Intermediation and Value Creation in an Incomplete Market: Implications for Securitization*

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Abstract

In an incomplete market economy, all claims cannot be priced uniquely based only on arbitrage. The prices of attainable claims (those that are spanned by traded claims) can be determined uniquely, whereas the prices of unattainable claims can only be bounded. The issue we consider is whether the values of the latter claims can be enhanced by undertaking appropriate transactions in the financial markets. We investigate this question for an economy consisting of firms, investors and financial intermediaries, in the specific context of securitization of claims by the intermediary, i.e., pooling of claims issued by firms and issuance of tranches on the pool to investors. We derive conditions under which the pooling of assets creates value, and firms willingly participate in the creation of the pool. We also determine the optimal tranching strategy for the pooled claims, and show that the additional value of creating new securities depends on the degree of dissimilarity among investors. Our framework can be extended to analyze several applications in financial economics for problems ranging from the structure of venture capital firms to the valuation of real options.

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1 Introduction

Many of the classical results in modern financial theory were originally derived in the context of a complete financial market in the sense of Arrow-Debreu. Although this framework has yielded rich results for a variety of problems in economics, the fact remains that financial markets are, in fact, incomplete. This, in turn, has implications for many interesting economic issues, in particular, the problem of valuation of assets, both real and financial. This paper studies the problem of simultaneous valuation of several claims in an incomplete market. Specifically, we investigate when it pays for an intermediary to pool cash flows originating from firms with unique investment opportunities and engage in capital market transactions to enhance the value of a pool. Our results have implications for financial decision making and securitization.

The problem we pose is a fairly common one. Firms often have opportunities that are unique to them, i.e., they generate future cash flows, which cannot be replicated by using traded securities. In standard financial theory, assets are generally valued on a stand-alone basis. Their prices are derived from the capital market based on the twin assumptions that the cash flows from the asset can be replicated in the financial market and that all agents are price-takers with respect to financial claims. However, when markets are incomplete, the cash flows from the unique investment opportunity may no longer be fully replicated in the financial market. This raises the question whether the value of this asset can be enhanced by transactions in the financial markets.

We investigate this question in the specific context of securitization of claims by an intermediary. We model an economy with firms, investors and financial intermediaries. There are primary securities traded in the financial market at fair (i.e., arbitrage-free) prices. Firms have opportunities to invest in unique assets, and their objective is to maximize the present value of the cash flows from these assets. Investors are utility maximizers. Intermediaries purchase claims from firms that are fully backed by their cash flows, and undertake two types of transactions. They can issue securities that are within the span of the financial market, or create new securities that are not spanned by the market to be sold to investors. Thus, intermediaries have the ability to not only match buyers and sellers for the same claim, but also pool assets and package claims to be issued against them in such a way as to take advantage of market incompleteness. Throughout our analysis, we assume that firms are not large enough to influence the prices of the securities traded in the market, i.e., they are price-takers. We study whether, in an arbitrage-free setting, such transactions create value for firms and investors.
We separate this question into two parts. First, we study the phenomenon whereby financial intermediaries pool cash flows from the assets of several firms or divisions of a firm and issue securities that are within the span of the market. Here, we address the following questions: Does pooling provide value enhancement to the contributing firms, i.e., does it facilitate affirmative decisions regarding projects whose values are ambiguous or even negative, due to market incompleteness? If firms have reservation prices for undertaking investments in opportunities that are unique to them, and can form coalitions with other firms, then what are the incentives of the firms to participate in the creation of the asset pool? Finally, how does the intermediary allocate the value of an asset pool among contributing firms? In answer to these questions, we determine conditions on the firms’ cash flows and reservation prices that are necessary and sufficient for value enhancement through pooling. These conditions are derived using duality theory. The incentives of firms to participate are addressed using a game theoretic formulation.

We then study the joint pooling and tranching problem, in which a financial intermediary first pools cash flows from several firms, and then issues new securities against the pool. It is clear that pooling and tranching will always provide more value than only pooling. We determine the incremental value of tranching and the incentives of firms to participate in this modified setting. Additionally, we uncover the optimal design of the tranches that maximize the value of the asset pool. Our research shows that incompleteness of the market places a premium on those assets that offer hedging possibilities, i.e., on assets that provide cash flows in states that are not spanned by the market. Due to this, more firms might be able to undertake unique investments, which, in turn, would lead to additional value creation via a “ratchet” effect. Financial intermediation, which exploits such synergies, increases the set of projects financed, and thus, creates additional value.

A number of typical problems of valuation in the context of incomplete markets can be analyzed within our framework. At a general level, the problem outlined above can be defined as the joint optimization of the investment and financing decisions of a firm in an incomplete capital market. Securitization, widely used in international financial markets, is a more specific example of this problem. It began in the 1970s with the securitization of mortgage loans by Fannie Mae, Ginnie Mae and Freddie Mac, with the objective of facilitating home ownership by providing a reliable supply of home mortgage financing. Securitization enabled mortgage originators to move mortgage loans off their balance sheets, freeing up capital for additional lending. Since then, it has expanded
to other industries, and, by 2002, had grown to a size of more than 2 trillion dollars (see Cummins (2004)). Examples of securitization, besides mortgage-backed securities (MBS), are automobile loans, credit card receivables, commercial mortgage loans, collateralized debt obligations (CDOs), etc. In all these cases, there are two aspects to the securitization structure: pooling and tranching. To take the example of CDOs, the basic structure is that a financial intermediary sets up a special purpose vehicle (SPV) that buys a portfolio of debt instruments - bonds and/or loans - and adds credit derivatives on individual “names.” This is the pooling that was referred to earlier. The SPV then issues various claims against the pooled portfolio, which enjoy different levels of seniority: e.g., a high-grade AAA claim, which has a negligible, virtually zero, probability of not meeting its promised payment; a medium-grade claim, say rated BBB+, which has a low but not negligible probability of such default; and an equity security, which is viewed as risky.\(^1\) This is referred to as tranching.

Traditionally, in the academic literature, the benefits of securitization have been discussed in the context of transaction costs and information asymmetry. In this paper, we consider an efficient market with no explicit problems of information asymmetry or transaction costs, and focus solely on the implications of market incompleteness for securitization. We explain the two aspects of securitization, pooling and tranching, by value creation in incomplete markets. For example, in the case of CDOs, market incompleteness implies that some states associated with poor, mediocre, or good performance by the group of firms whose bonds are being pooled, cannot be spanned by the available securities in the market. This creates an incentive for an intermediary to purchase a portfolio (pool) of junk bonds and then issue claims against the pool in various categories (tranches). The question that arises is: What is the optimal pooling and tranching strategy for the financial intermediary that contracts to purchase the CDOs? In other words, what debt instruments should be pooled and what tranches should be created to extract the maximum surplus from the transactions? Finally, because securitization in practice is affected by market incompleteness as well as information asymmetry, we compare our results with the types of tranches observed in practice.

Our results can be applied to other examples of financial intermediation in the context of market incompleteness, such as the choice of investments by a venture capitalist along with the optimal mix of claims issued against them to its investors, or the optimal asset-liability mix of a bank. They

\(^{1}\)Typically, there are more than three tranches in a CDO structure, but three would suffice to explain the essential principles involved. This issue is discussed in more detail in §5.
can also be applied to traditional corporate financial problems such as mergers and acquisitions, optimal financing, and the valuation of real options. We discuss a few of these examples in §6.

This paper is organized as follows. Section 2 reviews the related literature on incomplete markets and securitization. Section 3 presents the model setup and assumptions. Section 4 analyzes the conditions under which there is value in pooling and firms willingly participate in the creation of the asset pool. Section 5 analyzes the conditions under which there is value in pooling and tranching, and determines the optimal tranching strategy, and §6 concludes with a discussion of the implications of our analysis.

2 Literature Review

Our research is related to two strands in the literature. The first is the literature on valuation of assets when markets are incomplete, i.e., when the available securities do not span the state-space. The second is the literature on securitization, i.e., the issuance of securities in the capital market that are backed or collateralized by a portfolio of assets.

2.1 Valuation in Incomplete Markets

The approaches that have been proposed for pricing assets in incomplete markets fall into three categories. These are arbitrage free pricing, preference-based pricing, and pricing based on approximate arbitrage arguments. Under the arbitrage-based approach of Harrison and Kreps (1979),\textsuperscript{2} claims that cannot be spanned by the assets traded in the market, i.e., unspanned claims, cannot be priced uniquely. However, upper and lower bounds can be derived that preclude arbitrage opportunities. In the second approach, restrictions are placed on preferences (or the distributions of returns) to get exact prices, such as in the standard capital asset pricing model.\textsuperscript{3} Both the preference-based and arbitrage-based approaches to pricing have their respective shortcomings. The arbitrage-based approach can be faulted for yielding price bounds that are too wide and the preference-based approach can be criticized as being too specific and subject to misspecification error.

\textsuperscript{2}See also Ross (1976) and John (1981).

\textsuperscript{3}Another example of this approach is the literature on option pricing using such preference restrictions. See, for example, Perrakis and Ryan (1984), Levy (1985), Ritchken (1985), Ritchken and Kuo (1989) and Mathur and Ritchken (1999).
Recent research has focused on the middle ground between the two approaches either by imposing restrictions based on the reward-to-risk ratio, i.e., the Sharpe ratio, or by combining the arbitrage-based and the preference-based approaches. Examples of this hybrid approach are: Shanken (1992) who treats an investment opportunity with a high, but finite Sharpe ratio as an “approximate arbitrage,” Hansen and Jagannathan (1991) who show that a bound on the maximum Sharpe ratio is equivalent to a bound on the variance of the pricing kernel, Cochrane and Saa-Requejo (2000)’s good-deal bounds, and Bernardo and Ledoit (1999, 2000), who define approximate arbitrage as a zero-cost investment opportunity with a high ratio of expected gains to expected losses.\footnote{Other restrictions on the set of pricing kernels have been studied in the literature, for example, Snow (1992) who derives restrictions on the q-th moment of the pricing kernel; and Stutzer (1993) who presents an alternative restriction on the entropy of the pricing kernel by restricting the maximum expected utility attained by a CARA agent.}

We use the first of the above three valuation approaches, which is based on arbitrage-free pricing. However, it should not be construed that we are advocating only this approach. Instead, we believe that our methodology shows how the set of projects that can be undertaken in an incomplete market broadens due to intermediation. Indeed, one can derive an alternative formulation of our framework, yielding more specific conclusions, if we impose the additional restrictions on investor preferences or the reward-to-risk ratio in the market.

We illustrate these ideas with an example: Consider a firm that wishes to undertake a project requiring an immediate investment, which results in uncertain and unspanned cash flows in the future. From the preceding discussion, it is clear that the firm cannot place an exact value on its cash flows, based on arbitrage-free pricing. The ambiguity regarding the value of the project is the source of the problem considered in this paper. We do not rule out the possibility that adopting one or the other more specific approaches described above might resolve the decision problem unequivocally. However, in case one wishes to use an arbitrage-free framework, the question we ask is whether an intermediary can enhance the “value” by pooling assets from different firms and tranching them for sale to investors.

### 2.2 Securitization

Research on securitization has focused on the rationale for the widespread use of pooling and tranching in the asset-backed securities market. This rationale is explained through market imperfections,
such as transaction costs and information asymmetry. Specific examples of securitization include the literature on “supershares” (i.e., tranches of the portfolio of all securities in the market), primes and scores (i.e., income and capital gains portions of a stock), and “bull” and “bear” bonds. More recently, specific examples of securitization have been analyzed by researchers, e.g., the assets of insurance companies (Cummins (2004)), and those of firms in financial distress (Ayotte and Gaon (2004)).

Two alternative economic explanations have been proposed in the literature for the securitization of assets. The first relates transaction costs to the welfare improvement that can be achieved by designing, creating and selling securities to meet the preferences of particular clienteles of investors or issuers. The other type of explanation has to do with some aspect of information asymmetry and the ability of an financial intermediary to reduce the agency costs resulting from it. The first type of explanation is typified by Allen and Gale (1991), who examine the incentive of a firm to issue a new security when there are transaction costs. Allen and Gale examine the incentive of a firm to issue claims that increase the spanning of states. They study an exchange equilibrium that results in an incomplete market. In their model, firms do not behave as price takers and also incur a cost of splitting the return from their asset into financial claims. Clearly, if the value of the firm is unaffected by splitting the return, as is the case in a complete market, the firm has no incentive to do so. In an incomplete market, it may be possible to split the return into financial claims, i.e., innovate, and benefit from selling the new claims to investors. But not every firm needs to innovate. Even if a single firm amongst many similar ones, or a financial intermediary, for that matter, does so, investors can benefit if short sales are permitted. The new claims result in readjustment of consumption by investors, which, in turn, leads to a change in asset prices that may benefit the firm. There are two implications of this: a) the ex-post value of similar firms may be equal, thus reducing the incentive of any one firm to innovate, and b) the firm has an incentive to innovate new claims only if the prices change, i.e., if competition is imperfect.

Despite some similarities with Allen and Gale (1991), the model we develop in this paper is different in many ways. First, our model does not use a general equilibrium approach. The reason is that we are interested in obtaining more specific results without considering the complex feedback effects that a general equilibrium analysis would entail. Second, we also does not explicitly model

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5See Hakansson (1978), and Jarrow and O’Hara (1989) for details. For example, Hakansson (1978) argues that options or supershares on the market portfolio improve the allocational efficiency of an existing market structure, even if the market portfolio itself is not efficient.
the cost of issuing claims, since we wish to focus on value creation in a frictionless market. Third, our aim is to explicitly introduce a third type of agent into the exchange equilibrium and study how firms can benefit from intermediation. Moreover, in our framework, the problem for firms is not just whether to issue new claims against returns from existing assets; rather, the problem is also to decide whether to undertake new projects. In order to study the effect of intermediation and whether it helps more firms to undertake investments (or firms to invest in more projects), we have to necessarily limit short sales by investors - otherwise, firms and investors can intermediate. Therefore, we confine the financial innovation activity to designated financial intermediaries.

Many researchers have studied the effect of information asymmetry between issuers and investors in the context of securitization [see, for example, DeMarzo and Duffie (1999), DeMarzo (2005) and Leland and Pyle (1977)]. Pooling assets is considered beneficial to both an uninformed issuer and an uninformed investor. The benefit to an uninformed issuer is that it reduces the issuer’s incentive to gather information (Glaeser and Kallal (1997)). The benefit to uninformed investors is that pooling reduces their adverse selection problem when competing with informed investors (DeMarzo 2005).

In this context, Subrahmanyam (1991) shows that security index baskets are more liquid than the underlying stocks. DeMarzo (2005) also shows that an informed issuer (or intermediary) does not prefer pure pooling, because it destroys the asset-specific information of the informed issuer. Instead, an informed intermediary prefers pooling and tranching to either pure pooling or separate asset sales because pooling and tranching enable an intermediary to design low-risk debt securities that minimize the information asymmetry between the intermediary and uninformed investors. DeMarzo calls this the “risk diversification effect” of pooling and tranching. Pooling and tranching are also beneficial to uninformed investors. For example, Gorton and Pennachi (1990) show that uninformed investors prefer to split cash flows into a risk-less debt and an equity claim.

To summarize, the differences between our paper and the prior work on valuation as well as securitization are as follows. Even though our work in this paper is based on the arbitrage-free pricing approach, we restrict the ability of some agents to take advantage of arbitrage opportunities. In our framework, only those who are designated as financial intermediaries are able to take full advantage of these opportunities. Also, our model assumes an incomplete securities market, but we do not consider issues relating to transaction costs and information asymmetry, except in the indirect sense that the financial intermediaries in our model can undertake certain transactions that other agents cannot.
The second difference is that our paper analyzes the simultaneous enhancement of the expected utility of investors and the expansion of the set of value increasing projects undertaken by firms, due to the intermediaries’ intervention through the securitization of unspanned claims. We, therefore, consider multiple firms that have different assets (unlike Allen and Gale, who consider multiple firms that have the same asset). We relate securitization to the problem of financing projects and also to satisfying the needs of investors. Therefore, we examine the problem both from the supply and the demand side.

3 Model Setup

We consider an Arrow-Debreu economy in which time is indexed as 0 and 1. The set of possible states of nature at time 1 is \( \Omega = \{ \omega_1, \omega_2, \ldots, \omega_K \} \). For convenience, the state at time zero is denoted as \( \omega_0 \). All agents have the same informational structure: The true state of nature is unknown at \( t = 0 \) and is revealed at \( t = 1 \). Moreover, the \( K \) states are a complete enumeration of all possible events of interest, i.e., the subjective probability of any decision-maker is positive for each of these states and adds up to one when summed over all the states.

3.1 Securities Market

We start with a market in which \( N \) primary securities are traded via a financial exchange. Security \( n \) has price \( p_n \) and payoff \( S_n(\omega_k) \) in state \( k \). These securities are issued by firms and purchased by investors through the exchange. The securities market is arbitrage-free and frictionless, i.e., there are no transaction costs associated with the sale or purchase of securities. To keep the analysis uncluttered, cash flows are not discounted, i.e., the risk-free rate of interest is zero.

From standard theory, the absence of arbitrage is equivalent to postulating that there exists a set, \( \Theta \), of risk neutral pricing measures over \( \Omega \) under which all traded securities are uniquely priced, i.e., \( E_q[S_n] = p_n \), for all \( n \) and for all \( q \in \Theta \). It is well known that the set \( \Theta \) is spanned by a finite set of independent linear pricing measures. These are labelled \( \{ q_l, l = 1, \ldots, L \} \). In particular,

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6The model described below can be extended to a multi-period setting with some added complexity in the notation. However, the basic principles and results derived would still obtain.

7A linear pricing measure is a probability measure that can take a value equal to zero in some state, whereas a risk neutral probability measure is strictly positive in all states. Thus, the set \( \Theta \) is the interior of the convex set spanned by the set of independent linear pricing measures. The maximum dimension of this set equals the dimension of the solution set to a feasible finite-dimensional linear program, and thus, is finite. See Pliska (1997).
when the set $\Theta$ is a singleton the market is complete, else it is incomplete.

We need the following notation in the sequel. Not every claim can be priced uniquely in an incomplete market. When a claim cannot be priced uniquely, the standard theory provides bounds for the price of a claim $Z$ that pays $Z(\omega_k)$ in state $k$. Let $V^-(Z) = \max\{E[S] : S \leq Z, S \text{ is attainable}\}$, and let $V^+(Z) = \min\{E[S] : S \geq Z, S \text{ is attainable}\}$. $V^-(Z)$ and $V^+(Z)$ are well-defined and finite, and correspond to the lower and upper bound on the price of the claim $Z$. Given that the set $\Theta$ is spanned by a finite set of independent linear pricing measures labelled $\{q_l, l = 1, \ldots, L\}$, this can be formalized in the following Lemma. (All proofs are in the Appendix.)

**Lemma 1.** (i) $V^+(Z) = \max_{l \in L} E_{q_l}[Z]$.

(ii) $V^-(Z) = \min_{l \in L} E_{q_l}[Z]$.

(iii) If the payoffs from the claim $Z(\omega_k)$ are non-negative in all states, then these bounds are unaffected by the inability of agents to short sell securities.

This lemma is needed for several proofs in the Appendix.

### 3.2 Agents

We consider three types of agents in our model: investors, firms, and intermediaries. Investors are utility maximizers. Their decision problem is to construct a portfolio of primary securities (subject to budget constraints), so as to maximize expected utility. Investors can buy securities, but cannot short or issue securities. Firms own (real) assets and issue primary securities that are fully backed by the cash flows from these assets. Firms can also create new assets and sell claims against the cash-flows from these assets to intermediaries. They negotiate with intermediaries to get the highest possible value for their assets that is consistent with the prices prevailing in the financial market. Intermediaries facilitate transactions between firms and investors by repackaging the claims purchased from the firms and issuing secondary securities traded on the over-the-counter securities market. We stipulate that the claims sold by firms to the intermediaries must be fully backed by their asset-cash flows, and the claims issued by the intermediaries should be fully backed by the assets purchased from firms. We also do not allow for short sales by any agent. These assumptions enable us to isolate the roles of the three types of agents, and explicitly study the phenomenon of securitization through the intermediaries. A secondary reason for these assumptions is to avoid transactions that permit default in some states, because that would lead to complex questions relating to bankruptcy and renegotiation, that are outside the purview of this paper.
Having broadly described the agents, we set out the details of their decision making problems as below.

**Investors:** We model investors by classifying them into investor types. The set of investor types is finite and denoted as $I$. Each investor of type $i$ has endowments $e_i(\omega_k)$ in state $k$. The utility derived by type $i$ investors is given by a von Neumann-Morgenstern function $U_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$. $U_i$ is assumed to be concave, strictly increasing and bounded above. Investors maximize their expected utility, subject to the constraint that consumption is non-negative in every state.

Denote the consumption of type $i$ investors in state $k$ as $x_{ik}$ and let the subjective probability of state $k$ be $P_i(\omega_k)$. Then, the investor derives expected utility equal to $\sum_{k=0}^{K} P_i(\omega_k) U_i(x_{i0}, x_{ik})$. The portfolio of primary securities held by a type $i$ investor is denoted as the $N$-tuple of real numbers $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iN})$, where $\alpha_{in}$ is the amount of security $n$ in the portfolio. The type $i$ investor’s decision problem can be written as

$$\max \sum_{k=0}^{K} P_i(\omega_k) U_i(x_{i0}, x_{ik})$$

subject to

$$x_{ik} = e_i(\omega_k) + \sum_{n=1}^{N} \alpha_{in} S_n(\omega_k), \quad \forall \ k = 1, 2, \ldots, K$$

$$x_{i0} = e_i(\omega_0) - \sum_{n=1}^{N} \alpha_{in} p(n)$$

$$x_{ik} \geq 0, \quad \forall \ k = 0, 1, 2, \ldots, K$$

$$\alpha_{in} \geq 0, \quad \forall \ n = 1, 2, \ldots, N.$$  

The first constraint equates the consumption in each state at time 1 to the cash flow provided by the portfolio and the endowment. The second specifies the budget constraint for investment in primary securities at time 0. The third constraint specifies that the cash flow in each state at time 1 should be non-negative, while the last one restricts the investor to zero or positive holdings of the primary securities, since short-selling is prohibited.

Denote the derivatives of $U_i$ with respect to $x_{i0}$ and $x_{ik}$, $k \geq 1$ as $U_{i1}$ and $U_{i2}$ respectively. We shall assume, as customary, that the current period consumption is strictly bounded away from zero for investor types. It follows that, at optimality,

$$\sum_{k=1}^{K} P_i(\omega_k) \frac{U_{i2}(x_{i0}, x_{ik})}{\sum_{k=1}^{K} P_i(\omega_k) U_{i1}(x_{i0}, x_{ik})} S_n(\omega_k) \leq p_n.$$
Here, we obtain an inequality because of short sales restrictions. Investors can either hold positive or zero amounts of a primary security. In cases where they would like to sell a security, but cannot due to the short sales constraint, they do not hold it at all. The inequality suggests that, in state $k$, type $i$ investors are willing to buy an infinitesimal amount of consumption at a price, $m_{ik}$ given by

$$m_{ik} = P_i(\omega_k) \frac{U_{i2}(x_{i0}, x_{ik})}{\sum_{k=1}^{K} P_i(\omega_k)U_{i1}(x_{i0}, x_{ik})}.$$ 

These values serve as state prices (also called reservation prices) of primary securities for investors. We require that each security be present in the optimal portfolio of at least one investor type.

We note that the reservation price for an unspanned state may differ amongst investor types due to the incompleteness of the market. We further assume that an investor of type $i$ is willing to buy not only consumption that is specific to state $k$, but also secondary securities issued by the intermediary if the price of the secondary security is below that given by valuing its state dependent cash flows, using the investor’s reservation prices.

**Firms:** Firms maximize the time 0 expected values of their investments. Firm $j$ can create an asset that is unique to it, and that provides a positive cash flow of $X_j(\omega_k)$ in each state $k$, at time $t = 1$. The firm can sell claims issued against $X_j$ to the intermediary. Claims issued against $X_j$ should be fully backed by $X_j$; in other words, their sum should not exceed the value of $X_j$ in any state of nature. We assume that firm $j$ has a reservation price $r_j$ on $X_j$. The reservation price could be comprised of financial, physical and transaction costs, as well as opportunity costs of the key decision-makers of the firm required to create the asset. The firm invests in the asset, if the net present value, given by the difference between the selling price offered by the intermediary and the reservation price, is positive. Additionally, firms cannot trade with other firms directly and also cannot issue claims that are not fully backed by their assets. Let $J$ denote the number of firms that wish to undertake investment projects at time 0.

We assume that the total cash flow available from this set of firms in any state $k$, $\sum_{j=1}^{J} X_j(\omega_k)$, is small relative to the size of the economy. Each firm, therefore, behaves as a price-taker in the securities market. However, when the asset cannot be priced precisely, it negotiates with the intermediary for obtaining the highest possible price for securitization of the asset.

**Intermediaries:** Intermediaries are agents who have knowledge about the firms’ and investors’ asset requirements. Notice that such knowledge is different from receiving a private signal regarding
the future outcome. The intermediaries purchase assets from firms and repackage them to sell to investors. They seek to exploit price enhancement through securitization operations that increase the spanning of available securities. They use this superior ability to negotiate with the firms for the prices of their assets. They use the knowledge about the investors’ preferences to create new claims and price them correctly. An important aspect of the model is that intermediaries act fairly by paying the same price for a claim, independent of which firm is selling it to the pool, and charging the same price for the same product even though it is sold to different customers. The rationale for these fairness requirements is the possibility of entry and competition from other intermediaries. However, we do not explicitly model competition amongst intermediaries beyond imposing the fairness requirements (and participation constraints by firms that are discussed below). Hence, in what follows, we consider the securitization problem from the viewpoint of a single intermediary. The intermediary’s decision problem is the main topic of this paper and, therefore, analyzed separately in the next section.

### 3.3 Definition of the Securitization Problem

Intermediaries purchase claims from firms, pool them and package them into different tranches and sell them as collateralized secondary securities. Pooling is defined as combining the cash flows from claims issued by different firms. Tranching is defined as splitting this portfolio into sub-portfolios to be sold to different groups of investors, with the constraint that the sub-portfolios be fully collateralized, i.e., fully backed by the claims purchased from the firms. We study the problem for an intermediary in the following sequence:

1. The intermediary considers the set of claims available from the firms and determines how much to purchase of each claim, and at what price. It ensures that the prices offered to the firms are fair, i.e., the price paid for a claim is the same whether it is purchased from one firm or another. Initially, we assume that firms sell claims to the intermediary if their reservation price is met or exceeded. We study whether the pooling of assets creates value in the following sense: Does pooling lead to a superset of projects being financed, compared to the case when there is no intermediary? We then examine whether firms can be induced to pool their assets by formulating the firms’ problem as a cooperative game. We study the pooling problem in §4.

2. We study the tranching problem in §5. The intermediary determines the optimal tranches
to construct from the pooled asset and the prices at which to offer these tranches to the investors. Similar to fairness to firms, fairness to investors can be defined by the criterion that the same security sold to different investors has to be sold at the same price. We show that this condition need not be enforced at the outset, and that the optimal tranching solution automatically ensures fairness to investors. We also compare the value of pure pooling with the value of pooling and tranching.

An important assumption of our analysis is related to the previous literature, namely, the price-taking behavior of the sellers of claims. The problem of valuing a new asset that is introduced into an incomplete market goes beyond the valuation of assets. We require that firms and intermediaries use prices that prevail before the introduction of the new asset in their decision making. Even in a complete market, such an action might cause the prices of other assets to change. It is possible to derive the new equilibrium prices and consumptions by considering the model in a general equilibrium framework. However, this would be too general to obtain useful insights into the securitization aspects of the problem. Hence, the assumption made by us and others is that the firm is a price-taker, that is, the number of investors of each type is large, and that the cash flows \( \{X_j\} \) available from the firms are small compared to the demand from the investors. Therefore, prices and consumptions are undisturbed by the issuance of new claims against \( \{X_j\} \). This assumption, which implies that our equilibrium is partial in nature, enables us to check for arbitrage conditions at the margin, and determine prices at which to securitize claims such that the intermediary and the firm earn a profit and the investors are better off.

4 Value of Pooling

We attribute the role played by the intermediary to two factors: the value enhancement provided by pooling alone, and the incremental value provided by tranching. In this section, we consider the former. We analyze the problem of pooling the cash flows of some or all firms and valuing the pooled asset by replicating its cash flows in the securities market. We use the lower bound \( V^-(\cdot) \) as a measure of value, and thus, compute the lowest price at which the pooled asset can be sold under the no-arbitrage conditions. We say that there is value to pooling, if the lower bound on the market price of the pooled assets exceeds the sum of the reservation prices of the firms that have contributed to the asset pool, even though some of the lower bounds on the individual firms' claims may be less than the corresponding reservation prices.
We first characterize the conditions under which there is value in pooling. Consider any given firm \( j \). If \( r_j \leq V^-(X_j) \) then, clearly, firm \( j \) can profitably invest in asset \( X_j \) without pooling. If \( r_j \geq V^+(X_j) \), then it does not make sense for the firm to invest in the asset \( X_j \). The interesting case is the one where \( V^+[X_j] \geq r_j \geq V^-(X_j) \) because in this case, the decision to invest in \( X_j \) is ambiguous. Let the cash flows of the pooled asset be given by \( X(\omega_k) = \sum_j X_j(\omega_k) \) for all \( k \). Clearly, we have \( V^-(X) \geq \sum_j V^-(X_j) \).

In words, pooling improves the spanning of cash flows across states, and thus, provides value enhancement. However, we still need to consider the reservation prices of firms to determine if pooling reduces the ambiguity regarding investment in assets. We say that pooling creates additional value if there is a linear combination of assets with weight \( 0 \leq \alpha_j \leq 1 \) for asset \( j \) such that even though \( \sum_j V^-(X_j) < \sum_j r_j \), we obtain \( V^-(\sum_j (\alpha_j X_j)) \geq \sum_j (\alpha_j r_j) \).

Theorem 1 determines the necessary and sufficient condition under which there is value in pooling.

**Theorem 1.** (i) If there is a \( q \in \Theta \) such that \( r_j \geq E_q[X_j] \forall j \), then additional value cannot be created by pooling the \( X_j \)'s.

(ii) Conversely, if there is no \( q \in \Theta \) such that \( r_j \geq E_q[X_j] \forall j \), then additional value can be created by pooling the \( X_j \)'s.

Theorem 1(i) states that if there exists a *common* benchmark pricing measure, say \( q \in \Theta \), such that the expectation under \( q \) of every firm’s cash flows is smaller than the reservation price then additional value cannot be created through pooling. Theorem 1(i) also suggests how pooling enlarges the set of projects that can be undertaken profitably. To see this, note that without pooling, firm \( j \)'s project cannot be taken up unambiguously if \( r_j \geq V^-(X_j) \equiv E_{q_j}[X_j] \), where \( q_j \) denotes the linear pricing measure that attains \( V^-(X_j) \). However, with pooling, we only require the expected value of the project under the common pricing measure \( q \) to be less than the reservation price \( r_j \) for all \( j \) to definitely assert that there is no scope for additional value creation.

If the condition in part (i) of the theorem fails to hold, then part(ii) states the positive part of the result, that is, there is scope for enhancing value through pooling. The key to proving this result is that if, for each pricing measure \( q \in \Theta \), \( r_j \leq E_q[X_j] \) for at least one firm \( j \), then it is possible to show that we can create a pooled asset such that the lower bound on the value of this

---

The left hand side is given by minimizing the sum of the cash flows from all assets over the set of probability measures; whereas the right hand side the sum of the minimum of each individual cash flow. The minimum of the sum is always larger than or equal to the sum of minimums.
asset exceeds the total of reservation prices involved in creating the asset. In other words, there
exists a vector of weights, \((\alpha_j)\), where \(\alpha_j \geq 0\) for all \(j\) and \(\sum_j \alpha_j = 1\), such that for the pooled
asset given by \(\sum_j \alpha_j X_j\), we have \(V^- (\sum_j \alpha_j X_j) \geq \sum_j \alpha_j r_j\). The existence of weights \((\alpha_j)\) shows
that the conditions in Theorem 1 are necessary and sufficient to characterize the value of pooling
when the investment opportunities are scalable.\(^9\)

**Remark:** When the investment opportunities are non-scalable, even when Theorem 1(i) does not
hold, there may not be value in pooling since fractional values of \((\alpha_j)\) cannot be realized. This is
due to the restriction placed on the set of feasible solutions due to the integer constraint.

The important implication of Theorem 1 is that firms can undertake more projects when pooling
is facilitated by intermediaries. However, even when the value of the pooled asset exceeds the sum of
the reservation prices, i.e., even when there are sufficient profits in the pool to meet the reservation
prices of the firms, the firms may still not willingly participate in the asset pool. For example, this
may be the case if one or more firms have very high reservation prices and the rest of the firms
are better off keeping such firms out of the pool. This naturally leads to the following questions:
what is the highest value of the pooled assets that can be obtained by selling in the market; for
what reservation prices is there sufficient incentive to pool assets; can we characterize the set of
reservation prices that permits pooling of all assets; and can a fair price be set for each \(X_j\)? Theorem
2 below answers some of these questions fully and some partially.\(^{10}\)

We formulate the firms’ participation problem as a cooperative game. Let \(J_\alpha\) denote a subset
of the set of all firms, \(J\), and \(J_\alpha^c \equiv J - J_\alpha\) denote the complement of \(J_\alpha\). Also, let \(X(J_\alpha) = \sum_{j \in J_\alpha} X_j\). We consider the cooperative game in which the value of each coalition, \(V(J_\alpha)\), is
defined as \(V^- (X(J_\alpha))\). Following standard terminology, there is a solution to this game, i.e., its
core is non-empty, if the grand coalition of all firms cannot be blocked. The theorem below provides
sufficient conditions for the core of the game to be non-empty as well as conditions that guarantee
that payments can be made to the firms to cover their reservation prices.

**Theorem 2.** (i) If there are no reservation prices, then the core of the game is non-empty.

(ii) Assume that Theorem 1(ii) holds. Then the core is non-empty in the cooperative game

\(^9\)We still require an upper bound on the size of each investment to respect the assumption of price-taking behavior
of firms. However, this does not influence the choice of \((\alpha_j)\).

\(^{10}\)Adding firms to the pool decreases the losses due to payoffs that are not spanned by the market (reduces the
mismatch between marketable claims and the pooled asset). However, adding firms imposes additional costs, so that
the asking price may exceed the enhancement in value.
stated above and all firms can be paid more than their reservation prices if and only if for every
subset \( J_\alpha \) of \( J \) we have \( V(J) \geq \max(V(J_\alpha), \sum_{j \in J_\alpha} r_j) + \max(V(J^c_\alpha), \sum_{j \in J^c_\alpha} r_j) \).

Theorem 2(i) follows from a result due to Owen (1975) and Samet and Zemel (1984) that games
with the given structure always have a non-empty core. The ability to support the core comes from
the fact that each firm brings cash flows with some associated value to the “market.” In games of
this type, the value function is superadditive. The proof consists of showing this, i.e., that the value
of the pool can be partitioned using an allocation mechanism such that every coalition obtains a
higher value than it would do on its own. This result is proved using duality theory. An important
aspect of the result is that the dual variables in the optimal solution give a pricing measure that
can be used to price each firm’s asset in the pool. We elaborate on this solution in Corollary 1
below.

Theorem 2(ii) states that the necessary condition for the payments to firms to support the core
is also sufficient to guarantee its existence. The necessary condition is that there is enough money
to pay any coalition their value, \( V(J_\alpha) \), or the sum of the reservation prices, \( \sum_{j \in J_\alpha} r_j \), whichever
is greater without affecting the ability to provide a similar compensation to the firms not in the
coalition. It is important to differentiate this from the payment suggested by the solution to the
dual problem, alluded to above, which is a “linear” payment scheme, i.e., the payment is a linear
function of the cash flows in different states of the world. The dual solution need not satisfy the
individual firm’s requirement that the payment cover its reservation price. Therefore, even though
the total payment to all firms is determined by the optimal solution to the primal (or dual) its
break-up for individual firms has to be determined by solving another optimization problem. If this
problem is feasible, it is not certain whether all firms are better off, or some firms will prefer to be
left out of the coalition. The condition in part (ii) of the theorem is both necessary and sufficient
to ensure that all firms are better off when they participate in creating the pool.

The necessary part of Theorem 2(ii) is immediate, because under every solution in the core,
each coalition \( J_\alpha \) gets at least \( \max(V(J_\alpha), \sum_{j \in J_\alpha} r_j) \). If this condition does not hold, then either
some coalition does not get its value (and can do better on its own) or the payment to the firms in
some coalition cannot cover the sum of the reservation prices. Therefore, the condition is necessary
for firms to participate in the asset pool. The surprising conclusion is that the condition is sufficient
as well.

There is an unanswered part of the questions above whether the scheme for paying firms is
simple and fair. The theorem does not address this: The theorem states that the core is non-empty and it can support payments to all firms to cover their reservation prices. It turns out that it is very difficult to work with the fairness condition in full generality. Moreover, even if all the inequalities in part (ii) of Theorem 2 are satisfied, the payment scheme is yet to be determined. For example, a complex, non-linear pricing scheme may ensure that the core is non-empty. However, if we restrict ourselves to payments made using a linear pricing measure, then we can sharpen the results. A case can be made that if all firms are paid the same price for a unit cash flow in state $k$, then the scheme is surely fair.

**Corollary 1.** If the payment made to firms for their assets needs to be computed using the same linear pricing measure, then the core is non-empty when

(i) A pricing measure $q_p$ can be found which is either an extreme point of the set of risk neutral probability measures, or a convex combination of such extreme points, such that $\sum_j E_{q_p} X_j = V - (\sum_j X_j)$ and

(ii) The reservation prices satisfy: $r_j \leq E_{q_p} X_j$.

The interpretation of Corollary 1 is that the value enhancement provided by pooling can be characterized as the change in the pricing measure necessary to value the assets correctly. This is readily seen by assuming that $r_j \geq V - [X_j] = E_{q_j} [X_j]$, that is, firm $j$ cannot decide whether to invest in the project based on the minimum valuation. Notice that the measure to determine the minimum value of each firm’s asset, $q_j$, depends on the cash flow of the asset itself. This provides the lower bound on the value. The measure to determine the value when the project is considered to be part of the asset pool depends on the cash flow of the entire asset pool. This yields a higher value. The firm gains when the reservation price lies within these two bounds. Moreover, when we are restricted to compensate firms using the same pricing measure, we are assured that the gain from pooling can be used to induce all firms to participate when $r_j \leq E_{q_p} X_j$. There are other interesting aspects to the corollary. The scheme is fair because it uses the same pricing formula for each firm. The measure also prices the traded securities correctly. Thus, the firms can get a market benchmark to assure themselves that the intermediary is fair.
5 Value of Pooling and Tranching

In this section, we consider the problem of maximizing the value of the asset pool when, in addition to permitting sales of asset-backed claims in the financial market, the intermediary can also sell tranches directly to investors. The tranches may be replicas of primary securities already traded in the securities market, or new secondary securities issued to investors by the intermediary. The objectives of the securitization problem are to

1. determine the number of tranches into which the pooled assets will be divided;
2. design the tranches and price them;
3. determine if all firms will participate in the pool;
4. determine if each firm is better off with tranching to investors compared to tranching as primary securities, $S_1, \ldots, S_n$, to be sold in the securities market.

In general, the cash flows $\sum_j X_j$ can be split into several tranches, and each tranche offered to every investor type. However, the following observation simplifies this problem considerably and enables us to solve for the optimal tranches.

**Theorem 3.** Given a pool of assets, the optimal tranching strategy is obtained by solving:

$$V_T(J) = \max \sum_i \sum_k m_{ik}Y_{ik} + \sum_n p_n\beta_n$$

such that

$$\sum_i Y_{ik} + \sum_n \beta_nS_{nk} \leq \sum_{j \in J} X_{jk} \text{ for all } k$$

$$Y_{ik} \geq 0, \ \beta_n \geq 0 \text{ for all } i, k, n.$$  

where $Y_{ik}$ is the tranche for investor type $i$, and $\beta_n$ is the quantity of the tranche sold as primary security $n$.

In the optimization problem, $Y_i$ are the tranches created and sold as secondary securities, $Y_{ik}$ is the payoff from tranche $Y_i$ in state $k$, and the decision variables $\beta_n$ are quantities of tranches sold as primary security $n$. Recall that $m_{ik}$ denotes the state price of investor type $i$ for a unit consumption in state $k$. The objective is to maximize the combined value of these tranches. The first set of constraints specifies that the tranches should be fully backed by the asset pool. The
second set of constraints specifies that short sales are not allowed and the tranches should only have positive components. The latter is justified by recalling that we do not want the consumption to be negative in any state.

The main idea of Theorem 3 is that we can model the tranche design problem as a linear program. Otherwise, without this result, it is a complex non-linear optimization problem that provides no hint as to the nature of its solution. The theorem provides a useful relationship between the degree of dissimilarity between investor types and their ability to sell short. If the state prices are the same for all investors, then the intermediary should create a single tranche (which is optimal, essentially, if transaction costs, which are not explicitly modelled here, increase with the number of tranches). It can be shown that the state prices will be equal across investors if investors are permitted short sales. Therefore, the creation of multiple tranches not only signals market incompleteness (in complete markets, state prices will equalize) but also indicates the inability of investors to short sell.

Clearly, the value of pooling and tranching is always greater than or equal to the value of pooling alone, since the value of pooling can be obtained from the above linear program by setting $Y_{ik}$ to zero for all $i, k$. There are two sources of the incremental value obtained from tranching: (i) due to satisfying demand for cash flows that cannot be obtained using primary securities, (ii) due to differences in reservation prices amongst investors. The following corollary answers questions 1 and 2 posed at the beginning of this section.

**Corollary 2.** (i) The maximum value of the pool is

$$
\sum_k \left\{ \max_i m_{ik} \right\} \sum_j X_{jk}.
$$

(ii) The optimal tranches can be assumed to form a partition of the asset pool, i.e., if $Y_{ik} \geq 0$ for some investor type $i$ and state $k$, then $Y_{i'k} = 0$ for all $i' \neq i$. Thus, the cash flows in each state $k$ need not be split amongst multiple tranches.

The first part of the above corollary states that the value of the pool is maximized by selling the claim in each state to the investor type that values it the most. The second part shows that the pooled asset is neatly split across investor groups. No cash flow in a state is split amongst more than one investor group and all cash flows are sold to some investor group. The surprising conclusion is that the tranches can be assumed to partition the asset pool. This is a sharp result that is in
keeping with the assumption that the investors and the intermediaries are fully informed about the cash flows. Obviously, the number of tranches is related to the number of distinct investor types.

Comparing Corollary 2(i) with Corollary 1, we find that under pure pooling, the intermediary could only extract value from the component of the asset pool that could be priced in the securities market. However, due to the existence of demand for cash flows that cannot be obtained in the securities market, the intermediary is able to extract additional value from the entire pool through tranching. Moreover, due to differences between the reservation prices across investor types, the value of the pool is no longer determined by a pricing measure. This is so because the intermediary has the ability to package cash flows in each state to the investor type that values them the most.

Corollary 2(ii) gives the structure of the optimal tranches. We find that when there are differences between reservation prices across investor types, then the intermediary may not strip away any marketable component of the cash flows. Instead, it prefers to create new secondary securities. In general, the intermediary would create more than one tranche (secondary security), each tailored to an investor type. We may even hypothesize that the number of distinct tranches indicates the number of ‘broad’ investor classes. In practice, we do find that typical securitization structures have three or more tranches, with the senior tranche typically being sold to a risk-averse investor like an insurance company, the mezzanine tranche to a hedge fund, and the most junior tranche being retained by the structuring investment bank, due to its upside potential.

However, there is one difference between the tranching observed in practice and the optimal solution derived in Corollary 2(ii). In practice, the cash flows in each state are divided among the tranches in order of their seniority, whereas in our solution, the cash flows in each state are sold only to the investor class that values them the most. The reason for this difference is that our results are derived in an economy without transaction costs or issues of verifiability or contractibility. In the presence of transaction costs and/or non-verifiability of individual states, it may be optimal to package two or more state-contingent claims into a security catering to a particular investor class. If investors are ordered by risk aversion, the tranches may range from those that pay off in the most states to those that pay off in only a few. There is a vast literature in corporate financial theory that has focused on this very issue. For example, Townsend (1979) and Gale and Hellwig (1985), among others, show that when the verification of states is costly, debt is an optimal financial contract for a form to issue. In a similar setting in the securitization problem analyzed above, we would observe a nesting of securities from those that pay off only in a few states, to those that pay
off in more and more. This roughly corresponds to the hierarchy of tranches observed in practice ranging from the senior-most to the junior-most based on their risk characteristics.

It is useful to examine the effect of allowing short sales on the tranching problem. If short selling is allowed, all investors would have identical reservation prices as mentioned above. Thus, \( \{ \max_i m_{ik} \} \) becomes a pricing measure, that is, its values in different states add up to one. In this case, the optimal solution reduces to a single tranche that is offered to all investors and another one that is sold in the securities market. The intermediary no longer derives any value by issuing multiple secondary securities for different investor types. Thus, the deviation of \( \{ \max_i m_{ik} \} \) from being a pricing measure indicates the degree to which short sales are a restriction on the decision problem of investors.

The finance literature contains many examples where basic arbitrage conditions are violated. An explicit version of this is in the context of martingale restriction on asset prices extracted from options, where Longstaff (1995), Brenner and Eom (2001) and many others have shown that this is violated for options on the stock market index. More recent work on the limits to arbitrage conjectures that such departures occur more indirectly, for example, through consistent violation of put-call parity, due to restrictions on short-sales (see Ofek, Richardson and Whitelaw (2004) as an illustration).

The conditions to determine whether each firm will participate in the pool are analogous to those in Theorem 2(ii). Specifically, each firm will participate in the pool if the following inequalities are satisfied: for every partition \( J_\alpha, J_\beta \) of \( J \), we have \( V^T(J) \geq \max(V^T(J_\alpha), \sum_{j \in J_\alpha} r_j) + \max(V^T(J_\beta), \sum_{j \in J_\beta} r_j) \). Once again these are complex constraints and also do not specify the payments to the firms. However, if we restrict ourselves to linear pricing kernels, then the kernel \( \{ \max_i m_{ik} \} \) can be used (see corollary below). It is indeed interesting to see that the deviation of the pricing kernels from being a measure goes towards value creation for firms.\(^{11}\)

There are counterexamples that show that not every firm is better off with pooling and tranching even though the total value of the asset pool increases. This is not surprising by itself. This is similar to classic examples in which some consumers get worse off when new securities are introduced to increase spanning. However, the following corollary provides a condition under which pooling and tranching is “Pareto improving” for the value of all firms compared to pure pooling, if we restrict

\(^{11}\)This is similar to the situation, where, in a market where short sales restrictions are systematically violated, an intermediary that steps in and replaces long stock positions with positions in the call and put options can create value.
ourselves to linear pricing kernels.

**Corollary 3.** All firms are better off with pooling and tranching than with pure pooling if

\[
\sum_k \left\{ \max_i m_{ik} \right\} X_{jk} \geq \max \left\{ E_{q_p} [X_j], r_j \right\},
\]

where \( q_p \) is the risk neutral probability measure defined in Corollary 1.

The corollary implies that if the state price vectors of the investors span a set that is as large as \( \Theta \), then surely no firm is worse off in terms of value. In fact, we only require the smallest hypercube that contains the set spanned by the state price vectors to contain \( q \). Therefore, it suggests that if state prices are the same for all investor types, then the likelihood of the corollary holding is smaller than when state prices differ. Additional value gets created in the latter case due to the increased spanning provided by the tranches.

### 6 Discussion

Our paper adds to the literature on valuation by examining how incompleteness causes the market to place a premium on assets or asset combinations that augment the set of traded claims. We illustrate this in the specific context of securitization.\(^\text{12}\) Our results show that there is a benefit from pooling if the reservation prices of firms are not simultaneously larger than the values of their assets under every market pricing measure. Moreover, we show that the necessary condition to induce firms to participate in the pool is also sufficient to create additional value such that all firms (and also every possible coalition of firms) are better off. We also discuss fairness issues related to the payment and provide conditions under which a ‘fair’ linear pricing measure can be used to reimburse the firms.

The value creation by pooling can be augmented by tranching the pooled asset and issuing new securities to investors. A new insight provided by solving the tranching problem is that the inducement is stronger if the investor classes are dissimilar. Moreover, the dissimilarity of the investor classes goes to the advantage of the firms. Another surprising result is that the tranches fully partition the pooled asset, i.e., tranching results in “no wastage” so that the pooled asset is fully utilized in one or the other of the tranches.

\(^{12}\)Our model may also explain how venture capitalists add value both by pooling investments across the firms they invest in, as well as by selling claims of different classes to diverse groups of investors. The value creation is again due to the venture capitalists’ ability to improve the spanning of cash flows across states both by pooling and tranching.
Our results are related to the securitization phenomenon observed in practice. For example, Cummins (2004) describes the general structure of asset-backed securities (ABS) that applies across industries and asset types. According to him, securities issued by a financial intermediary are structured to appeal to various classes of investors in recognition of their different investment tastes. Thus, ABS transactions include several tranches of securities, with different degrees of seniority with respect to the underlying cash flows. Our model also determines the optimal design of tranches, given demand from investors for cash flows in different states as shown in §5. However, there is a subtle difference between the tranches derived in our analysis and those that are typically observed because we assume complete information and no transaction costs. We find that the optimal tranches partition the cash flows in different states, whereas securitization observed in practice allocates cash flows in each state to multiple tranches in a particular order of priority. However, securitization also helps to create new classes of securities based on events that are not otherwise traded in the securities market, such as prepayment, credit, or catastrophe risk. Securities based on such risks would be consistent with the predictions from §5.

Our model and results have several applications to common financial problems relating to the valuation of assets, real and financial. One such problem relates to mergers. It has been documented in the academic and practitioner literature that the synergies from a merger of two companies may come from several sources, economies of scale or scope from their operations, improvement in their debt capacity, etc. All these explanations deal with the benefits created in individual states of nature where the cash flow of the combined firm are larger than the sum of the cash flows of the parts taken separately. We provide a different rationale. Because a merger is equivalent to the pooling of cash flows of two firms, our results show that in an incomplete market, if the merger produces synergies across states rather than within states, then it may be worthwhile to merge firms that are not currently traded. Another application of our framework is the valuation of real options in incomplete markets. We show that, due to market incompleteness, the existing assets of a firm can affect the value of new investments (even when they do not affect the cash flows of new investments). This further influences the firm’s investment decisions, and leads to more projects being undertaken.

While the results in this paper are obtained under the strict definition of arbitrage, our analysis could be combined with price bounds derived under approximate arbitrage as in the recent literature (see Bernardo and Ledoit 2000, Cochrane and Saa-Requejo 2000). Under approximate
arbitrage, market incompleteness should still continue to provide a rationale for seeking value enhancement through pooling and tranching. However, the imposition of a constraint that precludes “approximate arbitrage”, instead of arbitrage, would further restrict the set of feasible solutions to the optimization problems considered in this paper. Additional analysis is required to determine the optimal pooling and tranching strategies when subjected to the tighter constraints.

In the analysis in this paper, we have not dealt with the problems of information asymmetry. For instance, in our model, the tranches constructed by the intermediary need to be verifiable for our results to hold. If investors can verify whether the claim has positive payoffs in a state and if investors value consumption equally in every state, then the resulting partition of claims will resemble the tranches offered in the CLO market. Of course, given the issues relating to verifiability of the states, intermediaries need to handle the associated agency problems and be more innovative in creating tranches – for example, those that pay when the economy is doing well and those when it is doing poorly. We defer these issues to subsequent research.
References


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Proof of Lemma 1. We prove part (i). The proof for part (ii) is similar. Consider the linear program:

\[
\begin{align*}
\min & \quad z \\
n \text{subject to} & \quad z \geq \sum_{k=1}^{K} q_l(\omega_k)Z(\omega_k) \quad l = 1, \ldots, L \\
& \quad z \text{ unsigned.}
\end{align*}
\]

If \( z \geq \sum_k q_l(\omega_k)Z(\omega_k) \) for all \( l \), then \( \sum_l \delta_l z \geq \sum_l \sum_k \delta_l q_l(\omega_k)Z(\omega_k) \) for all \( \delta_l \geq 0, \sum_l \delta_l = 1 \). Thus, \( z \geq \sup_{q \in \Theta} E_q[Z(\omega_k)] \). Therefore, the optimal solution to the linear program must be greater than or equal to \( V^+(Z) \). On the other hand, \( z = \max_{l \in L} E_q[Z] \) is a feasible solution to the linear program. But \( \max_{l \in L} E_q[Z] \leq \sup_{q \in \Theta} E_q[Z(\omega_k)] \). Thus, \( V^+(Z) = \max_{l \in L} E_q[Z] \).

For the proof of part (iii), assume that there are no redundant securities, i.e., the payoffs from the securities are independent vectors (if this is not the case simply drop the securities that can be expressed as linear combinations of other securities and replace them by these combinations). Consider the problem of maximizing the minimum marketable value of a claim \( Z \) that pays \( Z(\omega_k) \) is state \( k \).

\[
\begin{align*}
\max & \quad \sum_n \alpha_n p_n \\
n \text{subject to} & \quad \sum_n \alpha_n S_n(\omega_k) \leq Z(\omega_k) \quad \forall \ k \\
& \quad \alpha_n \geq 0 \quad \forall \ n
\end{align*}
\]

The objective is to maximize the market value of a portfolio that is less than that of the claim \( Z \), subject to the constraint that the portfolio pays less than the claim in every state \( k \). The “no short sales” constraints require that the \( \alpha_n \) should be non-negative. The dual of this problem is given by:

\[
\begin{align*}
\min & \quad \sum_k \lambda_k Z(\omega_k)
\end{align*}
\]
subject to
\[ \sum_k \lambda_k S_n(\omega_k) \geq p_n \forall n \]
\[ \lambda_k \geq 0. \]

Notice that if the first inequality is non-binding in the optimal solution to the dual problem for some primary security \( n \), then by the complementary slackness theorem, the corresponding security is not in the optimal portfolio in the primal problem. Therefore, for the securities that enter at a strictly positive value in the optimal solution to the primal problem, the corresponding constraint in the dual problem should be binding.

Assume, without loss of generality that the securities \( 1, 2, \ldots, h \) are at a strictly positive level in the optimal solution to the primal problem. (\( h \) can be zero, in which case the cash flows from \( Z \) are completely orthogonal to the payoffs spanned by the primary securities. If \( h = N \) we do not need the construction given below.) Consider adding one unit of primary security \( n = h+1, h+2, \ldots, N \) to the claim, so that the claim is now the bundle, say \( Z' = Z + S_{h+1} + S_{h+2} + \ldots + S_N \). By doing so, the value of the claim \( Z \) itself has not increased. In other words, the value of the bundle \( Z' \) is the value of \( Z \) plus the market value of one unit each of the securities \( S_{h+1} \) through \( S_N \). Re-solve the primal problem with this augmented claim \( Z' \) instead of \( Z \) in the right hand side of the primal problem. Recall that \( Z \geq 0 \), thus, none of the added securities cancel out any cash flows from \( Z \). Also note that none of the securities that were originally in the solution can span the payoffs from the added primary securities as there are no redundant securities. It follows that the optimal solution to the primal problem will consist of the original solution plus one unit of each security \( n = h+1, h+2, \ldots, N \) in the optimal portfolio.\(^{13}\)

For this modified primal problem, all the dual constraints will be binding by the complementary slackness condition, i.e., we require, \[ \sum_k \lambda_k^* S_n(\omega_k) = p_n \] for every security \( n \) in the optimal dual solution. But, that condition, by definition, implies that the optimal dual solution is a pricing measure that belongs to the set \( \Theta \). The proof follows by applying part (i) of the Lemma. The validity of the upper bound can be proved similarly. \( \Box \)

\(^{13}\)As a technical note, observe that the number of primal constraints is greater than the number of primary securities because the market is incomplete. Thus, a basic solution to the primal problem can have all primary securities strictly positive in the optimal solution. In other words, all the variables can be positive in a basic optimal solution because: (i) the number of variables is less than the number of rows, and (ii) the columns representing the cash flows from the primary securities are independent and thus can be part of the basis.
Proof of Theorem 1. To prove (i) of the theorem, we show the equivalent statement that if value can be created by pooling, then there does not exist any \( q \in \Theta \) such that \( E_q[X_j] \leq r_j \) for all \( j \). Consider the linear program:

\[
\max v - \sum_j \alpha_j r_j
\]

subject to

\[
-\sum_k q_l(\omega_k) \sum_j \alpha_j X_j(\omega_k) + v \leq 0, \quad l = 1, \ldots, L
\]

\[
\alpha_j \geq 0, \quad j = 1, \ldots, J.
\]

Here, the vector \((\alpha_j)\) denotes the proportion in which the assets \((X_j)\) are pooled together, and \( v \) denotes the value of the asset pool in the securities market. The value of the asset pool is defined as \( V - (\sum_j \alpha_j X_j) \) because this is the minimum price that the asset pool commands in the securities market. We have used Lemma 1 by requiring the expected value under each extreme pricing measure, \( q_l \), be greater than or equal to the value \( v \) (each of the first \( L \) constraints). The linear program seeks to obtain the combination of assets that will maximize the difference between its value \( v \) and the combination of reservation prices required to create the asset pool, \( \sum_j \alpha_j r_j \).

If value can be created by pooling, then there exist weights \( \alpha_j \) such that the linear program is feasible and

\[
v - \sum_j \alpha_j r_j > 0. \quad (6)
\]

Let \( \theta_l \geq 0 \) be any set of weights such that \( \sum_l \theta_l = 1 \). Multiply each of the \( L \) constraints with the corresponding weight \( \theta_l \) and add. Because the linear program is feasible, we get

\[
-\sum_k \sum_l \theta_l q_l(\omega_k) \left( \sum_j \alpha_j X_j(\omega_k) \right) + v \leq 0.
\]

(7)

Here, \( \sum_l \theta_l q_l \) is a pricing measure in \( \Theta \), which we denote by \( q \). Hence, (7) can be rewritten as

\[
-E_q \left[ \sum_j \alpha_j X_j \right] + v \leq 0.
\]

(8)

Combining (6) and (8), we get \( E_q \left[ \sum_j \alpha_j X_j \right] > \sum_j \alpha_j r_j \). Equivalently, there exists \( j \) such that \( E_q[X_j] > r_j \).

Since \( \theta_l \) are arbitrary and there is a one-to-one mapping between the sets \( \{(\theta_l) : \theta_l \geq 0, \sum_l \theta_l = 1\} \) and \( \Theta \), we conclude that, if value can be created by pooling, then there does not exist any \( q \in \Theta \) such that \( E_q[X_j] \leq r_j \) for all \( j \).

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To prove (ii) of the theorem, consider the dual of the above linear program. The dual variables \( \mu_l \) will be associated with each of the constraints related to the expected value under extreme pricing measure \( q_l \). The dual problem is:

\[
\begin{align*}
\min & \quad 0 \\
\text{subject to} & \quad \sum_l \mu_l = 1 \\
& \quad \sum_k \sum_l \mu_l q_l(\omega_k) X_j(\omega_k) \leq r_j, \quad j = 1, \ldots, J \\
& \quad \mu_l \geq 0.
\end{align*}
\]

We wish to show that if no value can be created by pooling, then there exists \( q \in \Theta \) such that \( E_q[X_j] \leq r_j \) for all \( j \). Notice that by choosing all \( \alpha_j = 0 \), the primal problem is always feasible and has a lower bound of zero. The only question is whether the primal has a bounded solution – which by strong duality theorem can only be zero from the dual program’s objective function – or an unbounded solution. The former situation is the one where pooling does not create value (and the dual is feasible), and the latter situation is the one where pooling leads to value creation (and the dual is infeasible). Thus, if no value can be created by pooling, then the primal has a bounded solution and the dual is feasible. From the dual constraints, we observe that dual feasibility implies that there exist weights \( \mu_l \) such that under the pricing measure \( \sum_l \mu_l q_l \), we have \( E[X_j] \leq r_j \) for all \( j \). This proves the converse.

\[ \Box \]

**Proof of Theorem 2.** The proof of part (i) of the theorem follows from the work of Owen (1975) and Samet and Zemel (1984). We sketch the proof for completeness. Consider the problem of maximizing the value of the portfolio formed from the assets of coalition \( J_\alpha \) by selling tranches of primary securities against it. The maximum value is given by solving the linear program:

\[
\begin{align*}
\max & \quad \sum_n \alpha_n p_n \\
\text{subject to} & \quad \sum_n \alpha_n S_n(\omega_k) \leq \sum_{j \in J_\alpha} X_j(\omega_k), \quad k = 1, \ldots, K \\
& \quad \alpha_n \geq 0, \quad i = 1, \ldots, N.
\end{align*}
\]
The dual to this problem is
\[
\min \sum_k \lambda_k \sum_{j \in J} X_j(\omega_k)
\]
subject to
\[
\sum_k \lambda_k S_n(\omega_k) \geq p_n, \quad n = 1, \ldots, N
\]
\[
\lambda_k \geq 0, \quad k = 1, \ldots, K.
\]

Notice that the constraints to the dual program do not depend on the coalition formed because the \(X_j\)'s enter only the objective function. Moreover, the dual is feasible because the market is arbitrage-free, that is, any \(q \in \Theta\) will satisfy the dual constraints, i.e., \(\sum_k q_k S_n(\omega_k) = p_n, \forall n, q \in \Theta\). Finally, as \(X(\omega_k) \geq 0\), the solution to the dual program is finite, as it cannot drop below zero.

Solve the problem for the grand coalition of all firms and obtain the optimal dual solution \(\lambda^*_k\). As \(X(\omega_k) \geq 0\), by applying the same reasoning as in Lemma 1(iii), we can also assume that these dual values constitute a pricing measure in \(\Theta\).

Consider the following solution to the cooperative game: Let firm \(j\) receive the payment \(\sum_k \lambda^*_k X_j(\omega_k)\). By definition, the coalition \(J_\alpha\) receives the sum of the payments to the firms in the coalition. This sum equals or exceeds the maximum value obtained by solving the linear program for just the coalition because: (a) the \(\lambda^*_k\)'s constitute a dual feasible solution to the problem for all \(J_\alpha \subseteq J\) because, as noted earlier, the constraints of the dual problem do not depend on the coalition formed; and (b) all dual feasible solutions are greater than or equal to the primal optimal solution (by weak duality). This proves part (i).

For the proof of part (ii), the problem is to demonstrate the existence of a payment scheme that works for all coalitions simultaneously. Redefine the value of a coalition without loss of generality to be \(V(J_\alpha) = \max(V^-(J_\alpha), \sum_{j \in J_\alpha} r_j)\). We first show that if the condition stated in part (ii) applies to partitions comprised of two subsets, then it also applies to any arbitrary partition. That is, if for every subset \(J_\alpha\) of \(J\), we have \(V(J) \geq \max(V(J_\alpha), \sum_{j \in J_\alpha} r_j) + \max(V(J^c_\alpha), \sum_{j \in J^c_\alpha} r_j)\), then for every partition \(J_1, J_2, \ldots, J_k\) of \(J\), the same inequalities hold. (Note that the reverse statement can also be proven, implying that the two conditions are equivalent.) The proof is by contradiction. Assume that the condition does not hold for some partition, \(J_1, J_2, \ldots, J_k\). Thus, by assumption,
\[
V(J) < \sum_i \max\left(V^-(J_i), \sum_{j \in J_i} r_j\right).
\]
Without loss of generality, assume that for $i = 1, 2, \ldots, h$, \( \max(V^-(J_i), \sum_{j \in J_i} r_j) = V^-(J_i) \), and for $i = h + 1, h + 2, \ldots, k$, \( \max(V^-(J_i), \sum_{j \in J_i} r_j) = \sum_{j \in J_i} r_j \). Then, by super-additivity of the value function (which follows from the definition of $V^-$),

\[
V^-(\bigcup_{i=1}^{h} J_i) \geq \sum_{i=1}^{h} V^-(J_i).
\]

Let $J_\alpha = \bigcup_{i=1}^{h} J_i$. By the condition given in part (ii) of Theorem 2, the definition of $V(\cdot)$, and the discussion above, we have

\[
V(J) \geq V(J_\alpha) + V(J^c_\alpha)
\]

\[
\geq \sum_{i=h}^{k} V^-(J_i) + \sum_{j=h+1}^{k} \sum_{j \in J_i} r_j
\]

\[
= \sum_{i} \max(V^-(J_i), \sum_{j \in J_i} r_j).
\]

This provides the necessary contradiction. The proof of part (ii) now appears to be immediate because, under every solution in the core, each coalition $J_\alpha$ gets at least $\max(V(J_\alpha), \sum_{j \in J_\alpha} r_j)$. Thus, the payment is sufficient to cover the reservation price. However, it must further be shown that this can be done simultaneously for every coalition and not just coalition by coalition. Redefine the value of a coalition, without loss of generality, to be $V(J_\alpha) = \max(V^-(J_\alpha), \sum_{j \in J_\alpha} r_j)$.

Consider the primal problem:

\[
\min 0
\]

subject to

\[
\sum_{j \in J_\alpha} \pi_j \geq V(J_\alpha), \quad \text{for all } J_\alpha \subseteq J,
\]

\[
\sum_{j} \pi_j = V(J).
\]

This program if feasible determines the payment schedule for the firms, i.e., firm $j$ receives a payment $\pi_j$. The dual problem is:

\[
\max \sum_{J_\alpha \subseteq J} \lambda_{J_\alpha} V(J_\alpha) + \lambda V(J)
\]

subject to

\[
\sum_{J_\alpha : j \in J_\alpha} \lambda_{J_\alpha} + \lambda \leq 0, \quad j = 1, \ldots, J,
\]

\[
\lambda_{J_\alpha} \geq 0, \quad \lambda \text{ unsigned.}
\]
The dual variables $\lambda_{J_a}$ correspond to the first set of constraints in the primal problem, and the dual variable $\lambda$ corresponds to the second constraint. Obviously, the dual problem is always feasible (set all variables equal to zero). The dual solution will equal zero. Moreover, $\lambda$ has to be less than or equal to zero. All we need to show is that zero is the maximum possible solution to the dual. If not, then the dual will be unbounded (by scaling all variables as large as desired), and therefore, the primal will be infeasible. We proceed to show that the solution to the dual problem is bounded.

Consider the constraint to the dual corresponding to $j = 1$. This constraint along with $\lambda \leq 0$ implies that:

$$\sum_{J_a: \lambda \in J_a} \lambda_{J_a} V(J_a) + \lambda \max(V(J_a) : 1 \in J_a, J_a \subseteq J) \leq 0.$$ 

Similarly, the constraint corresponding to $j = 2$ yields

$$\sum_{J_a:2 \in J_a \text{ and } 1 \in J_a^c} \lambda_{J_a} V(J_a) + \lambda \max(V(J_a) : 2 \in J_a \text{ and } 1 \in J_a^c, J_a \subseteq J) \leq 0.$$ 

We can write analogous inequalities for larger values of $j$. In general, we have

$$\sum_{J_a:j \in J_a \text{ and } \{1, \ldots, j-1\} \subseteq J_a^c} \lambda_{J_a} V(J_a) + \lambda \max(V(J_a) : j \in J_a \text{ and } \{1, \ldots, j-1\} \subseteq J_a^c, J_a \subseteq J) \leq 0.$$ 

The sets where the maximum is attained over $(J_a : j \in J_a \text{ and } \{1, \ldots, j-1\} \subseteq J_a^c, J_a \subseteq J)$ are disjoint and their union is less than or equal to $J$. Adding up these inequalities gives

$$\sum_{J_a \subseteq J} \lambda_{J_a} V(J_a) + \lambda (\max(V(J_a) : 1 \in J_a, J_a \subseteq J) + \max(V(J_a) : 2 \in J_a \text{ and } 1 \in J_a^c, J_a \subseteq J) + \ldots) \leq 0.$$ 

Recalling that $V(J)$ is greater than equal to the sum of the $V(J_i)$’s over any partition of $J$ we obtain

$$\sum_{J_a \subseteq J} \lambda_{J_a} V(J_a) + \lambda V(J) \leq 0.$$ 

Therefore, the optimal value of the dual problem is bounded above by zero. This implies that the dual problem is feasible and bounded, and therefore, has an optimal solution. Therefore, by strong duality theorem, the primal has a feasible solution.
**Proof of Corollary 1.** The choice of \( q_p \) in part (i) follows from Lemma 1. Notice that when the lower bound, \( V^-(\sum_j X_j) \) is achieved at several extreme points then a linear mixture of these measures also gives the same lower bound. The second part follows from part (ii) of Theorem 2. To see this, the optimal solution to the full coalition’s problem is the highest value that can be obtained by pooling all assets, which must equal \( E_{q_p}(\sum_j X_j) \). Moreover, any linear pricing measure that supports the core must be an optimal dual solution to the problem of determining \( V^-(\sum_j X_j) \). Also, all optimal solutions to the dual problem are obtained as convex combinations of the optimal extreme points solutions. Thus, if one such pricing measure can be found that not only supports the core but also gives a value of each \( X_j \) larger than \( r_j \) then all firms will willingly participate in creation of the pool.

\[ \square \]

**Proof of Theorem 3.** We begin the proof with the observation that even if the optimal tranching solution had several tranches for the same investor type, we can combine different tranches offered to an investor type into a single tranche. This offers a useful insight into the secondary market. This market can tailor to individual investor requirements instead of offering a variety of securities from which the investor has to create a portfolio.

Therefore, the problem reduces to determining at most one tranche per investor type. Let \( Y_i \) be the claim that pays \( Y_{ik} \) in state \( i \). By the observation above we can assume that this claim is sold only to investors of type \( i \). The price that this type of investor is willing to pay for the claim is given by \( \sum_k m_{ik} Y_{ik} \), where \( m_{ik} \) is the state price for state \( k \). The rest of the theorem follows after this.

\[ \square \]

**Proof of Corollary 2.** It is clear that once we fix the values of \( \beta_n \), the optimal solution is to sell the remaining cash flows state by state at the highest price. If we can show that we never need to sell to the primary market, i.e., \( \beta_n = 0 \), we are finished with the proof. But, we know that for each security \( n \) there is an investor, say \( i \), who holds it in his or her portfolio. Therefore, \( p_n = \sum_k m_{ik} S_n(\omega_k) \). This implies by selling the same cash flows state by state, the intermediary obtains,

\[ \sum_k \max_i m_{ik} S_n(\omega_k) \geq p_n. \]
Thus, it is sufficient to consider the case when no tranches are sold in the primary security market. The result hinges on the assumption that security \( n \) is in some investor type’s portfolio.

The second part follows by the fact that it is sufficient to sell the cash flows in a given state to the investor type that values them the most in that state. Thus, cash flows in each state are sold to a particular investor type that values them the most. Assembling the cash flows over all states for a particular investor type, say \( i \), yields the tranche that is specifically constructed for investors of type \( i \). Obviously, these optimal tranches partition the asset pool.

\[ \square \]

**Proof of Corollary 3.** The proof follows from Corollaries 1 and 2. \( E_{q_i}(X_j) \) is the value firm \( j \) would obtain under pooling. If the firm obtains the higher of this amount and its reservation price, then it is either surely better off or not worse off with pooling and tranching.

\[ \square \]