Harmful Addiction†

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Abstract

We construct an infinite horizon consumption model and use it to define and analyze harmful addiction. Consumption is compulsive if it differs from what the individual would have chosen had commitment been available. A good is addictive if its consumption leads to more compulsive consumption of the same good in the future. We analyze two types of drug policies. A policy is prohibitive if it decreases the maximal feasible drug consumption. A price policy is one that increases the opportunity cost of drug consumption. We show that purely prohibitive policies make agents better off and pure price policies make the agent worse-off.

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1. Introduction

Substantial resources are spent to reduce the availability of and the demand for drugs. These efforts are justified by the belief that drug addiction is a serious health and social problem. What is special about drugs that could justify restricting its supply and demand?

Standard economic analysis uses the individuals’ choice behavior as a welfare criterion. Alternative \( x \) is deemed to be better for the agent than alternative \( y \) if and only if given the opportunity, the agent would choose \( x \) over \( y \). Restricting the available options for an individual can never be welfare improving in a standard economic model.

While typical in economic analysis, the identification of welfare and choice is certainly not the norm in discussions of addiction. Instead, addiction is often viewed as a disease that impedes the agent’s decision-making ability.\(^1\) It is believed that after being struck by the disease, a person can no longer be trusted to make the right decision for his “true” self.\(^2\) The role of intervention is to “cure” (i.e. induce abstinence) or at least “control” (i.e. reduce consumption) the disease.

Viewing addiction as a disease creates a wedge between choice and welfare. This wedge makes room for desirable interventions that modify the addict’s choices but also creates the need for a new welfare criterion for evaluating the costs and benefits of these interventions. Consider a costly treatment that, if successful, would remove the agent’s drug dependency (i.e. cure the disease). If the probability of success is sufficiently high then the treatment is desirable regardless of whether the agent thinks so or not. Conversely, if the probability of success is sufficiently small then the treatment is undesirable. How can the planner determine whether the probability of success justifies the cost of the treatment in the wide range of intermediate cases?

In this paper we provide a model of addiction that is consistent with the view that addicts may benefit from interventions that modify their choices. At the same time, the model offers clear guidance for welfare comparisons. Our model builds on previous work (Gul and Pesendorfer (2000a)) in that it allows the agent’s welfare to depend both on what

\(^1\) “Is alcoholism a disease? Yes. Alcoholism is a chronic, often progressive disease with symptoms that include a strong need to drink despite negative consequences, such as serious job or health problems.” (cited from: National Institute on Alcohol Abuse and Alcoholism. http://silk.nih.gov/silk/niaaa1/questions/q-a.htm#question2)

\(^2\) There are numerous criticisms of the disease model of drug addiction (see for example, Davies (1992)).
he chooses and on the set of options from which the choice is made. This set may contain tempting alternatives that reduce the agent’s welfare either by distorting his choice or by necessitating costly self-control or both. In particular, drug consumption constitutes a temptation. Moreover, current drug consumption affects how the agent will respond to temptations in future periods. Specifically, the agent is more likely to give in to tempting drug consumption if he consumed an addictive drug in the past.

To give a more precise description of our concept of a harmful addiction, we introduce the notion of *compulsive consumption*. An individual is *compulsive* if his choice differs from what he would have chosen had commitment been possible. An agent is more compulsive after consumption history A than after consumption history B if for every decision problem in which the agent is compulsive after B he is also compulsive after A. The drug is *addictive* if an increase in drug consumption makes the agent more compulsive. Hence, a harmful addiction is defined as a widening of the gap between the individual’s choice and what he would have chosen before experiencing temptation.

To see how our model works, consider an agent who must choose between drug consumption ($d$) and non-drug consumption ($c$) from a budget set $B_t$ in period $t$. For simplicity, we assume that there is no saving. The dynamic program below characterizes the agent’s utility as a function of last period’s drug consumption. Let $W(d_{t-1})$ denote the utility (value) function in period $t$, then

$$W(d_{t-1}) = \max_{\{(c,d)\in B_t\}} \{u(c,d) + \sigma(d_{t-1})v(d) + \delta W(d)\} - \max_{\{(c',d')\in B_t\}} \sigma(d_{t-1})v(d')$$

We interpret $\sigma(d_{t-1})v$ as the temptation utility and call $u + \delta W$ the commitment utility. To understand this terminology, note that if all options were equally tempting; that is, resulted in the same $v$, then the $v$-terms in equation 1 would drop-out. Therefore, such consumption problems would be evaluated according to $u + \delta W$. In particular, if $B_t$ consists of a single choice $(c,d)$, then the overall utility of the current decision problem is the commitment utility $u + \delta W$, of the single option $(c,d)$. The commitment utility $u + \delta W$ is independent of past drug consumption while the temptation utility $\sigma(d_{t-1})v$ depends on last period’s drug consumption.
The individual’s choice \((c, d, x)\) maximizes \(u + \sigma(d_{t-1})v + \delta W\). This choice reflects the compromise between the commitment utility and temptation. An individual is compulsive if his choice \((u + \sigma(d_{t-1})v + \delta W\) maximizer) does not maximize his commitment utility. A drug is addictive if an increase in drug consumption leads to more compulsive drug consumption.

In Proposition 1, we show that an increase in \(\sigma\) implies that the agent is more compulsive. Hence, if \(\sigma\) is an increasing function then the drug is addictive. An increase in \(\sigma\) implies that the agent’s choices place a smaller weight on the commitment utility and hence the gap between optimal commitment choices and actual choices widens. Proposition 2 shows that consumption of an addictive drug is reinforcing, that is, higher drug consumption in the current period leads to higher drug consumption in future periods.

Section 3 examines the effect of price changes on drug demand. We show that drug demand decreases if the current price of the drug increases. If the drug is addictive, drug demand also decreases if the future price of the drug increases.

Section 4 analyzes the welfare effects of drug policies. We assume that the agent faces a fixed budget set in every period. The government can affect this budget set by changing the price of the drug (price policy) or by reducing the maximally feasible drug consumption (prohibitive policy). Many actual policies will change the price of the drug and the maximally feasible drug consumption simultaneously. We separate these two effects in order to identify the source of welfare effects. An example of a policy with mostly prohibitive effects is a ban on drug consumption. A tax on the drug will have mostly price effects if drug consumption is a relatively small part of an agent’s budget and hence the tax does not affect the maximally feasible drug consumption in a given period. A pure price policy refers to a policy that has only price effects whereas a purely prohibitive policy refers to a policy with only prohibitive effects.

We show that a pure price policy always reduces the agents welfare. A pure price policy makes it more costly to consume the drug but does not change the most tempting alternative. In response to a pure price policy the agent will consume less of the drug and exercise more costly self-control. By a simple revealed preference argument, the increased cost of self-control is always larger than the possible utility gain from reduced drug consumption. The key feature of a pure price policy is that it does not remove temptations
from the agent’s choice set while it increases the cost of drug consumption. Such a policy will reduce drug consumption but also decrease welfare.

To examine the effect of a prohibitive policy, we focus on the special case where the optimal drug consumption is zero if the agent can commit. We show that if the drug is addictive then a prohibitive policy increases welfare. A prohibitive policy changes the most tempting alternative without affecting the price of the drug. We also examine how drug demand changes when a prohibitive policy is introduced. If the prohibitive policy is not binding then a reduction in the maximally allowed drug consumption will increase drug demand. To see the intuition for this result note that current drug consumption makes self-control more costly in future periods. If the maximally feasible drug consumption is reduced then this cost is smaller and hence current drug consumption is more attractive.

Together, our welfare results show that welfare improvement stems from the commitment effect of a policy while the price effects reduce welfare. Moreover, if a policy reduces drug demand then this cannot be taken as an indication that the policy “works” in the sense of welfare improvement. This is true even though the drug is unambiguously “bad”, that is, the optimal drug consumption under commitment is zero.

Section 5 analyzes a decision problem in which the agent has the option of checking into a “rehabilitation center”. In our model, rehabilitation centers provide temporary commitment to zero drug consumption. This commitment is costly in terms of non-drug consumption. Drug treatment programs offer a variety of treatments, many of them go beyond simple commitment. However, making drugs difficult to acquire seems to be a common feature of most treatment programs. Treatment programs remove patients from their familiar surroundings, ensure that drugs are not available on the premises, and closely monitor the activities of their patients. In this way, they offer temporary commitment. Clearly, voluntary treatment programs cannot offer permanent commitment since agents can leave at any moment. For our purposes the key feature is that immediate drug consumption is not possible.

Section 5 considers a decision problem in which the agent faces a fixed budget in every period. In addition, the agent can choose to go enter a rehabilitation center. This choice results in a commitment to zero drug consumption for the subsequent period. Rehabilitation is costly in terms of the agent’s non-drug consumption. Our model predicts that
agents enter rehab after their drug consumption has reached its peak. Moreover, we show that after the visit to the rehabilitation center agents can be expected to follow a pattern of increasing drug consumption followed by another visit to the rehabilitation center. We also demonstrate that the expectation of attending a rehabilitation center in the next period increases current drug demand. Hence, agents will “go on a binge” just before entering rehab.

Section 6 provides a foundation for the preferences analyzed in this paper. We provide axioms that imply the representation used in the text. The key difference to the representation found in our earlier work is that preferences may depend on past (drug) consumption.

1.1 Evidence

The economics literature on addiction has focused on the comparative statics of the demand for drugs. The key comparative static is that the demand for the drug decreases as the future price of the drug increases. Becker, Grossman and Murphy (1994) found that consumption of addictive goods in the current period decreases if future prices go up. Gruber and Koszegi (2001) replicate this result in a recent paper. As we show in section 3, our model are consistent with this prediction.

Our model suggests that addicts should seek commitment opportunities. We observe addicts seeking commitment by enrolling in voluntary rehabilitation programs. Treatment programs provide commitment by making drugs difficult to procure. Prohibition of certain drugs also provides a form of commitment, albeit an involuntary commitment. The fact that prohibitive drug policies have strong public support also suggests that agents benefit from commitment.

A sophisticated form of commitment is achieved through the use of drugs to combat addiction. For example, consider an addict who seeks treatment for alcohol addiction and is given the drug disulfiram. Disulfiram is a deterrent medication that is used to fight alcohol addiction. Disulfiram produces a sensitivity to alcohol which results in a highly unpleasant reaction when the patient under treatment ingests even a small amount of alcohol. This
effect lasts up to 2 weeks after ingestion of the last dose.\textsuperscript{3} Hence, the patient is committed to abstaining from alcohol as long as the drug is effective (Chick 1992). Similarly, the opiate antagonist naltrexone blocks the opioid receptors in the brain and hence the euphoric effects of these drugs for up to 3 days after the last dose. Naltrexone is voluntarily used by some heroin and morphine addicts. Further evidence for the demand for commitment devices are the recent efforts by pharmaceutical companies to develop \textit{vaccines} for nicotine (Pentel, et al. (2000)) and cocaine.\textsuperscript{4} The function of these vaccines is to prevent the drug from reaching the brain, so as to eliminate its effects and provide commitment for individuals. A novel feature of these vaccines is their long term effectiveness, and hence their ability to provide commitment over many months.

\subsection{1.2 Related Literature}

The economics literature has typically identified addiction with inter-temporal complementarities. Becker and Murphy (1986) view the consumption of an addictive good much like an investment that increases the return of future consumption. The preferences analyzed by Becker and Murphy are “standard” in the sense that individuals can never benefit from the elimination of some alternatives. Therefore, an individual who voluntarily acquires costly commitment devices is inconsistent with the Becker and Murphy preferences.

Becker and Murphy’s treatment of addiction as an investment ensures that drugs are never “bad” in terms of individual welfare, and therefore their model does not leave room for a welfare improving drug policy. However, Becker and Murphy do distinguish between addictions that are harmful and those that are beneficial: an addiction is harmful if it leads to a utility penalty in future periods. If there is no such penalty then it is beneficial.

\textsuperscript{3} “Disulfiram plus even small amounts of alcohol produces flushing, throbbing in head and neck, throbbing headache, respiratory difficulty, nausea, copious vomiting, sweating, thirst, chest pain, palpitation, dyspnea, hyperventilation, tachycardia, hypotension, syncope, marked uneasiness, weakness, vertigo, blurred vision, and confusion. In severe reactions, there may be respiratory depression, cardiovascular collapse, arrhythmias, myocardial infarction, acute congestive heart failure, unconsciousness, convulsions, and death.” (cited from: http://www.mentalhealth.com/drug/)

\textsuperscript{4} “When injected in laboratory animals, the vaccine stimulates the immune system to produce antibodies that bind tightly to nicotine. The antibody-bound nicotine is too large to enter the brain, thereby preventing nicotine from producing its effects. The antibody-bound nicotine is eventually broken down to other harmless molecules.” cited from http://pharmacology.about.com/health/pharmacology/library/99news/bl9n1217a.htm
By contrast, we define an addiction to be harmful if current drug consumption leads to more compulsive behavior in future periods.

To distinguish a harmful addiction from a beneficial addiction in the Becker-Murphy model, we need to observe the utility an agent experiences. Harmful and beneficial addictions are distinguished only by the implied utility flows and not by observable behavior. By contrast, our definition of harmful addiction centers around observable behavior. A harmful addiction leads to greater divergence between commitment demand and actual demand and to a greater demand for commitment devices.

O’Donoghue and Rabin (1997) and Gruber and Koszegi (2001) offer models of addiction that merges the approach of Becker and Murphy with the time inconsistent $\beta - \delta$ model. In their model, the individual may consume more than his past selves would like because of a presence-bias in his preferences. As in our approach, this model implies that agents will utilize commitment opportunities (at least if they are sophisticated). However, their notion of a harmful addiction is based on that of Becker and Murphy. This makes it difficult to distinguish harmful from beneficial addictions in terms of observed behavior. The inability to distinguish harmful from beneficial addictions is less of an issue in the Becker-Murphy framework because there are no policy implications associated with this distinction. By contrast, the our model as well as the models of O’Donoghue-Rabin and Gruber-Koszegi (2001) leave room for beneficial intervention and hence it becomes important to identify which goods constitute harmful addictions.

This paper and the literature on $\beta - \delta$ preferences employ different models of the individual. The $\beta - \delta$ model identifies with each time period a different self for the agent. The various selves of the agent have competing interests and use their resources (i.e., their control of the decision-making for the current period) to maximize their own utility. In contrast, in our model there is a single agent with a consistent preference who maximizes a single consistent utility function. The main consequence of this difference is that welfare statements are typically ambiguous for the $\beta - \delta$ model while our model offers clear welfare statements.

It is difficult to imagine what kind of “evidence” one could provide in favor of either multi-selves or the single-self view. However, we note that the idea of a consistent preference corresponding to the agent’s true welfare seems to permeate our informal, everyday
analysis of struggles with temptations. Consider the example of a smoker. Suppose, in period 0 he has decided to quit and thrown out his last pack of cigarettes. In period 1, he visits a friend who offers him a cigarette which he accepts. After the visit, his friend is reproached by the friend’s spouse who asks: “Why did you do that? You know he was trying to quit!” To this the friend responds: “It was his period 0 self that wanted to quit. Obviously, the period 1 self did not, since it accepted the cigarette that I offered. Why should I be concerned with the welfare of the period 0 self? After all, it was the period 1 self that was nice enough to pay us a visit.” Should we consider this an adequate defence of the friend’s actions? If we take the multi-selves view literally, we may have to. By contrast, our model takes the view that the agent is harmed by the availability of the cigarettes in both periods. The agent’s decision to smoke when cigarettes are available only indicates that it has become too costly to exercise self-control.

Gruber and Koszegi conduct an empirical study of the \( \beta - \delta \) model of addiction. They argue that the comparative statics findings on drug demand are consistent with the \( \beta - \delta \)-model and Becker-Murphy model. Our model offers the same comparative statics and therefore passes this test as well.

Gruber and Koszegi also examine optimal taxes on cigarettes. They find that cigarette taxes can increase the welfare of a \( \beta - \delta \) decision maker. Their finding is in contrast to the theoretical result presented here where a (small) tax on an addictive good reduces the welfare of the agent. We discuss the reason for this difference in detail in section 4.

2. Model

We consider an environment with 2 goods and let \( C = [0, 1]^2 \) denote the set of possible consumption vectors. A consumption bundle is denoted \((c, d)\) where \(d\) will be interpreted as the consumption of the addictive good, the “drug”.

An agent is confronted with a dynamic decision problem. Every period \( t = 1, 2, \ldots \) the agent must take an action. This action results in a consumption for period \( t \) and constrains future actions. Dynamic decision problems can be described recursively as a set of alternatives where each alternative is a lottery over current consumption and continuation decision problems.\(^5\) Let \( Z \) denote the set of all decision problems. We use

\[^5\] See Gul and Pesendorfer (2000b) for a detailed discussion of dynamic decision problems.
x, y or z to denote generic decision problems (i.e., elements of Z). Generic choices (i.e., elements of a given z) are denoted μ, ν or η. A choice μ is a lottery over C × Z, where c ∈ C represents the realization of current consumption and x ∈ Z represents the realized continuation decision problem. A deterministic choice yields a particular consumption (c, d) and a particular deterministic continuation problem z with certainty and is denoted (c, d, z). Most of the analysis in this paper focuses on the set of deterministic decision problems, ¯Z ⊂ Z. Each z ∈ ¯Z is a (compact) set of alternatives of the form (c, d, x) where c denotes current consumption and x ∈ ¯Z denotes the deterministic continuation problem.

The set of decision problems Z serves as the domain of preferences for the agent. This allows us to describe agents who struggle with temptation. For example, the agent may strictly prefer a decision problem in which some alternatives are unavailable because these alternatives present temptations that are hard to resist. Even when the agent makes the same ultimate choice from two distinct decision problems he may have a strict preference for one decision problem because making the same choice from the other requires more self-control. Choice problems are the natural domain for identifying these phenomena. Below we represent the individual’s preferences over decision problems by a utility function. This utility function is analogous to the indirect utility function in standard consumer theory. The traditional indirect utility function is defined for decision problems that can be represented by a budget set. In contrast, our utility function is defined for a broader class of decision problems.

The preferences analyzed in this paper depend on the agent’s past consumption. To capture this dependence, we index the individual’s preferences by s ∈ S, the state in the initial period of the decision problem. The state s represents the relevant consumption history prior to the initial period of analysis. For simplicity, we assume that only drug consumption in the last period influences the agents preferences and set S := [0, 1]. We refer to the indexed family of preferences ≥ := {≥s}s∈S simply as the agent or the preference ≥. We say that the utility function W : S × Z → IR represents the preference ≥ if, for all s, x ≥s y iff W(s, x) ≥ W(s, y).

In section 6 we provide axioms under which ≥ can be represented by a continuous function W of the following form:

\[
W(s, z) = \max_{(c, d, x) \in z} \left[ u(c, d) + \sigma(s)v(d)\delta + W(d, x) \right] - \max_{(c', d', x') \in z} \sigma(s)v(d')
\] (1)
where \( u \) is a continuous function, \( v \) is continuous and strictly increasing, \( \sigma \) is a continuous and strictly positive function, and \( \delta \in (0, 1) \).

Straightforward application of results from dynamic programming imply that for every \((u, v, \sigma, \delta)\) with \( u, v \) continuous, \( \delta \in (0, 1) \), there is a unique \( W \) that satisfies equation 1. We say that \((u, v, \sigma, \delta)\) represents the preference \( \succeq \) if the unique \( W \) that satisfies equation 1 represents \( \succeq \). Below, we simply refer to \((u, v, \sigma, \delta)\) as the preference. The preference \((u, v, \sigma, \delta)\) is regular if \( u \) is not constant. (Recall that \( v \) is strictly increasing and therefore \( u \) and \( v \) are not constant for a regular preference).

Equation (1) implies that if the agent is committed to a single choice (i.e., \( z = \{(c, d, x)\}\)) then \( W(z) = u(c, d) + \delta W(x) \). Therefore, we refer to \( u + \delta W \) as the commitment utility of a particular choice. Note that the commitment utility is independent of the state \( s \).

Consider a deterministic decision problem that does not offer commitment and assume that \((c, d, x)\) is the unique maximizer of the commitment utility \( u + \delta W \) and \((c', d', y)\) is the unique maximizer of \( v \) in \( z \). In this case, it follows from equation 1 that removing \((c', d', y)\) from the choice set would increase the agent’s welfare. We refer to alternatives \((c', d', y)\) that have this property as temptations. Temptations create a preference for commitment; that is, situations where the agent strictly prefers the decision problem \( x \) over \( z \) even though \( x \subset z \).

The agent’s choice from \( z \) in state \( s \) maximizes the function

\[
u + \sigma(s)v + \delta W
\]

If \((c, d, x)\) is the choice from \( z \) and \((c', d', y)\) maximizes \( v \) in \( z \) then the agent incurs a self-control cost of

\[
-\sigma(s)[v(d) - v(d')]
\]

This cost is zero if the choice maximizes \( v \). Otherwise it is positive. In our model, past consumption affects current behavior by changing the cost of self control.
The optimal choices from $z$ is denoted $\mathcal{D}(s, z)$ and $C(z)$ denotes the maximizers of the commitment utility. For any function $f : C \times Z \to \mathbb{R}$ let $E_\mu[f]$ be the expectation of $f$ with respect to $\mu$. Then,

$$\mathcal{D}(s, z) := \{\mu \in z | E_\mu[u + \sigma(s)v + \delta W] \geq E_\nu[u + \sigma(s)v + \delta W], \forall \nu \in z\}$$

$$C(z) := \{\mu \in z | E_\mu[u + \delta W] \geq E_\nu[u + \delta W], \forall \nu \in z\}$$

When the agent chooses alternatives that do not maximize the commitment utility it means that behavior is affected by temptations. We call such choices compulsive. This motivates the following definition.

**Definition:** $s$ is compulsive at $z$ iff $\mathcal{D}(s, z) \setminus C(z) \neq \emptyset$.

Below, we offer criteria for ranking states with respect to how compulsive the agent is.

**Definition:** A preference $\succeq$ is more compulsive at $s$ than at $s'$ (denoted $s \prec_{comp} s'$) if $s'$ is compulsive at $z$ implies $\succeq_s$ is compulsive at $z$.

The notion of compulsive consumption plays a central role in the clinical definition of addiction and the definition we present below. What distinguishes addiction from other types of compulsive behavior is the fact that the compulsiveness associated with an addictive substance is “caused” (or made worse) by past consumption (or higher levels of past consumption) of the same substance.

Psychologists and health professionals commonly refer to an individual as addicted if, after repeated self-administration of a drug, the individual develops a pattern of compulsive drug seeking and drug-taking behavior.\(^6\) The clinical definition emphasizes a lack of control on the part of addicted subjects and suggest a conflict between what the addict ought to consume and what he actually consumes.

In our model, the agent is compulsive when the choice is different from the $u + \delta W$ optimal alternative. Thus, an agent is compulsive if behavior would change were commitment possible. Similar to the clinical definition above, we define an increase in drug consumption to be addictive if higher current drug consumption makes the more

\(^6\) See Robinson and Berridge (1993), pg. 248.
compulsive; that is, following the increase in drug consumption there are more situations in
which the agent makes a choice that does not maximize $U$. The definition below expresses
this idea.

**Definition:** The drug is addictive if $s'C_s$ for all $s' > s$ and $\succeq_1 \neq \succeq_0$.

**Proposition 1:** Let $(u,v,\sigma,\delta)$ be a regular preference. (i) $s'C_{s'}$ if and only if $\sigma(s) \geq 
\sigma(s')$; (ii) the drug is addictive if and only if $\sigma$ is non-decreasing with $\sigma(1) > \sigma(0)$.

Proposition 1 relates our definition of addiction to our representation of preferences. It shows that the function $\sigma$ measures how compulsive the agent is and therefore the drug
is addictive when $\sigma$ is increasing.

For $z \in \bar{Z}$, let $D(s,z)$ denote the individual’s current period drug demand in state $s$;
that is, $d \in D(s,z)$ if and only if there exists $c,x$ such that $(c,d,x) \in D(s,z)$. We write
$D(s,x) \geq D(s',y)$ if $d \in D(s,x), d' \in D(s',y)$ implies $d \geq d'$. Proposition 2 shows that
an increase in $\sigma$, leads to higher drug demand in every decision problem. Proposition 2
assumes that the temptation utility is non-decreasing in drug consumption.

**Proposition 2:** Let $(u,v,\sigma,\delta)$ be a regular, preference. If $\sigma(s) \geq \sigma(s')$ then $D(s,z) \geq 
D(s',z)$ for all $z \in \bar{Z}$.

Psychologists use the term reinforcement to describe the fact that an increase in
current drug consumption leads to an increase in future drug consumption. If $\epsilon > 0$ and
$\sigma(d + \epsilon) > \sigma(d)$ then the $\epsilon$ increase is reinforcing. In particular, an addictive increase in
drug consumption is always reinforcing.

### 3. Addiction and Drug Demand

Our next objective is to analyze the implications of addiction on drug demand. We
consider a simple stationary consumption problem. The individual cannot borrow or lend
and can at most consume 1 unit of the drug in every period. Let $p = p_1,...,p_t,...$ denote
the sequence of prices. The individual is endowed with one unit of wealth and must choose
consumption $(c,d)$ in the budget set

$$B_t = \{(c,d) \in C| c + p_t d \leq 1\}$$
We assume that the $p_t < 1$. Since $d \leq 1$ the maximally feasible drug consumption is 1 in every period independent of the price of the drug. We denote with $x(p)$ the corresponding dynamic decision problem and let $D(s, p)$ denote the individual’s current period drug demand in state $s$.

The comparative statics results in this and the subsequent sections impose further structure on the preferences. The following assumptions will be used in this and the subsequent sections. Assumption 1 requires that $u$ does not depend on drug consumption.

**Assumption 1:** $u(\cdot, d)$ is strictly increasing with $u(\cdot, d) = u(\cdot, d')$ for all $d, d'$.

When Assumption 1 is satisfied we write $u(c)$ instead of $u(c, d)$. Assumption 2 imposes curvature restrictions on $u, v$ and $\sigma$ with $u$ satisfying Assumption 1. The assumptions on $u, v$ are analogous to the standard curvature and differentiability assumptions in demand theory. The function $\sigma$ plays a role similar to a cost function in a standard optimization problem. Therefore concavity of the objective function in the decision problems below is guaranteed when $\sigma$ is convex.

**Assumption 2:** $u, v, \sigma$ are twice differentiable with $u'', v'' < 0$ and $\sigma'' > 0$.

The final assumption requires that $\sigma$ be strictly increasing. Hence, Assumption 3 implies that the drug is addictive.

**Assumption 3:** $\sigma$ is strictly increasing.

When $u$ depends only on non-drug consumption then the objective function for the agent’s choice from $x(p)$ simplifies to

$$u(c_t) + \sigma(d_{t-1})v(d_t) + \delta W(d_t, x(p))$$

For an interior solution the first order condition for an optimal drug demand in period $t$ (denoted $d_t$) is given by:

$$p_t u'(c_t) = \sigma(d_{t-1})v'(d_t) + \delta \sigma'(d_t)(v(d_{t+1}) - v(1))$$

To understand the above equation consider a marginal increase in drug consumption. This implies a reduction in current (non-drug) consumption and the left hand side captures the
utility consequence of this reduction. The first term on the right hand side captures the current period utility change from the increase in drug consumption. Since \( v \) is increasing this is positive. The second term captures the effect of the increase in drug consumption on future utility. This effect works through a change in the cost of self-control in the next period. If \( \sigma \) is increasing (as in the case of an addictive drug) then the increase in the current drug consumption implies a higher self-control cost next period and the second term on the left hand side is negative. If \( \sigma \) is decreasing, then the increase in drug consumption implies a smaller self-control cost in the next period and the term is positive.

The next proposition analyzes the change in demand as a function of current and future prices.

**Proposition 3:** (i) If \((u, v, \sigma)\) satisfy Assumptions 1 and 2 then \(D(s, \cdot)\) is a differentiable function satisfying \(\partial D(s, p)/\partial p_1 \leq 0\) with strict inequality if \(0 < D < 1\); (ii) if \((u, v, \sigma)\) satisfy Assumptions 1-3 (and hence the drug is addictive) then \(D(s, p)/\partial p_t \leq 0\) for all \(t \geq 1\). A strict inequality holds if drug demand is interior for all \(t' \in \{1, \ldots, t\}\).

**Proof:** (i) Assume that \(0 < D < 1\). The necessary first order condition is

\[-p_1 u'(1 - p_1 d_1) + \sigma(s) v'(d_1) + \delta \sigma'(v(d_2) - v(1)) = 0\]

Taking the total derivative we find

\[\partial D/\partial p_1 = \frac{p_1^2 u''(c_1) + \sigma(s) v''(d_1) + \delta \sigma''(d_1)(v(d_2) - v(1))}{u'(c_1) - p_1 u''(c_1)} < 0\]

by Assumption 2. The weak monotonicity for boundary solutions is equally straightforward.

(ii) Assume \(d_t, d_{t+1}\) are interior. Then

\[\frac{\partial d_t}{\partial d_{t+1}} = \frac{p_1^2 u''(c_t) + \sigma(d_{t-1}) v''(c_t) + \delta \sigma''(v(d_{t+1}) - v(1))}{-\sigma'(d_t) v'(d_{t+1})} > 0\]

by Assumptions 2 and 3. Part (i) and a straightforward application of the chain rule implies the result for the interior case. Weak monotonicity for the case of boundary solutions is equally straightforward.

\[\square\]
Proposition 3(i) shows that the drug is a normal good under our assumptions. Proposition 3(ii) shows that drug demand decreases if the future price of the drug increases. This connection between current demand and future prices has been documented in the literature on drug demand. Hence, Proposition 3 shows that our model is consistent with this empirical finding.

To understand the argument for Proposition 3(ii), note that by part (i) drug demand in period $t$ decreases as $p_t$ increases. As a result the period $t$ cost of self-control, given by $\sigma(d_{t-1})(v(1) - v(d_t))$, increases. Since the drug is addictive it follows that $\sigma$ is increasing. But this implies that drug consumption in period $t - 1$ becomes less attractive, since it is associated with a greater marginal increase in the cost of self-control in period $t$. Hence, drug demand in period $t - 1$ decreases. Proceeding inductively, we conclude that drug demand in period 1 must decrease.

Empirical work on drug demand has found support for the result described in Proposition 3. Becker, Grossman and Murphy (1994) and Gruber and Koszegi (2001) find support for the prediction that drug demand decreases as future prices increase. Note that results analogous to Proposition 3 have been shown for other models of addiction. Becker, Grossman and Murphy (1994) show this result for quadratic utility for the Becker-Murphy model. Gruber and Koszegi (2001) analyze a decision problem very similar to the one analyzed in this section. They give conditions under which Proposition 3(ii) holds in a $\beta - \delta$ model of addiction with quadratic utility.

4. Drug Policy and Welfare

Drug policies affect consumers along two dimensions. On the one hand, they affect the availability of the drug - and hence the feasible drug consumptions. On the other hand, they affect the price of the drug and hence the opportunity cost of drug consumption.

A moderate tax on a drug such a cigarettes will affect the opportunity cost of drug consumption without changing the feasible drug consumption in the current period if the period length is not too long. The reason is that cigarette consumption is a relatively small part of a typical consumer’s budget. If we assume that the length of a period is a week then a typical consumer can afford the maximally feasible cigarette consumption in a period before and after a moderate tax increase.
By contrast, the prohibition of drug consumption will affect the maximally feasible drug consumption for a typical consumer. Often prohibitive policies will be accompanied by a higher opportunity cost of drug consumption. For analytical clarity we separate the prohibitive effects of a policy from the price effects in the analysis below.

A drug policy is a pair \((\tau, q)\) where \(\tau \geq 0\) is a per unit tax on the drug and \(q \in [0, 1]\) is the maximum feasible drug consumption. Let

\[ B(\tau, q) = \{(c, d) \in [0, 1]^2 | c + (p + \tau)d \leq 1, d \leq q\} \]

denote the individual’s opportunity set under the policy \((\tau, q)\). We assume that the agent faces a stationary decision problem in which he must choose \((c, d)\) from \(B(\tau, q)\) in every period. Note that in this section we assume that prices (and the parameter \(q\)) are constant across time. This is done for simplicity. We denote with \(y(\tau, q)\) the corresponding decision problem.

Any policy \((0, q)\) with \(q < 1\) is a purely prohibitive policy since it reduces the maximum feasible drug consumption but does not affect the opportunity cost of drugs. A pure price policy is a policy \((\tau, 1)\) with \(p + \tau \leq 1\). In this case, the maximum feasible drug consumption remains 1 in every period but the opportunity cost of the drug is increased to \(p + \tau\). If the tax is high enough, in particular, if \(p + \tau > 1\) then the policy \((\tau, 1)\) also has a prohibitive effect since it decreases the maximal drug consumption to \(\frac{1}{p+\tau}\).

Propositions 5 and 6 examine the welfare effects of prohibitive and price policies. Proposition 5 considers the case where the \(u\) does not depend on drug consumption (Assumption 2) and assumes the drug is addictive. Under those assumptions, Proposition 5 shows that a more restrictive prohibitive policy leads to higher welfare than a less restrictive prohibitive policy.

**Proposition 5:** Assume \(u\) and \(v\) satisfy Assumption 1. If the drug is addictive then

\[ W(s, y(\tau, q)) > W(s, y(\tau, q')) \text{ for } q' > q \]

**Proof:** Let \(s = d_0 = d'_0\) be the initial state let \(\{(c_t, d_t)_{t \geq 1}\}\) denote the optimal consumption plan for the decision problem \(y(0, q)\) at state \(s\). Similarly, let \(\{(c'_t, d'_t)_{t \geq 1}\}\) denote the optimal consumption plan for the decision problem \(x(0, q')\) at state \(s\). Define
\( \hat{c}_t = 1 - pd_t' \) and \( \hat{d}_t = \min\{d_t', q\} \) for all \( t \geq 1 \). Clearly, \( \hat{d}_t \leq d_t' \) for all \( t \geq 1 \). Since Assumption 3 is satisfied we write \( u(c) \) instead of \( u(c, d) \). We have \( u(\hat{c}_t) \geq u(c'_t) \) and \( \sigma(\hat{d}_{t-1})[v(\hat{d}_t) - v(q)] \geq \sigma(d'_{t-1})[v(d'_t) - v(q')] \). Moreover, at least one of the two preceding inequalities is strict. To see this, note that if \( v(\hat{d}_t) - v(q) < 0 \) or \( v(d'_t) - v(q') < 0 \) then the second inequality is strict. If \( v(\hat{d}_t) - v(q) = v(d'_t) - v(q') = 0 \) then \( \hat{c}_t > c'_t \) so the first inequality is strict. Hence,

\[
W(s, y(0, q)) \geq \sum_{t=0}^{\infty} \delta^t [u(\hat{c}_t, \hat{d}_t) + \sigma(\hat{d}_{t-1})v(\hat{d}_t) - \sigma(\hat{d}_{t-1})v_0(q)] \\
> \sum_{t=0}^{\infty} \delta^t [(u(c'_t, d'_t) + \sigma(d'_{t-1})v(d'_t) - \sigma(d'_{t-1})v(q'))] \\
= W(s, y(0, q'))
\]

A prohibitive policy has two effects; it reduces self-control costs and it may render the previous level of drug consumption infeasible. The reduction in self-control costs always increases welfare. Assumption 3 ensures that the level of drug consumption that maximizes the commitment utility is zero. Hence, the reduction in consumption increases utility in the current period. Moreover, if the good is addictive, this reduction in consumption leads to a decrease in level of addiction which reduces future self-control costs. Thus, a purely prohibitive policy on an addictive drug always increases welfare.

To see why it is important for the drug to be addictive, consider an agent who is in state \( s = .5 \) in period 1. Suppose that abstaining \( (d = 0) \) or binging \( (d = 1) \) for one period will cause all temptation to go away in the next period but consuming intermediate levels will cause temptation to persist. Moreover, assume that the cost of self-control in the current state is very high. Then, it may be optimal for the agent to binge in the current period and abstain thereafter. In such a situation, a policy that reduces the maximal feasible level of drug consumption from 1 to \( q = .5 \) may reduce the agents welfare by forcing him to either incur the (reduced but still) high cost of self-control in the current period or remain addicted.

Proposition 6 below shows that a pure price policy can never increase welfare. This result is derived with a simple revealed preference argument and therefore does not require any additional assumptions.
Proposition 6: If \( p + \tau' < 1 \) and \( \tau' > \tau \) then \( W(s, y(\tau, q)) \geq W(s, x(\tau', q)) \) for all \( s \).

Proof: Let \( s = d_0 \) denote the initial state and \( \{(c_t, d_t)_{t \geq 1}\} \) be the optimal consumption plan for the problem \( y(\tau, 1) \). Since \( \{(c_t, d_t)_{t \geq 1}\} \) is a feasible choice from \( x(\tau, 1) \) we have

\[
W(s, y(\tau, 1)) \geq \sum_{t=0}^{\infty} \delta^t \left( u(c, d) + \sigma(d_{t-1})v(d^t) - \sigma(d^{t-1})v(1) \right)
\]

\[
= W(s, y(\tau', 1))
\]

A pure price policies does not affect the maximal feasible drug consumption and therefore does not reduce self-control costs. This implies that it cannot improve the agent’s welfare.

Proposition 6 stands in contrast to the findings of Gruber and Koszegi (2001) for the \( \beta - \delta \) model of addiction. One difficulty with the \( \beta - \delta \) model is that clear welfare statements are not possible and therefore it is difficult to make a direct comparison of the results. The reason why pure price policies cannot improve welfare in our setting is that pure price policies cannot eliminate temptations. By definition a pure price policy does not change the maximally feasible drug consumption. Since we assume that the temptation utility depends only on current drug consumption this implies that a price policy cannot eliminate temptations. More generally, a policy can improve welfare only if it can eliminate temptations.

In section 6, Theorem 1, we provide a representation theorem that allows for a more general specification of the temptation utility. In that model, the temptation utility depends not only on current drug consumption but also on non-drug consumption and the continuation problem. For that general model, it is possible to find specifications under which a price policy can increase welfare. We analyze the simpler model to capture the idea that the temptation associated with drugs is focused on current consumption of the drug.

Ultimately, it is an empirical question which specification of the temptation utility is more appropriate. To distinguish various specifications for the temptation utility one needs to examine the (policy) choices of consumers. Our specification would predict that
smokers voluntarily seek commitment but vote against an increase in cigarette taxes. By contrast, a formulation for the temptation utility that replicates the welfare effects in Gruber and Koszegi (2001) predicts that smokers seek voluntary commitment but also vote for an increase in cigarette taxes.

Next, we analyze the impact of prohibitive policies on the demand for drugs. Current period drug demand in state $s$ under the policy $(\tau, q)$ is denoted $D(s, y(\tau, q))$. Consider a purely prohibitive policy $(0, q)$. If the prohibitive policy is binding, that is, if $D(s, y(0, q)) = q$ then a reduction in the maximum allowed drug consumption $q$ will obviously lead to a reduction in drug demand. Proposition 7 shows that if the policy is not binding then a reduction in $q$ will lead to an increase in drug demand.

**Proposition 7:** If $(u, v, \sigma)$ satisfy Assumptions 1-3 and $0 < D(s, y(0, q)) < q$ then $\partial D(s, y(0, q))/\partial q < 0$.

**Proof:** Since the optimal consumption is interior, the necessary first order condition is

$$0 = -pu'(1 - pd_1) + \sigma(s)v'(d_1) + \sigma'(d_1)(v(d_2) - v(q)) \equiv A(d_1)$$

Taking the total derivative we get

$$dd_1 A'(d_1) - dq\sigma'(d_1)v'(q) = 0$$

From Assumption 1, we infer that $A'(d_1) < 0$. Since $\sigma' > 0$ and $v' > 0$, the desired result follows.

A prohibitive policy effectively reduces the cost of drug consumption by reducing the future cost of self-control associated with current drug consumption. For this reason, drug demand increases as the prohibitive policy becomes more stringent. By contrast, as Proposition 3 shows, a price policy reduces demand, that is, drug demand is decreasing in $\tau$.

Assumption 1 implies that the agent (in period 0) would choose not to consume the drug in any period if perfect commitment were available. Hence, the fact that the drug is available is unambiguously “bad” for the consumer. Nevertheless, as our results show, policies that reduce drug consumption may reduce welfare while policies that increases drug consumption may increase welfare.
5. Rehabilitation

This section analyzes a situation where the agent can choose to “check into a rehabilitation center.” In our interpretation, rehabilitation centers offer short term commitment to zero drug consumption.

As in the previous sections we consider a simple decision problem that rules out intertemporal transfers of resources. The agent is either in or out of the rehabilitation center. If the agent is out then he faces the budget set

\[ B^o := \{(c, d) | c + d = 1\} \]

If the agent is in then he is committed to a zero drug consumption and hence the choice set is

\[ B^i(a) := \{(c, d) | c = 1 - a, d = 0\} \]

The parameter \( a \in [0, 1] \) represents the cost of commitment.

The agent’s decision problem is as follows. In each period \( t \geq 1 \) the agent finds himself either in and hence choosing from \( B^i(a) \) or out and choosing from \( B^o \). In addition, the agent must choose whether to be in or out in the next period. The decision problems \( x^i(a), x^o(a) \) represent the two situations.

\[ x^o(a) := \{(c, d, x) | (c, d) \in B^o, x \in \{x^o(a), x^i(a)\}\} \]

\[ x^i(a) := \{(c, d, x) | (c, d) \in B^i(a), x \in \{x^o(a), x^i(a)\}\} \]

In period 1 the agent faces the decision problem \( x^0 \).

The following propositions characterize optimal rehabilitation strategies for the addict. To simplify the notation below, we write \((c, d, j)\) with \( j \in \{i, o\} \) for a choice from \( x^k(a) \), \( k = i, o \). An optimal policy for \( x^k(a) \) is a sequence \((c_t, d_t, j_t), t = 1, 2, \ldots \). We assume that the drug is addictive, that \( v \) is increasing and that there is a unique optimal policy. Proposition 8 establishes that under these conditions only three patterns of behavior can emerge. If the cost of rehab is too high the addict never utilizes the program. If rehab is very inexpensive, then agent eventually enters rehab once he is in he stays in. Between these
two extremes, we observe a cycle of addiction and rehabilitation, where the agent increases his drug consumption as long as he is not in rehab, then he enters rehab for one period and afterwards restarts the cycle of increasing drug consumption.

**Proposition 8:** Suppose the drug is addictive and \( v \) is increasing in drug consumption. Further assume that \((c_t, d_t, j_t)\) is the unique optimal policy for the decision problem \(x^o(a)\). Then, \((c_t, d_t, j_t)\) satisfies one of the following:

(i) \( j_t = i \) for all \( t \) and \( d_t = 0 \) for all \( t > 1 \);

(ii) \( j_t = o \) for all \( t \) and \( d_t \leq d_{t+1} \) for all \( t \);

(iii) there is \( N \in \{2, \ldots \} \) and \((\hat{c}_n, \hat{d}_n, \hat{j}_n), n = 1, \ldots, N \) such that \((c_t, d_t, j_t) = (\hat{c}_n, \hat{d}_n, \hat{j}_n)\) if \( t = kN - 1 + n \) for some \( k = 0, 1, \ldots \). Moreover, \( 0 = \hat{d}_1 < \hat{d}_2 < \ldots < \hat{d}_N \) and \( j_n = i \) if and only if \( n = N \).

**Proof:** Let \((c_t, d_t, i_t)\) denote the unique optimal policy. Note that

\[
W(s, x^i(a)) = W(0, x^i(a))
\]

since the agent is committed to zero drug consumption is zero in \( x^i(a) \). Further, note that \( u(c, d) + \sigma(s)(v(d) - v(1)) + \delta \max\{W(d, x^i(a)), W(d, x^o(a))\} \) is non-increasing in \( s \) since \( \sigma \) is increasing and \( v(d) \leq v(1) \) for all \( d \). Therefore, \( W(s, x^o(a)) \) is non-increasing in \( s \).

First, consider the case where \( W(0, x^i(a)) > W(0, x^o(a)) \). Hence, the agent prefers to be “in” when the state is 0. By the above argument it follows that \( W(s, x^i(a)) > W(s, x^o(a)) \) for all \( s \). This implies that the agent chooses \( j_t = i \) for all \( t = 1, 2, \ldots \). Hence, case (i) applies.

Next, consider the case where \( W(d_t, x^i(a)) < W(d_t, x^o(a)) \) for all \( d_t \). In that case the agent chooses \( j_t = o \) for all \( t = 1, 2, \ldots \). Note that the agent’s consumption plan is optimal for the stationary decision problem in which he faces the budget set \( B^o \) in every period. Let \( x = \{(c, d, x) \mid (c, d) \in B^o\} \) denote the corresponding decision problem. By Proposition 2, the drug demand from \( x \) is monotonically increasing in \( s \). This in turn implies that \( d_t \) is non-decreasing and case (ii) applies.
It remains to show that if \( W(0, x^0(a)) > W(0, x^i(a)) \) and \( W(d_t, x^i(a)) > W(d_t, x^0(a)) \) for some \( d_t \), then case (iii) applies. Since \( W(0, x^0(a)) > W(0, x^i(a)) \) it follows that \( j_t = i \) implies \( j_{t+1} = o \). Since \( W(d_t, x^i(a)) > W(d_t, x^0(a)) \) for some \( d_t \) along the optimal path, it follows that \( j_t = i \) for some \( t \). Let \( \tau \) be the first period \( t \) such that \( j_t = i \). Note that in period \( \tau + 2 \) the agent makes an optimal choice from the decision problem \( x^0(a) \) in state 0 which is the same state and the same decision problem the agent faces in period 1. Since there is a unique optimal policy it follows that optimal choices in periods \( \tau + 2, ..., 2\tau + 1 \) are identical to the choices in periods 1, ..., \( \tau \). It remains to show that \( 0 < d_1 < ... < d_\tau \). Note that by Proposition 2 the drug demand in \( x^0(a) \) is non-decreasing in the state. Hence, it follows that \( 0 \leq d_1 \leq ... \leq d_\tau \). Because the optimal policy is unique, the drug demand must be strictly increasing. To see this first suppose \( d_t = d_{t+1} \), \( t < \tau \). Then \((c_t, d_t, o)\) is an optimal choice from \( x^0(a) \) at state \( d_t \). But this contradicts the uniqueness of the optimal choice and the fact that \( j_t = i \) for some \( t \). If \( d_{\tau-1} = d_\tau = d \) then it must be that \( W(d, x^i(a)) = W(d, x^0(a)) \) again contradicting the uniqueness of the optimal choice. 

The example above also illustrates that cheaper rehabilitation centers may increase drug consumption in some periods. The following proposition demonstrates that this effect is present for all utility functions that satisfy Assumptions 1-3. More precisely, suppose the cost of rehab is so high that in state \( s \) it is not optimal to choose \( i \). Now assume that this cost is lowered such that \( i \) is the optimal choice. As a consequence of the less expensive rehab option, drug consumption in the current period increases.

**Proposition 9:** Let \((u, v)\) satisfy Assumptions 1-3 and let \( a < a' \). If \((c, d, i)\) is an optimal choice from \( x^0(a) \) in state \( s \) and \((c', d', o)\) is an optimal choice from \( x^0(a') \) in state \( s \) then \( d \geq d' \). A strict inequality holds if \( d > 0 \).

**Proof:** Note that \((c, d)\) maximizes \( u(\hat{c}) + \sigma(s)v(\hat{d}) \) since next period the agent is committed to a zero drug consumption. The pair \((c', d')\) maximizes

\[
u(\hat{c}) + \sigma(s)v(\hat{d}) + \delta\sigma(\hat{d})(v(\hat{d}) - v(1))\]

where \( \hat{d} \) is the optimal drug consumption in the next period. Note that \( \hat{d} < 1 \) since otherwise \((c, d, i)\) leads to higher utility than \((c', d', o)\). Hence,

\[
u' + \sigma(s)v' + \delta\sigma'(v(\hat{d}) - v(1)) < u' + \sigma(s)v'
\]
If \( d = 1 \) then we are done. If \( 0 < d < 1 \) then the first order condition at \( d \) holds with equality and hence \( d' > d \). If \( d = 0 \) then \( d' = 0 \).

Although drug demand in period 0 may increase as a result of less expensive rehab the agent’s welfare increases as rehab becomes cheaper.

**Proposition 10:** If \( u(\cdot, d) \) and \( v \) are non-decreasing and \( a > a' \) then \( W(s, x^o(a')) \geq W(s, x^o(a)) \).

**Proof:** Let \((c_t, d_t, j_t)\) denote an optimal policy for the decision problem \( x^o(a) \). We have

\[
W(s, x^o(a')) = \sum_{t=1}^{\infty} \delta^{t-1}(u(c_1, d_1) + v(d_1) - v^\text{max}_t)
\]

where \( v^\text{max}_t \) denotes the maximally feasible drug consumption in period \( t \). Note that \( v^\text{max}_t = 1 \) in period 1. In all other periods it is 1 if \( j_{t-1} = o \) and 0 if \( j_{t-1} = i \). Since \( a < a' \) there is a feasible policy \((\hat{c}_t, \hat{d}_t, \hat{j}_t)\) for \( x^o(a') \) policy with \( \hat{d}_t = d_t, \hat{j}_t = j_t \) and \( \hat{c}_t \geq c_t \). The utility of this policy is

\[
\sum_{t=1}^{\infty} \delta^{t-1}(u(\hat{c}_1, \hat{d}_1) + v(\hat{d}_1) - v^\text{max}_t) \geq W(s, x^o(a))
\]

Hence \( W(s, x^o(a')) \geq W(s, x^o(a)) \).

As in the previous section we find that the success of “treatment options” cannot be determined by examining their effect on drug consumption. The option of checking into a rehabilitation center is unambiguously welfare improving for the agent even though it may actually increase overall drug consumption.

**6. Representation Theorems**

In this section we provide two representation theorems. Theorem 1 axiomatizes a representation that is more general than the one used in the applications above. In particular, the representation allows for a general (finite) state space and a general specification of the temptation utility. Theorem 2 provides additional axioms that yield the representation used in the applications above.
The set of consumptions in each period is \( C = [0,1]^2 \) and \( b \in C \) denotes a generic consumption vector. For any subset \( X \) of a metric space, we let \( \Delta(X) \) denote the set of all probability measures on the Borel \( \sigma \)-algebra of \( X \) and \( K(X) \) denote the set of all nonempty compact subsets of \( X \). An infinite horizon decision problem (denoted \( z \in Z \)) can be identified with an element in \( K(\Delta(C \times Z)) \) and conversely each element in \( K(\Delta(C \times Z)) \) identifies a decision problem \( z \in Z \). For formal definitions of \( Z \) and the map that associates each element of \( Z \) with its equivalent recursive description as an element of \( K(\Delta(C \times Z)) \), we refer the reader to Gul and Pesendorfer (2000b). In what follows only the recursive definition is used and hence without risk of confusion we identify the sets \( Z \) and \( K(\Delta(C \times Z)) \).

The individual’s preferences are defined on \( Z \) and are indexed by \( s \in S \), the state in the initial period of the decision problem. The state \( s \) represents the relevant consumption history prior to the initial period. We assume that there is a finite number \( K \) such that consumption in only the last \( K \) periods influences the agents preferences and therefore \( S := C^K \). Without loss of generality, we assume that \( K \) is the minimal length of the individual’s consumption history that allows us to describe \( \geq \).

For any state \( s = (b_1, \ldots, b_K) \) let \( sb \) denotes the state \( (b_2, \ldots, b_K, b) \). We impose the following axioms on \( \succeq_s \) for every \( s \in S \).

**Axiom 1:** (Preference Relation) \( \succeq_s \) is a complete and transitive binary relation.

**Axiom 2:** (Strong Continuity) The sets \( \{ x \mid x \succeq_s z \} \) and \( \{ x \mid z \succeq_s x \} \) are closed in \( Z \).

**Axiom 3:** (Independence) \( \{\mu\} \succ_s \{\nu\} \) implies \( \{\alpha \mu + (1-\alpha)\eta\} \succ_s \{\alpha \nu + (1-\alpha)\eta\} \) \( \forall \alpha \in (0,1) \).

Axioms 1-3 are standard. In Axiom 4 we deviate from standard choice theory and allow the possibility that the adding options to a decision problem makes the consumer

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\(^7\) That is, there is a pair of states, \( (s = (b_1, \ldots, b_K), \hat{s} = (\hat{b}_1, \ldots, \hat{b}_K)) \) that differ only in their first component \( (b_1 \neq \hat{b}_1, b_t = \hat{b}_t, t \geq 2) \) and lead to different preferences \( (\succeq_s \neq \succeq_{\hat{s}}) \).
strictly worse off. For a detailed discussion of Axiom 4, we refer the reader to our earlier paper (Gul and Pesendorfer 2000a).

**Axiom 4:** *(Set Betweenness)* \( x \succeq_s y \) implies \( x \succeq_s x \cup y \succeq_s y \).

Next, we make a separability assumption. For \( z \in Z \) let \( bz \in Z \) denote the decision problem \( \{(b, z)\} \), that is, the degenerate decision problem that yields \( c \) in the current period and the continuation problem \( z \). Thus \( b_1b_2 \ldots b_Kz \) is a degenerate decision problem that yields the consumption \( (b_1, \ldots, b_K) \) in the first \( K \) periods and the continuation problem \( z \) in period \( K + 1 \). For \( s = (b_1, \ldots, b_K) \) we write \( sz \) instead of \( b_1b_2 \ldots b_Kz \). Axiom 5 considers decision problems of the form \( \{(b, sz)\} \) and requires that preferences are not affected by the correlation between current consumption \( c \) and the \( K + 1 \) period continuation problem \( z \).

**Axiom 5:** *(Separability)* \( \{(\frac{1}{2}(b, sz) + \frac{1}{2}(c', sz'))\} \sim_s \{(\frac{1}{2}(b, sz') + \frac{1}{2}(c', sz))\} \).

Axiom 6 requires preferences to be stationary. Consider the degenerate lotteries, \( (b, x) \) and \( (b, y) \), each leading to the same period 1 consumption \( c \). Stationarity requires that \( \{(b, x)\} \) is preferred to \( \{(b, y)\} \) in state \( s \) if and only if the continuation problem \( x \) is preferred to the continuation problem \( y \) in state \( sb \).

**Axiom 6:** *(Stationarity)* \( \{(b, x)\} \succeq_s \{(b, y)\} \) iff \( x \succeq_{sb} y \).

Note that Axiom 6 implies that the conditional preferences at time \( K + 1 \) after consuming \( s \) in the first \( K \) periods is the same as the initial preference \( \succeq_s \). Together, Axioms 5 and 6 restrict the manner in which past consumption influences future preferences. Axiom 5 ensures that correlation between consumption prior to period \( t - K \) and the decision problem in period \( t \) does not affect preferences whereas Axiom 6 ensures that the realization of consumption prior to period \( t - K \) does not affect preferences in period \( t \).

Axiom 7 requires individuals to be indifferent as to the timing of resolution of uncertainty. In a standard, expected utility environment this indifference is implicit in the assumption that the domain of preference is the set of lotteries over consumption paths. Our domain of preferences are decision problems and in this richer structure a separate assumption is required to rule out agents that are not indifferent to the timing of resolution of uncertainty as described by Kreps and Porteus (1978).
Consider the lotteries $\mu = \alpha(b, x) + (1 - \alpha)(b, y)$ and $\nu = (b, \alpha x + (1 - \alpha)y)$. The lottery $\mu$ returns the consumption $c$ together with the continuation problem $x$ with probability $\alpha$ and the consumption $c$ with the continuation problem $y$ with probability $1 - \alpha$. By contrast, $\nu$ returns $c$ together with the continuation problem $\alpha x + (1 - \alpha)y$ with probability 1. Hence, $\mu$ resolves the uncertainty about $x$ and $y$ in the current period whereas $\nu$ resolves this uncertainty in the future. If $\{\mu\} \sim_s \{\nu\}$ then the agent is indifferent as to the timing of the resolution of uncertainty.

**Axiom 7:** (Indifference to Timing) $\{\alpha(b, x) + (1 - \alpha)(b, y)\} \sim_s \{(b, \alpha x + (1 - \alpha)y)\}$.

**Definition:** $\succeq_s$ is regular if there exists $x, x', y, y' \in Z$ such that $x' \subset x, y' \subset y, x \succ_s x'$ and $y' \succeq_s y$. $\succeq$ is regular if each $\succeq_s$ is regular.\(^8\)

Theorem 1 below establishes that all regular preferences that satisfy Axioms 1 – 7 can be represented as a discounted sum of state-dependent utilities minus state-dependent self-control costs. We say that the function $W : S \times Z \rightarrow \mathbb{R}$ represents $\succeq$ when $x \succeq_s y$ iff $W(s, x) \geq W(s, y)$ for all $s$. For any $\mu \in \Delta$, let $\mu_1$ denote the marginal of $\mu$ on $C$. Axioms 1-7 yield the following representation.

**Theorem 1:** If $\succeq$ is regular and satisfies Axioms 1 – 7, then there exists $\delta \in (0, 1)$, continuous functions $u : S \times C \rightarrow \mathbb{R}$, $V : S \times C \times Z \rightarrow \mathbb{R}$, $W : S \times Z \rightarrow \mathbb{R}$ such that

$$W(s, z) = \max_{\mu \in \Delta} \int \left[ u(s, b) + \delta W(sb, z) + V(s, b, z) \right] d\mu(b, z) - \max_{\nu \in \Delta} \int V(s, b, z) d\nu(b, z)$$

for all $s \in S, \nu \in \Delta$ and $W$ represents $\succeq$. For any $\delta \in (0, 1)$, continuous $u, V$ there exists a unique function $W$ that satisfies the equation above and the $\succeq$ represented by this $W$ satisfies Axioms 1 – 7.

The two main steps of the proof of Theorem 1 entail showing that a preference relation (over decision problems) that satisfies continuity, independence, set betweenness, stationarity and indifference to timing of resolution of uncertainty has a representation of the form

$$W(s, z) = \max_{\mu \in \Delta} \{ U(s, \mu) + V(s, \mu) \} - \max_{\nu} V(s, \nu)$$ (2)

\(^8\) In section 2, we presented the definition of a regular SSC preference. It can be shown that the current general definition and the one offered in section 2 are equivalent for SSC preferences.
and then using stationarity and separability to show that $U$ is of the form $U = u + \delta W$.

In Gul and Pesendorfer (2000b) we offer a related proof under stronger stationarity and separability axioms, yielding a representation of state-independent preferences.

Next, we provide assumptions under which preferences can be represented by a utility function $W$ that satisfies equation (2) (i.e., characterize simple SSC preferences).

Assumption I below is taken from Gul and Pesendorfer (2003). It requires that two alternatives, $\nu, \eta$, offer the same temptation if they have the same marginal distribution over current consumption. For any $\mu \in \Delta(C \times Z)$ we denote by $\mu^1$ the marginal on the first coordinate (current consumption) and by $\mu^2$ the marginal on the second coordinate (the continuation problem).

**Assumption I:** (Temptation by Immediate Consumption) For $\mu, \nu \in \Delta$ suppose $\nu^1 = \eta^1$. If $\{\mu\} \succ \{\mu, \nu\} \succ_s \{\nu\}$ and $\{\mu\} \succ_s \{\mu, \eta\} \succ \{\eta\}$ then $\{\mu, \nu\} \sim_s \{\mu, \eta\}$.

To understand Assumption I, note that $\{\mu\} \succ \{\mu, \nu\} \succ \{\nu\}$ represents a situation where the agent is tempted by $\nu$ but chooses $\mu$ from $\{\mu, \nu\}$. Similarly, $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ means that the agent is tempted by $\eta$ but chooses $\mu$. Hence, the agent makes the same choice in both situations. If $\nu^1 = \eta^1$ then immediate temptation means that the agent experiences the same temptation in the two situations and therefore is indifferent between them; $\{\mu, \nu\} \sim \{\mu, \eta\}$.

Assumption N below ensures that goods other than $d$ are neutral, i.e., cause no temptation and have no dynamic effects. That is, only good $d$ is tempting and only past consumption of $d$ affects future rankings of decision problems.

**Assumption N:** Let $b = (c, d)$ and $b' = (c', d')$. If $d = d'$ then $\{(b, z), (b', z')\} \succeq_s \{(b, z)\}$ and $\succeq_{sb} \succeq_{sb'}$. If $d' > d$ and $\{(b, z)\} \succeq_s \{(b', z')\}$ then $\{(b, z)\} \succeq_s \{(b, z), (b', z')\}$.

The first statement in Assumption N ensures that there is no temptation so long as the options differ only with respect to current consumption of non-drugs. The second statement means that future preferences are the same so long as the current state and current consumption of drugs are the same. Finally, the third statement implies that higher current drug consumption is always tempting.

---

9 This follows from a straightforward application of the representation in (2).
To state the final assumption, we first define what it means for the agent to have the same preference for commitment at two states.

**Definition:** $\succeq_s$ has a preference for commitment at $z$ if there is $x \subset z$ such that $x \succeq_s z$; $\succeq$ has the same preference for commitment at $s'$ and at $s$ if $\succeq_s$ at $z$ iff $\succeq_{s'}$ at $z$.

Assumption P says that the agent’s preference for commitment does not change as the state changes. In other words, which alternatives constitute a temptation is independent of the state.

**Assumption P:** *The agent has the same preference for commitment at all $s$.***

**Theorem 2:** *Let $\succeq$ be a regular SSC preference satisfying I, N and P. Then, (i) $S = [0, 1]$, and (ii) there are continuous functions $v, \sigma : [0, 1] \rightarrow \mathbb{R}$, $u : C \rightarrow \mathbb{R}$, and $\delta \in (0, 1)$, such that for $z \in \bar{Z}$

$$W(s, z) = \max_{(c,d,x) \in z} \{u(c,d) + \sigma(s)v(d) + \delta W(c,d,x)\} - \max_{(c',d',y) \in z} \sigma(s)v(d')$$

and $W$ represents $\succeq$. (iii) $v$ is strictly increasing; $\sigma > 0$; and $s$ is the previous period’s drug consumption.

**Proof:** See Appendix.

To illustrate the role of the assumptions in Theorem 2, consider the representation of SSC preferences provided in Theorem 1. Then, Assumption I ensures that $V(s, \cdot)$ depends only on current consumption. Assumption N guarantees that $V(s, \cdot)$ depends only on current drug consumption and is strictly increasing in $d$. Finally, Assumption P implies that $U = u + \delta W$ is independent of the state and that the state is equal to last period’s drug consumption.
7. Conclusion

Most studies on drug abuse emphasize that addiction should be considered a disease.\(^\text{10}\) In our approach drug abuse is identified with the discrepancy between what the agent would want to commit to, as reflected by maximizing \(U\), and what he ends-up consuming by maximizing \(U + V\). We provide straightforward choice experiments for measuring this discrepancy. Our approach is silent on the question of whether addiction is a disease or a part of the “normal” variation of preferences across individuals.

While our approach is compatible with the disease concept of addiction, there are important differences between the two. Consider the following example: the opiate antagonist naltrexone blocks the opioid receptors in the brain and hence the euphoric effects of these drugs for up to 3 days after the last dose. Naltrexone is used in the treatment of heroin and morphine. However, with the exception of highly motivated addicts such as parolees, probationers and health care professionals, most addicts receiving naltrexone tend to stop taking their medicine and relapse. Addicts often report that they stop taking naltrexone because it prevents “getting high”. Doctors call this as a “compliance problem” with naltrexone. For them, this is simply a limitation on the usefulness naltrexone, the same way that toxicity might be a limitation on the usefulness of some other medication.

In our model, there can be two reasons for an addict to discontinue naltrexone and resume heroin consumption: either 3 days is not the right time horizon for commitment or the addict does not wish to commit. The former would suggest a need for longer acting drugs while the latter would mean that there is neither a need nor any room for treatment of this addict. In fact, by our definition, an individual who is unwilling to commit to reducing his drug consumption, for any length of time, at any future date is not an addict. Hence, where the disease model of addiction finds a compliance problem our model suggests that there may be no problem at all.

The fact that naltrexone continues to be used by the most motivated addicts, those who are more likely to abstain even without commitment, suggests a reduction of the cost of self-control as a possible motive taking naltrexone.

\(^{10}\) To emphasize the organic basis of the condition the term “disease of the brain” is often used.
Economists interpret behavior as a reflection of the agents’ stable interests and desires. In standard economic analysis there is no room for the notion of a behavioral problem, except to the extent that the behavior is a problem for someone else. Consequently, there is no role for therapy aimed at controlling problem behavior. In contrast, psychologists often view behavior to be independent of and even an impediment to the agent’s welfare. Our model of temptation and self-control provides a potential bridge between these two approaches. Like standard models in economics, we take as given agents’ interests and desires (i.e. utility functions) and accept the hypothesis that behavior is motivated by these interests and desires (i.e. utility maximization). But, we extend the domain of utility functions to include temptation. Without the aid of some outside agency, it is difficult and often very costly for the individual to commit, that is; reduce temptation. Hence, our model leaves room for welfare enhancing treatments and policy. In our interpretation, the role of treatment and policy is to develop commitment devices and opportunities for the agent.

Our model provides a framework for the analysis of both the purposeful actions (e.g. decisions made in the stock market) studied by most economists as well as the compulsive and detrimental behavior (e.g. addiction) studied by many psychologists and health care professionals. We have analyzed the interaction of these two types of behavior and evaluated policy alternatives. Our focus was on psychoactive drugs but the model presented in this paper can also be applied to other types of compulsive behavior such as over-eating and other forms of dependency.
8. Appendix

Proof of Proposition 1: To prove the “if” part, let \( \sigma(\hat{s}) \geq \sigma(s) \) and let \( \mu \in D(\hat{s}, z) \cap C(z) \). Then

\[
\int (u(c', d') + \sigma(\hat{s})v(d') + \delta W(d', z))d\mu(c', d', z) \geq \\
\int (u(c', d') + \sigma(\hat{s})v(d') + \delta W(c', d', z))d\nu(c', z) \\
\int (u(c', d') + \delta W(d', z))d\mu(c', d', z) \geq \\
\int (u(c', d') + \sigma(\hat{s})v(d') + \delta W(c', d', z))d\nu(c', z)
\]

for all \( \nu \in z \). Since \( \sigma > 0 \) there is \( \alpha \in (0, 1] \) such that \( \sigma(c) = \alpha \sigma(\hat{c}) \). Taking a convex combination of the above two inequalities we conclude that \( \mu \in D(s, z) \cap C(z) \). Hence, if \( \succeq \hat{s} \) is not compulsive then \( \succeq s \) is not compulsive. Obviously \( \succeq s \neq \succeq \hat{s} \) if \( \sigma(\hat{s}) > \sigma(s) \).

To prove the “only if” part we can repeat the argument of Lemma 12 from Gul and Pesendorfer (2000a) in the current setting to obtain the following fact.

**Fact:** \( \succeq \) is more compulsive at \( \hat{s} \) than at \( s \) only if for some \( \beta \in \mathbb{R}_+^3 \)

\[
U + \sigma(s)v = \beta_1(U + \sigma(\hat{s})v) + \beta_2\sigma(\hat{s})v + \beta_3
\]

for all \( \mu \).

Note that

\[
U + \sigma(s)v = \frac{\sigma(\hat{s})}{\sigma(s)}U + \frac{1 - \sigma(s)}{\sigma(\hat{s})}(U + \sigma(\hat{s})v)
\]

and therefore the only if part follows from the fact.

8.1 Proof of Proposition 2

Let \( d \) be the maximal element in \( D(s, z) \) and let \( (c, d, x) \in D(s, z) \) be a corresponding choice. Then,

\[
u(c, d) + \sigma(s)v(d) + \delta W(d, x) \geq u(c', d', x') + \sigma(s)v(d') + \delta W(c', d', x')
\]
for all \((c', x') \in z\). Suppose that \(\sigma(\hat{s}) > \sigma(s)\). Then, for any \((c', d', z') \in z\) with \(d' < d\)

\[
u(c, d) + \sigma(\hat{c})v(d) + \delta W(d, x) > \nu(c', d') + \sigma(\hat{c})v(d') + \delta W(d', x')
\]

Hence \(D(\hat{s}, z) \geq D(s, z)\).

\[\square\]

9. Proof of Theorems 1 and 2

9.1 Proof of Theorem 1

It is easy to show that if \(\succeq\) satisfies Axioms 3, 6 and 7 then it also satisfies the following stronger version of the independence axiom:

**Axiom 3**: \(x \succ s y, \alpha \in (0, 1)\) implies \(\alpha x + (1 - \alpha)z \succ s \alpha y + (1 - \alpha)z\).

Theorem 1 of Gul and Pesendorfer (2000) establishes that \(\succeq_s\) satisfies Axioms 1, 2, 4 and 3* if and only if there exist \(W(s, \cdot), U(s, \cdot), V(s, \cdot)\) such that

\[
W(s, z) := \max_{\mu \in z} \{U(s, \mu) + V(s, \mu)\} - \max_{\nu \in z} V(s, \nu)
\]

for all \(z \in Z\) and \(
\hat{W}\) represents \(\succeq_s\). Moreover, the functions \(W(s, \cdot), U(s, \cdot), V(s, \cdot)\) are continuous and linear in their second arguments. We refer to the triple \((U(s, \cdot), V(s, \cdot), W(s, \cdot))\) as a representation of \(\succeq_s\). The content of Theorem 1 is that we may choose functions \((U, V, W)\) that are continuous in \(s\) such that \((U(s, \cdot), V(s, \cdot), W(s, \cdot))\) is a representation of \(\succeq_s\) for each \(s\) and \(U(s, \cdot)\) satisfies

\[
U(s, \mu) = \int [u(s, b) + \delta W(sb, z)]d\mu(c, z)
\]

for some continuous function \(u\) and \(\delta \in (0, 1)\).

Fix \(\bar{s}\) and let \((\hat{W}(\bar{s}, \cdot), \hat{U}(\bar{s}, \cdot), \hat{V}(\bar{s}, \cdot))\) be a representation of \(\succeq_{\bar{s}}\). Define \(W\) to be the following function:

\[
W(s, y) := \hat{W}(\bar{s}, sy)
\]

Observe that \(W\) is well defined and continuous in both arguments since \(\hat{W}\) is continuous in its second argument. In the following Lemmas, the function \(W\) is the function defined in (**).
Lemma 1: \( W \) represents \( \succeq \). Moreover, there exist continuous functions \( U, V \) such that

\[
W(s, z) := \max_{\mu \in z} \{U(s, \mu) + V(s, \mu)\} - \max_{\nu \in z} V(s, \nu)
\]

and \( W, U, V \) are linear in their second arguments.

Proof: Axiom 6 implies \( W(s, x) \geq W(s, y) \) iff \( \check{W}(s, x) \geq \check{W}(s, y) \). Therefore, \( W \) represents \( \succeq \). Note that \( \check{W} \) is linear in its second argument. Let \( z = \alpha x + (1 - \alpha) y \). Axiom 7 and linearity of \( \check{W} \) in its second argument imply that

\[
W(s, z) = \check{W}(s, \alpha x + (1 - \alpha) y) = \alpha \check{W}(s, x) + (1 - \alpha) \check{W}(s, y)
\]

Thus, \( W \) is linear in its second argument. It follows that \( W(s, z) = \alpha(s) \check{W}(s, z) + \beta(s) \) for some \( \alpha, \beta : S \to \mathbb{R} \) such that \( \alpha(s) \geq 0 \). Since \( \succeq \) is regular, \( \alpha(s) > 0 \) for all \( s \). Hence, \( U = \alpha \check{U} + \beta, V = \alpha \check{V} \) and the \( W \) have the desired properties.

\[\Box\]

Lemma 2: Let \( \check{W}(s, \cdot) \) represent \( \succeq \). Then,

\[
\check{W}(s, b_1 \ldots b_l \check{z}) - \check{W}(s, b_1 \ldots b_l \check{z}) = \check{W}(s, \check{b}_1 \ldots \check{b}_l \check{z}) - \check{W}(s, \check{b}_1 \ldots \check{b}_l \check{z})
\]

for all \( l, (\check{b}_1, \ldots \check{b}_l), (b_1, \ldots b_l) \in C^{l+1}, \check{z} \in C^K, z, \check{z} \in Z \).

Proof: Note that by Axiom 5,

\[
\frac{1}{2}(\check{b}_1, b_2 \ldots b_l \check{z}) + \frac{1}{2}(b_1, b_2 \ldots b_l \check{z}) \sim_s \frac{1}{2}(b_1, b_2 \ldots b_l \check{z}) + \frac{1}{2}(b_1, b_2 \ldots b_l \check{z})
\]

Assume that the assertion holds for \( l' \leq l - 1 \). Then, Axiom 6 implies that

\[
\frac{1}{2}(\check{b}_1, b_2 \ldots b_l \check{z}) + \frac{1}{2}(b_1, b_2 \ldots b_l \check{z}) \sim_s \frac{1}{2}(b_1, b_2 \ldots b_l \check{z}) + \frac{1}{2}(b_1, b_2 \ldots b_l \check{z})
\]

and hence the assertion holds for \( l' \leq \check{l} \). Observe that Axiom 5 implies that the Lemma holds for \( l = 1 \). \[\Box\]
Lemma 3: \( W(s', sx) - W(s', sy) = W(s'', sx) - W(s'', sy) \) for all \( s', s'' \).

Proof: Recall that
\[ W(s', sx) = \hat{W}(\bar{s}, s' sx) \]
for some \( \hat{W} \) such that \( \hat{W}(\bar{s}, \cdot) \) represents \( \succeq_{\bar{s}} \). Lemma 2 implies that
\[ \hat{W}(\bar{s}, s' sx) - \hat{W}(\bar{s}, s' sy) = \hat{W}(\bar{s}, s'' sx) - \hat{W}(\bar{s}, s'' sy) \tag{\dagger} \]
Substituting \( W \) for \( \hat{W} \) in equation (\dagger) then proves the Lemma.

Lemma 3: There exist \( \delta : S \times C \to (0, \infty) \) and \( u : S \times C \to IR \) such that \( U(s, \nu) = \int [u(s, b) + \delta(s, b)W(sb, z)]d\nu(b, z) \) for all \( s \in S, \nu \in \Delta \).

Proof: Since \( U(s, \cdot) \) is linear and continuous, it has an integral representation. That is,
\[ U(s, \nu) = \int U(s, b, z)d\nu(b, z) \]
By Axiom 6, \( U(s, b, \cdot) \) and \( W(sb, \cdot) \) yield the same linear preferences over \( Z \). By regularity, neither function is constant. It follows that \( U(s, b, \cdot) \) is a strictly positive affine transformation of \( W(sb, \cdot) \). Hence, for some \( u, \delta \),
\[ U(s, b, \cdot) = u(s, b) + \delta(s, b)W(sb, y) \]
where \( \delta(s, b) > 0 \) for all \( s \in S, b \in C \). Therefore,
\[ U(s, \nu) = \int [u(s, b) + \delta(s, b)W(sb, y)]d\nu(b, z) \]
as desired.

Lemma 4: The function \( \delta(\cdot) \) in Lemma 3 is constant.

Proof: Suppose \( \delta \) is not constant. Let \( k \in 1, \ldots, K + 1 \) denote the smallest integer such that \( \delta(b_1, \ldots, b_{K+1}) = \delta(\bar{b}, \ldots, \bar{b}_{K+1}) \) for all \( (b_1, \ldots, b_{K+1}), (\bar{b}_1, \ldots, \bar{b}_{K+1}) \) with \( b_n = \bar{b}_n \) for \( n \leq k \). Then, it is straightforward to show that there exist \( (s, b_{K+1}) = (b_1, \ldots, b_{K+1}) \) and \( (s^*, b^*_{K+1}) = (b^*_1, \ldots, b^*_{K+1}) \) such that \( b_n = b^*_n, n \neq k \) and \( \delta(b_1, \ldots, b_{K+1}) > \delta(b^*_1, \ldots, b^*_{K+1}) \).
Pick any $b \in C$. Let $s' = (b, \ldots, b, b_1, b_2, \ldots, b_{k-1})$. Fix any $\tilde{s}$. By regularity there are $y_h, y_l \in Z$ such that $W(\tilde{s}, y_h) > W(\tilde{s}, y_l)$. Let $y_{hh} = b_k \ldots b_{K+1}\tilde{s}y_h, y_{hl} = b_k \ldots b_{K+1}\tilde{s}y_l$ and $y_{lh} = b_k \ldots b_{K+1}\tilde{s}y_h, y_{ll} = b_k \ldots b_{K+1}\tilde{s}y_l$. Let $x = .5y_{hh} + .5y_{hl}$ and $z = .5y_{hl} + .5y_{hl}$. By Lemma 2, $W(s', x) = W(s', z)$.

Applying Lemma 4 repeatedly and using the fact that $\delta(s, b) = \delta(\tilde{s}, \tilde{b})$ for $(s, b), (\tilde{s}, \tilde{b})$ with $b_n = \tilde{b}_n, n \leq k$ establishes $W(s', x) = W(s', z) = 0$ iff
\[ \delta(s, b_{K+1})W(sb_{K+1}, \tilde{s}y_h) + \delta(s^*, b_{K+1}^*)W(s^*b_{K+1}^*, \tilde{s}y_l) = \]
\[ \delta(s, b_{K+1})W(sb_{K+1}, \tilde{s}y_l) + \delta(s^*, b_{K+1}^*)W(s^*b_{K+1}^*, \tilde{s}y_h) \]
Rearranging, this implies
\[ \delta(s, b_{K+1})(W(sb_{K+1}, \tilde{s}y_h) - W(sb_{K+1}, \tilde{s}y_l)) = \]
\[ \delta(s^*, b_{K+1}^*)(W(s^*b_{K+1}^*, \tilde{s}y_h) - W(s^*b_{K+1}^*, \tilde{s}y_l)) \]
Observe that $W(s, \tilde{s}y_h) - W(s, \tilde{s}y_l) > 0$ by construction and hence Lemma 3 implies the desired contradiction.

**Claim 5:** Let $\delta \in IR$ denote the constant function in Claim 3. Then, $0 < \delta < 1$.

**Proof:** That $\delta > 0$ has already been established. Pick any $b \in C$ and let $s = (b, b, \ldots, b)$. Let $z_b$ denote the unique $z = \in Z$ such that $z = \{b, z\}$. Pick $y_1 \in Z$ such that $W(s, y_1) \neq W(s, z)$. By regularity, such a $y_1$ exists. Define $y_n \in Z$ inductively as $y_n = \{(b, y_n-1)\}$ and note that $y_n$ converges to $z$. Hence, by continuity, $W(s, z) - W(s, y_n)$ must converge to 0. But, by Lemmas 4 and 5 $W(s, z) - W(s, y_n) = \delta^{n-1}(W(s, y_1) - W(s, z)) \neq 0$. Hence, $\delta < 1$.

Lemmas 1-5 establish that there is a continuous representation $(U, V, W)$ that satisfies $U(s, \mu) = u(\mu^1) + \int \delta W(sb, z)d\mu(b, z)$.

To conclude the proof, let $\delta \in (0, 1)$ and $u : S \times C \to IR$ and $V : S \times C \times Z \to IR$ be continuous functions.

**Lemma 6 (A Fixed-Point Theorem):** If $B$ is a closed subset of a Banach space with norm $\|\cdot\|$ and $T : B \to B$ is a contraction mapping (i.e., for some integer $m$ and scalar $\alpha \in (0, 1)$, $\|T^m(W) - T^m(W')\| \leq \alpha \|W - W'\|$ for all $W, W' \in B$), then there is a unique $W^* \in B$ such that $T(W^*) = W^*$.

Let \( W \) be the Banach space of all continuous, real-valued functions on \( S \times Z \) (endowed with the sup norm). The operator \( T : \mathcal{W}_b \to \mathcal{W}_b \), where

\[
TW(s, z) = \max_{\mu \in \mathcal{Z}} \left\{ \int [u(s, b) + V(s, b, z) + \delta W(sb, x)] d\mu(b, x) - \max_{\nu \in \mathcal{Z}} \int V(s, b, z) d\nu(b, z) \right\}
\]

is well-defined and is a contraction mapping. Hence, by Lemma 6, there exists a unique \( W \in \mathcal{W} \) such that \( T(W) = W \).

For any \( W, u, v, \delta \) such that

\[
W(s, z) = \max_{\mu \in \mathcal{Z}} \left\{ \int [u(s, b) + V(s, b, z) + \delta W(sb, x)] d\mu(b, x) \right\} - \max_{\nu \in \mathcal{Z}} V(s, b, z) d\mu(b, z)
\]

define \( \succeq_s \) by \( x \succeq_s y \) iff \( W(s, x) \geq W(s, z) \). Verifying that \( \succeq_s \) satisfies Axioms 1 - 7 is straightforward.

9.2 Proof of Theorem 2

By Theorem 1, \( \succeq \) can be represented by a continuous \( \hat{W} \) where

\[
\hat{W}(s, z) = \max_{\mu \in \mathcal{Z}} \left\{ \int [\hat{u}(s, \mu \uparrow) + \delta \int \hat{W}(sb, x) d\mu(b, x) + \hat{V}(s, \mu)] \right\} - \max_{\nu \in \mathcal{Z}} \hat{V}(s, \nu)
\]

for some continuous \( u, v \) and \( \delta \in (0, 1) \). Moreover, \( \hat{W}, \hat{u}, \hat{V} \) are linear in their second argument. Let \( \hat{U}(s, \mu) = \int (\hat{u}(s, b) + \delta \hat{W}(sb, x) d\mu(b, x) \)

Lemma 6: \( \hat{V}(s, \mu) = \hat{V}(s, \nu) \) if \( \mu^1 = \nu^1 \).

Proof: If \( \hat{V}(s, \cdot) = \alpha \hat{U}(s, \cdot) + \beta \) for some \( \alpha \leq -1 \), then \( x \succeq_s y \) for all \( x \subset y \) contradicting regularity. If \( \hat{V}(s, \cdot) = \alpha \hat{U}(s, \cdot) + \beta \) for some \( \alpha \geq 0 \) then \( x \succeq_s y \) for all \( y \subset x \in Z \) and \( \succeq \) is not regular. Hence, for each \( s \in S \) there are two possibilities: either \( \hat{V}(s, \cdot) \) is not an affine transformation of \( \hat{U}(s, \cdot) \) or there exists \( \alpha \in (-1, 0) \) such that \( \hat{V}(s, \cdot) = \alpha \hat{U}(s, \cdot) + \beta \). Then, regularity implies that there exist \( \mu^s, \nu^s \in \Delta \) such that \( \hat{U}(s, \mu^s) + \hat{V}(s, \mu^s) > \hat{U}(s, \nu^s) + \hat{V}(s, \mu^s) \) and \( \hat{V}(s, \mu^s) < \hat{V}(s, \nu^s) \).
Take any $\nu, \hat{\nu} \in \Delta$ such that $\nu^1 = \hat{\nu}^1$. There exists $\alpha > 0$ small enough so that

\[
\hat{U}(s, \mu^s) + \hat{V}(s, \mu^s) > \hat{U}(s, \alpha \nu + (1 - \alpha)\nu^s) + \hat{V}(s, \alpha \nu + (1 - \alpha)\nu^s)
\]

Continuity and Assumption I then imply $\{\alpha \nu + (1 - \alpha)\nu^*, \mu^s\} \sim_s \{\alpha \hat{\nu} + (1 - \alpha)\nu^*, \mu^s\}$. Since $\hat{W}$ represents $\nu$ we have $\hat{V}(s, \alpha \nu + (1 - \alpha)\nu^s) = \hat{V}(s, \alpha \hat{\nu} + (1 - \alpha)\nu^s)$. Since $\hat{V}$ is linear, we conclude $\hat{V}(s, \nu) = \hat{V}(s, \hat{\nu})$ as desired.

By Lemma 6, there is a function $\hat{v} : S \times \Delta(C) \to \mathbb{R}$ such that $\hat{V}(s, \mu) = \hat{v}(s, \mu^1)$. Regularity implies that neither $\hat{U}(s, \cdot)$ nor $\hat{v}(s, \cdot)$ is constant. Moreover, (since $\delta > 0$) this implies that $\hat{v}(s, \cdot)$ is not an affine transformation of $\hat{U}(s, \cdot)$. Hence, we may apply Theorem 7 of Gul and Pesendorfer (2000a) to yield the following implications:

**Fact 1:** (Theorem 7 (Gul and Pesendorfer (2000a))) $\hat{S}P$s iff for some $\alpha_u, \alpha_v \in [0, 1], \gamma > 0, \gamma_u, \gamma_v \in \mathbb{R}$

\[
\gamma \hat{U}(s, \mu) = \alpha_u \hat{U}(s, \mu) + (1 - \alpha_u)\hat{v}(s, \mu^1) + \gamma_u
\]

\[
\gamma \hat{v}(s, \mu^1) = \alpha_v \hat{U}(s, \mu) + (1 - \alpha_v)\hat{v}(s, \mu^1) + \gamma_v
\]

for all $\mu$.

By Assumption P and Fact 1 it follows for all $s \in S$

\[
\hat{U}(s, \mu) = \alpha(s) \hat{U}(s_0, \mu) + \gamma_u(s)
\]

\[
\hat{v}(s, \mu^1) = \beta(s) \hat{v}(s_0, \mu^1) + \gamma_v(s)
\]

for some functions $\alpha, \beta, \gamma_u, \gamma_v$ such that $\alpha(s) > 0, \beta(s) > 0$ for all $s$. Note that $\hat{U}$ and $\hat{v}$ are continuous and hence $\alpha, \beta, \gamma_u, \gamma_v$ are continuous.

Combining (3) and (4) yields,

\[
\int [\hat{u}(s, b) + \delta \hat{W}(sb, z)]d\nu(b, z) = \int [\alpha(s)\hat{u}(s_0, b) + \gamma_u(s) + \alpha(s)\delta \hat{W}(s_0b, z)]d\nu(b, z)
\]
The only terms on either side of (4) that depend on \( \nu_2 \) are \( \delta \hat{W}(sb, z) \) and \( \alpha(s) \delta \hat{W}(s_0 b, z) \). Since regularity implies that neither of these terms is constant it follows that

\[
\hat{W}(sb, \cdot) = \alpha(s) \hat{W}(s_0 b, \cdot) + A(s, b)
\]

Lemma 2 (in the proof of Theorem 1) then implies that \( \alpha(s) = 1 \) for all \( s \). It follows that \( \hat{W}(s_0 b, \cdot) \) represents \( \hat{sb} \). Hence, \( K = 1 \). That is, \( sb = b \) for all \( s, b \). Henceforth, we write \( b \) instead of \( sb \).

Let \( W(b, z) = \hat{W}(b, z) - \gamma_u(b) \), \( u(b) = \hat{u}(s_0, b) + \delta \gamma_u(b) \) for all \( b \). Let \( v(\cdot) = \hat{v}(s_0, \cdot) \). Then,

\[
W(b, z) = \hat{W}(b, z) - \gamma_u(b) = \max_{\mu \in z} \{ \hat{U}(b, \mu) + \hat{v}(b, \mu) \} - \max_{\nu \in z} \hat{v}(b, \nu) - \gamma_u(b)
\]

\[
= \max_{\mu \in z} \{ \hat{U}(s_0, \mu) + \beta_v(b) \hat{v}(s_0, \mu) \} - \max_{\nu \in z} \beta_v(b) \hat{v}(s_0, \nu)
\]

\[
= \max_{\mu \in z} \int [\hat{u}(s_0, b') + \beta_v(b) \hat{v}(s_0, b') + \delta \hat{W}(b', x)] d\mu(b', x)
\]

\[
- \max_{\nu \in z} \beta_v(b) \hat{v}(s_0, \nu)
\]

\[
= \max_{\mu \in z} \int [u(b') + \beta_v(b) v(b') + \delta W(b', x)] d\mu(b', x)
\]

\[
- \max_{\nu \in z} \beta_v(b) \hat{v}(\nu^1)
\]

Define \( \sigma(b) := \beta_v(b) \). By Assumption N \( v(c, d) = v(\hat{c}, \hat{d}) \) if \( d = \hat{d} \). Assumption N also implies that \( v(c, d) \) is strictly increasing in \( d \). Finally, Assumption N implies that \( \sigma(c, d) = \sigma(\hat{c}, \hat{d}) \) if \( d = \hat{d} \). Hence, \( u, v, \sigma, \delta \) satisfy all the desired properties. \( \Box \)
References


