How to Make an Offer? Bargaining with Value Discovery

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Abstract

Firms in industrial markets use personal selling to reach prospective customers. Such person-to-person interaction between salespersons and customers allows for communication on product attributes and flexibility on price. In this paper, I study a continuous-time game in which a buyer and a seller discover their value from trade while simultaneously bargain for price. The match between the product's attributes and the buyer's preferences is revealed gradually over time. The seller makes repeated price offers bounded by a self-imposed list price, and the buyer decides whether to accept or wait. Players incur flow costs during the game and can leave at any moment. I find that the list price plays an important role in balancing the buyer's incentive to stay and the seller's bargaining power. But the players always trade at a discounted price, due to the seller's incentive to close the sale early. The costly discovery of match provides a rationale for the use of list price and discount, which is absent in the bargaining literature. I also examine whether the seller should commit to a fixed price or allow bargaining. The paper finds that bargaining benefits the seller by shortening the process and increasing ex-ante success rate. When the seller's flow cost is high, both players are willing to participate in the game only if bargaining is allowed. In such cases, bargaining leads to a Pareto improvement over the fixed price, which can explain the prevalent use of bargaining in many industries. If the buyer has private information on his outside option, the model predicts that, counter-intuitively, the buyer with higher value for the product pays lower price. More fundamentally, the paper expands the bargaining literature by adding a discovery process that makes product value stochastic. This leads to departure from standard results.

Keywords: bargaining, sales force, matching, continuous-time game.

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1 Introduction

Sales force is an indispensable marketing instrument for many firms, especially those that serve other businesses. Firms rely on salespersons to provide product information, learn about prospective customers' needs, and persuade them to buy. Typical activities include sales pitch, Q&A meeting, product demonstration, etc. This process of interaction that leads up to transaction is referred to as the sales process (or the selling process). Mantrala et al. (2010) put the sales process at the core of their framework for sales force modelling. They state that "the firm's decisions surrounding the selling process...are critical and impact response functions, operations, and, ultimately, strategies."

This paper focuses on two main functions that this person-to-person interaction allows: (1) to discover the total surplus from trade through discovery of product match; and (2) to determine the split of that surplus through bargaining. In many markets, value from trade can be relationship-specific. Vendors differ in the products or services that they offer, and customers differ in their needs for attributes. Potential trading partners that lack adequate information about each other have to communicate to find out how well the product matches the customer's needs. The buyer acquires product information to guide his purchasing decision, and the seller learns about the buyer's needs to better fine-tune her selling strategy. This view is consistent with earlier works on the role of selling (see, e.g., Wernerfelt 1994a).

The second important aspect of the sales process is bargaining. For example, Mukerjee (2009) explains that in industrial goods markets, "[f]or initial proposals forwarded to the buyer by the seller, the list price is mentioned...negotiations over these prices happen to reach the final figures." Similar practices can be observed for certain consumer goods as well, such as automobiles and housing. A survey of sales forces by Krafft (1999) found 72% of sampled companies allow their salespersons to adjust price offers. The number rose to 88% for industrial-goods companies in the survey. Anecdotal evidence suggests that giving the buyer a discount has become the norm in B2B markets. (Caprio 2015 and Wang 2016)

The matching and bargaining that takes place during the sales process can be viewed as dynamic, interdependent, and simultaneous. Information on product fit arrives gradually and the parties take time to reach agreement on price. Product match affects bargaining strategy and vice versa. For example, intuitively, finding out that the product is a good fit is a good news for the buyer. However, if the seller observes that the buyer really values the product, the seller may charge a higher price. This reduces the buyer's interest to find out about product fit in the first place. Most importantly, there are no natural "stages" that separate matching and bargaining. The buyer and the seller may continue to discuss about the product or reach a deal on price at any moment during the process. Such observation pushes us to view matching and bargaining as happening simultaneously.

In this paper, I study the game where a buyer and a seller discover their product match over time while simultaneously bargain for price. I solve the optimal selling and buying strategies in continuous time. The model provides novel insights on the use of list price and price discount, both of which have been ignored in previous bargaining models. For example, if we allow the seller to set the list price in Fudenberg et. al. (1985), it is optimal as long as the list price is higher than valuation of the highest type. Thus, "list price" and "discount" are meaningless terms in such models. This paper also look at the impact of bargaining versus committing to a fixed price. Answering these questions provide important implications on managerial issues such as optimal pricing and delegation of pricing authority. On the theoretical side, the paper expands the existing bargaining literature by adding a simultaneous matching process. The matching process causes product value to be stochastic, as well as introducing a hold-up problem to the bargaining framework.

More specifically, a buyer and a seller trade over a product with uncertain value to the buyer. The seller's product's attributes and the buyer's preference for these attributes are revealed to each other over time. The seller can publish a list price before the sales process that acts as a price ceiling. At each moment, players discover more about their match, the seller makes a price offer without future commitment, and the buyer decides whether to take that offer. Continuing the process is costly to both parties and they can choose to quit. For example, the seller's cost can come from the salesperson salary and product demonstration cost, etc, while buyer has opportunity cost when dedicating employees to talk to salesperson instead of working.

I show that the list price acts as an important instrument for the seller, yet players always trade at a discounted price. The seller uses the list price to balance the buyer's incentive to engage and the seller's bargaining power. If the seller sets the list price too high, then the buyer may leave immediately due to a hold-up problem. Even if the product match is good, the buyer cannot get any utility since the seller can charge a high price once the good match is revealed. In order to encourage the buyer to continue for a sufficiently long time, the seller needs to set a low enough list price to limit the impact of such hold-up. However, a lower list price also indirectly decreases the seller's bargaining power by increasing the buyer's continuation value. The optimal list price thus has to balance these two effects. Surprisingly though, I find that the parties always trade below the list price, regardless of what the list price is. The intuition is that the seller does not want to wait until the buyer is fully convinced to buy at the original price. Instead, the seller prefers to convince the buyer to buy earlier by offering a price discount. This finding provides a rational explanation for the "list price - discount" pattern that we observe in real world.

Who should leave the process if the product fit is poor? Surprisingly, it's always the buyer who leaves, otherwise the seller should increase the list price. The intuition is that, raising the list price has two effects: it discourages buyer engagement and improves surplus extraction. However, if the seller quits first, discouraging the buyer is costless. This makes raising the list price strictly beneficial to the seller.

Should the seller commits to a fixed price or be open to bargaining? I find that bargaining leads to a lower final price, a shorter sales process, and a higher ex-ante probability of trade than under fixed price. Bargaining always benefits the seller and increases overall welfare, but it's effect on buyer's utility depends on the ratio of the players' costs. However, when the seller's cost is high, bargaining is necessary for both players to participate in the game. If the price is fixed, then one of the player quits immediately, regardless of the level of the price. The flexibility from allowing players to bargain improves overall efficiency by saving time and cost and by increasing ex-ante success rate. In this case, bargaining is welfare-enhancing for both the buyer and the seller.

I also examine the case where the buyer has private information on his outside option. The seller only knows the buyer's valuation for the product up to a distribution. This model is similar to an one-sided bargaining model with incomplete information (such as Fudenberg et al. 1985) but with a stochastic product value resulting from the gradual discovery of product match. In equilibrium, the seller can separate different types of buyers, in contrast to previous models. One counter-intuitive finding is that the buyer with a higher valuation for the product buys at a lower price. This is due to a combination of the seller's incentive to stop early and the buyer's information rent.

1.1 Literature Review

The paper is related to literature on decision making with gradual learning (e.g., Roberts and Weitzman 1981, Moscarini and Smith 2001, Branco et al. 2012, Ke et al. 2016). Similar to these papers, I use a Brownian motion to capture the effect of gradual arrival of product information on product value. The main difference is that this paper introduces bargaining to the process. Whereas previous papers focus on single-agent decision-making, the addition of bargaining in this paper turns the problem into a dynamic game between the buyer and the seller. This greatly increases the complexity of the problem.

Other papers have also looked at bargaining problems with stochastic features. Daley and Green (2016) look at a bargaining game with asymmetric information and gradual signals. The seller knows the true quality, and the buyer receives noisy signals of the quality over time. In contrast, this paper features a symmetric learning that leads to a stochastic product value. Ortner (2017) solves a bargaining model where the seller's marginal cost change over time. Fuchs and Skrzypacz (2010) look at bargaining with a stochastic arrival of events that can end the game.

The use of list price to encourage buyer engagement in this paper is related to the use of published price to reduce hold-up in the search literature. Wernerfelt (1994b) shows that, when product quality is uncertain and requires search/inspection, then the seller wants to inform the buyer about the price before search. Similarly, to encourage visit, multi-product retailers want to advertise the prices of some products to put a bound on the total price of a basket. (Lal and Matutes 1994) This paper introduces a similar hold-up problem into the bargaining game, which helps to rationalize the use of list price in bargaining situations.

This paper is organized as follows. Section 2 presents the model. Section 3 solves the model with a costless seller, and discusses the core findings and their intuitions. Then section 4 solves the equilibrium where selling is costly, and proves the necessity of bargaining when selling cost is high. Section 5 gives the buyer private outside options, and discusses how the model differs from other one-sided bargaining models. Section 7 offers concluding remarks.

2 The Model

I first give an intuitive description of the game, with two players simultaneously discover product match and bargain for price.

2.1 Description

Consider two players, a Buyer (b) and a Seller (s). Throughout the paper, we will refer to Buyer as "he" and refer to Seller as "she". Seller owns a product that can be sold to Buyer. The product has an infinite number of attributes, and each attribute has equal importance to the Buyer. In each period, the players discover a little more on their match and then bargain for price.

The discovery of match goes as follows: At the beginning of each period, Seller reveals one attribute of the product, and observes Buyer's preference for the attribute $z_t = +\sigma\sqrt{dt}$

or $-\sigma\sqrt{dt}$, where dt is the length of the period (alternatively can be viewed as the size of each attribute). The expected value of the product at time t then can be written as $x_t = x_0 + \sum_0^t z_s$, where x_0 is Buyer's expected value of the product prior to the sales process.

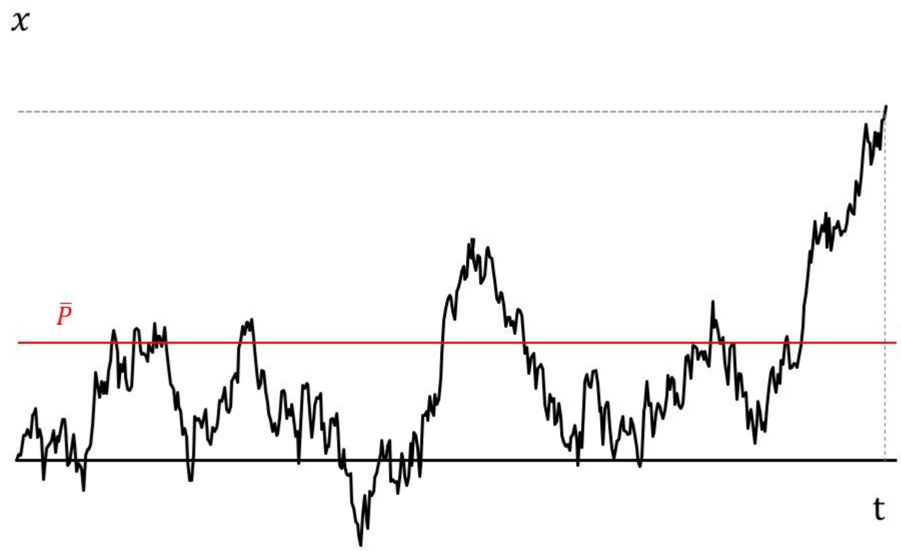
The bargaining goes as follows: Before the game starts, Seller can set a list price $\bar{P}$, which is a commitment that Buyer can always buy the product at this price (setting $\bar{P} = \infty$ is equivalent to not setting a list price). Then in each period during the sales process, after observing z_t, Seller can make a price offer P_t with no guarantee for future prices. Buyer decides whether to accept the offer and receive utility $x_t - P_t$. If Buyer rejects the offer, the game continues. Notice that Seller cannot effectively offer any price above $\bar{P}$. Also, if Seller does not make an offer, it is equivalent to making an offer at $\bar{P}$, since $\bar{P}$ is always available. So there's a standing offer at each period, bounded above by $\bar{P}$.

Both players incur flow costs during the game. Players can choose to quit the sales process at the end of each period. If either party quits, the game ends and the players receives their outside options, π_b and π_s, respectively.

As we shrink the length of each period (or size of each attribute), the game approaches to continuous time. Buyer's expected value for the product, x_t, approaches to a Brownian motion with $dx = \sigma dW$, where W is the Wiener process.

Figure 1 is an example of how the value of the product evolves. Horizontal axis is time t, and vertical axis is product value x. The solid line in the middle represents the list price $\bar{P}$. For now assume that Seller never makes a price offer, so the price is always at $\bar{P}$. If Seller does not take a action, the game becomes Buyer's single agent optimization problem. In the beginning, Buyer optimally chooses to wait and find out more about product match. Waiting is optimal even beyond the list price because of the optional value of buying a well-matched product is greater than the utility he gets if he buys when x_t reaches $\bar{P}$ (which is 0). Buyer chooses to buy when product value reaches a threshold. In retrospect, however, Seller might prefer to close the sale earlier at a discounted price. Closing the sale earlier saves time and cost, while also eliminating the possibility of subsequently discovering a bad

Figure 1:



match and never making the sale. This point is shown in Figure 2.

However, solving the equilibrium in this game is difficult for two reasons. The first reason is that Seller cannot commit to future prices. Buyer's decision has to depend on Seller's entire pricing strategy P_t in all future states of the world. Conversely, Seller's optimal price offer depend on Buyer's buying strategy in all future states of the world. The game is infinite horizon so we cannot use backward induction. The second difficulty is that when product match is bad, the game ends with one party quitting, but we do not know which player quits. This effectively takes away one optimality condition, since we don't know whether quitting at a given threshold $\underline{x}$ is Buyer's optimal choice or Seller's optimal choice.

2.2 Formal Model

The game is continuous-time with infinite horizon. There are two players: a Buyer (b) and a Seller (s). Product value x_t is observable to both players and follows Brownian motion $dx = \sigma dW$, with initial position x_0. Let $(\Sigma, \mathcal{F}, \mathrm{P})$ be the probability space that supports the

Figure 2:

Wiener process W, and $\mathrm{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration process generated by W satisfying usual assumptions. As explained earlier, this is the continuous-time limit of a product with infinite number of small, independent attributes, which players learn over time.

Before the game starts, Seller chooses a list price $\bar{P}$. This is a commitment that Buyer can always buy the product at $\bar{P}$. Effectively, this puts a upper bound on the price that Seller can offer to the Buyer. At all time $t \geq 0$, players take actions in the following order:

1. Seller makes price offer P_t subject to $P_t \leq \bar{P}$.

2. Buyer chooses buying decision $a_t \in \{0, 1\}$, where $a_t = 1$ indicates buying product with value x_t at price P_t.

3. (If Buyer does not buy) Buyer and Seller choose whether to quit with $q_{b,t} \in \{0, 1\}$ and $q_{s,t} \in \{0, 1\}$, respectively.

Because there is an infinite number of attributes with equal importance, the number of attributes already revealed should be irrelevant conditional on product value x_t. Thus I restrict attention to stationary Markov equilibrium with state variable $x \in \mathbb{R}$.

I consider pure strategies only. Seller's action is characterized by $(\bar{P}, P(x, \bar{P}), q_s(x, \bar{P}))$, with list price $\bar{P} \in \mathbb{R}^+$, price offer $P(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$, and quitting decision $q_s(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$. Buyer's action is characterized by $(a(x, P, \bar{P}), q_b(x, \bar{P}))$, with buying decision $a(x, P, \bar{P}) : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \{0, 1\}$, and quitting decision $q_b(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$. Since everything depends on list price $\bar{P}$, which is fixed throughout the game, we'll drop $\bar{P}$ from notation going forward.

I make a simplifying assumption on players' quitting strategies. In Appendix I show that this assumption is innocuous. I assume that $q_s(x, \bar{P}) \geq q_s(x', \bar{P})$ and $q_b(x, \bar{P}) \geq q_b(x', \bar{P})$ for $x \geq x'$. This means that players' quitting function can only have a single jump. For each player, there exits a threshold such that the player quits whenever x reaches that threshold or below, and remains in the game when x is above such a threshold. This type of stopping rule is common in optimal stopping problems. Intuitively, the parties should continue the sales process when product fit is good, and leave only when the product fit is sufficiently bad. If Seller or Buyer is willing to continue the process for a certain product value x, he or she should be willing to continue if x is even higher. Conversely, if Seller or Buyer wants to leave when at a low x, he or she should leave when x is even lower. Let $\underline{x_s}$ and $\underline{x_b}$ denotes the quitting thresholds for seller and buyer respectively. With this assumption, seller's action can be characterized by $(\bar{P}, P(x), \underline{x_s})$, and buyer's action can be characterized by $(a(x, P), \underline{x_b})$.

Buyer (Seller) incurs flow cost c_b (c_s) during the game. Game ends if players trade or if either player quits. If buyer and seller agree to trade at time τ, Buyer receives utility $x_\tau - P_\tau$ and Seller receives P_τ. If either player quits, game ends and players get outside option of π_b and π_s, respectively. Players are risk neutral and perfectly patient with discount factor $r_b = r_s = 0$. As an extension, I show that having time discounting instead of flow costs does not affect core results.

Let $u(x, \bar{P}, P(\cdot), \underline{x_s} \mid a(\cdot), \underline{x_b})$ denote seller's expected utility at state x, and

$v(x, a(\cdot), \underline{x_b} \mid \bar{P}, P(\cdot), \underline{x_s})$ denote buyer's expected utility at state x. Let $U(x,t)$ represent the equilibrium value function for Seller at state x and time t, and $V(x,t)$ represent the equilibrium value function for Buyer. For state x such that players choose to continue, we can write recursively:

$$U(x,t) = -c_s dt + e^{-r_s dt}\mathbb{E}U(x+dx, t+dt)$$
$$V(x,t) = -c_b dt + e^{-r_b dt}\mathbb{E}V(x+dx, t+dt)$$

Under stationarity and $r_s = r_b = 0$, they become:

$$U(x) = -c_s dt + \mathbb{E}U(x+dx)$$
$$V(x) = -c_b dt + \mathbb{E}V(x+dx)$$

Furthermore, by taking Taylor expansion and applying Ito's Lemma on $\mathbb{E}U$ and $\mathbb{E}V$ terms, we can reduce the the expressions to the following second order conditions:

$$U''(x) = \frac{2c_s}{\sigma^2}$$
$$V''(x) = \frac{2c_b}{\sigma^2}$$

which implies that, for some coefficients A_s, B_s, A_b, B_b:

$$\begin{aligned} U(x) &= \frac{c_s}{\sigma^2}(x-\bar{P})^2 + A_s(x-\bar{P}) + B_s \\ V(x) &= \frac{c_b}{\sigma^2}(x-\bar{P})^2 + A_b(x-\bar{P}) + B_b \end{aligned} \tag{1}$$

We can solve the coefficients later by applying appropriate boundary conditions.

In this model, x_t is observable to both players. The idea is that Seller truthfully reveals attributes and Buyer truthfully reveals his preference for each attribute. In the real world, potential buyers may hide their preferences in order to barter a lower price.[1] In section 5, I extend the model by giving Buyer private information on his outside option, and discuss what happens in such environment.

There can be many equilibrium strategies that result in the same equilibrium outcome. What determines the outcome is the earliest points that players trade or quit; what happen beyond those states do not show up on equilibrium path. Define $\bar{x} = inf\{\ x \geq x_0 \mid a(x, \bar{P}, P(x)) = 1\}$ as the lowest x starting from initial position that Buyer and Seller trade. Define $\underline{x} = sup\{\ x \leq x_0 \mid q_b(x, \bar{P}) = 1\ or\ q_s(x, \bar{P}) = 1\}$ as the highest x to initial position that either Seller or Buyer quits. The process continues as long as $\underline{x} < x < \bar{x}$. On the equilibrium path, the game never proceed above $\bar{x}$ or below $\underline{x}$. An equilibrium outcome is defined as the triplet $(\bar{x}, P(\bar{x}), \underline{x})$. We will focus on the solution of this triplet. Note that we can easily come up with different equilibrium strategies for the same equilibrium outcome. For example, for values of x such that Buyer does not buy, Seller can charge any price above Buyer's willingness to pay, and the resulting strategies still constitute as an equilibrium.

3 Costless Selling

In this section, let the selling activity be costless ($c_s = 0$). We can assume that Seller never quits on equilibrium path. This simplifies the problem and help us to get core intuitions. We want to solve the trading threshold $\bar{x}$, and the price that they trade at, $P(\bar{x})$. But in order to do so, we have to also specify pricing and buying strategies for off equilibrium path points $x > \bar{x}$ in order to establish SPE.

When Buyer and Seller trade at $\bar{x}$, they split a total surplus of size $\bar{x}$. Seller receives the price $P(\bar{x})$, and Buyer receives the rest of the surplus, $\bar{x} - P(\bar{x})$. Alternatively, we can think of Buyer as receiving his equilibrium utility $V(\bar{x})$, and Seller receives the rest of the

[1] We can motivate the truthful revelation in a simple model. Suppose that Buyer can choose whether to reveal his preference for each attribute. Buyer always chooses to reveal if he does not like the attribute, then even if Buyer does not reveal when he likes the attribute, Seller can infer that Buyer indeed likes it from Buyer's silence.

surplus with $\bar{x} - V(\bar{x})$. Thus the highest price that Seller can charge at $\bar{x}$ in equilibrium corresponds to the lowest equilibrium utility that Buyer can get at $\bar{x}$. In other words, the lower the Buyer's continuation value is, the more he is willing to pay now. If Seller wants to trade now, she prefers a strategy that makes Buyer as miserable as possible from continuing the game.

Since the product is always available at list price $\bar{P}$, Buyer cannot be worse off than if Seller never makes an offer below $\bar{P}$. Thus we can get a lower bound on Buyer's equilibrium value by solving for Buyer's value function facing a fixed price of $\bar{P}$. Since Seller never quits, this is Buyer's single-agent optimal stopping problem with stopping utility $\max\{0,\ x - \bar{P}\}$. Each moment, Buyer decides between buying at $\bar{P}$, continuing, or quitting. Such problem has been solved for investment under uncertainty (eg., Dixit 1993) as well as consumer search (e.g., Roberts and Weitzman 1981, Moscarini and Smith 2001, Branco et al. 2012).

Let $\widetilde{V}(x)$ denote the Buyer's value function for a fixed price of $\bar{P}$. Let $\bar{\bar{x}}$ denote the earliest state in which Buyer decides to purchase, and let $\underline{\underline{x}}$ denote the earliest state in which Buyer decides to quit. By equation 1, Buyer's value function must be of the form $\widetilde{V}(x) = \frac{c_b}{\sigma^2}(x - \bar{P})^2 + \alpha(x - \bar{P}) + \beta$ for some coefficients α and β. The value function must satisfy the following boundary conditions:

$$
\begin{cases}
\widetilde{V}(\bar{\bar{x}}) = \bar{\bar{x}} - \bar{P} \\
\widetilde{V}(\underline{\underline{x}}) = \pi_b \\
V'(\bar{\bar{x}}) = 1 \\
V'(\underline{\underline{x}}) = 0
\end{cases}
$$

The first two conditions ensure that the value function must match the stopping value, when Buyer buys or quits. The last two conditions, often referred to as "smooth-pasting" conditions, ensure that the stopping time is optimal. See Dixit (1993) for more details.

Solving to the system of equation shows that Buyer buys at

$$\bar{\bar{x}} = \bar{P} + \frac{\sigma^2}{4c_b} + \pi_b \tag{2}$$

and quits at

$$\underline{\underline{x}} = \bar{P} - \frac{\sigma^2}{4c_b} - \pi_b \tag{3}$$

Buyer's value function is

$$\widetilde{V}(x|\bar{P}, \pi_b) = \begin{cases} \pi_b, & x - \pi_b < \bar{P} - \frac{\sigma^2}{4c_b} \\ \frac{c_b}{\sigma^2}(x - \pi_b - \bar{P})^2 + \frac{1}{2}(x - \pi_b - \bar{P}) + \frac{\sigma^2}{16c_b} + \pi_b, & x - \pi_b \in \left[\bar{P} - \frac{\sigma^2}{4c_b}, \bar{P} + \frac{\sigma^2}{4c_b}\right] \\ x - \bar{P}, & x - \pi_b > \bar{P} + \frac{\sigma^2}{4c_b} \end{cases} \tag{4}$$

Although $\widetilde{V}(x)$ is a lower bound on Buyer's equilibrium value function, it' not clear whether it is binding. We do not know if there exists an equilibrium with $V(x) = \widetilde{V}(x)$, because Seller cannot credibly commit to charge $\bar{P}$ in the future. Lemma 1 states that there indeed exists an equilibrium where Buyer's continuation value is $\widetilde{V}(x)$. Buyer players the game as if price is fixed at $\bar{P}$.

Lemma 1. *There exists an equilibrium with $V(x) = \widetilde{V}(x, \bar{P})$. In this equilibrium, $\bar{x} \leq \bar{\bar{x}}$. Strategies $P(x)$ and $a(x, P)$ have the following properties:*

- $a(x, P(x)) = 1$ *if* $P(x) = x - \widetilde{V}(x, \bar{P})$.
- $a(x, P(x)) = 0$ *if* $P(x) > x - \widetilde{V}(x, \bar{P})$.

Proof. See appendix. □

Furthermore, $\widetilde{V}(x)$ is the unique limit of discrete-time equilibrium value functions. Specifically, if we solve the discrete-time analog of this game as described in Section 2.1, with the additional refinement that players do not quit unless quitting is strictly preferred, then

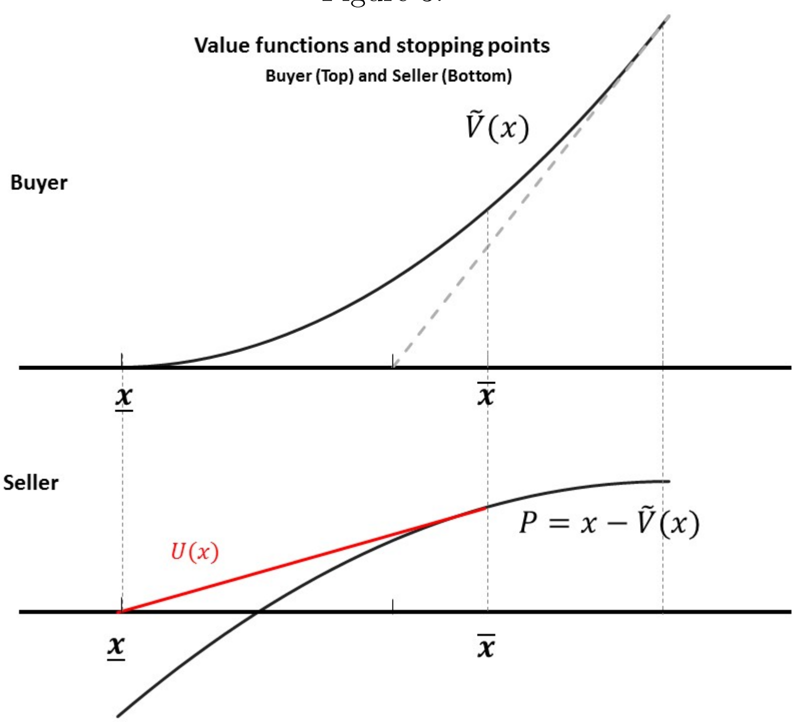
Buyer's equilibrium value functions must converge to $\widetilde{V}(x)$ as the game approaches continuous time. Full specification of the discrete-time game and the proof are in the Appendix. The intuition of the proof is similar to backward-induction and trembling-hand.

Lemma 2. *$\widetilde{V}(x)$ is the unique limit of Buyer's value functions in discrete-time equilibria where players do not quit unless quitting is strictly preferred,*

Lemma 1 says that even though Seller cannot commit to future price, Seller can still enforce Buyer's continuation value to be $\widetilde{V}(x)$. Consider the following strategy: Seller offers price $P(x) = x - \widetilde{V}(x)$ if she wants to trade, and offers $P(x) > x - \widetilde{V}(x)$ if she does not want to trade (subject to $P(x) \leq \bar{P}$). If Seller follows this strategy, Buyer never receives more than $\widetilde{V}(x)$ utility when he buys. Thus Buyer's optimal response is to comply to Seller. Buyer buys if $P(x) = x - \widetilde{V}(x)$, which is the price that makes Buyer indifferent between buying now, or continuing with a value of $\widetilde{V}(x)$. If $P(x) > x - \widetilde{V}(x)$, Buyer rejects because he gets less utility than continuing, since continuation value must be at least $\widetilde{V}(x)$. Thus by following this pricing strategy, Seller can fully control when they trade (subject to $\bar{x} \leq \bar{\bar{x}}$). Intuitively, at every moment in time, Seller is choosing between two options. She can continue the sales process, or she can close the sale right now by offering a discount. If Seller choose to close the sale, she offers a discounted price of $P(x) = x - \widetilde{V}(x)$, and receives that as her utility. This transforms the game into Seller's optimal stopping problem. By solving Seller's optimal stopping problem, we also solve the equilibrium, since Buyer's optimal actions is always to comply with Seller's stopping decision. Thus the new questions are when Seller should make the final offer, and what offer Seller should make. Note that Seller can only control the stopping point between $\bar{\bar{x}}$ and $\underline{\underline{x}}$, which are Buyer's stopping points facing a fixed price of $\bar{P}$. At $\underline{\underline{x}}$, Buyer quits and ends the game. At $\bar{\bar{x}}$, Buyer buys the product even if it will be forever priced at the highest price $\bar{P}$, so Seller cannot delay trade beyond this point in equilibrium.

Figure 3 illustrates Seller's problem graphically. The vertical axis is utility, and horizontal axis is state x. Buyer's value function $V(x) = \widetilde{V}(x)$ is shown on top, and Seller's value function $U(x)$ is the straight line on the bottom chart ($U(x)$ is straight line due to zero flow cost). Seller's value hits zero at $\underline{x}$ when Buyer quits, and Seller gets $P(x) = x - \widetilde{V}(x)$ when she sells. To maximize $U(x)$, the trading point $\bar{x}$ must make $U(x)$ and stopping value $P(x)$ tangent.

Figure 3:



Proposition 1 provides the closed-form solution of the equilibrium outcome $(\bar{x}, \underline{x}, P(\bar{x}))$. Definition 1 simplifies the notation by normalizing outside options into x and P.

Definition 1. *Define* $x_n = x - \pi_b - \pi_s$, $x_{n,0} = x_0 - \pi_b - \pi_s$, $P_n = P - \pi_s$, *and* $\bar{P}_n = \bar{P} - \pi_s$.

The notation x_n represents the total increase in surplus from trade, and the notation P_n captures Seller's gain from trade. This normalization gets rid of outside options but does not affect value functions. It's easy to check that $\widetilde{V}(x|\bar{P}, \pi_b) = \widetilde{V}(x_n|\bar{P}_n, \pi_b)$. We can solve the equilibrium outcomes using x_n and P_n, then back out the original solution. For the rest of the section, we simply work with x_n and P_n.

Proposition 1 (Costless Selling)**.** *If* $\bar{P}_n > \frac{1}{4}\frac{\sigma^2}{c_b}$, *the games ends immediately. Otherwise, players continue for* $\bar{P}_n - \frac{1}{4}\frac{\sigma^2}{c_b} < x_n < \bar{P}_n + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}_n\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$*; Buyer and Seller trade at* $\bar{x}_n =$

$\bar{P}_n+\frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4}-\bar{P}_n\frac{c_b}{\sigma^2}}-\frac{1}{4}\right]$ *at price* $P_n(\bar{x}_n)=\frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4}-\bar{P}_n\frac{c_b}{\sigma^2}}-2\left(\frac{1}{4}-\bar{P}_n\frac{c_b}{\sigma^2}\right)\right]$*; and Buyer quits at* $\underline{x}_n=\bar{P}_n-\frac{1}{4}\frac{\sigma^2}{c_b}$*. The size of the price discount is* $\bar{P}_n-P_n(\bar{x}_n)=\frac{\sigma^2}{c_b}\left[-\sqrt{\frac{1}{4}-\bar{P}_n\frac{c_b}{\sigma^2}}+\frac{1}{2}-\bar{P}_n\frac{c_b}{\sigma^2}\right]$

Proof. Since $V(x_n)=\widetilde{V}(x_n)$, Buyer must quit at $\underline{x}_n=\underline{\underline{x}}_n$ from equation 3. For $\bar{x}_n$ and $P(\bar{x}_n)$, solve Seller's optimal stopping problem, with stopping value $P_n(x_n)=x_n-\widetilde{V}(x_n)$ and an exogenous exit at $\underline{x}_n$.

See Appendix for details. □

Proposition 1 gives two interesting findings. The first finding is that the list price plays an crucial role in the game. There exits an optimal list price for Seller. The second finding is that the final trading price is always lower than the list price, regardless of what the list price is. Thus Proposition 1 predicts that every trade comes at a discounted price.[2]

Figure 4:

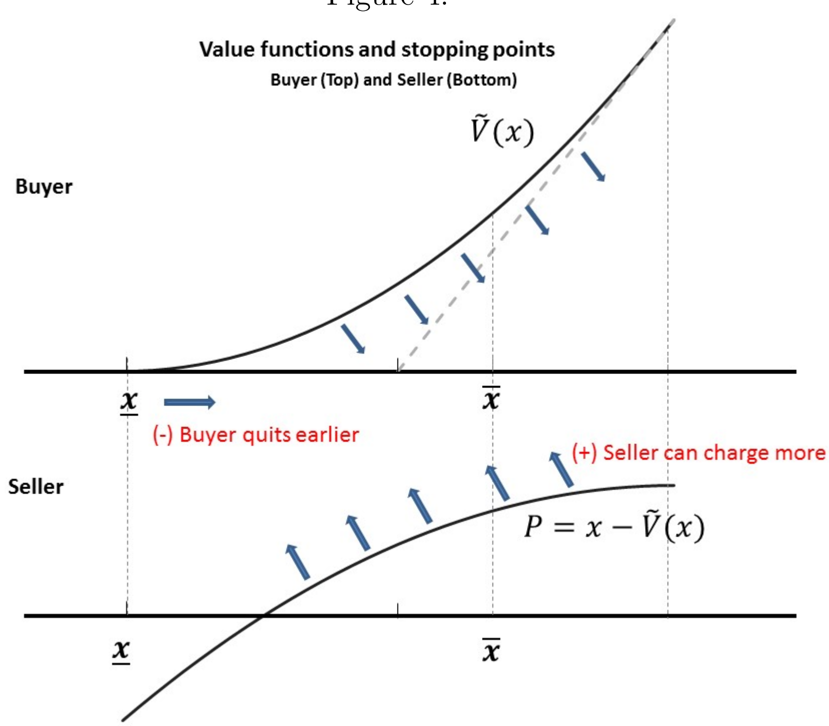


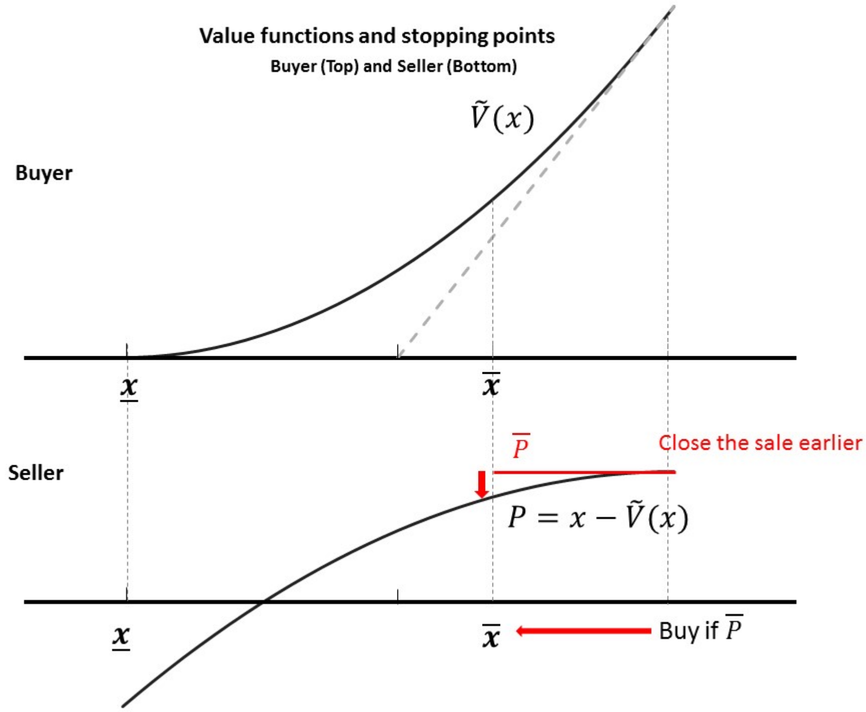
Figure 4 illustrates that Seller's choice of list price need to balance two effects. A higher

[2]In reality, trade could happen at the list price for in some customers. Note that this strong prediction that every trade happens at a discounted price is due to having only a single type of Buyer in this model. If there are different types of Buyers with different costs or starting positions, then some Buyers may buys at the list price in equilibrium.

list price discourages Buyer from engaging. As $\bar{P}$ increases, $\underline{x} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$ increases, which means that Buyer is quicker to leave the sales process when he receives unfavorable information about product match. In the extreme case where list price is too high, Buyer leaves immediately, because the expected gain from trade is not enough to justify the cost of staying in the sales process. This is conceptually similar to the hold-up problem in Wernerfelt (1994b). It is costly for Buyer to find out about product match (or quality), and Seller can hold Buyer up by charging a price equal to the product value after Buyer spent the cost. This gives Buyer a negative utility ex-ante, and as a result, Buyer chooses to not spend any effort in the first place. Thus in order to encourage Buyer to stay in the sales process, Seller has to impose a low enough list price, which increases the optional value of discovery for Buyer. On the downside, a lower list price restricts Seller's ability to bargain. Buyer's higher continuation value means that Seller has to offer more discount in order to close the sale, since $P(x) = \bar{x} - \widetilde{V}(x, \bar{P})$ decreases as $\bar{P}$ increases. Seller's choice of list price must balance these two effects. List price can be viewed as an instrument that balances Buyer's incentives to engage and Seller's bargaining power.

Figure 5 illustrates why it's never optimal for Seller sell at list price. If Seller never offers a price discount, Buyer will continue to learn about the product until product value reaches $\bar{P} + \frac{\sigma^2}{4c_b}$ (the vertical dotted line on the right). As Buyer gets close to his buying threshold, Buyer's value function $\widetilde{V}(x)$ becomes increasingly tangent to the function $y = x$. As a result, $P(x)$ becomes increasingly tangent to $\bar{P}$. This means that Buyer is willing to accept the trade with discount that approaches 0. Thus Seller can close the deal earlier by sacrificing very little on price. Closing the sale earlier is beneficial because it increases the ex-ante success rate and saves cost (in this case, only Buyer's cost is saved, but Seller can extract that saving through price). The required amount of price discount to close the sale increases at a faster rate as we move the buying threshold to the left, so it's not optimal to close the deal too early. In another word, some discovery of product match is valuable to Seller, due to the possibility of selling at a higher price in case the product match is good.

Figure 5:



Proposition 1 gives the optimal timing and size of the price offer, which balances the risk of losing the sale with the potential of selling at a higher price.

Proposition 2 solves the optimal list price by maximizing Seller's ex-ante utility.

Proposition 2. *(Optimal List Price and Outcome at $t = 0$)*

- *For intermediate values of $x_{n,0}$ ($-\frac{1}{4}\frac{\sigma^2}{c_b} < x_{n,0} < \frac{1}{16}\frac{\sigma^2}{c_b}$), the optimal list price is $\frac{1}{3}x_{n,0} - \frac{1}{18}\frac{\sigma^2}{c_b}\sqrt{1 - 12x_{n,0}\frac{c_b}{\sigma^2}} + \frac{7}{36}\frac{\sigma^2}{c_b}$. Game continues beyond $t = 0$.*

- *For higher $x_{n,0}$, any list price above $x_{n,0} + \frac{\sigma^2}{4c_b}$ is optimal. Parties trade at $P_n = x_{n,0}$ at $t = 0$.*

- *For lower $x_{n,0}$, any non-negative list price is optimal. Buyer quits at $t = 0$.*

As Proposition 2 shows, the sales process only takes place if the initial surplus from trade, $x_{n,0} = x_0 - \pi_b - \pi_s$ is not too big or too small. If the initial surplus is big enough ($x_{n,0} \geq \frac{1}{16}\frac{\sigma^2}{c_b}$), then Seller does not want Buyer to learn more about product match. Seller publishes a list price high enough to deter Buyer from continuing, and offers a monopoly spot price that takes all surplus. On the other hand, if the initial trade surplus is too low ($x_{n,0} \leq -\frac{1}{4}\frac{\sigma^2}{c_b}$), matching is no longer socially efficient. Buyer will not engage in the game even if list price is 0. Note that since only Buyer has cost, Buyer's action is socially optimal when list price is 0.

Given that the majority of firms in the real world allows sales persons to bargain with customers (Kraft 1994), it is important to understand the effect of bargaining. Should Seller come to the market committing to a fixed price or allowing price to be negotiated? Here we can examine the effect of bargaining by comparing results from Proposition 1 to equilibrium where Seller commits to a fixed price $\bar{P}$.

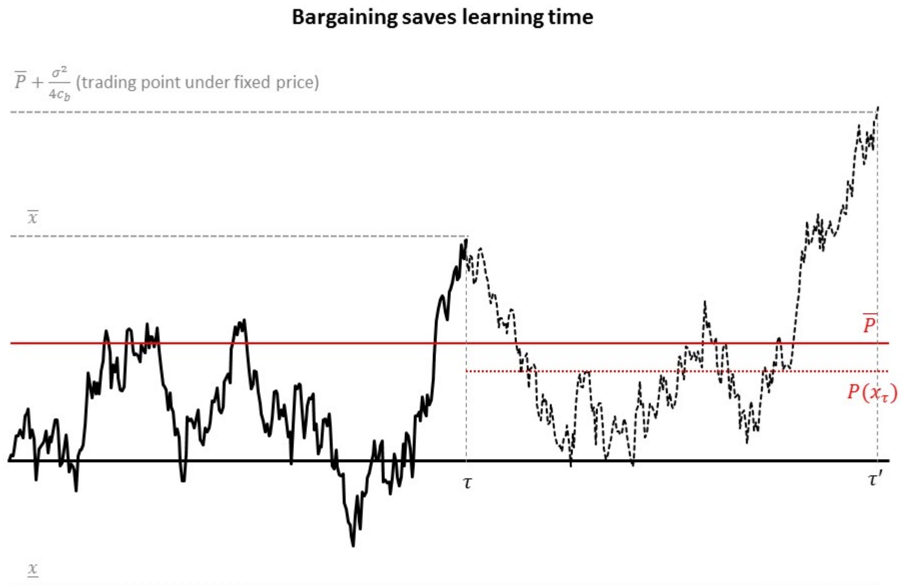
Corollary 3 (Comparison to Fixed Price). *Optimal list price under bargaining is higher than optimal fixed price. Final price under bargaining is lower than optimal fixed price. Expected length of the game is shorter under bargaining. Ex-ante social welfare is higher but Buyer's utility is lower.*

Proof. For prices and ex-ante utilities, simply compare solutions from Proposition 1 and Proposition 2 to the fixed price case, which is solved in equations 2 through 4.

Let T denote the total length of the game. Buyer's ex-ante utility must satisfy $V(x_{n,0}) = \frac{x_{n,0}-\underline{x}_n}{\bar{x}_n-\underline{x}_n}(\bar{x}_n - P(x_n)) - c_b\mathbb{E}[T]$, where $\frac{x_{n,0}-\underline{x}_n}{\bar{x}_n-\underline{x}_n}$ is the probability that players trade. Expected length of the sales process is thus calculated as $\mathbb{E}[T] = \frac{1}{c_b}\left[\frac{x_{n,0}-\underline{x}_n}{\bar{x}_n-\underline{x}_n}(\bar{x}_n - P(x_n)) - V(x_{n,0})\right]$ □

Figure 6 and 7 show two sample paths of product value, and illustrate the value of bargaining to the Seller. In both graphs, trade happens at time τ, with price $P(x_\tau)$ slightly below sticker price. However, the small discount significantly decreases Buyer's buying threshold from $\bar{P} + \frac{\sigma^2}{4c_b}$ to $\bar{x}$. The dotted path in Figure 6 simulates what happens if Seller commits to the list price. Buyer continues to discover the match for a period of time, before deciding to purchase when product value reaches the higher threshold. In this example, Seller's ability to bargain significantly decreases length of the process. The cost saving is captured by Seller

Figure 6:



through price and results in a higher ex-ante utility.

Figure 7 simulates a different continuation trajectory. Players subsequently discover that the product match is bad, so that game ends without a trade. In this example, the lower trading threshold from bargaining increases the ex-ante success rate. Note that neither graphs shows the full price path. We only know that price is $P(x_\tau)$ at point of trade. There are infinite price paths leading up to time τ that produces this equilibrium outcome. Price strategies before time τ only need to be high enough so that Buyer does not want to buy. Keeping the price at the list price before time τ, for example, would work.

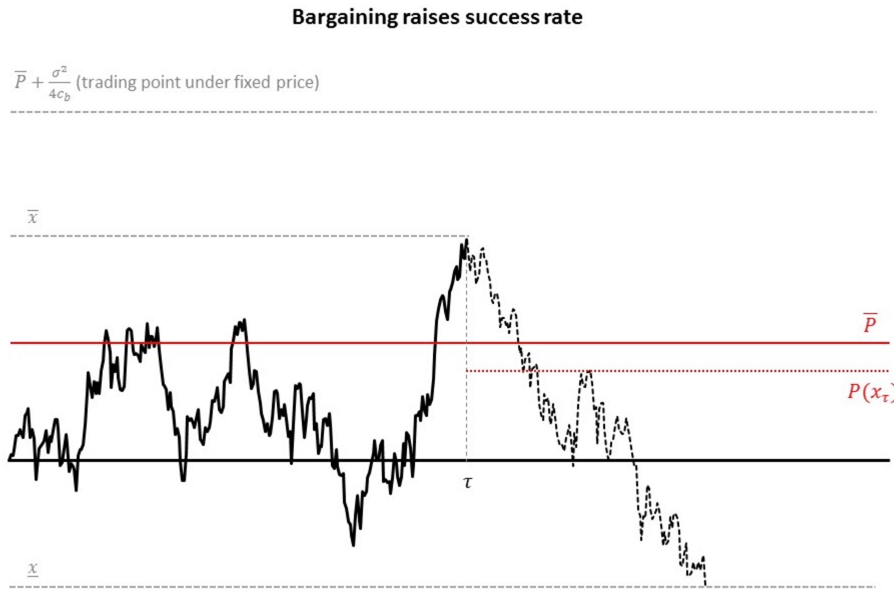
Next we look at comparative statics.

Corollary 4. *(Comparative Statics w.r.t c_b, π_s, π_b)*

For $-\frac{1}{4}\frac{\sigma^2}{c_b} < x_{n,0} < \frac{1}{16}\frac{\sigma^2}{c_b}$*:*

- *List price decreases in Buyer's cost, c_b; increases in Seller's outside option, π_s; and decreases in Buyer's outside option, π_b.*

Figure 7:



- *Final price decreases in Buyer's cost, c_b; increases in Seller's outside option, π_s; and decreases in Buyer's outside option, π_b.*
- *Size of price discount decreases in Buyer's cost, c_b, and outside options, π_s and π_b.*
- *Ex-ante probability of trade is unaffected by Buyer's cost, c_b, and decreases in outside options π_s and π_b.*
- *Expected length of the game decreases in Buyer's cost, c_b, and has inverse U-shape in outside options, π_s and π_b.*

The most interesting comparative statics is that expected length of the game is not monotonic in outside options. Intuitively, one would expect that better outside options make players less interested to engage with each other. However, as outside options become increasingly poor, surplus from trade gets higher. As a result, Seller is more inclined to close the deal early. Seller imposes a high list price to discourage Buyer from continuing, and offer a discount to close the deal. Thus the length of the sales process is lower for both good and bad outside options.

Another finding is that the probability of trade is unaffected by Buyer's cost. This is somewhat counter-intuitive. For a given list price, probability of trade decreases in c_b. However, higher cost for Buyer pushes Seller to charge a lower list price, which increases the probability of trade. In equilibrium, these two effects negate each other. Note that this is not true if price is fixed. With a fixed price, we can show that the ex-ante probability of trade is monotonically decreasing in cost. As discussed earlier, bargaining improves ex-ante probability of trade.

4 Costly Selling

Now we look at the case where selling is costly and Seller may quit before Buyer does. Let $\underline{x} = \max\{\underline{x}_b, \underline{x}_s\}$ denotes the threshold where the first player quits. When product value hits this point, Buyer and Seller get their respective outside options, π_b and π_s. We can assume WLOG that $\pi_b = \pi_s = 0$, as Definition 1 shows that they can be normalized into x and P. The difficulty in solving the equilibrium is that we don't know which player quits at $\underline{x}$. Intuitively, the player with higher flow cost is more likely to quit earlier. However, this is not true. If Seller optimally chooses the list price, then Buyer always quit before Seller does. This result is proved in Proposition 5 below.

As in Section 3, I first derive the Buyer's equilibrium value function. The following Lemma is a generalization of Lemma 1 to the case of $c_s \geq 0$. Given that one of the players quit at $\underline{x}$, we can solve for the Buyer's value function if price is fixed at $\bar{P}$. This gives a lower bound on the Buyer's equilibrium value function. We can then show that there exists an equilibrium where Buyer receives exactly this lower bound. Furthermore, this is the limit of discrete-time equilibria where players do not quit unless quitting is strictly preferred. If Buyer quits first in equilibrium, then his value function is same as when Seller is costless. If

Seller quits first, then Buyer's value function is shifted down and left.

Lemma 3. *In equilibrium, Buyer's value function* $V(x) \geq \widetilde{V}(x+\eta, \bar{P}) - \eta$*, where* $\eta = \frac{1}{4}\frac{\sigma^2}{c_b} - (\underline{x} - \bar{P}) - \sqrt{-\frac{\sigma^2}{c_b}(\underline{x} - \bar{P})}$ *. There exists an equilibrium with* $V(x) = \widetilde{V}(x+\eta, \bar{P}) - \eta$*. Strategies* $P(x)$ *and* $a(x, P)$ *have the following properties:*

- $a(x, P(x)) = 1$ *if* $P(x) = x - \widetilde{V}(x+\eta, \bar{P}) + \eta$
- $a(x, P(x)) = 0$ *if* $P(x) > x - \widetilde{V}(x+\eta, \bar{P}) + \eta$*.*

Proof. See Appendix. □

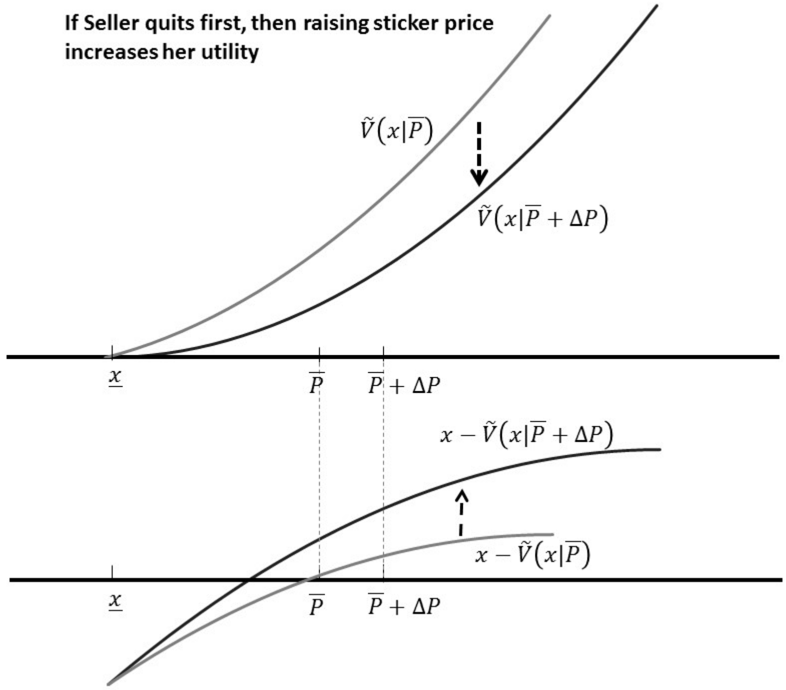
Using Lemma 3 we can prove that Buyer always quits before Seller in equilibrium. The intuition can be traced back to Figure 5. Suppose Seller sets a list price, and under such list price Seller quits first. Consider the effect of raising the list price. As Figure 5 shows, increasing the list price has both positive and negative effects on Seller's utility. The positive effect is that higher list price lowers Buyer's continuation value, which increases the surplus that Seller can extract in every state. The negative effect is that Buyer's lower continuation value pushes him to quit earlier. However, if Seller is quitting earlier than Buyer does in equilibrium, then pushing Buyer to quit earlier has no impact on the outcome and both players' utilities. Thus raising the list price is strictly beneficial to the Seller. Thus the original list price must be sub-optimal.

Proposition 5. *If Seller quits earlier than Buyer, then sticker price* $\bar{P}$ *is sub-optimal.*

Proof. See Appendix. □

Figure 8 outlines the proof of Proposition 5. If Seller quits first at $\underline{x}$, then increasing list price by $\Delta P = \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b} - \bar{P}$ is strictly better for the Seller. This implies that the original list price was sub-optimal. Setting list price to $\bar{P} + \Delta P$ leads to an optimal stopping problem with same quitting threshold but a higher stopping payoff at every point. Thus we know that Seller's ex-ante utility must be strictly higher under the new list price. Note that this

Figure 8:



new list price $\bar{P}+\Delta P$ is **not** the optimal list price. The optimal list price must be even higher, so that Buyer is quitting strictly earlier than Seller.

Proposition 5 implies that we can solve the equilibrium by assuming that $\underline{x} = \underline{x_b} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, then maximizes $U(x_0)$ over $\bar{P}$. We can then verify that $\underline{x_b}(\bar{P}^*) \geq \underline{x_s}(\bar{P}^*)$.

For tractability, I restrict to the case of $x_0 = 0$ in remainder of the section.

Proposition 6 (Costly Selling with $x_0 = 0$)**.** *Let $k = \frac{c_s}{c_b}$. The optimal list price is $\bar{P} = \frac{1}{4} - \frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2}$. Buyer and Seller trade at $\bar{x} = \bar{P} + \frac{\sigma^2}{c_b}\left[\frac{3-\sqrt{\frac{9+k}{1+k}}}{4k} - \frac{1}{4}\right]$ at price $P(\bar{x}) = \bar{P} - \frac{\sigma^2}{c_b}\left(\frac{3-\sqrt{\frac{9+k}{1+k}}}{4k} - \frac{1}{2}\right)^2$; and Buyer quits at $\underline{x} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$.*

Proof. Buyer's value function must be $\widetilde{V}(x)$. Seller's value function is $U(x) = \frac{c_s}{\sigma^2}(x-\bar{P})^2 + A_s(x-\bar{P}) + B_s$ for some coefficients A_s and B_s. Assuming $\underline{x} = \bar{P} - \frac{\sigma^2}{4c_b}$, we can solve $\bar{x}$ and $P(\bar{x})$ by two value-matching conditions: $U(\underline{x}) = 0$, $U(\bar{x}) = P(\bar{x})$, and one smooth-pasting condition: $U'(\bar{x})$. Then maximize $U(x_0)$ with respect to $\bar{P}$, and verify that $\underline{x_b}(\bar{P}) \geq \underline{x_s}(\bar{P})$.

See Appendix for full proof. □

What's the effect of bargaining when selling is costly? Corollary 7 shows that bargaining is necessary for the game to move past time 0 when Seller's cost is high. If the ratio of c_s to c_b exceeds a certain threshold, the game ends immediately if price is fixed, regardless of the level of the price. However, if bargaining is allowed, no matter how high Seller's cost is, the length of the sales process and the chance of a trade are always positive. The intuition is that bargaining lowers both player's costs. Suppose Seller charges a fixed price. When Seller' cost is very high, Seller needs to charge a high price to make up for her cost. But a higher price pushes Buyer to wait further more before buying, which makes the process more costly for Seller, who in turn has to charge an even higher price. If Seller charges too high a price, Buyer quits immediately. Thus there may not exist any fixed price such that both players are willing to engage in the sales process. If Seller is open to bargaining, then Seller can close the sales earlier, curbing her cost of selling. This flexibility makes trade possible. Also, Seller always benefits from Buyer engaging in the sales process for at least some positive amount of time. If product value increases in that period, Seller can finish the sale at a positive price. If it decreases, Buyer will quit and Seller gets 0. Thus the option value of matching at time 0 is always positive regardless of Seller's cost. This effect guarantees that Seller does not charge a list price too high, so that both players are willing to continue at $t = 0$. Also, Buyer prefers fixed price over bargaining when the ratio of costs are low. When the ratio $k = \frac{c_s}{c_b}$ exceeds a certain threshold, both Buyer and Seller gain from bargaining.

Corollary 7 (Comparison to Fixed Price for $x_0 = 0$)**.** *For $k >= 1$, the game ends at $t = 0$ under any fixed price, and ends at $t > 0$ with positive chance of trade under bargaining. Both Buyer's and Seller's ex-ante utilities are higher under bargaining.*

Proof. See Appendix. □

This can explain the prevalence of bargaining in selling. In industries that rely on sales force to win customers, firms have to incur significant cost in employing and training salespersons. Corollary 7 shows that if this cost is high relative to customer's cost of being in the sales process, then the firm must be open to bargaining. Otherwise, the sales process will not take place regardless of the price.

Corollary 8 (Comparative Statics w.r.t $k = \frac{c_s}{c_b}$)**.** *Both the list price, $\bar{P}$, and the size of discount, $\bar{P} - P(\bar{x})$, increase in k. The buying threshold, $\bar{x}$, and the final price, $P(\bar{x})$, decrease in k. Expected length of the game decreases in k. Both Buyer's and Seller's ex-ante utility decreases in k.*

It is interesting that a higher $k = \frac{c_s}{c_b}$ leads to higher $\bar{P}$ but lower $P(\bar{x})$. This means that when Seller's cost increases, she issues a higher list price, but gives a bigger discount and actually sells at a lower price than before. Why would Seller settle for lower price if her cost is high? Again this is to reduce the length of the sales process and saves selling cost. Note that if Seller is not able to give a discount (as under a fixed price), Seller would want charge a higher price when cost increases. This leads to the no-sales-process result in Corollary 7.

5 Private Outside Option

Previous sections assume that Buyer's expected value for the product, x, is fully observed by the Seller. In this section we consider the case where Buyer has private information on his outside option. Seller still learns Buyer's preference for each attribute. However, Seller does not know Buyer's full WTP for the product due to his private outside options. This gives rise to the uncertainty from Seller's point of view. If we normalize outside option into product value per Definition 1, the state x now represents the product value net of Buyer's outside option. At each moment, Seller is facing a distribution of Buyers with different levels of x. Seller observes how this distribution shifts over time, but does not observe where Buyer falls within this distribution.

In this section we solve the case of having two types of Buyers with equal probability. In Appendix, I extend the results to N evenly spaced discrete types, and derive the limiting case of uniform distribution as N approaches infinity. Outside options are normalized into product value and thus are set to 0. I assume selling to be costless. Positive selling cost should have limited impact on the qualitative nature of the equilibrium outcome, as shown in Section 4

We will call the Buyer with the **better** outside option as the **Low(L)** type, and the Buyer with the **worse** outside option as the **High(H)** type. The reason is that the outside option is a negative component of Buyer's normalized product value. Intuitively, for the same attributes and preferences, the buyer with better outside option has a lower WTP.

Formally, let $i \in \{H, L\}$ denote Buyer's type. Nature draws types with $Prob(i = H) = Prob(i = L) = \frac{1}{2}$. Define $x_H = x + \frac{\epsilon}{2}$ and $x_L = x - \frac{\epsilon}{2}$, where ϵ is the gap in outside option between the two types. Note that x_i is the product value (net of outside option) for type i, whereas x is the common state variable and the midpoint between x_i's. Both H type and L type incur flow cost $c_b > 0$. Seller has no cost.

I look for stationary Markov equilibrium with pure strategy, with Seller's belief as an additional state variable. Let $\mu \in \{\{H\}, \{L\}, \{H, L\}\}$ denote Seller's belief about which types she's facing. Strategy functions now depend on μ: $P(x, \mu)$, $a_i(x, \mu, P)$, and $\underline{x_i}(\mu)$. Type i's value function is $V_i(x, \mu)$, where $x = \frac{1}{2}(x_H + x_L)$ is the common state variable for both types. Note that if $\mu = \{H\}$ or $\mu = \{L\}$, then we are back to a game with single type, which has been solved in Section 3 and 4. This happens in a separating equilibria. Finally, we can simplify notation by denoting $\mu_H = \{H\}$, $\mu_L = \{L\}$, and $\mu_{HL} = \{H, L\}$. I assume that Seller does not update her belief off the equilibrium path.

Proposition 9 summarizes the equilibrium outcome. The proof is roughly structured as follows. First, Lemma 4 proves that H type must buy before L type does in equilibrium. The intuition is that H type suffers from Seller finding out about his type. H type is always better off taking L type's offer than waiting. If he waits, he becomes the only remaining type, and receives Buyer's payoff in Proposition 1. This is strictly inferior than buying at the same time as L type. Since H type buys earlier than L type, L type must get the same deal as he does when there's no uncertainty about types. This is proved in Lemma 5. As a result, the strategy and outcome for L type should be the same as in Proposition 1. Seller cannot use $\bar{P}$ as a threat to H type, since H type can take L type's offer, which is strictly better than facing a fixed price at $\bar{P}$. Thus L type's offer becomes the new boundary condition on H type's value function. Seller then faces the new optimal stopping problem regarding when to sell to H type and at what price.

Lemma 4. *In equilibrium, if $a_L(x, \mu_{HL}, P(x, \mu_{HL})) = 1$, then $a_H(x, \mu_{HL}, P(x, \mu_{HL})) = 1$.*

Proof. See Appendix. □

Lemma 5. *In equilibrium, $V_L(x, \mu_{HL}) = \widetilde{V}(x_L)$. L type quits at $x_L = \underline{x}_L = \bar{P} - \frac{\sigma^2}{4c_b}$, and buys at $x_L = \bar{x}_L = \bar{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ at price $P_L = P(x_L = \bar{x}_L) = \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}} - 2\left(\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}\right)\right]$.*

Proof. See Appendix. □

We only need to consider the case of $\bar{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b} = x_{L,t=0} + \frac{\sigma^2}{4c_b}$. By Lemma 5, L type should act the same way as when there's only one type. As Proposition 1 shows, when $\bar{P} \geq x_0 + \frac{\sigma^2}{4c_b}$, Buyer buys immediately if $x_0 > 0$ or Buyer quits immediately if $x_0 \leq 0$. By Lemma 5, H type should buy immediately if L type buys immediately. If L type quits immediately and the H type remains, then we are back to a single type player game. So the only interesting case is if $\bar{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$

Proposition 9 (Private Outside Options for $\bar{P} < -\frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$)**.** *Buyer with type $i \in \{H, L\}$ quits at $\underline{x}_i = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, and buys at $\bar{x}_i = \bar{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$. H type buys earlier and pays less than L type: $P(x = \bar{x} - \frac{\epsilon}{2}, \mu_{HL}) < P(x = \bar{x} + \frac{\epsilon}{2}, \mu_L) = \bar{x} - \widetilde{V}(\bar{x})$.*

Proof. See Appendix. □

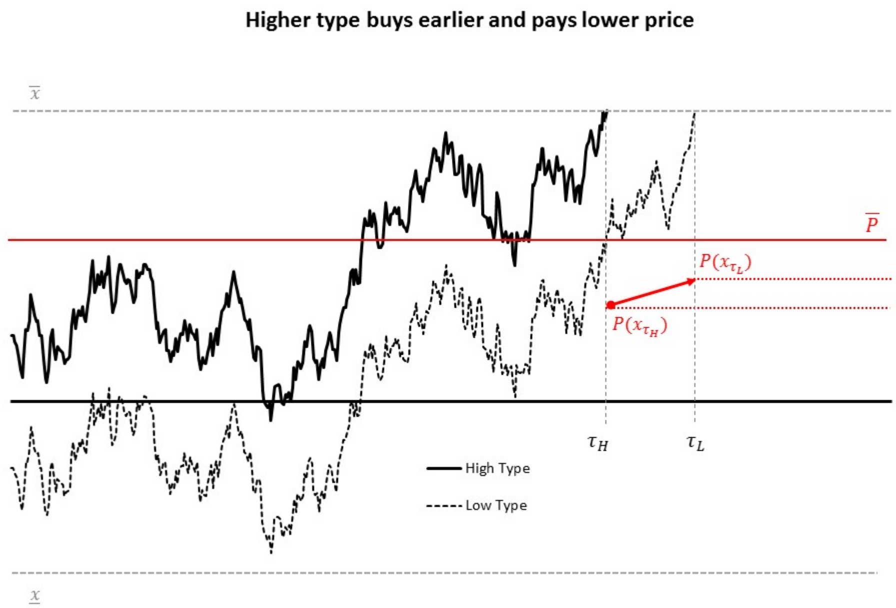
Surprisingly, having private types does not effect Buyer's buying and quitting thresholds in the equilibrium. Not only the two types buy and quit at the same thresholds of net product values, they are also the same threshold as in Proposition 1 when there's only a single type.

This implies that H type buys before L type does, since H type has a higher product value than L type and must reach the threshold earlier. The second surprising finding is that H type pays less than L type does, even though they are buying the product at the same value net of outside option. L type Buyer pays the same amount as in single type case, but H type Buyer receives a bigger discount. Thus on equilibrium path, we would actually observe trading price rising overtime. This is counter-intuitive. In typical bargaining models, types that buy earlier are expected to pay higher prices.

Why would L type choose to wait if waiting is costly and price is rising? The idea is that, when Seller makes an offer to the H type, L type is facing a product with lower value (by ϵ), thus strictly preferring to wait and discover more. On the flip side, Seller has to offer H type a lower price because H type has the option to pretend to be L type, but Seller does not want H type to pretend. H type can wait till x_L reach buying threshold, then pays the same price as L type. However, Seller prefers H type to buy earlier, which saves cost as well as increasing the ex-ante success rate. Intuitively, if the Buyer already comes into the sales process with good intention to buy, then discovery is less valuable, and Seller prefers to close the sale early. But in order for the Buyer to voluntarily buy earlier, Seller needs to make a more generous offer. This is also analogous to designing different contracts to different types of buyers. H type buyer can take L type's contract, but the seller does not want H type to take L type's contract. The binding incentive constraint means that the seller has to concede more utility to the H type. In this game, Seller wants Buyers to self-sort into sales processes of different lengths: a shorter process for H type and a longer process for L type. The price cut for H type is needed to satisfy the IC condition so that H type does not undertake L type's sales process. The price difference between the two offers represents type H's information rent. The size of this rent is $P(\bar{x} + \frac{\epsilon}{2}, \mu_L) - P(\bar{x} - \frac{\epsilon}{2}, \mu_{HL}) = \widetilde{V}(\bar{x}) - \frac{c_b}{\sigma^2}(\bar{x} - \frac{\epsilon}{2} - \bar{P})^2 - \alpha_H(\bar{x} - \frac{\epsilon}{2} - \bar{P}) - \beta_H$, where α_H and β_H are solutions to equations (7) in the proof.

Figure 9 shows one example of the equilibrium outcome. Both types trade when their respective product value reaches $\bar{x}$. H type reaches earlier, and pays a lower price. Trading price increases from time τ_H to τ_L.

Figure 9:



This model expands on previous bargaining models with one-sided incomplete information, such as Fudenberg et al. (1985). The novelty is that, with the additional discovery of product match, we allow the product value to follow a public diffusion process. Fudenberg et al.(1985) shows that, when Seller does not have any price commitment, equilibrium satisfies Coase conjecture. All types buy immediately at price equal to value of the lowest type. If players incur flow cost, as Fudenberg et al. points out, the bargaining game is similar to a one-shot monopoly pricing, since no Buyers will wait beyond time 0. In either case, trade is immediate, and Seller cannot separate different types. After adding discovery of match to the bargaining game, we find that the outcome is different from both Coase conjecture and one-shot monopoly. Proposition 9 shows that trade happens with delay, and Seller can separate Buyers of different types. Buyer with the higher ex-ante value are offered better price due to information rent.

Another novelty is that the discovery of match rationalizes the use of list price in a bargaining framework. In most bargaining models (eg., Fudenberg et al. (1985) and Rubinstein (1982), etc.), list price is meaningless as long as it is high enough. Setting the list price at infinity is always optimal for the seller. With the discovery of match, list price is not only meaningful, but required. If the seller sets the list price too high, the buyer would not engage in the sales process at all. Our model can also be viewed as a special case of the hold-up problem in Wernerfelt (1994b) and Grossman and Hart (1986). Instead of ex-ante investment and ex-post bargaining, our model features simultaneous investment and bargaining, which makes the hold-up problem more dynamic.

One implication of our model is that the relationship between length of the game and trading price can be positive. In bargaining models without stochastic value, price path is declining and often deterministic in time. In our model, the relationship between trading time and trading price is probabilistic, and the average trading price is increasing in time. This increase in price is of course partially due to our assumption that the product has infinite number of attributes. If we assume that the product has finite mass of attributes, then the average trading price can increase in the short term then decline. We explore this case more in Section 6.

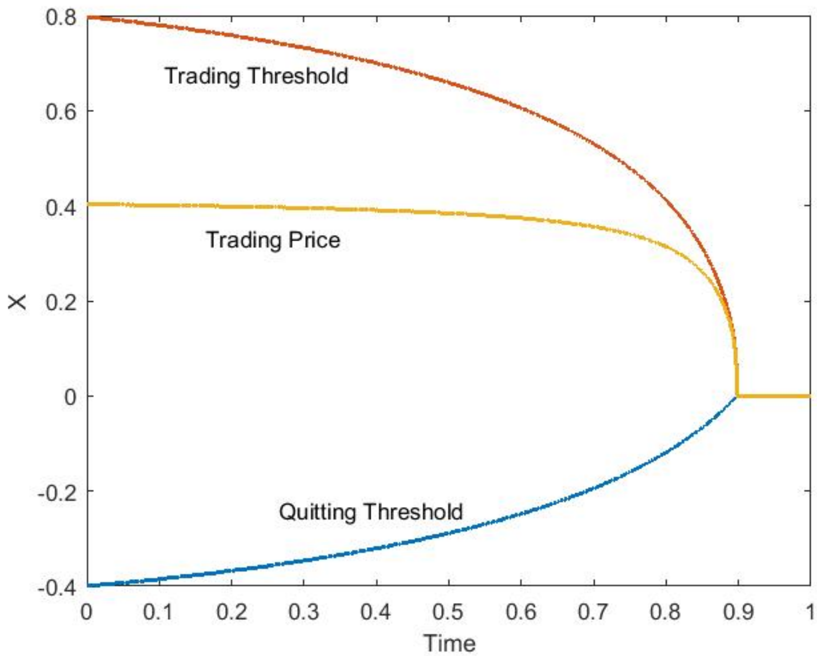
6 Extensions

6.1 Time Discounting

6.2 Finite Mass of Attributes

Suppose that the product a finite mass of attributes, T, then players can only discovery their match up to time T. One can show that, in general, both trading threshold and price decline over time. Also, players never wait till they exhaust all attributes. The following chart shows the case for $T = 1$, $c_b = 0.25$, $\sigma = 1$, $x_0 = 0$, at a list price of 0.5.

Figure 10:



7 Conclusion

This paper presents a bargaining model with a stochastic value discovery process. A seller and a buyer find out their product match over time while simultaneously bargain for price.

The paper provides an explanation for the use of list price and price discount in bargaining, which is absent in previous literature. The buyer does not participate in the sales process if the list price is too high due to a hold-up problem. A lower list price incentivizes the buyer to stay in the game longer, but reduces the seller's bargaining power. The optimal list price balances these two effects. However, the final price is always below the list price, because the seller prefers to shorten the process and secure the trade. The seller can push the buyer to buy earlier by offering a small discount, and some level of discount is always optimal regardless of the list price. Thus the model explains the common "list price - discount" pattern that we observe in industry. Additionally, the buyer should always quit before the seller does, otherwise the seller should increase the list price.

Should the seller commit to a fixed price or be open to bargaining? The model shows that bargaining leads to smaller deal size, shorter game, and higher ex-ante probability of trade. Allowing bargaining is always beneficial to the seller, but can hurts the buyer if buyer's flow cost is high relative to seller's. When seller's flow cost is relatively high, bargaining is required for both players to participate in the sales process. This result highlights the importance of allowing salespersons and customers to bargain, and is supported by the widespread practice of delegating pricing authority to salespersons.

If the buyer has private information on his outside option, then the seller only knows buyer's valuation of the product up to a distribution. This is a one-sided bargaining model with incomplete information, with a simultaneous matching process. The stochastic nature of the product value enables the seller to separate different types of buyers even when the seller has no commitment power. Surprisingly, I find that buyer with higher valuation (or lower outside option) for the product pays lower price due to information rent and seller's incentive to trade early.

This paper contributes to literature on bargaining and sales force management by solving the optimal selling and pricing strategies in response to a strategic customer in a dynamic environment. The paper develops novel insights into the roles of list price and bargaining in the sales process that match real world observations. Last but not least, the model expands existing bargaining literature by adding a simultaneous matching process which makes the product value stochastic.

This paper is limited in several ways. First, the model ignores many aspects of the sales process, such as prospecting and service delivery. Also, the model only looks at one potential buyer with one potential seller. Future research can explore how competition in either side of the market affects outcome. Last but not least, the model assumes that all communication to be honest. The seller truthfully reveals product information, and the buyer truthfully reveals his preference for attributes. Future research can consider what happens if players have choice on what message to send, or whether to send message at all.

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Appendix

Proof of Lemma 1

Let $\Gamma = \{x|a(x, P(x)) = 1\}$ be the set of x such that players trade in equilibrium.

To show that $\widetilde{V}$ is credible, we only need to show that Buyer indeed buys $\forall x \in \Gamma$, if $P(x) = x - \widetilde{V}(x|\bar{P}, \pi_b)$ for $x \in \Gamma$ and $P(x) > x - \widetilde{V}(x|\bar{P}, \pi_b)$ for $x \notin \Gamma$. That is, the described strategy produces Γ. If $P(x) = x - \widetilde{V}(x|\bar{P}, \pi_b)$ for $x \in \Gamma$, then Buyer receives $V(x) = \widetilde{V}(x|\bar{P}, \pi_b)$ for $x \in \Gamma$. This implies $V(x) = \widetilde{V}(x|\bar{P}, \pi_b)$ for all x. This is because $V(x)$ and $\widetilde{V}$ have same diffusion process with same flow cost. Buyer is indifferent between buying and not buying for $x \in \Gamma$. Thus Buyer buys at x if and only if $P(x) \leq x - \widetilde{V}(x|\bar{P}, \pi_b)$. Thus for $x \notin \Gamma$, we must have $P(x) > x - \widetilde{V}(x|\bar{P}, \pi_b)$. For $x \in \Gamma$, if $P(x) < x - \widetilde{V}(x|\bar{P}, \pi_b)$, then Buyer can profitably deviate by charging $P(x) = x - \widetilde{V}(x|\bar{P}, \pi_b)$. Thus $\widetilde{V}$ is attainable in equilibrium. Such equilibrium must have the stated properties.

Proof of Proposition 1

Fix any $\bar{P}_n \leq x_{n,0} + \frac{1}{4}\frac{\sigma^2}{c_b}$. Given Lemma 1, Buyer has same value function as if she's facing a committed price at $\bar{P}$. Since $V(x) = \widetilde{V}(x, \bar{P})$, Buyer's quitting strategies must be same. So $\underline{x_n} = \underline{\underline{x_n}} = \bar{P}_n - \frac{1}{4}\frac{\sigma^2}{c_b}$.

Since Seller controls when to trade, the trading threshold $\bar{x}$ must solve the optimal stopping problem with stopping value $x - \widetilde{V}(x_n, \bar{P}_n)$. By Taylor expansion and Ito's Lemma, Seller's value function satisfies $rU(x_n) = -c_s + \frac{\sigma^2}{2}U''(x_n)$. $r = 0$ and $c_s = 0$ implies linear value function for the Seller $U(x_n) = \alpha_s(x_n - \bar{P}_n) + \beta_s$. We have three boundary conditions. First two conditions match values at quitting and trading points. Third condition is a first-order "smooth-pasting" condition ensuring the stopping time is optimal (Dixit 1993).

$$\begin{cases} \alpha_s\Big(-\frac{1}{4}\frac{\sigma^2}{c_b}\Big) + \beta_s & = 0 \\ \alpha_s(\bar{x_n} - \bar{P}_n) + \beta_s & = \bar{x_n} - \frac{c_b}{\sigma^2}(\bar{x_n} - \bar{P}_n)^2 - \frac{1}{2}(\bar{x_n} - \bar{P}_n) - \frac{\sigma^2}{16c_b} \\ \alpha_s & = 1 - \frac{2c_b}{\sigma^2}(\bar{x_n} - \bar{P}_n) + 1/2 \end{cases} \tag{5}$$

Solving the system of equations, we get $\bar{x_n} = \bar{P_n} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P_n}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ as the point of trade, with $P(\bar{x_n}) = \bar{x_n} - \widetilde{V}(\bar{x_n}, \bar{P_n}) = \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P_n}\frac{c_b}{\sigma^2}} - 2\Big(\frac{1}{4} - \bar{P_n}\frac{c_b}{\sigma^2}\Big)\right]$. Optimal sticker price is found by

$$\underset{\bar{P_n}}{\operatorname{argmax}}\, U(0) = \underset{\bar{P_n}}{\operatorname{argmax}}\, \alpha_b(-\bar{P_n}) + \beta_b = \underset{\bar{P_n}}{\operatorname{argmax}}\, \big(\frac{1}{4}\frac{\sigma^2}{c_b} - \bar{P_n}\big)\big(1 - 2\sqrt{\frac{1}{4} - \bar{P_n}\frac{c_b}{\sigma^2}}\big)$$

We can prove that Seller wants to trade for all $x \geq \bar{x_n}$. Suppose there exists $\tilde{x_n} > \bar{x_n}$ such that $a(\tilde{x_n}, P(\tilde{x_n})) = 0$. Let $x_{left} = sup\{x_n | x_n \in \Gamma \;\&\; x_n < \tilde{x_n}\}$, and $x_{right} = inf\{x_n | x_n \in \Gamma \;\&\; x_n > \tilde{x_n}\}$. Since there's no trade between x_{left} and x_{right}, $U(\tilde{x_n})$ is on a convex function connecting $U(x_{left})$ and $U(x_{right})$. Because the stopping value $x_n - \widetilde{V}(x_n, P(x_n))$ is a concave function connecting $U(x_{left})$ and $U(x_{right})$, we must have $U(\tilde{x_n}) \leq \tilde{x_n} - \widetilde{V}(\tilde{x_n}, P(\tilde{x_n}))$. Thus Seller should trade at $\tilde{x_n}$, a contradiction.

If $\bar{P} > \frac{1}{4}\frac{\sigma^2}{c_b}$, then $\underline{x_n} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b} > 0$. At $\underline{x_n}$, Seller would offer $P(\underline{x_n}) = \underline{x_n}$ and Buyer buys. The only difference now is that Seller receives utility of $\underline{x_n}$ instead of 0 at Buyer's quitting threshold. We can re-solve the above system equations by swapping out the first condition of equations (5) with

$$\alpha_b\Big(-\frac{1}{4}\frac{\sigma^2}{c_b}\Big) + \beta_b = \bar{P_n} - \frac{1}{4}\frac{\sigma^2}{c_b}$$

and we get $\bar{x_n} = \underline{x_n} = \bar{P_n} - \frac{1}{4}\frac{\sigma^2}{c_b}$. Furthermore, for $0 \leq x_n \leq \underline{x_n}$, Seller would offer $P(x_n) = x_n$ and Buyer accepts, otherwise Buyer would quit. Thus game ends immediately for all $x_n \in \mathbb{R}^+$

Proof of Lemma 3 For any stationary equilibrium that satisfy our refinement, let $\Gamma = \{x | a(x, P(x)) = 1\}$ be the set of x such that trade happens. Since $\bar{P}$ is upper bound on price, Buyer cannot be worse off than if price is committed at $\bar{P}$. However, Buyer's value function if price is fixed at $\bar{P}$ is no longer $\widetilde{V}(x, \bar{P})$ as in Lemma 1, since Seller may quit earlier than Buyer wants to. We show that $\widetilde{V}(x+\eta, \bar{P}) - \eta$ is the lower bound on $V(x)$, and offering the lower bound to Buyer at point of trade is SPE.

First let's assume $\underline{x_b} \geq \underline{x_s}$ in equilibrium. Then $\bar{x_s}$ does not affect equilibrium outcome.

Buyer's value function with fixed price is same as in Lemma 1 with $\widetilde{V}(x, \bar{P}) = \widetilde{V}(x+\eta, \bar{P}) - \eta$, where $\eta = 0$ since $\underline{x_b} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$ per Proposition 0.

Now assume $\underline{x_s} > \underline{x_b}$ in equilibrium. $\underline{x} = \underline{x_s}$ is an exogenous stopping point for the Buyer. Given $\underline{x}$, let $V_{\underline{x}}^{\bar{P}}(x)$ denotes Buyer's value function if price is fixed at $\bar{P}$. By Ito's Lemma, $V_{\underline{x}}^{\bar{P}} = -c_b + \frac{\sigma^2}{2}\frac{d}{dx^2}V_{\underline{x}}^{\bar{P}}$, which implies $V_{\underline{x}}^{\bar{P}} = \frac{c_b}{\sigma^2}(x - \bar{P})^2 + \alpha(x - \bar{P}) + \beta$ for some α and β. Let $\bar{x}$ denotes the point that Buyer chooses to buy if price is fixed at $\bar{P}$ and game ends at $\underline{x}$. Then we have following value-matching and smooth-pasting conditions.:

$$\begin{cases} \frac{c_b}{\sigma^2}(\bar{x} - \bar{P})^2 + \alpha(\bar{x} - \bar{P}) + \beta = \bar{x} - \bar{P} \\ \frac{2c_b}{\sigma^2}(\bar{x} - \bar{P}) + \alpha = 1 \\ \frac{c_b}{\sigma^2}(\underline{x} - \bar{P})^2 + \alpha(\underline{x} - \bar{P}) + \beta = 0 \end{cases}$$

Solving these 3 conditions shows that $V_{\underline{x}}^{\bar{P}} = \widetilde{V}(x + \eta, \bar{P}) - \eta$, with $\eta = \frac{1}{4}\frac{\sigma^2}{c_b} - (\underline{x} - \bar{P}) - \sqrt{-\frac{\sigma^2}{c_b}(\underline{x} - \bar{P})}$.

We have shown now $\widetilde{V}(x + \eta, \bar{P}) - \eta$ is the lower bound on $V(x)$. The rest of the proof is same as Lemma 1. In equilibrium, Seller offers Buyer $V(x) = \widetilde{V}(x + \eta, \bar{P}) - \eta$ if Seller wants to trade and offers less if Seller does not want to trade. Buyer will buy at x if $P(x) = x - \widetilde{V}(x + \eta, \bar{P}) + \eta$ since she is indifferent between buying and waiting.

Proof of Proposition 5

We prove by showing that, if Seller quits earlier than Buyer, then there's a profitable deviation in $\bar{P}$. Suppose in equilibrium, $\underline{x} = \max\{\underline{x_b}, \underline{x_s}\} = \underline{x_s} > \underline{x_b}$. We know that $\bar{P} = \underline{x_b} + \frac{1}{4}\frac{\sigma^2}{c_b}$ from Lemma 1 and Proposition 1, thus $\bar{P} < \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$. We'll show that increasing $\bar{P}$ to $\underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$ is strictly better for the Seller ex-ante.

As an interim step, we first prove that $\underline{x_s}$ decreases as $\bar{P}$ increases. To find $\underline{x_s}$, assume $\underline{x_b} = -\infty$. Let $\bar{\bar{x}}$ denotes the point where Buyer would buy if price is committed at $\bar{P}$, and let $\bar{x}$ denotes the point where trade happens in equilibrium. We know that Buyer's value function is in the from of $V(x) = \frac{c_b}{\sigma^2}(x - \bar{P})^2 + \alpha_b(x - \bar{P}) + \beta_b$ and Seller's value function is in the form of $U(x) = \frac{c_s}{\sigma^2}(x - \bar{P})^2 + \alpha_s(x - \bar{P}) + \beta_s$. Then $\bar{\bar{x}}$, $\bar{x}$, and $\underline{x}$ must satisfy the following 7 boundary conditions, 3 for the Buyer and 4 for the Seller:

$$\begin{cases} \frac{c_b}{\sigma^2}(\bar{\bar{x}} - \bar{P})^2 + \alpha_b(\bar{\bar{x}} - \bar{P}) + \beta_b = \bar{\bar{x}} - \bar{P} \\ 2\frac{c_b}{\sigma^2}(\bar{\bar{x}} - \bar{P}) + \alpha_b = 1 \\ \frac{c_b}{\sigma^2}(\underline{x} - \bar{P})^2 + \alpha_b(\underline{x} - \bar{P}) + \beta_b = 0 \\ \frac{c_s}{\sigma^2}(\underline{x} - \bar{P})^2 + \alpha_s(\underline{x} - \bar{P}) + \beta_s = 0 \\ 2\frac{c_s}{\sigma^2}(\underline{x} - \bar{P}) + \alpha_s = 0 \\ \frac{c_s}{\sigma^2}(\bar{x} - \bar{P})^2 + \alpha_s(\bar{x} - \bar{P}) + \beta_s = \bar{x} - V(x) \\ 2\frac{c_s}{\sigma^2}(\bar{x} - \bar{P}) + \alpha_s = 1 - \frac{d}{dx}V(x) \end{cases}$$

Using the last 6 conditions we can derive $\underline{x} = -\frac{c_b}{c_s}\bar{P}$, Thus $\underline{x_s} = \underline{x}$ decreases as $\bar{P}$ increases.

Now we prove that increasing $\bar{P}$ to $\underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b}$ is strictly better for the Seller ex-ante. Fix a $\underline{x}$ in equilibrium, we can think of Seller as facing an optimal stopping problem regarding when to sell, with a lower boundary at $\underline{x}$. Her utility from stopping is $P(x) = x - V(x) = x - \widetilde{V}(x + \eta, \bar{P}) + \eta$. Now suppose sticker price is increased from $\bar{P}$ to $\bar{P} + \Delta P$ where $\Delta P = \underline{x} + \frac{1}{4}\frac{\sigma^2}{c_b} - \bar{P}$. Seller's quitting threshold is decreased as argued above, and Buyer's quitting threshold increases to $\bar{P} + \Delta P - \frac{1}{4}\frac{\sigma^2}{c_b} = \underline{x}$, so the lower boundary of the game is unchanged. With the new sticker price $\bar{P}+\Delta P$, η becomes 0, and Seller's payoff from trading is $x-\widetilde{V}(x, \bar{P}+\Delta P)$, which is strictly higher than her selling price under $\bar{P}$: $x-\widetilde{V}(x+\eta, \bar{P})+\eta$, for $x > \underline{x}$. Thus Seller is facing the same lower boundary under the two sticker prices, and a stopping utility under $\bar{P} + \Delta P$ that strictly dominates $\bar{P}$. Thus regardless of where she wants to trade, Seller's ex-ante utility is strictly higher under $\bar{P} + \Delta P$ than under $\bar{P}$. Thus $\bar{P}$ is sub-optimal.

Proof of Proposition 6 From Proposition 5, we know that Buyer quits first in equilibrium. Thus given $\bar{P}$, Lemma implies that Buyer's value function must be $V(x) = \widetilde{V}(x) =$

$\frac{c_b}{\sigma^2}(x-\bar{P})^2+\frac{1}{2}(x-\bar{P})+\frac{\sigma^2}{16c_b}$, and Buyer's quitting threshold is $\underline{x}=\bar{P}-\frac{\sigma^2}{4c_b}$.

Since $c_s>0$, by $U''(x)=\frac{2c_s}{\sigma^2}$, we know that $U(x)=\frac{c_s}{\sigma^2}(x-\bar{P})^2+A_s(x-\bar{P})+B_s$ for some coefficients A_s and B_s. At $\underline{x}$, we must have (1) $U(\underline{x})=0$. Since $\bar{x}$ is Seller's optimal stopping points, $\bar{x}$ must satisfy: (2) $U(\bar{x})=P(\bar{x})=\bar{x}-V(\bar{x})$, and (3) $U'(\bar{x})=1-V'(\bar{x})$. From these 3 conditions, we can solve $\bar{x}$, A_s, and B_s by solve the following system of equations:

$$\begin{cases} \frac{c_s}{\sigma^2}(-\frac{\sigma^2}{4c_b})^2+A_s(-\frac{\sigma^2}{4c_b})+B_s=0 \\ \frac{c_s}{\sigma^2}(\bar{x}-\bar{P})^2+A_s(\bar{x}-\bar{P})+B_s=\bar{x}-\frac{c_b}{\sigma^2}(\bar{x}-\bar{P})^2-\frac{1}{2}(\bar{x}-\bar{P})-\frac{\sigma^2}{16c_b} \\ 2\frac{c_s}{\sigma^2}(\bar{x}-\bar{P})+A_s=1-\frac{2c_b}{\sigma^2}(\bar{x}-\bar{P})-\frac{1}{2} \end{cases}$$

From which we get

$$\begin{cases} \bar{x}=\bar{P}-\frac{\sigma^2}{4c_b}+\frac{\sigma^2}{c_b}\frac{1}{\sqrt{1+k}}\sqrt{\frac{1}{4}-r} \\ A_s=\frac{2+k}{2}-2\sqrt{1+k}\sqrt{\frac{1}{4}-\bar{P}\frac{c_b}{\sigma^2}} \\ B_s=(\frac{1}{4}+\frac{k}{16})\frac{\sigma^2}{c_b}-\frac{\sqrt{1+k}}{2}\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4}-\bar{P}\frac{c_b}{\sigma^2}}. \end{cases}$$

Plug A_s and B_s into $U(x_0)$, and let $r=\bar{P}\frac{c_b}{\sigma^2}$, we get

$$U(x_0)=kr^2(\frac{\sigma^2}{c_b})^2-(1+\frac{k}{2}-2\sqrt{1+k}\sqrt{\frac{1}{4}-r})r(\frac{\sigma^2}{c_b})^2+(\frac{1}{4}+\frac{3k}{16})\frac{\sigma^2}{c_b}-\frac{\sqrt{1+k}}{2}\frac{\sigma^2}{c_b}\sqrt{\frac{1}{4}-r}+k\frac{\sigma^2}{c_b}{x_0}^2-2krx_0+(1+$$

Maximize $U(0)$ with respect to r produces optimal r^*, and $\bar{P}^*=r^*\frac{\sigma^2}{c_b}$. For $x_0=0$, we get $\bar{P}^*=\frac{1}{4}-\frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2}$. Plug $\bar{P}$ into $\bar{x}=\bar{P}-\frac{\sigma^2}{4c_b}+\frac{\sigma^2}{c_b}\frac{1}{\sqrt{1+k}}\sqrt{\frac{1}{4}-r}$ produces $\bar{x}$. Plug $\bar{P}$ into $P(\bar{x})=\bar{x}-V(\bar{x})$ produces $P(\bar{x})$. It's easy to check that $U'(\underline{x})>0$, which confirms that Seller does not want to quit before Buyer does.

Proof of Corrolary 7

We need is to solve for the equilibrium under fixed price. Let P^* be the optimal fixed price. We'll solve separately for the cases of (1) Buyer quits first and (2) Seller quits first.

Suppose Buyer quits first. We know Buyer's value function is $V(x)=\frac{c_b}{\sigma^2}(x-P^*)^2+$

$\frac{1}{2}(x-P^*)+\frac{\sigma^2}{16c_b}$, and Seller's value function is $U(x)=\frac{c_s}{\sigma^2}(x-P^*)^2+A_s(x-P^*)+B_s$ for some coefficients A_s and B_s. Let $\bar{x}$ and $\underline{x}$ denote Buyer's buying and quitting thresholds, respectively. Since both buying and quitting are Buyer's decision, we know $\bar{x}=P^*+\frac{\sigma^2}{4c_b}$ and $\underline{x}=P^*-\frac{\sigma^2}{4c_b}$ as in Section 3. Solve $U(\bar{x})=P^*$ and $U(\underline{x})=0$ simultaneously give us $A_s=P^*\frac{2c_b}{\sigma^2}$ and $B_s=-k\frac{\sigma^2}{16c_b}+\frac{1}{2}P^*$. Thus

$$U(x_0)=\frac{kc_b}{\sigma^2}(x_0-P^*)^2+P^*\frac{2c_b}{\sigma^2}(x_0-P^*)+-k\frac{\sigma^2}{16c_b}+\frac{1}{2}P^*$$

Take derivative with respect to P^* and set to zero, we get $P^*=\frac{1}{2-k}\frac{\sigma^2}{4c_b}+\frac{1-k}{2-k}x_0$. To make sure $\underline{x}<x_0$, we need $k<x_0\frac{4c_b}{\sigma^2}+1$, otherwise Buyer quits immediately.

Now suppose Seller quits first. Since quitting is Seller's decision, we do not know coefficients to Buyer's value function $V(x)=\frac{c_b}{\sigma^2}(x-P^*)^2+A_b(x-P^*)+B_b$. We still have $U(x)=\frac{c_s}{\sigma^2}(x-P^*)^2+A_s(x-P^*)+B_s$. To solve A_b, B_b, A_s, B_s, $\bar{x}$, and $\underline{x}$, we need to solve the following system of 6 equations:

$$\begin{cases} U(\bar{x})=P^* \\ U(\underline{x})=0 \\ U'(\underline{x})=0 \\ V(\bar{x})=\bar{x}-P^* \\ V'(\bar{x})=1 \\ V(\underline{x})=0 \end{cases}$$

This gives $\underline{x}=(1-\frac{1}{k})P^*$, and $U(x_0)=\frac{kc_b}{\sigma^2}(x_0-P^*)^2+\frac{2c_b}{\sigma^2}P^*(x_0-P^*)+\frac{kc_b}{\sigma^2}\left(\frac{P^*}{k}\right)^2$. Then maximize $U(x_0)$ with respect to P^*. If $k\leq x_0\frac{4c_b}{\sigma^2}+1$, then $\frac{d}{dP}U>0$ for all P. This implies Seller will raise price till Buyer quits first, i.e., $P^*-\frac{\sigma^2}{4c_b}\geq(1-\frac{1}{k})P^*$. If $k>x_0\frac{4c_b}{\sigma^2}+1$, we get $P^*=\frac{k}{k-1}x_0=\underline{x}$. Thus Seller quits immediately. Thus there does not exist an equilibrium with positive length such that Seller quitting strictly before Buyer. For $k<x_0\frac{4c_b}{\sigma^2}+1$, the

length fo the game is positive and Buyer quits first. For $k >= x_0 \frac{4c_b}{\sigma^2} + 1$, one of player stops immediately regardless of the price.

Now we can compare equilibrium outcome under bargaining to outcome under fixed price. Particularly, for $x_0 = 0$, we can compare $\bar{P} = \frac{\sigma^2}{c)b}\left[\frac{1}{4} - \frac{27+10k-6\sqrt{9+10k+k^2}}{16k^2}\right]$ with $P^* = \frac{1}{2-k}\frac{\sigma^*}{4c_b}$. There exists a $\tilde{k} < 1$ such that $\bar{P} > P^*$ for $k < \tilde{k}$ and $\bar{P} < P^*$ for $k > \tilde{k}$. There's no tractable analytical solution for $\tilde{k}$.

Proof of Lemma 4 Suppose both types are in the game and L type buys at x. Since L type cannot be worse than if price is committed at $\bar{P}$, we must have $V_L(x, \mu_{HL}) \geq \widetilde{V}(x_L)$, where $\widetilde{V}$ is from equation (3). So $P(x, \mu_{HL}) \leq x_L - \widetilde{V}(x_L)$. If H types buys at x, he gets utility at least $x_H - (x_L - \widetilde{V}(x_L)) = \widetilde{V}(x_L) + \epsilon$. If H type does not buy at x, then he's the only type remaining, and gets single-type value $V_H(x, \mu_H) = \widetilde{V}(x_H) = \widetilde{V}(x_L + \epsilon)$. Since $\widetilde{V}(x_L) + \epsilon \geq \widetilde{V}(x_L + \epsilon)$, H type buys.

Proof of Lemma 5 First we see that if $\mu = L$, the game is same as Proposition 1 in Section 3. Thus on equilibrium path L type buys for $x_L \geq \bar{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$ and quits for $x_L \leq \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$. Denote these two thresholds $\bar{x}^o$ and $\underline{x}^o$, respectively.

Now for $\mu = \{H, L\}$, by Lemma 4, H type must buy weakly earlier than L type in equilibrium. Suppose strictly earlier ($\bar{x}_H < \bar{x}_L + \epsilon$), then we have a separating equilibrium. After H type buys, L type buys immediately if $x_L \geq \bar{x}^o$. Thus H type must buy strictly before x_H reaches $\bar{x}^o + \epsilon$, otherwise there cannot be separation. As a result, on equilibrium path, L type buys at $x_L = \bar{x}^o$ without uncertainty on types. The existence of H type does not affect L type's equilibrium payoff. Thus $V_L(x, \mu_{HL}) = V_L(x, \mu_L) = \widetilde{V}(x_L)$ as in Lemma 1, and he buys at $\bar{x}_L = \bar{x}^o$ and quits at $\underline{x}_L = \underline{x}^o$ as in Proposition1.

Suppose for $\mu = \{H, L\}$, we have a pooling equilibrium where H and L types buy together ($\bar{x}_H = \bar{x}_L + \epsilon$). If Seller cannot separate the two types, then it is as if she's only dealing with L type. Using arguments in Lemma 1, Seller can charge up to $P(x, \mu) = P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \widetilde{V}(x_L)$ at point of trade. In equilibrium Seller charges this price if she wants to trade and above this price if she does not want to trade. L type behaves as if there's no high type.

Then we must have $\bar{x}_L = \bar{x}^o$, otherwise Seller can profitable deviates by charging $P(x, \mu) = P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \widetilde{V}(x_L)$ for $x_L \geq \bar{x}^o$ and charging $P(x, \mu) > P(x_L + \frac{\epsilon}{2}, \mu) = x_L - \widetilde{V}(x_L)$ for $x_L < \bar{x}^o$.

If $\bar{P} \geq x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, then $x_{L|t=0} < \underline{x}_L$, thus L type wants to quit immediately. If $\bar{P} < x_0 - \frac{\epsilon}{2} + \frac{\sigma^2}{4c_b}$, then we have shown above that L type behaves same as in Proposition 1.

Proof of Proposition 9 By previous Lemmas, L type buys and quits at same thresholds as in single type case. Thus through Proposition 1, we have $\underline{x}_L = \underline{x} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$, and $\bar{x}_L = \bar{x}$ solves

$$(\bar{x}_L - \bar{P} - \frac{\sigma^2}{4c_b})^2 + \bar{P} - \frac{\sigma^2}{4c_b} = 0 \tag{6}$$

and $V_L(x, \mu_{HL}) = V_L(x, \mu_L) = \widetilde{V}(x - \frac{\epsilon}{2}) = \widetilde{V}(x_L) = \frac{c_b}{\sigma^2}(x_L - \bar{P})^2 + \frac{1}{2}(x_L - \bar{P}) + \frac{\sigma^2}{16c_b}$.

When x drops to $\underline{x}_L + \frac{\epsilon}{2}$, L type quits and the game become single type Buyer with value $\underline{x}_L + \epsilon$. When x increases to $\bar{x}_L + \frac{\epsilon}{2}$, L type buys and by Lemma 3 H type buys too. Thus Seller has to decide when to sell to H type for $\underline{x}_L + \frac{\epsilon}{2} < x \leq \bar{x}_L + \frac{\epsilon}{2}$. As before, Seller need to offer H type no more utility than H type's lower bound. When L quits at $x_L = \underline{x}_L$, H's utility become $\widetilde{V}(\underline{x}_L + \epsilon)$. When L buys, H's utility becomes $\widetilde{V}(\bar{x}_L) + \epsilon$. These two conditions bound H type's value function from below. By Taylor expansion and Ito's Lemma, we have $V_H(x, \mu_{HL}) = V_H(x_H - \frac{\epsilon}{2}, \mu_{HL}) = \frac{c_b}{\sigma^2}(x_H - \frac{\epsilon}{2} - \bar{P})^2 + \alpha_H(x_H - \frac{\epsilon}{2} - \bar{P}) + \beta_H$. Thus the two conditions translate to:

$$\begin{cases} \frac{c_b}{\sigma^2}(\frac{\epsilon}{2} - \frac{\sigma^2}{4c_b})^2 + \alpha_H(\frac{\epsilon}{2} - \frac{\sigma^2}{4c_b}) + \beta_H = \frac{c_b}{\sigma^2}(\epsilon - \frac{\sigma^2}{4c_b})^2 + \frac{1}{2}(\epsilon - \frac{\sigma^2}{4c_b}) + \frac{\sigma^2}{16c_b} \\ \frac{c_b}{\sigma^2}(\bar{x}_L + \frac{\epsilon}{2} - \bar{P})^2 + \alpha_H(\bar{x}_L + \frac{\epsilon}{2} - \bar{P}) + \beta_H = \frac{c_b}{\sigma^2}(\bar{x}_L - \bar{P})^2 + \frac{1}{2}(\bar{x}_L - \bar{P}) + \frac{\sigma^2}{16c_b} + \epsilon \end{cases} \tag{7}$$

These 2 equations give α_H, β_H, and consequently $V_H(x, \mu_{HL})$. It's easy to check that $V_H(x) > V_L(x + \epsilon)$ for $\underline{x}_L + \frac{\epsilon}{2} < x < \bar{x}_L + \frac{\epsilon}{2}$. Thus H type has higher value function than L type, even when they face the same product value. It's also easy to check that $V_H(x) - \epsilon < V_L(x)$ for $\underline{x}_L + \frac{\epsilon}{2} < x < \bar{x}_L + \frac{\epsilon}{2}$; thus L type would not buy if he's offered price $P(x,\mu_{HL}) = x_H - V_H(x,\mu_{HL})$. Now given $V_H(x,\mu_{HL})$, Seller receives $P(x,\mu_{HL}) = x_H - V_H(x,\mu_{HL})$ when she trade with H type. Seller's decision of when to trade with H type must solve the following optimal stopping problem:

$$\begin{cases} \alpha_s(\bar{x}_H - \frac{\epsilon}{2}) + \beta_s = -(\bar{x}_H - \frac{\epsilon}{2} - \bar{P})^2 + (1-\alpha_H)(\bar{x}_H - \frac{\epsilon}{2} - \bar{P}) + \bar{P} - \beta_H \\ \alpha_s = -2(\bar{x}_H - \frac{\epsilon}{2} - \bar{P}) + (1-\alpha_H) \\ \alpha_s(\frac{\epsilon}{2} - \frac{\sigma^2}{4c_b}) + \beta_s = (1 - 2\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}})\epsilon \end{cases} \tag{8}$$

Using equations 7 and 8, we can derive $(\bar{x}_H - \bar{P} - \frac{\sigma^2}{4c_b})^2 + \bar{P} - \frac{\sigma^2}{4c_b} = 0$. This is same condition as Equation 6. Thus $\bar{x}_H = \bar{x}_L = \bar{x} = \bar{P} + \frac{\sigma^2}{c_b}\left[\sqrt{\frac{1}{4} - \bar{P}\frac{c_b}{\sigma^2}} - \frac{1}{4}\right]$. Note that even though they buy at same threshold, H type arrives the threshold strictly before L type does, since $x_H = x_L + \epsilon$. Also, this common threshold $\bar{x}$ is the same buying threshold when we only have single type.

Since $\widetilde{V}(\bar{x}_L + \epsilon) > 0$, H type does not quit when L is present. After L quits, H has same quitting threshold as L type. So $\underline{x}_i = \underline{x} = \bar{P} - \frac{1}{4}\frac{\sigma^2}{c_b}$

At time of trade, L type pays price $P_L = P(\bar{x} + \frac{\epsilon}{2}, \mu_L) = \bar{x} - V_L(\bar{x} + \frac{\epsilon}{2})$, and H type pays price $P_H = P(\bar{x} - \frac{\epsilon}{2}, \mu_{HL}) = \bar{x} - V_H(\bar{x} - \frac{\epsilon}{2})$. Thus $P_L - P_H = V_H(\bar{x} - \frac{\epsilon}{2}) - V_L(\bar{x} + \frac{\epsilon}{2})$. Since we showed above that $V_H(x) > V_L(x+\epsilon)$, this proves that $P_L > P_H$.

Discrete-time Game and Uniqueness of Limit

In this section, I prove Lemma 2, which states that $\widetilde{V}(x)$ from equation (4) is the unique limit of equilibrium value functions from discrete-time game. I'll first formally state the discrete-time game, then introduce a refinement that gets rid of trivial equilibria, then prove the uniqueness of the limit as the game approaches continuous time.

The discrete time game follows the description in Section 2.1. Let $\mathbb{G}(\Delta t)$ denote the game with period length of Δt. Time $t \in \{0, \Delta t, 2\Delta t, \cdots\}$. There are two players, a Seller (b) and a Buyer (b). x is the product value with $x = x_0$ at time $t = 0$. State variabel x_t evolve as a Markov chain, with $x_{t+\Delta t} = x_t + \sigma\sqrt{\Delta t}$ with probability $\frac{1}{2}$ and $x_{t+\Delta t} = x_t - \sigma\sqrt{\Delta t}$ with probability $\frac{1}{2}$. Let $\mathbf{X}(\Delta t) = \{x_0 + \alpha\sigma\sqrt{\Delta t} \mid \alpha \in \mathbb{N}\}$ denote the grid on x spanned by $\sigma\sqrt{\Delta t}$ from x_0. We must have $x_t \in \mathbf{X}(\Delta t)\ \forall t$. x_t is observable to both players.

Before game starts, Seller chooses a list price $\bar{P} \in \mathbb{R}^+$. Then at $t \geq 0$:

1. Seller chooses price P_t subject to $P_t \leq \bar{P}$.

2. Buyer chooses whether to buy $a_t \in \{0, 1\}$.

3. If $a_t = 0$, then both players chooses whether to quit with $q_{i,t} \in \{0, 1\}$. If no player quits, then Buyer incurs continuation cost and game moves to $t + \Delta t$.

I look for stationary Markov equilibrium with pure strategy. Seller's action is characterized by $(\bar{P}, P(x, \bar{P}), q_s(x, \bar{P}))$, with list price $\bar{P} \in \mathbb{R}^+$, price offer $P(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$, and quitting decision $q_s(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$. Buyer's action is characterized by $(a(x, P, \bar{P}), q_b(x, \bar{P}))$, with buying decision $a(x, P, \bar{P}) : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \{0, 1\}$, and quitting decision $q_b(x, \bar{P}) : \mathbb{R} \times \mathbb{R}^+ \mapsto \{0, 1\}$. Since everything depends on list price $\bar{P}$, which is fixed throughout the game, we'll drop $\bar{P}$ from notation going forward.

Buyer incurs continuation cost of $c_b * \Delta t$ at the end of the period if $a(x_t) = 0$ and $q_s(x_t) = q_b(x_t) = 0$. If $a(x_t) = 1$, Buyer receives utility $x_t - P_t$ and Seller receives P_t. If either player quits, game ends and players get outside option of π_b and π_s, which can be normalized to 0 WLOG (see Definition 1 in Section 3 for details).

Let $u(x, \bar{P}, P(\cdot), q_s(\cdot) \mid a(\cdot), q_b(\cdot)$ denote seller's expected utility at state x, and $v(x, a(\cdot), q_b(\cdot) \mid \bar{P}, P(\cdot), q_s(\cdot))$ denote buyer's expected utility at state x. Let $V_1(x)$ represent the equilibrium value function for Buyer at state x. Let $V_2(x)$ denote Buyer's expected value of staying in the game, conditional of $a(x_t) = 0$.

We can write $V_2(x)$ recursively as:

$$V_2(x) = -c_b\Delta t + \frac{1}{2}V_1(x + \sigma\sqrt{\Delta t}) + \frac{1}{2}V_1(x - \sigma\sqrt{\Delta t})$$

Given $V_2(x)$, in equilibrium Seller must charge price up to $\max\{x - V_2(x), \bar{P}\}$. This implies:

$$V_1(x) = \begin{cases} \max\{0, V_2(x)\} & \forall x \leq \bar{P} \\ \max\{V_2(x), x - \bar{P}\} & \forall x > \bar{P} \end{cases}$$

Before we proceed, we want to get rid of trivial equilibria such that $q_b(x) = q_s(x) = 1$ for some x. These equilibria are similar to failures in matching games. If the opponent is quitting, then the player is indifferent between quiting and not quitting, thus quitting is always optimal. Thus the opponent's quitting decision is optimal too. As a result, we can construct equilibrium with arbitrary quitting rules, as long as they quit at the same time. These equilibria exist due to the simultaneity of quitting decisions, and do not offer any insight. One example is $q_b(x) = q_s(x) = 1 \quad \forall x$, $P(x) = max\{0,\ min\{x, \bar{P}\}\ \}$, and $a(x, \bar{P}, P) = \mathbb{I}\{x \geq 0\}$ is an equilibrium. Player always quit no matter what x is; Seller charges the value of the product (if positive) as price; Buyer buys immediately since he's indifferent between buying and quitting. In this equilibrium, game ends immediately no matter what the initial position is.

To get rid of this simultaneous quitting problem, I introduce the following refinement based on the logic of trembling hand.

Equilibrium Refinement. $q_i(x) = 1$ *if only if quitting is strictly preferred.*

Under this refinement, players cannot quit simultaneously. If the opponent quits, then I'm indifferent and will choose to stay. This refinement is not restrictive. The logic is similar to trembling hand. Suppose we have an equilibrium with $\underline{x_s} = \underline{x_b}$. Now keep $\bar{P}$, $P(\cdot)$, and $a(\cdot)$ as before, and see whether $\underline{x_s} - \epsilon$ and $\underline{x_b}$ constitute a equilibrium, and whether $\underline{x_b} - \epsilon$ and $\underline{x_s}$ constitute a equilibrium. If neither is an equilibrium, that means that players quit only because the other player is quitting. If either perturbation on $\underline{x_s}$ or on $\underline{x_b}$ constitutes an equilibrium, then the new equilibrium has the same equilibrium outcome as before, thus is not affected by the restriction that $\underline{x_s} \neq \underline{x_b}$. Requiring $\underline{x_s} \neq \underline{x_b}$ also gives us a clear answer to who quits in equilibrium, which is a question of interest in this game.

Now we are ready to prove Lemma 2.

Claim 1. *These exists a $\underline{\underline{x}}$ such that $a(x) = 0$ and $q_b(x) = 1$ iff $x \leq \underline{\underline{x}}$.*

This states that there exists a threshold $\underline{\underline{x}}$ such that for all states below the threshold, the game will end as Buyer quits. First it's easy to see that if $x < 0$, then $a(x) = 0$. Secondly, we can see that Buyer must quit for some states that are low enough. When x is negative, there cannot be trade, and Buyer has to pay cost to continue the game. For x approaches $-\infty$, continuation value also goes $-\infty$ and Buyer will quit.

Next, we can show that if $q_b(x) = 1$, then $q_b(x - \sigma\sqrt{\Delta t}) = 1$. If $q_b(x) = 1$, then we must have $V_2(x) < 0$ and $V_1(x) = 0$. Suppose that $q_b(x - \sigma\sqrt{\Delta t}) = 0$, this implies that

$$V_2(x) = -c_b \Delta t + \frac{1}{2} V_1(x) + \frac{1}{2} V_1(x - 2\sigma\sqrt{\Delta t}) \geq 0$$

which means that $V_1(x - 2\sigma\sqrt{\Delta t}) > V_2(x - 2\sigma\sqrt{\Delta t} \geq 0$. By induction, we have $V_1(x') > 0$ for all $x' < x - 2\sigma\sqrt{\Delta t}$. Thus Buyer never quits for $x' < x$. This is a contradiction, as Buyer must quit as x approaches $-\infty$.

Lastly, if $x > \bar{P}$, then we cannot have $a(x) = 0$ and $q_b(x) = 1$, because if $q_b(x) = 1$, then Seller would charge $\bar{P}$ and Buyer should buy the product and get $x - \bar{P} > 0$. This concludes the proof.

Claim 2. *These exists a $\bar{\bar{x}}$ such that $a(x) = 1$ and $P(x) = \bar{P}$ iff $x \geq \bar{\bar{x}}$.*

Here we claim that there exists a threshold such that when x is above the threshold, Seller charges $\bar{P}$ and Buyer buys immediately.

First we see that if $P(x) = \bar{P}$ and $a(x) = 1$, then $x \geq \bar{P}$. If $x \geq \bar{P}$, then $V_1(x) = \max\{V_2(x), x - \bar{P}\}$. Seller must charge $P(x) = \bar{P}$ if $V_2(x) < x - \bar{P}$. Let $\tilde{P}(x) = \min\{x - V_2(x), \bar{P}\}$ denote the highest price that Buyer is willing to pay at $x > \bar{P}$. Buyer buys at $\bar{P}$ if and only if $\tilde{P} = \bar{P}$.

Now we prove that $\tilde{P}$ must be non-decreasing. Suppose $\tilde{P}$ decreases for some x. Then there exist x' such that $\tilde{P}(x' - \sigma\sqrt{\Delta t}) > \tilde{P}(x')$. This implies that $\tilde{P}(x') < \bar{P}$, which means $\tilde{P}(x') = x' - V_1(x') = x' - V_2(x') = \frac{1}{2}(\tilde{P}(x' - \sigma\sqrt{\Delta t}) + \tilde{P}(x' + \sigma\sqrt{\Delta t}) + c_b\Delta t$. By rearranging the terms we get: $\tilde{P}(x' + \sigma\sqrt{\Delta t}) - \tilde{P}(x') = \tilde{P}(x') - \tilde{P}(x' - \sigma\sqrt{\Delta t}) - 2c_b\Delta t < 0$. Thus $\tilde{P}(x' + \sigma\sqrt{\Delta t}) < \tilde{P}(x') - c_b\Delta t$. By induction, $\tilde{P} \to -\infty$ as $x \to \infty$. When $\tilde{P} < 0$, there cannot be trade since Seller can't charge negative price, and Buyer does not quit since $V_2(x) > 0$. Thus there cannot be trade or quit after x pass some threshold. However, this implies that $V_1(x)$ should be decreasing beyond that threshold, which is a contradiction.

Given that $\tilde{P}$ must be non-decreasing, we need to show that $\tilde{P}$ must hit $\bar{P}$ at some point. Suppose $\tilde{P}(x)$ never reach $\bar{P}$. Previously we show that if $x > \bar{P}$ and $\tilde{P}(x) < \bar{P}$, then $\tilde{P}(x + \sigma\sqrt{\Delta t}) - \tilde{P}(x) = \tilde{P}(x) - \tilde{P}(x - \sigma\sqrt{\Delta t}) - 2c_b\Delta t < 0$. Since $\tilde{P}$ is non-decreasing, $\tilde{P}(x) - \tilde{P}(x - \sigma\sqrt{\Delta t}) - 2c_b\Delta t > 0$, thus $\tilde{P}(x) - \tilde{P}(x - \sigma\sqrt{\Delta t}) > 2c_b\Delta t$ for all $x > \bar{P}$. Then $\tilde{P}$ must reach $\bar{P}$ for some x large. This concludes the proof. We can find $\bar{\bar{x}}$ by taking the $\min\{x \mid \tilde{P} = \bar{P}\}$.

Given the existence of $\bar{\bar{x}}$ and $\underline{\underline{x}}$, we can know specify $V_1(x)$ in three cases:

$$V_1(x) = \begin{cases} 0, & \forall x \leq \underline{\underline{x}} \\ -c_b\Delta t + \frac{1}{2}V_1(x + \sigma\sqrt{\Delta t}) + \frac{1}{2}V_1(x - \sigma\sqrt{\Delta t}), & \forall \underline{\underline{x}} < x < \bar{\bar{x}} \\ x - \bar{P}, & \forall x \geq \bar{\bar{x}} \end{cases} \tag{9}$$

By rearranging the terms, We can prove that equilibrium $V_1(x)$ must satisfy the following properties:

$$\begin{cases} \frac{2c}{\sigma^2} = \frac{\frac{V_1(x+\sigma\sqrt{\Delta t}-V_1(x)}{\sigma\sqrt{\Delta t}}+\frac{V_1(x)-V_1(x-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}}}{\sigma\sqrt{\Delta t}} \quad \text{for } \underline{\underline{x}} < x < \bar{\bar{x}} \\ V_1(x) = 0 \quad \forall x \leq \underline{\underline{x}} \\ 0 \leq \frac{V_1(\underline{\underline{x}}+\sigma\sqrt{\Delta t})-V_1(\underline{\underline{x}})}{\sigma\sqrt{\Delta t}} < \frac{2c}{\sigma}\sqrt{\Delta t}. \\ V_1(x) = x - \bar{P} \quad \forall x \geq \bar{\bar{x}} \\ 1 \geq \frac{V_1(\bar{\bar{x}})-V_1(\bar{\bar{x}}-\sigma\sqrt{\Delta t})}{\sigma\sqrt{\Delta t}} \geq 1 - \frac{2c}{\sigma}\sqrt{\Delta t} \end{cases}$$

Let $V(x)$ denote the limit of $V_1(x) \to V(x)$ as $\Delta t \to 0$. The above 5 condition converge to:

$$\begin{cases} V''(x) = \frac{2c}{\sigma^2} \quad \text{for } \underline{\underline{x}} < x < \bar{\bar{x}} \\ V(x) = 0 \quad \forall x \leq \underline{\underline{x}} \\ V'(\underline{\underline{x}}) = 0 \\ V(x) = x - \bar{P} \quad \forall x \geq \bar{\bar{x}} \\ V'(\bar{\bar{x}}) = 1 \end{cases}$$

Together they imply that $V(x)$ is C^1, and the only C^1 function that satisfy the five conditions is:

$$\widetilde{V}(x) = \begin{cases} 0, & x < \bar{P} - \frac{\sigma^2}{4c_b} \\ \frac{c_b}{\sigma^2}(x-\bar{P})^2 + \frac{1}{2}(x-\bar{P}) + \frac{\sigma^2}{16c_b}, & x \in \left[\bar{P} - \frac{\sigma^2}{4c_b}, \bar{P} + \frac{\sigma^2}{4c_b}\right] \\ x - \bar{P}, & x > \bar{P} + \frac{\sigma^2}{4c_b} \end{cases} \tag{10}$$

This proves Lemma 2, which states that $\widetilde{V}(x)$ is the unique limit of value functions in discrete-time game equilibria in which players do not quit unless quitting is strictly preferred.

Equilibrium with N types