{1-\pi_{t-}^*(H)}{1-\pi_{t-}^*(L)}; \tag{21}$$
$$\frac{q_t}{1-q_t} = \frac{p_0}{1-p_0} \cdot \frac{\varphi_t^H(X_t)}{\varphi_t^L(X_t)} \cdot \frac{d\pi_t^*(H)}{d\pi_t^*(L)}. \tag{22}$$

   where $\varphi_t^\theta(\cdot)$ is a c.d.f. of the normal $\mathcal{N}(\mu_\theta t, \sigma t)$ random variable.[5]

The entrepreneurial value in (20) is decomposed into two components. The first term is the present value of cash flows generated by the assets in place that the firm receives until the exercise of the growth option, and the second term is the discounted value of the stake in the firm that remains in the entrepreneur's ownership after new funds are raised and the growth option is exercised. The sum of these two components is integrated over the equilibrium strategy π^* that determines the timing of the investment.

To explore the evolution of the market's belief, note that as long as no investment has occurred, the market revises its beliefs based on two sources of information: publicly available cash flows and equilibrium investment strategies. The former gives rise to the non-strategic component of the belief

[5] As standard, $\pi_{t-}^*(\theta)$ stands for the left limit $\lim_{s\uparrow t} \pi_s^*(\theta)$, which exists because $\pi^*(\theta)$ is monotone. The formal definition of $d\pi_t^*(\theta)$ is omitted for brevity. We use this notation as a shorthand for two important scenarios: (i) if there is a discrete jump in the probability of investment, then $d\pi_t^*(\theta) = \pi_t^*(\theta) - \pi_{t-}^*(\theta)$; and (ii) when only a high (low) type firm invests with a positive probability, the ratio $d\pi_t^*(H)/d\pi_t^*(L)$ equals to $+\infty$ (0), and the exact meaning of $d\pi_t^*(L)$ ($d\pi_t^*(H)$) is irrelevant.

process, while the latter – to the strategic one. To disentangle these two components, as well as simplify further exposition, we introduce an auxiliary belief process P_t, which is defined as:

$$\frac{P_t}{1-P_t} = \frac{p_0}{1-p_0} \cdot \frac{\varphi_t^H(X_t)}{\varphi_t^L(X_t)}. \tag{23}$$

The process P_t encapsulates the probability of facing a high type firm, conditional *only* on the observed cash flows up to time t. In other words, P_t represents the non-strategic component of the belief process.

To demonstrate the effect of the second, strategic, source of information, define z_t as $z_t = \ln\left(\frac{p_t}{1-p_t}\right)$ and the corresponding auxiliary process Z_t as $Z_t = \ln\left(\frac{P_t}{1-P_t}\right)$. Then, equations (21) and (23)can be rewritten as:

$$z_t = Z_t + \ln\left(\frac{1-\pi_{t-}^*(H)}{1-\pi_{t-}^*(L)}\right), \tag{24}$$

$$Z_t = \ln\left(\frac{p_0}{1-p_0}\right) + \frac{\mu_H - \mu_L}{\sigma^2}\left(X_t - \frac{\mu_H+\mu_L}{2}t\right). \tag{25}$$

The second term in (24) captures the pure signaling effect on the market's belief. Obviously, $Z_t = z_t$ as long as there is no investment, i.e. $\pi_t(H) = \pi_t(L) = 0$.

As Z_t is monotone in P_t and z_t is monotone in p_t, we can also refer to Z_t and z_t as the "belief" processes. The market's belief process Z_t does not depend on the firm's type. The firm, however, knows its type, and the entrepreneur observes the "true" belief process Z_t, denoted Z_t^θ. Substituting (1) into (25), we can write:

$$dZ_t^H = \frac{h^2}{2}dt + hdB_t, \tag{26}$$

$$dZ_t^L = -\frac{h^2}{2}dt + hdB_t, \tag{27}$$

where $h = (\mu_H - \mu_L)/\sigma$. Path by path, Z_t coincides with Z_t^H if the firm is of the H type, and coincides with Z_t^L if the firm is of the L type. The difference between Z_t and Z_t^H (or Z_t^L) is that at any date s the conditional distribution of future realizations of Z_t, $t > s$, is given by (25) for the market's belief and by (26) for an H type firm (and by (27) for an L type firm). This difference in the assessment of future conditional distributions is *the* essential source of *dynamic* information asymmetry.

The advantage of using Z_t is that it is the only state variable that affects decision-making. Note that (25) establishes the linear relationship between three potential state variables: time t, cumulative cash flow X_t, and belief Z_t. Cash flow X_t and time t come into the players' decision only through their linear combination $X_t - (\mu_H + \mu_L)t/2$, and the dependence of Z_t on X_t and t are thus implicit.

The next proposition states the main theoretical result of this section. It derives a specific Perfect Bayesian equilibrium of the dynamic model that will turn out, as we demonstrate later, to be the only equilibrium that supports economically reasonable behavior within a broad class of admissible strategies.

Proposition 2 (Perfect Bayesian Equilibrium) *There exists a unique pair* $(\bar{z}^*, \underline{z}^*)$*,* $\bar{z}^* \geq \underline{z}^*$*, such that the following is a Perfect Bayesian Equilibrium:*

1. *The market's belief conditional on not observing issuance up to time* t *is:*

$$p_t = \frac{e^{z_t}}{1+e^{z_t}}. \tag{28}$$

2. *The strategies of a type-*θ *firm are given by:*

$$\pi_t(H) = \begin{cases} 1, & \text{if } \sup_{s\leq t} z_s \geq \bar{z}^*; \\ 0, & \text{if } \sup_{s\leq t} z_s < \bar{z}^*; \end{cases} \tag{29}$$

$$\pi_t(L) = \begin{cases} 1, & \text{if } \sup_{s\leq t} z_s \geq \bar{z}^*; \\ 1 - e^{-Y_t}, & \text{if } \sup_{s\leq t} z_s < \bar{z}^*, \end{cases} \tag{30}$$

where $Y_t = \max\left(\underline{z}^* - \inf_{s\leq t} Z_s, 0\right)$.

3. *If the firm decides to invest at time* t*, the market's belief is updated to:*

$$q_t = \begin{cases} 0, & \text{if } z_t < \bar{z}^*; \\ p_t, & \text{if } z_t \geq \bar{z}^*, \end{cases} \tag{31}$$

and the market demands a share $\lambda^*(z_t)$ *of the firm's equity in exchange for investment* I*:*

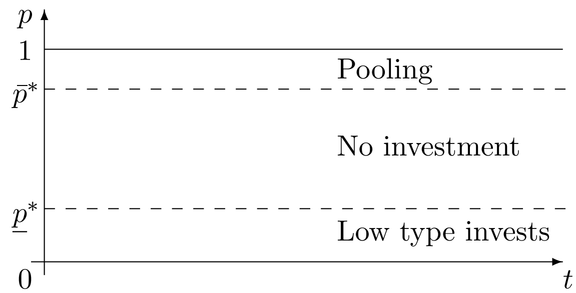
$$\lambda^*(z_t) = \begin{cases} \frac{Ir}{\mu_H+k}\left[1 + \frac{\mu_H-\mu_L}{\mu_L+k+(\mu_H+k)e^{z_t}}\right], & \text{if } z_t \geq \bar{z}^*; \\ \frac{Ir}{\mu_L+k}, & \text{if } z_t < \bar{z}^*. \end{cases} \tag{32}$$

The equilibrium described in Proposition 2 is a partial pooling one. We later characterize economically reasonable selection criteria, under which this equilibrium is unique.[6] Figure 2 shows three equilibrium regions that depend on the market's belief p.[7] When the market's belief reaches the upper threshold $\bar{p}^*$, i.e., when outside investors become sufficiently optimistic about the firm's quality, both types pool by issuing equity and invest. The upper barrier $\bar{p}^*$ is essentially chosen, as the threshold p^* in the static game, by a high type firm to maximize its value, taking into account a low type firm standing ready to imitate any high type's actions.

[6] It is important to point out that the proposition does not claim the uniqueness of the equilibrium. So far we have established that, given the threshold strategies (29) and (30) and the market's belief (31), there exists a unique pair $(\bar{z}^*, \underline{z}^*)$, such that (29)–(31) is an equilibrium. There could be other, economically implausible, equilibria that do not have this threshold structure.

[7] We can use the original belief variable p to describe the equilibrium regions, because there is a one-to-one correspondence between the thresholds $(\bar{z}^*, \underline{z}^*)$ and thresholds $(\bar{p}^*, \underline{p}^*)$.

Figure 2: Equilibrium beliefs with reflecting barrier at $\underline{p}^*$



When the market's belief reaches $\underline{p}^*$, only a low type firm exercises its investment strategy and, according to (30), it does so probabilistically. The lower barrier is chosen so that a low type firm is indifferent between investing now (and thus revealing its type) and postponing investing in hope of experiencing positive shocks that would lead the market's belief to hit the upper boundary $\bar{p}^*$ in the future. The equilibrium rate of mixing by a low type firm at the lower boundary forces the beliefs to be *reflecting*. That is, conditional on not observing investment, the market's belief immediately adjusts upwards, because a high type firm never invests at the low threshold. By (30), we can verify that in the equilibrium z_t is indeed the reflected version of Z_t at $\underline{z}^*$:

$$z_t = Z_t + Y_t = Z_t + \max\left(\underline{z}^* - \inf_{s \leq t} Z_s, 0\right). \tag{33}$$

For all the belief levels between the two thresholds $\bar{p}^*$ and $\underline{p}^*$ there is a region of optimal inaction: both types postpone a decision, a high type firm in expectation of the market's belief becoming more optimistic, which will reduce underpricing, and a low type firm in hope of relatively higher underpricing. This region of inaction is a new feature of the dynamic financing problem and resembles the optimal economic behavior in many real option contexts, in which firms face transaction costs of adjustment (e.g, see Stokey (2009)).

Because the proof of Proposition 2 is by construction, the thresholds $\bar{z}^*$ and $\underline{z}^*$ are implicitly derived (the equations that characterize their values are provided by equations (A14) and (A27) in Appendix). This enables us to introduce and derive, as a function of $\bar{z}^*$ and $\underline{z}^*$, fundamental contingent claims securities, which have the role similar to that of standard Arrow-Debreu claims. The next lemma gives the values of those fundamental securities that are useful in the later analysis.

Lemma 2 (Fundamental Contingent Claims) *Suppose that the equilibrium strategies and market beliefs are as in Proposition 2. Let $A_{P,\theta}(z)$ be the present value of \$1 paid off to a type-θ firm at the pooling threshold $\bar{z}^*$, conditional on separation has not occurred so far, and $A_{S,L}(z)$ be the present* *value of \$1 paid off to a low type firm at the time of the separation (i.e., when a low type firm invests at $\underline{z}^*$), conditional on pooling has not occurred so far. Then the values of these claims, given the market's current belief z, are:*

$$A_{P,L}(z) = \frac{v_1\, e^{v_1(z-\underline{z}^*)} - v_2\, e^{v_2(z-\underline{z}^*)}}{v_1\, e^{v_1(\bar{z}^*-\underline{z}^*)} - v_2\, e^{v_2(\bar{z}^*-\underline{z}^*)}}; \tag{34}$$

$$A_{P,H}(z) = \frac{u_2\, e^{u_1(z-\underline{z}^*)} - u_1\, e^{u_2(z-\underline{z}^*)}}{u_2\, e^{u_1(\bar{z}^*-\underline{z}^*)} - u_1\, e^{u_2(\bar{z}^*-\underline{z}^*)}}; \tag{35}$$

$$A_{S,L}(z) = \frac{e^{v_2(\bar{z}^*-\underline{z}^*)+v_1(z-\underline{z}^*)} - e^{v_1(\bar{z}^*-\underline{z}^*)+v_2(z-\underline{z}^*)}}{v_2 e^{v_2(\bar{z}^*-\underline{z}^*)} - v_1 e^{v_1(\bar{z}^*-\underline{z}^*)}}, \tag{36}$$

where

$$u_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2r}{h^2}}; \tag{37}$$

$$v_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2r}{h^2}}. \tag{38}$$

Note that the values of these claims take into account that pooling occurs only and only if the separation does not occur earlier and, conversely, the separation occurs only and only if the pooling is not reached first. Using the values of these fundamental claims, the next proposition derives the initial values of firms that can then be compared with their counterparts in the static problem. We also provide an intuitive decomposition of the firm values to the assets in place and the growth option.[8]

Proposition 3 (Entrepreneur's Initial Value) *Suppose that the equilibrium strategies and market beliefs are as in Proposition 2. The value of a type-θ firm at time 0, $E_\theta(z_0)$, with $z_0 = \ln\left(\frac{p_0}{1-p_0}\right)$, can be written as:*

$$E_H(z_0) = \frac{\mu_H}{r} + A_{P,H}(z_0)\left(E_H(\bar{z}^*) - \frac{\mu_H}{r}\right) \tag{39}$$

and

$$E_L(z_0) = \frac{\mu_L}{r} + A_{P,L}(z_0)\left(E_L(\bar{z}^*) - \frac{\mu_L}{r}\right) + A_{S,L}(z_0)\left(\frac{k}{r} - I\right). \tag{40}$$

The value of a high type firms consists of two components, the present value of the assets in place and the present value of the payoff from exercising of the growth option. The latter takes into account the dilution of the entrepreneur's stake due to pooling as well as the condition that the growth option will be exercised only if the pooling occurs before the separation. The value of a low type firm consists of three components, and the latter two give the value of the growth option in case of pooling and separation, respectively.

[8]Because the equilibrium in Proposition 2 is stationary and the current market belief z_t is the only state variable, from now on we will use $E_\theta(z_t)$ to denote the equity value of existing shareholders.

3.2.1 Equilibrium Selection

As typical in signaling games, the equilibrium established by Proposition 2 is just one of potentially myriad Perfect Bayesian equilibria in the absence of further restrictions on the off-equilibrium beliefs. Many of them, however, are economically implausible. For example, if the market threatens the firm that it will believe the firm is of low type with probability 1 in case there is no issuance at $t = 0$, we are back to the static model equilibria. However, such beliefs fail the following intuitive forward induction reasoning: if the market does not observe equity issuance at $t = 0$, this deviation is more likely to be caused by a high type firm (because a low type firm always prefers to invest under any belief). This reasoning suggests that the market's conditional probability the firm is a high type, p_t, should rise or even jump up right after $t = 0$, if there is no issuance. This in turn creates incentives for a low type firm to delay investment.

To exclude such economically meaningless equilibria, we require the beliefs to be *monotone* off the equilibrium path. This refinement is a natural extension of the Divinity refinement (Banks and Sobel (1987) to the continuous-time setting and has been used by Gul and Pesendorfer (2012) and Daley and Green (2012).

Definition 3 (Monotone Beliefs) *Belief process p_t is called monotone if for all $t > s$:*

$$p_t \geq \frac{\varphi^H_{t-s}(X_t - X_s)p_s}{\varphi^H_{t-s}(X_t - X_s)p_s + \varphi^L_{t-s}(X_t - X_s)(1 - p_s)}, \tag{41}$$

and, if observing investment at $t \geq 0$:

$$q_t = \begin{cases} 0, & \text{if an } H \text{ type firm would not want to sell for } \lambda_t \text{ offered at the belief level } p_t; \\ p_t, & \text{otherwise.} \end{cases} \tag{42}$$

Notice that if we replace p with P in (41), it will hold with equality. In other words, the first part of the belief monotonicity condition requires the market belief p_t to be at least as high as a non-strategic posterior P_t, given that they started from the same point $P_s = p_s$, in the absence of issuance between s and t. Hence, the delay of investment is interpreted as a signal which is more likely to be sent by a high type firm. The second part of the definition deals with another out-of-equilibrium deviation, when the market observes an unexpected investment. We use the analog of intuitive criterion to restrict out-of-equilibrium beliefs and place zero probability on the firm's type being high when a particular deviation is unprofitable for a high type firm.

We call a Perfect Bayesian equilibrium that is consistent with monotone beliefs a Monotone Perfect Bayesian Equilibrium. The following proposition shows that the equilibrium of Proposition 2 is of this type.

Proposition 4 (Monotone Perfect Bayesian Equilibrium) *The equilibrium of Proposition 2 is a monotone Perfect Bayesian Equilibrium.*

Are there any other reasonable monotone equilibria? In general, it is difficult to characterize the set of reasonable equilibria in this type of models, unless further restrictions on strategies are introduces. A natural restriction in our case follows from the following observation. Note that all strategies in the equilibrium of Proposition 2 are Markov and depend only on (Z_t, t). This is the case because X_t and Z_t are linearly related, X_t is a Markov process and, moreover, the Bayesian updated posterior beliefs process, P_t, is also a Markov process that depends only on the history of cash flows X_t. In other words, these strategies depend only on the beliefs process Z_t and do not have explicit time dependence, because Z_t is a homogeneous Markov process adapted to $(\mathcal{G}_t)_{t\geq 0}$.

Definition 4 (Stationary Strategies) *The firm's strategy $\pi^*(\theta)$ and the market's strategy λ^* are called* stationary *if they are functions of the belief process Z_t only.*

As the next proposition shows, the equilibrium in Proposition 2 is the unique monotone Perfect Bayesian equilibrium in stationary strategies:

Proposition 5 (Equilibrium Uniqueness) *The equilibrium of Proposition 2 is the unique Monotone Perfect Bayesian Equilibrium in stationary strategies.*

The equilibrium derived in Proposition 2 is, thus, the only reasonable equilibrium within a broad class of equilibria admitting only stationary strategies.[9] Therefore, to study the economic implications of the dynamic model and to compare its behavior to the static counterpart, we can focus on this equilibrium only.[10]

3.3 Discussion of the Dynamic Equilibrium

3.3.1 Behavior of Belief Thresholds

As in the static case, the economic behavior of the dynamic model is determined to a large extent by the properties of the belief thresholds. These properties are driven by the impact of parameters on

[9]Proposition 5 holds for a suitably adjusted definition of strategies that includes off-equilibrium behavior. The resulting equilibrium is equivalent to the one described in Proposition 2 on equilibrium path. See Appendix for details.

[10]Because a main objective of our analysis is to compare the static market with a dynamic one, focusing only on stationary strategies is likely to understate the importance of dynamics in determining investment and capital formation. Nonetheless, as we demonstrate in the next section, our dynamic equilibrium with stationary strategies already has substantially different implications from its static counterpart; allowing nonstationary, time-dependent strategies is likely to make the difference more stark.

two types of margins. The first margin is the difference in profitability between assets in place and the investment project, and the second one is the difference between the value of assets in place for a high type and a low type firms. The next proposition elucidates the relationship between dynamic and static thresholds and shows the limiting properties of dynamic thresholds.

Proposition 6 (Belief Thresholds) *The equilibrium belief thresholds $\underline{p}^*$, $\bar{p}^*$, and p^* have the following properties:*

(a) $\bar{p}^* > p^*$;

(b) There exists $\varepsilon > 0$ *such that if* $p^* > \frac{1}{2} + \varepsilon$, *then* $\underline{p}^* < p^*$; *and if* $p^* < \frac{1}{2} - \varepsilon$, *then* $\underline{p}^* > p^*$;

(c) $\bar{p}^* \to p^*$ *and* $\underline{p}^* \to p^*$ *as* $\sigma \to \infty$;

(d) $\bar{p}^* \to 1$ *and* $\underline{p}^* \to \frac{1}{2}$ *as* $\sigma \to 0$ *or* $\mu_H \to +\infty$.

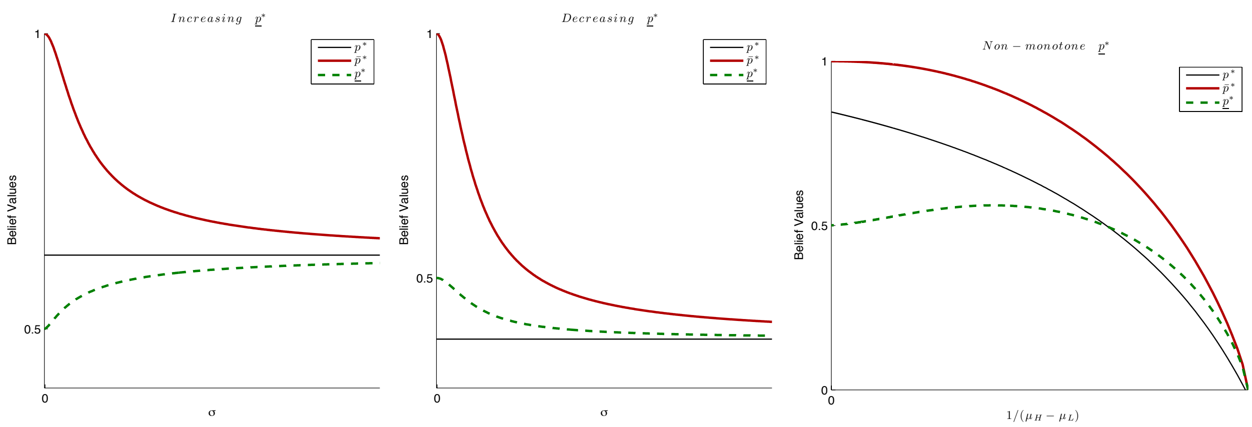
As discussed in Section 2, the threshold p^* is strategically chosen by a high type firm on the basis of the trade-off between investing in a positive NPV project and pooling-driven underpricing of assets in place. In the dynamic setup, a high type firm can relax the trade-off constraints and reduce underpricing of existing assets by waiting and investing in the positive NPV project at a later date. Therefore, it optimally sets $\bar{p}^* > p^*$ in the equilibrium.

A low type firm does not behave strategically in the static setup, because it always finds it profitable to invest. In the dynamic environment, a new strategic dimension is added, for a low type firm is torn now between investing in order to get the benefits of the investment project sooner and waiting in the hope to mimic a high type firm and obtain the overpriced value for assets in place. When p^* is sufficiently large, a high type firm effectively minimizes pooling, because existing assets are more valuable. The same force increases the desire of a low type firm to mimic and lowers $\underline{p}^*$. Conversely, as the investment project becomes relatively more valuable, p^* decreases and $\underline{p}^*$ increases. When these forces are strong enough, part (b) of the proposition follows.

Figures 3 plot the equilibrium thresholds as functions of σ and $\mu_H - \mu_L$. As $\sigma \to \infty$, the realized cash flows become increasingly uninformative, and investors can no longer learn the quality of assets in place over time. Without learning, the dynamic environment converges to the static one of Section 2, and the dynamic thresholds, $\bar{p}^*$ and $\underline{p}^*$, converge to the static threshold, p^*.

Conversely, as the cash flows become infinitely informative – i.e., as either $\sigma \to 0$ or as $\mu_H \to \infty$ – the true type of the firm is revealed in the immediate future. A high type firm sets the highest upper threshold (i.e. $\bar{p}^* \to 1$) to avoid underpricing. The less obvious limiting behavior of $\underline{p}^*$ is the result of two countervailing forces. A low type firm is torn between decreasing its threshold, because pooling becomes more attractive, and increasing it, because a high type firm's behavior implies a longer waiting time without corresponding benefits. In the equilibrium, these two forces cancel each other, making the lower threshold $\underline{p}^*$ converge to 1/2.

Figure 3: Equilibrium thresholds as functions of σ and $\mu_H - \mu_L$



(a) (b) (c)

Table 2 reports the comparative statics of the belief thresholds (both in p-space and z-space) with respect to the model parameters.[11] The upper threshold $\bar{p}^*$ behaves similarly to the static threshold p^*, consistent with intuition that both thresholds are effectively chosen by a high type firm. In the dynamic model, a higher μ_H and a lower μ_L have two effects on the upper threshold. First, they imply a worse underpricing for a high type firm. Second, they increase the signal-to-noise ratio and make cash flows more informative. Both effects encourage a high type firm to wait longer, resulting in higher $\bar{p}^*$. Conversely, an increase in the volatility parameter σ makes cash flows less informative and leads to lower $\bar{p}^*$.

A lower net present value of the investment project (i.e., higher I, lower k, or higher r) makes existing assets relatively more valuable to a high type firm and thus implies higher $\bar{p}^*$. The discount rate r has yet another effect on $\bar{p}^*$, however, by rendering future less valuable and therefore waiting more costly. In all the parameter constellations we have considered, the first effect dominates, leading to an overall positive impact of r on $\bar{p}^*$.

The effect of any parameter on $\underline{p}^*$ can be intuitively understood by noting that a low type firm faces two opposing forces. For example, higher μ_H increases the desirability of pooling, because the benefit from overpricing, conditional on pooling, is greater (direct effect). At the same time and for the same reason, higher μ_H causes a high type firm less willing to pool and leads to an increase in the upper threshold $\bar{p}^*$, making pooling more costly to a low type firm and leading correspondingly to higher $\underline{p}^*$

[11] Because the equilibrium thresholds, $\bar{p}^*$ and $\underline{p}^*$, are expressed through intricate implicit functions, most of the comparative statics are based on extensive numerical checks.

Table 2: Comparative Statics of the Dynamic Model

Variable	Sign of change in variable for an increase in: μ_H	μ_L	I	k	r	σ
p^*, z^*	+	−	+	−	+	0
$\bar{p}^*$, $\bar{z}^*$	+	−	+	−	+	−
$\underline{p}^*$, $\underline{z}^*$	$\diamond$	−	+	−	+	$\diamond$
$\bar{p}^* - \underline{p}^*$	+	−	$\diamond$	$\diamond$	$\diamond$	−
$\bar{z}^* - \underline{z}^*$	+	−	+	−	+	−
α_L, α	+	−	+	−	+	−

In this table, $\diamond$ means "increasing, decreasing, or non-monotone".

(indirect effect). It is because of these two countervailing forces that the behavior of $\underline{p}^*$ with respect to μ_H is not necessarily monotone. When the NPV of the investment project is relatively low, the direct effect dominates (as shown on Figure 3(a)), and when the investment project is relatively more valuable, the indirect dominates (as in Figure 3(b)). The impact of higher volatility σ is similar: lower $\bar{p}^*$ makes pooling more attractive, but a higher noise-to-signal ratio renders waiting more costly. In the case of an increase in μ_L or in NPV, $k/r - I$, the indirect effect (i.e., when a low type firm effectively chooses its threshold in the wake of the decision on the upper threshold by a high type firm) dominates the direct effect.

The penultimate two rows of Table 2 show the comparative statics of the inaction region, $\bar{p}^* - \underline{p}^*$ ($\bar{z}^* - \underline{z}^*$). Higher noise-to-signal ratio shrinks the inaction region, because it makes less opportune for firms to wait longer. In addition, higher $\mu_H - \mu_L$ increases the transfer from a high type firm to a low type firm in pooling and tends to widen the gap further. A higher NPV is expected to reduce the inaction region, because the cost of delay for both firms is large. Indeed, it does so in the log-likelihood space (the "z-space"), but the behavior in the "p-space" is more convoluted because of non-linearities.

3.3.2 Probabilities of Pooling and Separation

Eventually, either pooling or separation occurs in the dynamic model after an initial inaction period. We are interested both in the conditional separation probability, i.e., if the type of a firm is fixed, and the unconditional probability, i.e. the one that can be observed by the market. These probabilities are given in the next proposition.

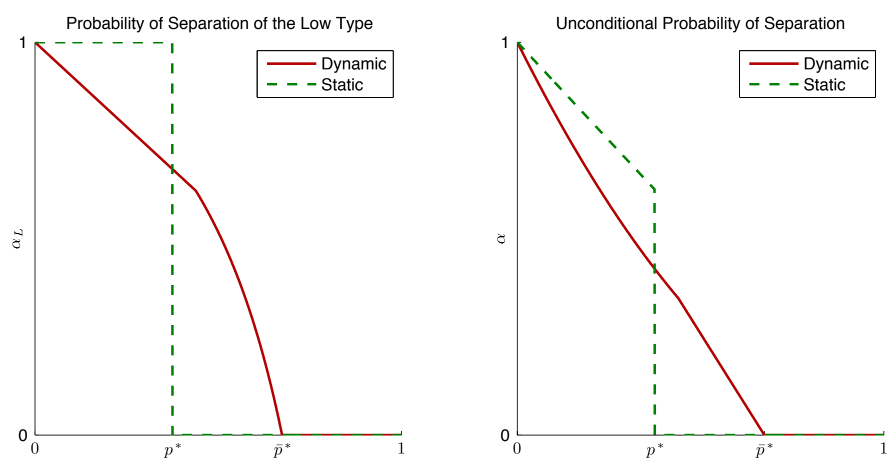
Proposition 7 (Pooling and Separation Probabilities) *If the current market belief is p, then probability of a low type firm eventually separating (investing at the lower threshold $\underline{p}^*$) is:*

$$\alpha_L(p) = \begin{cases} 0, & if\ p \geq p^*; \\ \frac{\bar{p}^* - p}{\bar{p}^*(1-p)}, & if\ p \in (\underline{p}^*, \bar{p}^*); \\ \frac{\underline{p}^* - p}{\underline{p}^*} + \frac{p}{\underline{p}^*} \cdot \frac{\bar{p}^* - \underline{p}^*}{\bar{p}^*(1-\underline{p}^*)}, & if\ p < \underline{p}^*. \end{cases} \tag{43}$$

The unconditional probability of separation is $\alpha(p) = (1-p) \cdot \alpha_L(p)$.

Compared to the static model, with its stark prediction that separation happens with certainty when the current level of beliefs p is below the threshold value p^* and does not happen at all otherwise, our model generates much smoother separation probability. Figure 4 shows the impact of current market beliefs p on separation probabilities for static and dynamic models. If p is sufficiently low (i.e., if $p < \underline{p}^*$), the rate of separation in dynamics is lower than in the static case, because of the pooling option for a low type firm that leads to temporary inaction. For higher values of current market beliefs (if $p^* < p < \bar{p}^*$), separation probability in dynamics is larger, because of the real option held by a high type firm that enables it to mitigate partially the adverse selection problem. Interestingly, the probability of separation in the inaction region, given by (eq:sepprob), does not directly depend on the lower threshold $\underline{p}^*$. In the dynamic model of the political process, Gul and Pesendorfer (2012) obtain a related result (in their model, however, the upper barrier is exogenously fixed). Intuitively, higher $\underline{p}^*$ makes the inaction region smaller, but also the rate of investment at the separation threshold lower. These two effects cancel making the probabilities of pooling and separation independent of the lower threshold in the inaction region.

Figure 4: Probability of Separation as a Function of Current Belief Level p



As can be seen from the last row of Table 2, both the unconditional and conditional (on type) probabilities of separation respond intuitively to the change of the model parameters. The reason for that is the fact that both α_L and α depend only on $\bar{p}^*$ in the inaction region, thus, with the change of $\bar{p}^*$ the whole curve shifts. The pooling threshold $\underline{p}^*$ plays a much simpler role - it only determines the cutoff at which α_L becomes linear with a slope satisfying $\alpha_L(0) = 1$. Lower volatility of cashflows (σ) and stronger adverse selection ($\mu_H - \mu_L$) increase the signal-to-noise ratio making in harder for the low type to pool, therefore, increasing the probability of separation. Higher NPV of the project decreases the relative importance of signaling for the high type which benefits the low type firm through the lower probability of separation.

4 Discussion

In this section we explore the economic implications of the dynamic environment and compare them to the implications of the static setting in Myers and Majluf (1984). We discuss the impact of dynamics on social welfare, firm market values, and investment delay.

4.1 Firm Values and Social Welfare

In the static environment, the society loses because in a market economy a positive NPV project is not implemented by a high type firm, if the likelihood of the market facing a low type is sufficiently large. In the dynamic environment, the investment opportunity is eventually implemented, but the social loss comes from the delay by both types, caused by the time playing the role of a quality signal. We explore here whether the society benefits ex-ante from enabling the time dimension.

As a first step, we analyze private benefits and losses experienced by high and low type firms. The next proposition compares firm values in both environments. For the purpose of this section, define $E^S_\theta(p)$ and $E^D_\theta(p)$ as the value of a type-θ firm, given the belief p, in the static and dynamic models, respectively.

Proposition 8 (Static and Dynamic Firm Values) *Firm values, $E^S_\theta(p)$ and $E^D_\theta(p)$, satisfy the following:*

1. *For a high type firm:*
   (a) $E^D_H(p) = E^S_H(p)$, *if* $p \geq \bar{p}^*$;
   (b) $E^D_H(p) > E^S_H(p)$, *if* $p < \bar{p}^*$.
2. *For a low type firm:*
   (a) $E^D_L(p) = E^S_L(p)$, *if* $p \geq \bar{p}^*$ *or if* $p \leq \min(\underline{p}^*, p^*)$;

(b) $E_L^D(p) < E_L^S(p)$*, if* $p \in (p^*, \bar{p}^*)$*;*

(c) $E_L^D(p) > E_L^S(p)$*, if* $\underline{p}^* < p^*$ *and* $p \in (\underline{p}^*, p^*)$*.*

A high type firm prefers an option to delay. Intuitively, because it is a high type firm that effectively chooses the upper trigger, by choosing the same upper trigger as the static trigger, a high type firm can guarantee itself a payoff that is not less than in the static case. The voluntary option to wait always has a non-negative value. This preference for the dynamic environment is not shared by a low type firm. When the market's belief is of intermediate quality (i.e., if $p \in (\max(p^*, \underline{p}^*), \bar{p}^*))$, which is arguably the most interesting case, the "evaporating" investment opportunity in the static economy forces a high type firm to invest immediately. This results in the wealth transfer from a high to a low type firms. Given the option to wait, a low type firm looses the benefit of an immediate wealth transfer. In this case, the option to wait is clearly harmful to a low type firm.

Let E^S and E^D be unconditional firm values in the static and the dynamic economies, respectively, $E^S(p) = pE_H^S(p) + (1-p)E_L^S(p)$ and $E^D(p) = pE_H^D(p) + (1-p)E_L^D(p)$. These values summarize the economy social welfare and their comparison enables us to explore welfare implications. As the next proposition shows, adding dynamics does not always lead to better society outcome.

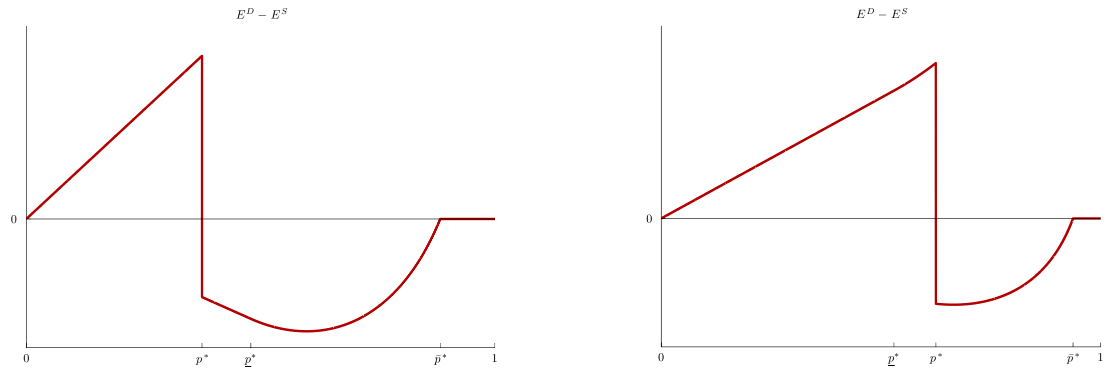
Proposition 9 (Static and Dynamic Social Welfare) *Unconditional firm values,* $E^S(p)$ *and* $E^D(p)$*, satisfy the following:*

1. $E^D = E^S$ *for* $p > \bar{p}^*$*;*
2. $E^D > E^S$ *for* $p < p^*$*;*
3. $E^D < E^S$ *for* $p \in (p^*, \bar{p}^*)$*.*

The proposition is illustrated in Figure 5, which shows two cases, depending on the relation between the lower and the static thresholds. The intuition is similar for both cases. Because immediate pooling achieves the first best outcome and the dynamic option enables a high type firm to delay, the inaction region (above the static threshold) leads to a strictly worse society outcome. On the other hand, because delay is preferred to a high type not investing at all, the welfare is improved, if the static outcome is separation. In other words, from the social planner's point of view the delay is inefficient, because the project is commonly known to have a positive NPV; thus, absence any adverse selection investment should be made right away. Dynamic equilibrium improves efficiency by expanding the region of beliefs, at which a high type firm invests; however, it does so at the expense of introducing the delay in the inaction region.

That delay leads to a welfare loss may hinge, of course, upon the economy structure. For example, even though we do not entertain this possibility, firms can vary the size of their investment, with a

Figure 5: The difference between Unconditional Static and Dynamic Firm Values as a function of p



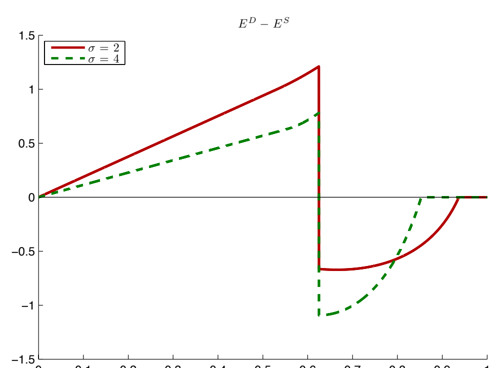
(a) Case $p^* < \underline{p}^*$

(b) Case $p^* > \underline{p}^*$

high type firm preferring a lower investment outlay in the pooling case (with the optimal size resulting from the same trade-off that leads it to choose the upper threshold trigger) and a larger project in the dynamic case. This may well affect the welfare implications. This suggests, however, that that the optimality of delay is an important characteristic of the industry and is worth further exploration.

The difference between unconditional dynamic and static firm values increases in cash flow levels and growth options's NPV. The impact of volatility is more subtle, as Figure 6 shows. For higher volatility, the inaction region is smaller but the loss of welfare is larger, because the delay is longer. At the same time, the advantage of the dynamics decreases in the static separation region, because higher volatility makes it harder for a high type firm to signal its type.

Figure 6: The difference between Unconditional Static and Dynamic Firm Values for various σ

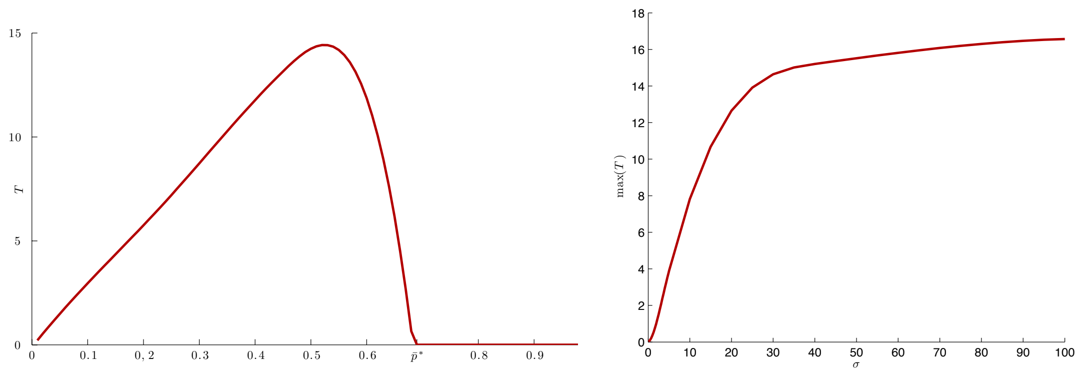


Baseline model parameters: $\mu_H = 3$, $\mu_L = 1$, $r = 0.1$, $k = 2, I = 17$.

4.2 Delay in Investment

Our discussion so far suggests that the delay plays a critical role in the dynamic environment. Figure 7(a) shows that the expected waiting time in the inaction region can be very high, especially if the NPV of the growth option is relatively small in comparison with the value of assets in place.

Figure 7: Unconditional Expected Time to Investment



(a) As a function of p for $\sigma = 20$

(b) Maximal delay as a function of σ

Model parameters: $\mu_H = 12$, $\mu_L = 5$, $r = 0.05$, $k = 2$, and $I = 30$

The expected waiting time is sensitive to the volatility parameter, although the total effect is ambiguous. On the one hand, as informativeness decreases, the dynamic upper threshold eventually approaches the static threshold. This implies that a high type firm is almost indifferent between investing right away and not investing at all. On the other hand, lower informativeness decreases the signal-to-noise ratio making it harder for the market to learn from the evolution of cash flows, thus, the belief process stays longer in any given interval and the maximal delay in investment increases as illustrated by Figure 7(b).

Notice that our results are robust to the following extension of the model: suppose that firms can disentangle the times of raising money and investment. In such a world, there would be no way for a high type firm to separate itself via a choice of the way the funds are raised. Obviously, only investment plays the signaling role and it is therefore suboptimal to raise money before investment is made. Given such a strategy of a high type firm, a low type firm would optimally mimic it at the pooling threshold and would raise the full cost of investment at the separating threshold as well. Therefore, waiting times in the model with several stages of fund raising are exactly the same as in the benchmark model of Section 3.

5 Conclusion

In this paper, we have extended the classical Myers and Majluf (1984) static setup to the dynamic environment, in which firms can optimally delay taking investment projects in an attempt to signal their quality. A real option to wait enriches the interplay between low and hight type firms and makes the trade-off between assets in place and growth options more subtle. The dynamic learning environment gives rise to the inaction region, in which both types delay, one to signal and another to mimic. A high type firm welcomes an option to wait, while a low type firm is more likely to lose. At the same time, delay, caused by the inaction region, has an ambiguous impact on social welfare.

Our paper belongs to an ever-increasing field that studies the impact of the truly dynamic environment on behavior and economic properties of equilibrium. The analysis of dynamic issues in earnest has just begun, and while more research is needed to understand these issues better, it becomes abundantly clear that the properties of the dynamic environment are often quite different from that of the static one.

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A Appendix: Proofs

A.1 Proof of Lemma 1

Proof. Because the share λ^* that is required by the market is uniquely defined by q_0 via (10), we can consider the best response of an H type firm as a function of q_0. L type firm always invests with probability 1.

If $q_0 < p^*$, then a low type firm invests with probability 1, and a high type firm does not invest at all. The Bayesian consistency of beliefs, given by (7), requires that $q_0 = 0$, which is less that p^*. Thus, the strategies given in part (1.) constitute an equilibrium.

If $q_0 > p^*$, both types invest with probability 1, therefore, the Bayes rule (7) implies $q_0 = p_0$. If $q_0 = p^*$, a low type firm invests with probability 1 and a high type firm is indifferent between investing and not investing. Under our tie-breaking assumption, a high type firm also invests with probability 1 and the previous case applies. Thus, the strategies given in part (2.) also constitute an equilibrium. ■

A.2 Proof of Proposition 2

Proof. The idea of the proof can be summarized in three steps. First, we show that, given the reflecting belief process with any lower boundary $\underline{z}$, there exists a well defined function $f(\underline{z})$, such that the barrier policy $\bar{z} = f(\underline{z})$ is optimal for a high type firm. Then, given any upper boundary $\bar{z}$, there exists a well defined function $g(\bar{z})$, such that a low type firm is indifferent between revealing itself at the lower boundary $\underline{z} = g(\bar{z})$ and waiting to pool. Finally, we show that there exists a unique fixed point.

Let the total value of type θ be $E_\theta(z)$. In the inaction region $E_\theta(z)$ satisfies:

$$\frac{h^2}{2}E_H'(z) + \frac{h^2}{2}E_H''(z) - rE_H(z) + \mu_H = 0, \tag{A1}$$

$$-\frac{h^2}{2}E_L'(z) + \frac{h^2}{2}E_L''(z) - rE_L(z) + \mu_L = 0. \tag{A2}$$

Part I Given the lower reflecting barrier of beliefs at $\underline{z}$, it is well known (e.g., Harrison (1985)) that the optimal issuance decision by a high type firm is of threshold type, and the threshold $\bar{z}$ should satisfy the following boundary condition:

$$E_H(\bar{z}) = E_H^P(\bar{z}), \tag{A3}$$

$$E_H'(\bar{z}) = E_H^{P\prime}(\bar{z}), \tag{A4}$$

$$E_H'(\underline{z}) = 0, \tag{A5}$$

where $E_H^P(\bar{z})$ is the value of equity a high type firm in the case of pooling:

$$E_\theta^P(z) = (1 - \lambda(z))\frac{\mu_\theta + k}{r}, \tag{A6}$$

where

$$\lambda(z) = \frac{(1 + e^z)Ir}{e^z(\mu_H + k) + (\mu_L + k)}. \tag{A7}$$

Equation (A1) has a general solution of the form:

$$E_H(z) = C_1 e^{u_1 z} + C_2 e^{u_2 z} + \frac{\mu_H}{r}, \tag{A8}$$

where C_1 and C_2 are constants, and u_1 and u_2 are, respectively, the positive and negative roots of the characteristic equation:

$$\frac{h^2}{2}u^2 + \frac{h^2}{2}u - r = 0, \tag{A9}$$

and their values are:

$$u_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2r}{h^2}}. \tag{A10}$$

Using (A8), we can rewrite (A3), (A4), and (A5) as:

$$C_1 e^{u_1 \bar{z}} + C_2 e^{u_2 \bar{z}} + \frac{\mu_H}{r} = E_H^P(\bar{z}), \tag{A11}$$

$$C_1 u_1 e^{u_1 \bar{z}} + C_2 u_2 e^{u_2 \bar{z}} = E_H^{P'}(\bar{z}), \tag{A12}$$

$$C_1 u_1 e^{u_1 \underline{z}} + C_2 u_2 e^{u_2 \underline{z}} = 0. \tag{A13}$$

Solving (A11) and (A12) for C_1 and C_2, and substituting the result into (A13), we obtain:

$$e^{(u_1-u_2)\underline{z}} = -\frac{u_2}{u_1}\frac{u_1(E_H^P(\bar{z}) - \mu_H/r) - E_H^{P'}(\bar{z})}{E_H^{P'}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r)} \cdot e^{(u_1-u_2)\bar{z}}. \tag{A14}$$

As we prove below, the right-hand side (RHS) of (A14) is strictly increasing in $\bar{z}$. Together with the limiting values and the implicit function theorem, this would imply the existence of a continuous and strictly increasing function $f(\underline{z})$.

The partial derivative of the RHS of (A14) w.r.t $\bar{z}$ is equal to (up to a positive multiplier):

$$\begin{aligned} &(u_1 - u_2)\frac{u_1(E_H^P(\bar{z}) - \mu_H/r) - E_H^{P'}(\bar{z})}{E_H^{P'}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r)} \\ &+ \frac{(u_1 E_H^{P'}(\bar{z}) - E_H^{P''}(\bar{z})(E_H^{P'}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r))}{(E_H^{P'}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r))^2} \\ &- \frac{(u_1(E_H^P(\bar{z}) - \mu_H/r) - E_H^{P'}(\bar{z}))(E_H^{P''}(\bar{z}) - u_2 E_H^{P'}(\bar{z}))}{(E_H^{P'}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r))^2}. \end{aligned} \tag{A15}$$

The second term is clearly positive, because $E_H^{P'} > 0$, $E_H^{P''} < 0$, $u_2 < 0$, and $E_H^P(\bar{z}) - \mu_H/r) > 0$. The sign of the sum of the first and the third terms is determined by the sign of

$$(u_1-u_2)[E_H^{P'} - u_2(E_H^P - \mu_H/r)] - [E_H^{P''} - u_2 E_H^{P'}] > -u_2[u_1(E_H^P - \mu_H/r) - E_H^{P'}] + (u_1-u_2)E_H^{P'} - E_H^{P''} > 0. \tag{A16}$$

As $\bar{z} \to +\infty$, the RHS of (A14) also approaches $+\infty$. Moreover, the RHS has a unique root $\bar{z}_r$:

$$u_1(E_H^P(\bar{z}_r) - \mu_H/r) - E_H^{P\prime}(\bar{z}_r) = 0. \tag{A17}$$

Hence, for any $\underline{z}$ there exists a well-defined, smooth, and strictly increasing best response function $\bar{z} = f(\underline{z})$.

Part II Next, we solve the problem for a low type firm. Given $\bar{z}$ and a reflecting belief process at $\underline{z}$, there exists a unique $\underline{z}$, such that a low type firm is exactly indifferent at the lower boundary. Therefore, it randomizes to sustain the reflecting beliefs. Similar to the above, (A2) has a general solution:

$$E_L(z) = D_1 e^{v_1 z} + D_2 e^{v_2 z} + \frac{\mu_L}{r}, \tag{A18}$$

where D_1 and D_2 are constants, and v_1 and v_2 are, respectively, the positive and negative roots of the characteristic equation:

$$\frac{h^2}{2} v^2 - \frac{h^2}{2} v - r = 0, \tag{A19}$$

and their values are:

$$v_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2r}{h^2}}. \tag{A20}$$

The value function has to satisfy (again, see Harrison (1985)) three conditions: indifference and reflecting at the lower boundary, and value matching at the upper boundary:

$$E_L(\underline{z}) = \frac{\mu_L + k}{r} - I, \tag{A21}$$

$$E_L'(\underline{z}) = 0, \tag{A22}$$

$$E_L(\bar{z}) = E_L^P(\bar{z}). \tag{A23}$$

Using (A18), we can re-write (A21) and (A22) as:

$$D_1 e^{v_1 \underline{z}} + D_2 e^{v_2 \underline{z}} + \frac{\mu_L}{r} = \frac{\mu_L + k}{r} - I, \tag{A24}$$

$$D_1 v_1 e^{v_1 \underline{z}} + D_2 v_2 e^{v_2 \underline{z}} = 0. \tag{A25}$$

Solving (A24) and (A25) for D_1 and D_2, we can write D_1 and D_2 in terms of $\underline{z}$:

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{1}{v_1 - v_2} \begin{pmatrix} -v_2 e^{(v_2 - 1)\underline{z}} \\ v_1 e^{(v_1 - 1)\underline{z}} \end{pmatrix} \left(\frac{k}{r} - I \right). \tag{A26}$$

Substituting the above equation into (A23), we obtain:

$$\frac{1}{v_1 - v_2} \left(\frac{k}{r} - I \right) \left(v_1 e^{(v_1 - 1)\underline{z} + v_2 \bar{z}} - v_2 e^{(v_2 - 1)\underline{z} + v_1 \bar{z}} \right) = \frac{k}{r} - \lambda(\bar{z}) \frac{\mu_L + k}{r}. \tag{A27}$$

It suffices to show that for any given $\bar{z}$ there exists a unique $\underline{z}$ such that the above equation holds. The RHS of (A27) is not a function of $\underline{z}$, while for the left-hand side (LHS):

$$\frac{d\ LHS}{d\underline{z}} = \frac{1}{v_1 - v_2}\left(\frac{k}{r} - I\right) v_1 v_2 \left(e^{v_1(\bar{z}-\underline{z})} - e^{v_2(\bar{z}-\underline{z})}\right) < 0, \forall \underline{z} \le \bar{z}. \tag{A28}$$

Because $v_1 + v_2 = 1$, as $\underline{z} \to \bar{z}$, $LHS \to \frac{k}{r} - I < RHS$, and as $\underline{z} \to -\infty$, $LHS \to \infty$. Thus, by the intermediate value theorem and monotonicity, we have a unique solution $\underline{z} = g(\bar{z}) < \bar{z}$.

Part III Re-writing (A27), using $\underline{z} = f^{-1}(\bar{z})$, we obtain:

$$\frac{1}{v_1 - v_2}\left(\frac{k}{r} - I\right)\left(v_1\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_2} - v_2\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_1}\right) - \frac{k}{r} + \lambda(\bar{z})\frac{\mu_L + k}{r} = 0. \tag{A29}$$

Any root of this equation greater than $\bar{z}_r$ (defined in (A17)) gives rise to a pair $(\bar{z}, \underline{z})$ that characterize an equilibrium. Here, $\bar{z} \ge \bar{z}_r$ is needed, because both sides of (A14) must be positive.

Notice that, as $\bar{z} \to \bar{z}_r+$, the LHS of (A29) approaches $+\infty$, because $f^{-1}(\bar{z})$, which is a positive multiple of the log of the RHS of (A14), approaches $-\infty$, and $v_1 > 0 > v_2$. By (A14), $\bar{z} - \underline{z} \to 0$ as $\bar{z} \to \infty$. Thus, as $\bar{z} \to +\infty$, the limit of the LHS of (A29) is:

$$\left(\frac{k}{r} - I\right) - \frac{k}{r} + \frac{Ir}{\mu_H + k}\frac{\mu_L + k}{r} < 0. \tag{A30}$$

The partial derivative of the LHS of (A29) is equal to:

$$\frac{v_1 v_2}{v_1 - v_2}\left(\frac{k}{r} - I\right)\left[\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_2 - 1} - \left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_1 - 1}\right] \cdot \frac{d}{d\bar{z}}\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right) + \lambda'(z)\frac{\mu_L + k}{r}. \tag{A31}$$

We already know that $\lambda' < 0$, $v_1 > 0 > v_2$, and $\frac{k}{r} - I > 0$. Differentiating $\frac{e^{\bar{z}}}{e^{\underline{z}}}$ from (A14), we find that $\frac{d}{d\bar{z}}\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right) < 0$. Finally, $\left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_2 - 1} < \left(\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}}\right)^{v_1 - 1}$, because $\frac{e^{\bar{z}}}{e^{f^{-1}(\bar{z})}} > 1$.

We have proved that the whole expression in (A31) is negative. Thus, the LHS of (A29) is strictly decreasing from $+\infty$ at $\bar{z}_r+$ to a negative value at $+\infty$. Therefore, there exists a unique $\bar{z}^* > \bar{z}_0$ that solves (A29). The other boundary $\underline{z}^*$ is given by $f^{-1}(\bar{z}^*)$ or, equivalently, by $g(\bar{z}^*)$.

Part IV We have shown that the pair $(\underline{z}^*, \bar{z}^*)$ constitutes a unique equilibrium, if we allow deviations within the class of two threshold (lower reflective and upper pooling, respectively) strategies. However, it does not account for arbitrary deviations that are allowed by the definition of $\pi(\theta)$. We now verify sub-optimality of arbitrary deviations by considering a modified version of our game. Define $E^*_\theta(z)$ as the value function of the original game:

$$E^*_\theta(z) = \sup_{\pi(\theta)} \mathbf{E}\left[\int_0^\infty \left(\int_0^t e^{-ru} dX_u + e^{-rt} E_\theta(q, t)\right) d\pi_t(\theta)\right]. \tag{A32}$$

Consider the following modification: at any moment the firm can be sold to an outsider for $E_\theta(z)$, where E_θ is the expected payoff to a θ-type player when the parties play $(\underline{z}^*, \bar{z}^*)$ strategies.

Define $\tilde{E}_\theta(z)$ as be the value function of the modified game:

$$\tilde{E}_\theta(z) = \sup_{\pi(\theta)} \mathbf{E}\left[\int_0^\infty \Big(\int_0^t e^{-ru}dX_u + e^{-rt}\max(E_\theta(q,t), E_\theta(z_t))\Big)d\pi_t(\theta)\right]. \tag{A33}$$

Clearly, $\tilde{E}_\theta(z) \geq E^*_\theta(z) \geq E_\theta(z)$. In addition, $E_\theta(z_t) \geq E_\theta(q,t)$, as can be easily seen from equations (A3)–(A5), (A21)–(A23), and the convexity of E_θ in z. Therefore:

$$\tilde{E}_\theta(z) = \sup_{\pi(\theta)} \mathbf{E}\left[\int_0^\infty \Big(\int_0^t e^{-ru}dX_u + e^{-rt}E_\theta(z_t)\Big)d\pi_t(\theta)\right]. \tag{A34}$$

In this case, there is no need for mixed strategies, because every player is facing a simple optimal stopping problem. Thus, we can re-write $\tilde{E}$ as:

$$\begin{aligned}\tilde{E}_\theta(z) &= \sup_{\tau} \mathbf{E}\left[\int_0^\tau e^{-ru}dX_u + e^{-r\tau}E_\theta(z_\tau)\right] \\ &= \sup_{\tau} \mathbf{E}\left[(1-e^{-r\tau})\frac{\mu_\theta}{r} + e^{-r\tau}E_\theta(z_\tau)\right] \\ &= \sup_{\tau} \mathbf{E}\, f_\theta(\tau, z_\tau).\end{aligned} \tag{A35}$$

Because E_θ is in $\mathbb{C}^2$, except for the two points $\{\underline{z}^*, \bar{z}^*\}$, we can use Itô's lemma and write:

$$\begin{aligned}df_\theta(t,z_t) &= re^{-rt}\frac{\mu_\theta}{r}dt - re^{-rt}E_\theta(z_t)dt + e^{-rt}dE_\theta(z_t) && \text{(A36)}\\ &= e^{-rt}\left(\mu_\theta - rE_\theta(z_t) + \text{sgn}(2\mu_\theta - \mu_H - \mu_L)\frac{h^2}{2}E'_\theta(z_t) + \frac{h^2}{2}E''_\theta(z_t)\right)dt && \text{(A37)}\\ &\quad + he^{-rt}E'_\theta(z_t)dB_t + e^{-rt}E'_\theta(\underline{z})dY_t && \text{(A38)}\\ &= \Gamma_\theta E_\theta(z_t)dt + he^{-rt}E'_\theta(z_t)dB_t + e^{-rt}E'_\theta(\underline{z})dY_t, && \text{(A39)}\end{aligned}$$

where $Y_t = \max(\underline{z}^* - \inf_{s\leq t} Z_t, 0)$ and Γ_θ is a second order differential operator that corresponds to equations (A1)–(A2).

Since $E'_\theta(\underline{z}^*) = 0$, the last term in (A39) disappears; and because $E'_\theta(z)$ is bounded, the second term is a martingale, and $\Gamma_\theta E_\theta(z) \leq 0$[12]. Using Optional Sampling Theorem[13] we conclude that:

$$\mathbf{E}\, f_\theta(\tau_b, z_{\tau_b}) \leq f_\theta(0, z_0) \tag{A40}$$

[12]For $z \in (\underline{z}^*, \bar{z}^*)$, $\Gamma_\theta E_\theta(z) = 0$ by (A1)–(A2). For $z > \bar{z}^*$, it can be shown that $\Gamma_\theta E_\theta(z) = \Gamma_\theta E^P_\theta(z_t) < 0$.

[13]See for example Peskir and Shiryaev (2006) Theorem 3.2.A.

for all bounded stoping times τ_b. Because every other stopping time τ can be obtained as a limit of $\tau_b = \min(\tau, b)$ when $b \to \infty$, we conclude that:

$$\tilde{E}_\theta(z) \le f_\theta(0, z) = E_\theta(z). \tag{A41}$$

Therefore:

$$\tilde{E}_\theta(z) = E^*_\theta(z) = E_\theta(z), \tag{A42}$$

which implies that E_θ is a maximal attainable equilibrium pay-off in the original game. Because this pay-off is guaranteed when players stick to the two threshold strategies $(\underline{z}^*, \bar{z}^*)$, the latter constitute an equilibrium. ■

A.3 Proof of Lemma 2

Proof. Let Z_t be a non-strategic belief process of the market. It is a (μ_z, σ_z) Brownian Motion, i.e.:

$$dZ_t = \mu_z dt + \sigma_z dB_t, \tag{A43}$$

where $\mu_z = \frac{h^2}{2}$ for $Z_t = Z^H_t$ and $\mu_z = -\frac{h^2}{2}$ for $Z_t = Z^L_t$, and $\sigma_z = h$ for both types. Let z_t be the corresponding Reflected (at the lower threshold) Brownian Motion, which corresponds to the true market belief process:

$$z_t = Z_t + Y_t, \tag{A44}$$

where $Y_t = \max\left(\underline{z}^* - \inf_{s \le t} Z_s, 0\right)$. Define $T = \inf\{t > 0 : z_t \ge \bar{z}^*\}$ to be the time of pooling. Then the value of $A_{P,\theta}(z)$ can be written as:

$$\begin{aligned} A_{P,\theta}(z) &= \mathbf{E}_z\left[e^{-rT}(1 - \pi_{T-}(\theta))\right] \\ &= \mathbf{E}_z\left[e^{-rT - m_\theta Y_T}\right], \end{aligned} \tag{A45}$$

where $m_\theta = 0$ for a low type firm and $m_\theta = 1$ for a high type firm.

For an arbitrary piece-wise twice continuously differentiable function with uniformly bounded first derivative $g(z)$ consider a process $e^{-rt - m_\theta Y_t} g(z_t)$:

$$d\left(e^{-rt - m_\theta Y_t} g(z_t)\right) = e^{-rt - m_\theta Y_t}\left(-rg(z_t) + \mu_z g'(z_t) + \frac{1}{2}\sigma_z^2 g''(z_t)\right) dt \tag{A46}$$

$$+ e^{-rt - m_\theta Y_t} g'(z_t)\sigma_z dB_t + e^{-rt - m_\theta Y_t}(g'(\underline{z}^*) - m_\theta g(\underline{z}^*))dY_t. \tag{A47}$$

Integrating and taking $\mathbf{E}_z$ of both sides yields:

$$\mathbf{E}_z\left[e^{-rT - m_\theta Y_T} g(z_T)\right] = g(z) + \mathbf{E}_z\left[\int_0^T e^{-rt - m_\theta Y_t}\left(-rg(z_t) + \mu_z g'(z_t) + \frac{1}{2}\sigma_z^2 g''(z_t)\right) dt\right] \tag{A48}$$

$$+ E_z\left[\int_0^T e^{-rt - m_\theta Y_t}\left(g'(\underline{z}^*) - m_\theta g(\underline{z}^*)\right) dY_t\right]. \tag{A49}$$

If

$$\begin{cases} g(z_T) = g(\bar{z}^*) = 1, \\ g'(\underline{z}^*) - m_\theta g(\underline{z}^*) = 0, \\ -rg(z) + \mu_z g'(z) + \frac{1}{2}\sigma_z^2 g''(z) = 0, \quad \forall\, z \in (\underline{z}^*, \bar{z}^*), \end{cases} \tag{A50}$$

then $g(z) = A_{P,\theta}(z)$ for $z \in (\underline{z}^*, \bar{z}^*)$.

Solving the ODE (A50) and substituting μ_z, σ_z, and m_θ for $\theta = H$ and $\theta = L$ we obtain:

$$A_{P,L}(z) = \frac{v_1\, e^{v_1(z-\underline{z}^*)} - v_2\, e^{v_2(z-\underline{z}^*)}}{v_1\, e^{v_1(\bar{z}^*-\underline{z}^*)} - v_2\, e^{v_2(\bar{z}^*-\underline{z}^*)}} \tag{A51}$$

and

$$A_{P,H}(z) = \frac{u_2\, e^{u_1(z-\underline{z})} - u_1\, e^{u_2(z-\underline{z}^*)}}{u_2\, e^{u_1(\bar{z}-\underline{z}^*)} - u_1\, e^{u_2(\bar{z}^*-\underline{z}^*)}}. \tag{A52}$$

Similarly, the value of v is given by:

$$A_{S,L}(z) = \mathbf{E}_z\left[\int_0^T e^{-rt} d\pi_t(L)\right] = \mathbf{E}_z\left[\int_0^T e^{-Y_t} e^{-rt} dY_t\right]. \tag{A53}$$

For an arbitrary piece-wise twice continuously differentiable function with uniformly bounded first derivative $g(z)$ consider a process $e^{-rt-Y_t}g(z_t)$:

$$d\left(e^{-rt-Y_t}g(z_t)\right) \;=\; e^{-rt-Y_t}\left(-rg(z_t) + \mu_z g'(z_t) + \frac{1}{2}\sigma_z^2 g''(z_t)\right)dt \tag{A54}$$

$$+e^{-rt-Y_t}g'(z_t)\sigma_z dB_t + e^{-rt-Y_t}(g'(0) - g(0))dY_t. \tag{A55}$$

Integrating and taking $\mathbf{E}_z$ of both sides yields:

$$\mathbf{E}_z\left[e^{-rT-Y_T}g(z_T)\right] \;=\; g(z) + \mathbf{E}_z\left[\int_0^T e^{-rt-Y_t}\left(-rg(z_t) + \mu_z g'(z_t) + \frac{1}{2}\sigma_z^2 g''(z_t)\right)dt\right] \tag{A56}$$

$$+E_z\left[\int_0^T e^{-rt-mY_t}\left(g'(0) - g(0)\right)dY_t\right]. \tag{A57}$$

If

$$\begin{cases} g(z_T) = g(\bar{z}^*) = 0, \\ g'(\underline{z}^*) - g(\underline{z}^*) = -1, \\ -rg(z) + \mu_z g'(z) + \frac{1}{2}\sigma_z^2 g''(z) = 0, \quad \forall\, z \in (\underline{z}^*, \bar{z}^*), \end{cases} \tag{A58}$$

then $g(z) = A_{S,L}(z)$ for $z \in (\underline{z}^*, \bar{z}^*)$.

Solving the ODE (A58) and substituting $\mu_z = -\frac{h^2}{2}, \sigma_z = h$, we obtain:

$$A_{S,L}(z) = \frac{e^{v_2(\bar{z}^*-\underline{z}^*)+v_1(z-\underline{z}^*)} - e^{v_1(\bar{z}^*-\underline{z}^*)+v_2(z-\underline{z}^*)}}{v_2 e^{v_2(\bar{z}^*-\underline{z}^*)} - v_1 e^{v_1(\bar{z}^*-\underline{z}^*)}}. \tag{A59}$$

■

A.4 Proof of Proposition 3

Proof. Let $T = \inf\{t > 0 : z_t \geq \bar{z}^*\}$ be the time of pooling. The equity value of a high type firm can then be written as:

$$\begin{aligned} E_H(z_0) &= \mathbf{E}_{z_0}\left[\int_0^\infty \Big(\int_0^t e^{-ru}dX_u + e^{-rt}E_H(z_t)\Big)d\pi_t^*(H)\right] \\ &= \mathbf{E}_{z_0}\left[\int_0^T e^{-ru}dX_u + e^{-rT}E_H(\bar{z}^*)\right] \\ &= \mathbf{E}_{z_0}\left[\frac{\mu_H}{r}(1-e^{rT}) + \int_0^T e^{-ru}\sigma dB_u + e^{-rT}E_H(\bar{z}^*)\right] \\ &= \frac{\mu_H}{r} + A_{P,L}(z_0)\left(E_H(\bar{z}^*) - \frac{\mu_H}{r}\right). \end{aligned} \tag{A60}$$

The equity value of a low type firm can be written as:

$$E_L(z_0) = \mathbf{E}_{z_0}\left[\int_0^\infty \Big(\int_0^t e^{-ru}dX_u + e^{-rt}E_L(z_t)\Big)d\pi_t^*(L)\right] \tag{A61}$$

$$= \mathbf{E}_{z_0}\left[\int_0^T \Big(\int_0^t e^{-ru}dX_u + e^{-rt}E_L(\underline{z}^*)\Big)d\pi_t^*(L)\right] \tag{A62}$$

$$+\mathbf{E}_{z_0}\left[\Big(\int_0^T e^{-ru}dX_u + e^{-rT}E_L(\bar{z}^*)\Big)(1-\pi_{T-}^*(L))\right] \tag{A63}$$

$$= \mathbf{E}_{z_0}\left[\int_0^T \Big(\frac{\mu_L}{r}(1-e^{rt}) + e^{-rt}E_L(\underline{z}^*)\Big)d\pi_t^*(L)\right] \tag{A64}$$

$$+\mathbf{E}_{z_0}\left[\Big(\frac{\mu_L}{r}(1-e^{rT}) + e^{-rT}E_L(\bar{z}^*)\Big)(1-\pi_{T-}^*(L))\right] \tag{A65}$$

$$= \mathbf{E}_{z_0}\left[\frac{\mu_L}{r}\pi_{T-}^*(L) + + \Big(E_L(\underline{z}^*) - \frac{\mu_L}{r}\Big)\int_0^T e^{-rt}d\pi_L^*(L)\right] \tag{A66}$$

$$+\mathbf{E}_{z_0}\left[\frac{\mu_L}{r}(1-\pi_{T-}^*(L)) + \Big(E_L(\bar{z}^*) - \frac{\mu_L}{r}\Big)e^{-rT}(1-\pi_{T-}^*(L))\right] \tag{A67}$$

$$= \frac{\mu_L}{r} + A_{S,L}(z_0)\left(E_L(\underline{z}^*) - \frac{\mu_L}{r}\right) + A_{P,L}(z_0)\left(E_L(\bar{z}^*) - \frac{\mu_L}{r}\right) \tag{A68}$$

$$= \frac{\mu_L}{r} + A_{S,L}(z_0)\left(\frac{k}{r} - I\right) + A_{P,L}(z_0)\left(E_L(\bar{z}^*) - \frac{\mu_L}{r}\right). \tag{A69}$$

■

A.5 Proof of Proposition 4

Proof. Inequality (41) in Definition 3 can be re-written as:

$$\frac{p_t}{1-p_t} \geq \frac{p_s}{1-p_s}\cdot\frac{\varphi_{t-s}^H(X_t - X_s)}{\varphi_{t-s}^L(X_t - X_s)}. \tag{A70}$$

Taking log of both sides and utilizing the z_t notation yields:

$$z_t \geq z_s + \ln\left(\frac{\varphi^H_{t-s}(X_t - X_s)}{\varphi^L_{t-s}(X_t - X_s)}\right) \tag{A71}$$

or, equivalently:

$$Z_t + Y_t \geq Z_s + Y_s + \ln\left(\frac{\varphi^H_{t-s}(X_t - X_s)}{\varphi^L_{t-s}(X_t - X_s)}\right). \tag{A72}$$

Notice that $Z_t = Z_s + \ln\left(\frac{\varphi^H_{t-s}(X_t - X_s)}{\varphi^L_{t-s}(X_t - X_s)}\right)$. Equation (A72) follows, because the process $Y = (Y_t)_{t\geq 0}$ is non-decreasing.

The condition in (42) holds trivially, because $E_H(z) > E^P_H(z)$ for all $z < \bar{z}^*$, implying that a high type firm would not want to pool at any level of beliefs below $\bar{z}^*$. ■

A.6 Proof of Proposition 5

Proof. Uniqueness of the equilibrium follows from the application of Theorem 5.1 by Daley and Green (2012). In our model, the seller and the buyer in the model by Daley and Green (2012) can be interpreted as the initial shareholders of the firm and the market, respectively. The expected benefits of the seller per unit of time (K_θ) is μ_θ/r in our case, while the buyers value per unit of time (V_θ) is $(\mu_\theta + k)/r - I$. Because the decision to invest is essentially the same as the decision to sell the asset (the existing shareholders are indifferent between receiving a single payment of $E_\theta(q_t, t)$ at the time of investment or receiving the cash flows in perpetuity), we can treat the equity value of the existing shareholders $E_\theta(q_t, t)$ as the sale price W in Daley and Green (2012). Finally, because we assumed that $p^* > 0$, our equation (14) ensures that the Static Lemons Condition of Daley and Green is satisfied. ■

A.7 Proof of Proposition 6

Proof. (a) At p^* the high type firm is indifferent between investing now and never investing. At $\bar{p}^*$ the high type firm is locally indifferent between investing now and waiting for dt. Because the option value of investing later dominates the zero profit of not investing at all, and because the pooling value for the high type firm is increasing in the market belief, it must be the case that $\bar{p}^* > p^*$.

(c) We first consider $\sigma \to \infty$. It is easy to see that both thresholds converge to the same limit. If this were not true, then there exists some $z \in (\underline{z}^*, \bar{z}^*)$. For this z, equation (A2) implies that $E_L(z) \to \frac{\mu_L}{r}$, which can not happen because $E_L(z) \geq \frac{\mu_L + k}{r} - I > \frac{\mu_L}{k}$. Given that $\lim_{\sigma\to\infty} \bar{p}^* = \lim_{\sigma\to\infty} \underline{p}^*$, the dynamic problem for the high type reduces to the static one: either to invest right away, or not to invest at all. Therefore, the static threshold p^* is the limit of $\bar{p}^*$ and $\underline{p}^*$.

(d) We next consider $\sigma \to 0$. Recall that $\lambda(p) = \frac{Ir}{p(\mu_H-\mu_L)+(\mu_L+k)}$. Hence,

$$\frac{d}{dp}\lambda(p) = -\frac{Ir(\mu_H - \mu_L)}{(p(\mu_H - \mu_L) + (\mu_L + k))^2} = -\lambda^2(p)\frac{\mu_H - \mu_L}{Ir}, \tag{A73}$$

and

$$\frac{d}{dz}\lambda(p(z)) = -\lambda^2(p(z))\frac{\mu_H - \mu_L}{Ir}\frac{e^z}{(1+e^z)^2}. \tag{A74}$$

Since $E_H^P(\lambda(z)) = [1 - \lambda(p(z))]\dfrac{\mu_H + k}{r}$,

$$\frac{d}{dz}E_H^P(\lambda(p(z))) = \lambda^2\frac{(\mu_H - \mu_L)(\mu_H + k)}{Ir^2}p(1-p). \tag{A75}$$

Using (A14) we write

$$e^{(u_1-u_2)(\underline{z}-\bar{z})} = -\frac{u_2}{u_1}\frac{u_1(E_H^P(\bar{z}) - \mu_H/r) - E_H^{P\prime}(\bar{z})}{E_H^{P\prime}(\bar{z}) - u_2(E_H^P(\bar{z}) - \mu_H/r)}. \tag{A76}$$

Recall that $\bar{z}^* > \bar{z}_r$, where $\bar{z}_r$ is a unique root of (A17). It is easy to see that $\bar{z}_r \to +\infty$ as $\sigma \to 0$. Thus, $\bar{z}^*$ converges to $+\infty$, i.e. $\bar{p}^*$ converging to 1. Plugging the limit of $\bar{z}^*$ back yields

$$E_H^P(\bar{z}^*) \to \frac{\mu_H + k}{r} - I \quad \text{and} \quad \frac{d}{dz}E_H^P(\bar{z}^*) \to 0+. \tag{A77}$$

Part II of Proposition 2 establishes that $\underline{z}^* = g(\bar{z}^*)$ for some continuous function g. Therefore, the lower threshold $\underline{z}^*$ also converges. This implies that the limit $\lim(\bar{z}^* - \underline{z}^*)$ exists. With this knowledge, we expand (A27):

$$\begin{aligned}
&\frac{1}{v_1 - v_2}\left(\frac{k}{r} - I\right)\left(v_1 e^{(v_1-1)\underline{z}^* + v_2\bar{z}^*} - v_2 e^{(v_2-1)\underline{z}^* + v_1\bar{z}^*}\right) - \frac{k}{r} + \lambda(\bar{z}^*)\frac{\mu_L + k}{r} \\
&\sim \left(\frac{k}{r} - I\right)\left(e^{-v_2(\underline{z}^* - \bar{z}^*)} - v_2 e^{-(\underline{z}^* - \bar{z}^*)}\right) - \frac{k}{r} + I\frac{\mu_L + k}{\mu_H + k} \\
&= \left(\frac{k}{r} - I\right)\left(e^{u_1(\underline{z}^* - \bar{z}^*)} + u_1 e^{-(\underline{z}^* - \bar{z}^*)}\right) - \frac{k}{r} + I\frac{\mu_L + k}{\mu_H + k},
\end{aligned} \tag{A78}$$

where the notation "A $\sim$ B" in the first equality above means that A and B have the same limit. If $\lim(\bar{z}^* - \underline{z}^*)$ were finite, then the last line of equation (A78) would converge to

$$-\frac{k}{r} + I\frac{\mu_L + k}{\mu_H + k} > 0,$$

contradicting the assumption (14). Thus, $\lim(\bar{z}^* - \underline{z}^*) = +\infty$.

Expanding the RHS of (A14), we obtain

$$
\begin{aligned}
e^{(u_1-u_2)(\underline{z}^*-\bar{z}^*)} &\sim \frac{\frac{k}{r}-I-\frac{\frac{d}{dz}E_H^P(\bar{z}^*)}{u_1}}{\frac{k}{r}-I} \\
&= 1-\frac{\frac{d}{dz}E_H^P(\bar{z}^*)}{u_1\left(\frac{k}{r}-I\right)} \\
&= 1-\frac{\lambda^2(\mu_H-\mu_L)(\mu_H+k)}{u_1 I r^2\left(\frac{k}{r}-I\right)}\frac{e^{\bar{z}^*}}{(1+e^{\bar{z}^*})^2} \\
&\sim 1-\frac{I(\mu_H-\mu_L)}{(\mu_H+k)\left(\frac{k}{r}-I\right)}\cdot\frac{1}{u_1 e^{\bar{z}^*}}
\end{aligned}
\tag{A79}
$$

Since $\lim e^{(u_1-u_2)(\underline{z}^*-\bar{z}^*)}=0$ it must be the case that

$$
\lim_{\sigma\downarrow 0} u_1 e^{\bar{z}^*}=\frac{I(\mu_H-\mu_L)}{(\mu_H+k)\left(\frac{k}{r}-I\right)}. \tag{A80}
$$

This fact in turn implies that $\lim u_1\bar{z}^*=\lim\left(u_1 e^{\bar{z}^*}\cdot\frac{\bar{z}^*}{e^{\bar{z}^*}}\right)=0$, hence, $\lim u_1\underline{z}^*=0$ as well. After plugging it back to (A78) we solve for $\lim e^{\underline{z}^*}$:

$$
\begin{aligned}
\lim_{\sigma\downarrow 0}\left[\left(\frac{k}{r}-I\right)\left(e^{u_1(\underline{z}^*-\bar{z}^*)}+u_1 e^{-(\underline{z}^*-\bar{z}^*)}\right)-\frac{k}{r}+I\frac{\mu_L+k}{\mu_H+k}\right]&=0\\
\left(\frac{k}{r}-I\right)\left(1+\frac{\lim u_1 e^{\bar{z}^*}}{\lim e^{\underline{z}^*}}\right)-\frac{k}{r}+I\frac{\mu_L+k}{\mu_H+k}&=0\\
\left(\frac{k}{r}-I\right)\left(1+\frac{I(\mu_H-\mu_L)}{(\mu_H+k)\left(\frac{k}{r}-I\right)}\cdot\frac{1}{\lim e^{\underline{z}^*}}\right)-\frac{k}{r}+I\frac{\mu_L+k}{\mu_H+k}&=0
\end{aligned}
\tag{A81}
$$

and find that $\lim e^{\underline{z}^*}=1$. Thus, $\underline{z}^*\to 0$ and $\underline{p}^*\to\frac{1}{2}$ as $\sigma\downarrow 0$.

We finally consider $\mu_H\to+\infty$. Parameter μ_H affects the equilibrium thresholds through two channels: directly and indirectly (through $h=(\mu_H-\mu_L)/\sigma$). As $\mu_H\to+\infty$ the signal-to-noise ratio $h\to+\infty$. Hence, the effect of $\mu_H\to\infty$ on h is essentially the same the effect of $\sigma\to 0$ on h. Therefore, all the approximations in part (b) of the proof remain valid because they only use $h\to+\infty$.

We need to take into account the additional effect of μ_H that is reflected not through h. As $\mu_H\to\infty$, the last equation in (A81) can be rewritten as:

$$
\left(\frac{k}{r}-I\right)\left(1+\frac{I}{\left(\frac{k}{r}-I\right)}\cdot\frac{1}{\lim\limits_{\mu_H\to+\infty}e^{\underline{z}^*}}\right)-\frac{k}{r}=0. \tag{A82}
$$

Once again we conclude that $\lim_{\mu_H\to+\infty}\underline{z}^*=0$, and $\lim_{\mu_H\to+\infty}\underline{p}^*=1/2$.

■

A.8 Proof of Proposition 7

Proof. Investing at the lower boundary occurs only if the firm is of the low type. Conditional on the firm being of the low type, the log-likelihood of the posterior belief is $z_t = Z_t^L + Y_t$, where $Y_t = \max(\underline{z}^* - \inf_{s \le t} Z_s^L, 0)$, and Z_t^L follows (27). Because the equilibrium is stationary, without loss of generality we let the current "time" be 0 and denote the current belief variable by z. Given the strategy of the low type (30), the probability of investing at the lower threshold is $\mathbf{E}_z(1 - e^{-Y_{\tau_{\bar{z}^*}}})$, where $\tau_{\bar{z}^*} = \inf\{t \ge 0 : z_t \ge \bar{z}^*\}$. We are interested in computing $f(z) \equiv \mathbf{E}_z\, e^{-Y_{\tau_{\bar{z}^*}}}$.

Applying Ito's Lemma, we obtain

$$e^{-Y_t} f(z_t) = f(z_0) + \int_0^t e^{-Y_s} \Gamma f(z_s) \mathrm{d}s + h \int_0^t e^{-Y_s} f'(z_s) \mathrm{dB}_s + \int_0^t e^{-Y_s}(f'(z_s) - f(z_s)) \mathrm{d}Y_s, \quad \text{(A83)}$$

where operator $\Gamma \equiv -\frac{h^2}{2}\frac{d}{dy} + \frac{h^2}{2}\frac{d^2}{dy^2}$.

Now we pick $t = \tau_{\bar{z}^*}$, at which point $z_{\tau_{\bar{z}^*}} = \bar{z}^*$, $Y_{\tau_{\bar{z}^*}} = 0$, and $f(z_{\tau_{\bar{z}^*}}) = 1$. Using the fact that $\mathbf{P}(\tau_{\bar{z}^*} < \infty) = 1$ and the boundedness of the integrand in the dB_s term above, we get

$$\mathbf{E}_z\, e^{-Y_{\tau_{\bar{z}^*}}} f(z_{\tau_{\bar{z}^*}}) = f(z) + \mathbf{E}_z \int_0^{\tau_b} e^{-Y_s} \Gamma f(z_s) \mathrm{d}s + \mathbf{E}_z \int_0^{\tau_{\bar{z}^*}} e^{-L_s}(f'(z_s) - f(z_s)) \mathrm{d}Y_s. \quad \text{(A84)}$$

But the left-hand side of the above equation is $f(z)$ by construction. Thus,

$$\mathbf{E}_z \int_0^{\tau_{\bar{z}^*}} e^{-Y_s} \Gamma f(z_s) \mathrm{d}s + \mathbf{E}_z \int_0^{\tau_{\bar{z}^*}} e^{-Y_s}(f'(z_s) - f(z_s)) \mathrm{d}L_s = 0. \quad \text{(A85)}$$

A sufficient condition for the above equation to hold is that

$$\begin{cases} \Gamma f(z) = 0, \quad \forall\, z \in (\underline{z}^*, \bar{z}^*), \\ f'(z_s) - f(z_s) = 0, \quad \forall\, z_s \in (\underline{z}^*, \bar{z}^*), \\ f(\bar{z}^*) = 1. \end{cases} \quad \text{(A86)}$$

This second-order ODE has a unique solution $f(z) = e^{z - \bar{z}^*}$.

Thus, the probability of investment at the lower boundary, conditional on the firm being of the low type, is $e^{z-\bar{z}^*}$. The firm is of the low type with the ex ante probability $\frac{1}{1+e^z}$, so the unconditional probability of observing an investment at the lower barrier is $\frac{1}{1+e^z}(1 - e^{z-\bar{z}^*})$. The statement of the proposition immediately follows if one changes the variables from the "z-space" to the "p-space" and accounts for the discrete chunk of separative investment when $z < \underline{z}^*$ ($p < \underline{p}^*$).

■

A.9 Proof of Proposition 8

Proof.

1. Recall that $E_H^D(p) = E_H^P(p)$ for $p \geq \bar{p}^*$, where E_H^P is a pooling value of equity for the H type, since in equilibrium the good firm invests immediately with probability 1 above the upper threshold. Since $\bar{p}^* \geq p^*$ the H type firm also invests in the static game, therefore, $E_H^S(p) = E_H^P(p)$ for $p \geq \bar{p}^*$ as well and part (a) of the proposition follows. Below the pooling threshold $\bar{p}^*$ the H type delays the investment, thus, the expected continuation payoff from delay should be greater than the immediate pooling payoff (it can't be less or equal because "invest right away" is always a feasible strategy).

2. (a) When $p > \bar{p}^*$ both static and dynamic environments predict immediate pooling, thus the expected pay-offs are equal.

   The logic is a little less straightforward for $p \leq \min(\underline{p}^*, p^*)$. Low type immediately invests and reveals itself in static game and invests probabilistically in dynamic game, i.e. with some probability investment will happen and the low type will be revealed while with the remaining probability no investment occurs and the game continues. Recall that whenever a player is playing a mixed strategy in the game he has to be indifferent between all the pure strategies in the support of the mixture, that is why the expected payoff from mixing to the low type in the dynamic setting is exactly the same as in static setting where he invests for sure.

   (b) The dynamic value $E_L^D(p)$ satisfies value matching conditions at $\bar{p}^*$ (being equal to the pooling value) and at $\underline{p}^*$ (being equal to the separating value), $E_L^D(p)$ is also strictly increasing in p in the inaction region, thus, $E_L^D(p) < E_L^P(p)$ for $p < \bar{p}^*$. Together with the fact that low type equity static value $E_L^S(p)$ is equal to the pooling value $E_L^P(p)$ for $p \geq p^*$ this implies the statement of the proposition.

   (c) In this part of the inaction region only the L type invests in the static setup. Inaction occurs precisely because the option to wait and be pooled with the H type firm has a positive value in dynamic setting.

■

A.10 Proof of Proposition 9

Proof. Parts 1 and 2 of the statement are an easy corollary of the Proposition 8. Part 3 follows from the observation that the immediate pooling is efficient at any initial level of beliefs, i.e. the social welfare maximizing planner would force both types of firms to invest at time 0. Since the market always breaks even it must be the case that $E^S(p)$ attains the maximal total value of the two firms for $p \geq p^*$. Dynamic equilibrium, however, is not efficient when $p < \bar{p}^*$ since it results in delay. Due to the positive discount rate a part of the social welfare gets wasted, thus, $E^D(p)$ is strictly suboptimal in this region. ■