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Embedded Nash Bargaining: Risk Aversion and Impatience

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In telling the tale of, and analyzing the decisions made by, an heir claimant to a large fortune, Lippman and McCardle (2004) introduce *embedded Nash bargaining*, an approach to modeling joint decision making. They embed several bargaining games in a joint decision tree and calculate the expected payoffs to the two sides if the Nash bargaining solution is used to generate the intermediate payoffs from bargaining. The purpose of the current paper is to provide theoretical underpinnings for that approach: we establish some general results regarding the existence, uniqueness, and comparative statics (with respect to costs, risk aversion, and time discounting) of the embedded Nash bargaining solution. In particular, when the disagreement payoff is random, we show that a decision maker's embedded Nash bargaining payoff decreases with both his risk aversion and impatience, and it increases with his opponent's risk aversion and impatience.

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1. Introduction

The scholarly development of game theory and expected utility theory diverged shortly after their joint presentation in von Neumann and Morgenstern (1947). Game theory developed under the aegis of microeconomic theory, studied and practiced mainly in economics departments. Noncooperative, as opposed to cooperative, game theory has dominated. The prevailing attitude toward cooperative game theory is articulated by Rasmusen (1989, p. 231): "Cooperative game theory may be useful for ethical decisions, but its attractive features are inappropriate for most economic situations, and the spirit of the axiomatic approach is very different from the utility maximization of current economic theory." At the same time, expected utility theory developed into decision analysis, mainly studied and practiced in operations research departments in engineering schools or quantitative methods departments in business schools. One purpose of the current paper is to reintroduce game theory and decision analysis to each other in a narrow context: the division of an asset between a pair of antagonists where there is uncertainty regarding each individual's claim. We are certainly not the first to attempt this reintroduction, but for the most part the matchmakers have tended to be economists (game theorists) come to borrow ideas from decision analysis. We, on the other hand, are decision analysts borrowing ideas from cooperative game theory.

The approach studied in this paper, embedded Nash bargaining, was introduced via example in Lippman and McCardle (2004). There we applied the basic mechanism to the analysis of the division of the contested estate of the late Larry Hillblom, founder and principal of DHL and apparent father of several illegitimate heirs. The current paper adds to the development of the approach in a more general and less sensational setting. As decision analysts, our view is that we are embedding a Nash bargaining model in a pair of matched decision trees, hence we call it *embedded Nash bargaining*. Alternatively, this could be viewed as imposing subgame perfection on a multistage Nash bargaining game with random transitions, in which case it would be called *subgame perfect Nash bargaining*.

The bargaining applications to which our approach can be applied entail two parties who must divide an asset. The asset could be physical, such as real estate, or it could be pecuniary, for example, an income stream associated with intellectual property. As time passes while the two parties negotiate, costs are incurred, the asset value shrinks because of discounting, and chance events alter the two parties' relative shares of the asset. The chance event might involve discovery of new information or a change in relative position.

The original paper on Nash bargaining is Nash (1950). Rubinstein (1982) introduced subgame perfection in a noncooperative bargaining model with alternating take-it-or-leave-it offers. Thompson (1994) provides an excellent review of the cooperative bargaining literature. For the basics on intertemporal preferences, see Nachman (1975), Prakash (1977), and Fishburn and Rubinstein (1982).

In the next section we introduce the basic model of a negotiation between two decision makers, DM1 and DM2, and we review a standard noncooperative approach in which one of the DMs has the ability to make take-it-or-leave-it offers to the other. In §3 we introduce embedded Nash bargaining. In contrast to the modeling perspective of a noncooperative game, the Nash bargaining approach is axiomatic: a set of axioms is proposed, and solutions that satisfy these axioms are analyzed. There is no attempt to model or constrain the give-and-take (or take-it-or-leave-it) process of the bargaining. An example shows that the embedded Nash bargaining solution provides a middle ground between the two alternative noncooperative models in which one of the DMs issues all of the take-it-or-leave-it offers. We focus on exponential utility in our numerical examples, but the basic results are established more generally.[1]

The embedded Nash bargaining model is first detailed for time-insensitive DMs. The penultimate section considers the effects of discounting. To provide comparative statics on the embedded Nash solution, we first prove a simple but important result: when the disagreement payoff for a standard Nash bargaining game is random, the Nash bargaining payoff to a DM decreases with increases in that DM's risk aversion.[2] Comparative statics on the embedded Nash bargaining solution are then established to show that the payoff to a DM increases with increases in the risk aversion, the costs, or the discount rate of the opponent. Along the way we consider an alternative bargaining solution concept, the dictatorial solution, which yields payoffs equivalent to the noncooperative solution discussed in the next section.

2. Model

There are two decision makers, DM1 and DM2, whose task is to divide between themselves an infinitely divisible asset with value $v > 0$. Let $X = \{(x_1, x_2): x_1 + x_2 \le v\}$ be the set of feasible divisions of the asset. The utilities of the DMs to their individual payoffs are denoted $u_1(x_1)$ and $u_2(x_2)$. We assume they are each risk averse: u_1 and u_2 are concave and strictly increasing. Let Ω be the nonempty set of possible utility pairs: $\Omega = \{(u_1(x_1), u_2(x_2)): (x_1, x_2) \in X\}$. If they agree to a division (x_1, x_2) that is feasible, i.e., $(x_1, x_2) \in X$, they each get the agreed-to shares and the game is over. If they fail to agree, the fallback position is the disagreement point (d_1, d_2), measured in utilities. We assume $(d_1, d_2) \in \Omega$. Denote the certainty equivalent of a gamble Y by $CE_i(Y)$ so $u_i[CE_i(Y)] = Eu_i(Y)$.

The decision makers' claims to the asset involve several uncertainties (chance events). For ease of exposition, we include only two chance events, A and B, each of which has two possible outcomes, a and $\tilde{a}$ and b and $\tilde{b}$, respectively. Event A precedes event B. The probabilities are completely specified by $P(a)$, $P(b \mid a)$, and $P(b \mid \tilde{a})$. Resolving each of the uncertainties requires time (and hence induces a delay) and furthermore imposes a cost on decision maker $i = 1, 2$ of $c_i(A)$ and $c_i(B)$, respectively. It is straightforward to allow the costs of resolving B to vary not only by decision maker but also based on the outcome of

[1] Exponential utility is widely used in decision analysis applications and models. For instance, Corner and Corner (1995) find it to be the most widely used model of utility in the applications they review. It exhibits constant absolute risk aversion, hence, is wealth independent and analytically tractable. It also allows easy comparison of the risk attitudes of the DMs.

[2] Researchers have suggested a variety of reasons as to why risk aversion might vary: Howard (1988) suggests corporate risk tolerances modeled with exponential utility vary with equity book value of the firm; in the studies reviewed by Kirkwood (2004), risk aversion varies depending on the size of the DM's customary budget; and Bickel (2006) analyzes the impact of varying levels of financial distress, the costs of external financing, and principal–agent concerns.

A: $c_i(B, a)$ need not equal $c_i(B, \tilde{a})$. Our results also follow through in a relatively straightforward manner to more complex situations than the two-uncertainty, two-states-per-uncertainty model detailed here.

The DMs can divide the asset prior to any of the uncertainties being resolved. If they fail to agree to a division, two things happen: they pay for uncertainty A to be resolved, and they wait for its resolution. Once event A is resolved, they can again attempt to reach an agreement on the division of the asset. Knowing how A was resolved, if they fail a second time to agree to a division of the asset, two things happen: they pay for uncertainty B to be resolved, and they wait for its resolution. Then the disagreement point is reached and payoffs are determined.

Throughout this paper, the value v of the asset does not vary. Nevertheless, in light of the twin burdens imposed by disagreement, namely, the direct costs $c_i(\cdot)$ and the time delays (and hence discounting), it is not incorrect to view disagreement as causing the value v of the asset to shrink.

2.1. Noncooperative Equilibrium

To model the division of the asset as a decision, we begin by arbitrarily assigning DM2 the power to make take-it-or-leave-it offers to DM1. To keep the discussion simple, we begin by assuming that both decision makers are time indifferent, an assumption we relax later.

We rely on the following assumptions:

Assumption vNM. *The preferences of the DMs satisfy the von Neumann and Morgenstern (1947) axioms.*

Assumption CK. *The probabilities, payoffs, and utilities of the DMs are common knowledge.*

Consider a standard decision analytic approach to this problem from DM1's point of view as presented in Figure 1. At the first decision node, DM1 has the choice between accepting and rejecting DM2's offer of x_1. If he accepts the offer, he gets x_1, DM2 gets the rest of v (i.e., $v - x_1$), and the game comes to an end. If he rejects the offer, he pays $c_1(A)$ and waits to observe the outcome of the chance event A. Once the outcome of event A has occurred and has been revealed to both DMs, DM2 makes a second offer, either $x_1(a)$ or $x_1(\tilde{a})$, at which time DM1 is faced with a second decision: accept or reject the second offer. If he accepts the second offer, the game comes to an end. If he rejects the second offer, he pays $c_1(B)$ to observe the outcome of the chance event B, either b or $\tilde{b}$, which determines his ultimate share of v. We assume disagreement entails a loss in value, i.e., all of the costs are nonnegative.

Let $x_1(a, b)$ denote the share of v received by DM1 if both chance events are resolved and $a \cap b$ occurs. Define $x_1(a, \tilde{b})$, $x_1(\tilde{a}, b)$, and $x_1(\tilde{a}, \tilde{b})$ analogously. Let $Ex_1(A, B)$ denote the expected share of v received by DM1 in disagreement, where the expectation is taken over A and B; let $Ex_1(a, B)$ denote the expected share received by DM1 in disagreement conditioned on the event $A = a$; and let $Ex_1(\tilde{a}, B)$ be the expected share to be received by DM1 in disagreement conditioned on the event $A = \tilde{a}$.

Once the probabilities and values for each of the branches have been specified, DM1's choices are simple and clear: accept an offer if and only if its utility is at least as large as the expected utility of rejecting it. Equivalently, DM1 accepts an offered share if and only if it is at least as large as the certainty equivalent of the payoff earned by rejecting it.

Turn now to the decisions faced by DM2. The structure of the uncertainties for DM2 is the same as for DM1, but rather than a decision between accepting and rejecting an offer, DM2 must choose a set of optimal offers. DM2, knowing that DM1 will accept any offer greater than or equal to DM1's certainty equivalent of the event that follows, will offer exactly that amount. The values of the offers are chosen in the usual backward, iterative, dynamic-programming fashion.[3]

For example, suppose that event $A = a$ has occurred. Using the probabilities at the end of the subtree following the "reject offer $x_1(a)$" branch of

[3] Alternative approaches are possible absent the assumption of common knowledge. Without common knowledge, DM2 would assign subjective probabilities to the events that DM1 accepts each potential offer and then would choose the offers that maximized the expected value. Kadane and Larkey (1982) take the subjective approach and provide a general criticism of Assumption CK and of game theoretic modeling more broadly; Harsanyi (1982) provides a rejoinder. Also see Brandenburger (1992) for a detailed review. For models of noncooperative bargaining absent Assumption CK, see Harsanyi and Selten (1972) and Chatterjee and Samuelson (1983).

Figure 1 Decision Tree from DM1's Perspective When DM2 Makes Take-It-or-Leave-It Offers

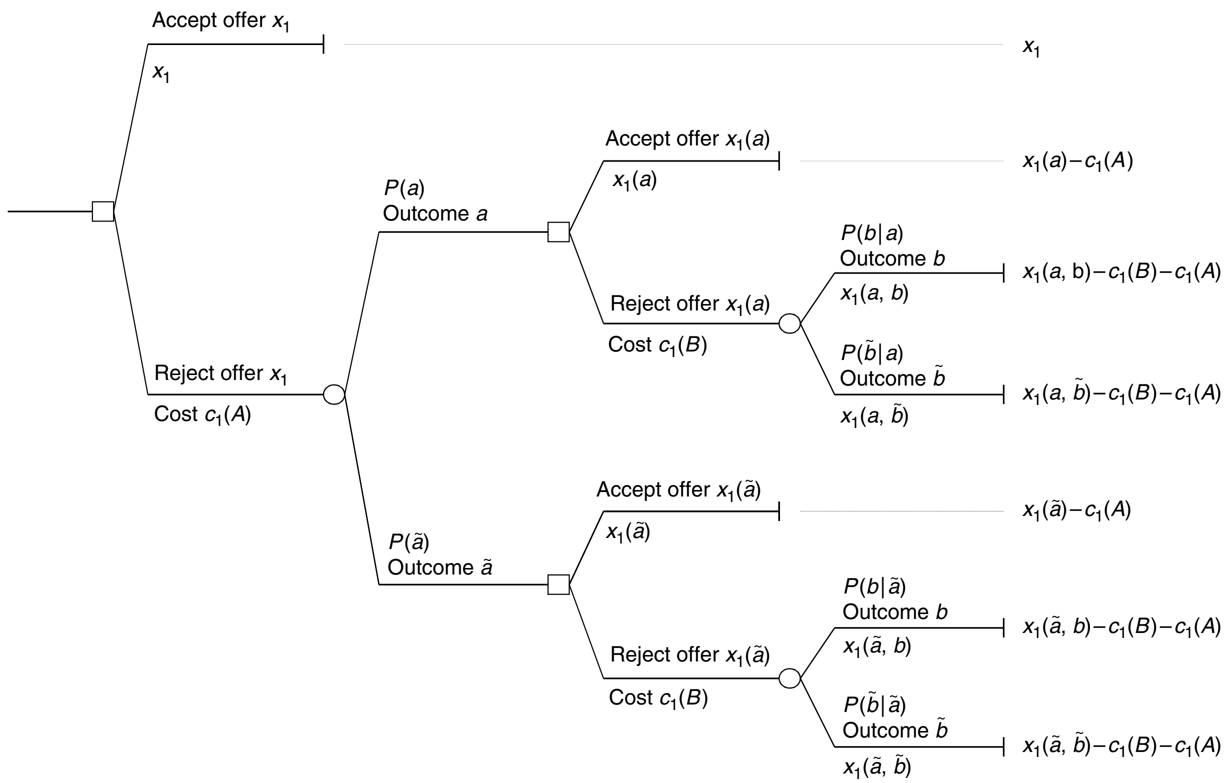

Figure 1, DM2 computes $d_1(a)$, the expected utility received by DM1 when DM1 rejects offer $x_1(a)$:

$$\begin{aligned} d_1(a) &= Eu_1[x_1(a,B)-c_1(B)-c_1(A)] \\ &= P(b\mid a)u_1[x_1(a,b)-c_1(B)-c_1(A)] \\ &\quad + P(\tilde{b}\mid a)u_1[x_1(a,\tilde{b})-c_1(B)-c_1(A)]. \end{aligned} \tag{1}$$

This characterization allows for wealth-dependent utility functions—the final utility is for the share of the asset net of costs. The specific utility function, however, depends on initial wealth; we ignore changes to wealth outside the asset division being modeled. If DM1 rejects the offer, DM2 gets the expected utility $d_2(a) = Eu_2[v - x_1(a,B) - c_2(B) - c_2(A)]$.

If DM1 accepts an offer of $x_1(a)$ after $A = a$ has been revealed, then his utility is $u_1[x_1(a) - c_1(A)]$. Thus, DM2 offers $x_1(a)$ that makes DM1 indifferent between accepting and rejecting: $u_1[x_1(a) - c_1(A)] = d_1(a)$ or

$$\begin{aligned} x_1(a) &= u_1^{-1}(d_1(a)) + c_1(A) \\ &= CE_1[x_1(a,B)-c_1(B)-c_1(A)] + c_1(A). \end{aligned} \tag{2}$$

DM2 retains the rest, $v - x_1(a)$ (though DM2 has already paid $c_2(A)$). The power to make take-it-or-leave-it offers enables DM2 to garner all of the surplus created by the foregone costs and the risk of resolving event B. Similar calculations yield the payoffs when the event $A = \tilde{a}$ occurs.

Taking one step back in the decision tree of Figure 1, we arrive at DM1's first decision node. If DM1 rejects the initial offer, he receives the disagreement payoff d_1 given by

$$\begin{aligned} d_1 &= Eu_1[x_1(A)-c_1(A)] \\ &= P(a)u_1[x_1(a)-c_1(A)] + P(\tilde{a})u_1[x_1(\tilde{a})-c_1(A)]. \end{aligned} \tag{3}$$

Once again, DM2 offers just enough to entice DM1 to accept: DM2 offers

$$x_1 = u_1^{-1}(d_1) \tag{4}$$

to DM1 at the first decision node. DM1 accepts that offer, whence DM2 retains $v - x_1$, the remainder.

Coupling the choices of the two decision makers results in a noncooperative game. Each decision maker's choice is a best response to the choice of the other; hence, their choices are in equilibrium. The dynamic nature of the choices leads to subgame perfection.

The solution presented in (2) is similar to the solution of Rubinstein's (1982) sequential offers game in that the person making the offers, in this case DM2, benefits from the avoidance of future costs. If, in the case just presented, DM1 rejects an offer, costs are incurred, and, because one of the outcomes in A will occur, the DMs face the risk of an unfavorable outcome. At each decision node, DM1 compares the utility of the proffered share of the not-yet-cost-reduced asset with the expected utility of the cost-reduced risky share yielded by disagreement. DM2 chooses offers that make DM1 indifferent, and then DM2 keeps all of the surplus (the associated costs of resolving the uncertainty) to himself. A similar solution obtains if we assume either that DM1 has the power to make take-it-or-leave-it offers that DM2 then accepts or rejects, in which case all of the potential surplus accrues to DM1, or that the decision makers make alternating offers, which are then accepted or rejected by the other decision maker, in which case the surplus is (unevenly) split between the two DMs.

2.2. A Numerical Example

Suppose the decision makers are litigating over an asset with value $v = \$1{,}000{,}000$. Event A reveals the outcome of legal discovery (the evidence potentially relevant to the outcome of a trial that is revealed before the start of the trial), and event B reveals the outcome of the trial itself. Suppose $P(a) = 0.2$ and $P(b \mid a) = P(b \mid \tilde{a}) = 0.5$, and let the payoffs to DM1 in the four states of the world be

$$\begin{aligned} &x_1(a,b) = 900{,}000;\; x_1(a,\tilde{b}) = 750{,}000;\\ &x_1(\tilde{a},b) = 500{,}000;\; x_1(\tilde{a},\tilde{b}) = 100{,}000. \end{aligned} \tag{5}$$

DM1's expected share is $Ex_1(A,B) = 405{,}000$, so the expected share of DM2 is 595,000. The outcome $A = a$ represents good news for DM1 in that his expected share is larger if $A = a$ occurs: $Ex_1(a,B) = 825{,}000 > 300{,}000 = Ex_1(\tilde{a},B)$.

Suppose both DMs have exponential utility, $u_i(x) = 1 - e^{-x/R_i}$, and DM1 is more risk averse: DM1 has a risk tolerance of $R_1 = 25{,}000$, whereas DM2 has a risk tolerance of $R_2 = 5{,}000{,}000$. Let the costs of discovery be $c_1(A) = 1{,}000$ and $c_2(A) = 1{,}200$, and let the costs of the trial be $c_1(B) = 10{,}000$ and $c_2(B) = 15{,}000$.

If the take-it-or-leave-it offers made by DM2 leave DM1 indifferent between accepting or rejecting, substitution in (2) yields the equilibrium offer of $x_1(a) = 757{,}266$, and a similar calculation yields $x_1(\tilde{a}) = 107{,}329$. Substitution in (4) then demonstrates that the subgame perfect payoffs are $x_1 = 111{,}907$ to DM1 and $x_2 = 888{,}093$ to DM2. Alternatively, if DM1 makes take-it-or-leave-it offers that DM2 either accepts or rejects, calculations yield equilibrium offers of $x_2(a) = 159{,}438$ and $x_2(\tilde{a}) = 681{,}001$; the subgame perfect payoffs are $x_1 = 428{,}955$ to DM1 and $x_2 = 571{,}045$ to DM2. In both versions the first offer is accepted so no costs are incurred.

The two payoffs to DM1 (111,907 if DM2 makes the offers and 428,955 if DM1 does) straddle his expected share of 405,000; the difference is determined by the costs and risk tolerances of each of the decision makers. Observe also that because DM2 is much less risk averse than DM1, the benefit to DM2 of being the offer maker $(888{,}093 - 595{,}000 = 293{,}093)$ relative to his expected share exceeds the relative benefit to DM1 of being the offer maker $(428{,}955 - 405{,}000 = 23{,}955)$.

3. Embedded Nash Bargaining

As is standard in noncooperative models of bargaining, the foregoing analysis assumes that one of the decision makers has the power to make take-it-or-leave-it offers to the other. In cooperative game theory, the negotiation process is left unmodeled; henceforth in this paper, we follow this course. Rather than a pair of actions, a solution is a mapping from the feasible set to a point, where the mapping satisfies a set of axioms. We impose the Nash bargaining solution. It is easiest to understand via example.

3.1. Nash Bargaining Solution for Final Decision Nodes

Suppose that at the decision node following the event $A = a$, the decision makers fail to agree to a division of the asset. From Equation (1), disagreement leads to utility $d_1(a) = Eu_1[x_1(a,B) - c_1(B) - c_1(A)]$, or equivalently, the share $y_1(a) = u_1^{-1}(d_1(a))$ for DM1 and $y_2(a) = u_2^{-1}(d_2(a))$ for DM2, where $d_2(a) = Eu_2[x_2(a,B) - c_2(B) - c_2(A)]$. If they can agree at the decision node, they can share the surplus $v - [y_1(a) + y_2(a)] \geq c_1(B) + c_2(B)$.

Assume for the moment that the DMs are risk neutral, so $y_1(a) = d_1(a) = Ex_1(a, B) - c_1(B) - c_1(A)$, and $y_2(a) = d_2(a) = Ex_2(a, B) - c_2(B) - c_2(A)$. The potential surplus is $c_1(B) + c_2(B)$. The Nash bargaining solution for risk-neutral DMs evenly divides this surplus and adds it to each DM's disagreement payoff. For example, DM1 receives

$$s_1(a) = d_1(a) + \frac{c_1(B) + c_2(B)}{2}, \tag{6}$$

and DM2 receives

$$s_2(a) = d_2(a) + \frac{c_1(B) + c_2(B)}{2}. \tag{7}$$

As explained below, the idea is similar for risk-averse decision makers: evenly split the surplus from avoiding disagreement, where the even splitting is utility adjusted.

To arrive at his solution, Nash (1950) begins by assuming that the decision makers are rational (i.e., their preferences satisfy Assumption vNM). The Nash bargaining solution is the unique outcome that satisfies an additional set of axioms: (1) the solution is invariant to positive linear transformations of the individual utilities; (2) the solution is not Pareto dominated; (3) the solution satisfies independence of irrelevant alternatives; and (4) the solution does not depend on the labeling of the players. Let $s(\Omega, d)$ be a solution to the bargaining problem (Ω, d). In terms of s, the four axioms just listed are as follows:

Axiom N1. *Let $\alpha_i > 0$ and β_i be arbitrary constants, $i = 1, 2$. Let $\Omega' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2)\colon (u_1, u_2) \in \Omega\}$ and $d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$. If $s = (s_1, s_2)$ is a solution to (Ω, d), then $s' = (\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2)$ is a solution to (Ω', d').*

Axiom N2. *If $s = (s_1, s_2)$ is a solution to (Ω, d), then there does not exist a pair $(u_1, u_2) \in \Omega \cup \{d\}$ such that $u_i \geq s_i$ for $i = 1, 2$ and $u_i > s_i$ for at least one $i = 1, 2$.*

Axiom N3. *Let (Ω_j, d) and (Ω_k, d) be two bargaining games with the same disagreement point and proposed solutions s_j and s_k, respectively. If $\Omega_j \subset \Omega_k$ and $s_k \in \Omega_j$, then $s_j = s_k$.*

Axiom N4. *Let s be a solution to (Ω, d). Let $\tilde{\Omega}$ and $\tilde{d}$ be derived from Ω and d by a reversal of the indices: $\tilde{\Omega} = \{(u_2, u_1)\colon (u_1, u_2) \in \Omega\}$, and $\tilde{d} = (d_2, d_1)$. Then $\tilde{s} = (s_2, s_1)$ is a solution to $(\tilde{\Omega}, \tilde{d})$.*

Nash proposed a solution $s = (s_1, s_2) = \arg\max\{(u_1 - d_1)(u_2 - d_2)\colon (u_1, u_2) \in \Omega\}$; that is, the Nash bargaining solution maximizes the product of the respective distances in utilities to the disagreement point. In the risk-neutral case, this means an even split of the surplus as given in (6) and (7).

Proposition 1 (Nash 1950). *The Nash bargaining solution is the unique solution satisfying Assumption* vNM *and Axioms* N1–N4.

In our model of the division of an asset with value v and share x_1 going to DM1, with risk-averse DMs, the Nash product to be maximized is $[u_1(x_1) - d_1] \cdot [u_2(v - x_1) - d_2]$.

3.2. Rolling Back the Nash Bargaining Solution

Subgame perfection is a central concept in noncooperative game theory. Our proposal is to import this concept and mix it with Nash bargaining. Subgame perfection is commonly asserted in noncooperative games (as in Rubinstein's (1982) alternating-offers bargaining model and its variations), but we are unaware of its application to a cooperative bargaining model. Note that our structure is different from the Rubinstein (1982) model.

To compute the possible payoffs at the first joint decision node, the decision makers must look ahead to the uncertainties and the potential bargaining games that ensue if they disagree. Consequently, they will need to determine the Nash bargaining solutions to the subgames. Those solutions are probability weighted and rolled back into the current bargaining situation as the disagreement payoff.

Thus far we have assumed the DMs are time insensitive. More commonly, economic analyses assume that decision makers have a multiattribute utility function over time and payoffs. In the most widely used special case represented by discounted expected utility, the multiattribute utility is additively separable across time.[4]

[4] Nachman (1975) developed a general model of time and risk preferences and introduces the notion of temporal risk aversion.

There are three time periods in our model: $t_0 = 0$, the starting time or the time of the first bargaining attempt; t_A, the time of resolution of event A which is concurrent with the time of the second bargaining attempt; and t_B, the time of resolution of event B. Naturally, $0 < t_A < t_B$. Bargaining is instantaneous and takes place at times t_0 and t_A. The DMs' time preferences are represented by constant discounting with rate δ_i: decision maker i's preference for a series of three lotteries $Y(j)$ that pay off in time periods t_j can be represented by the utility function

$$U_i(Y) = \sum_{j=0,\,A,\,B} Eu_i[\delta_i^{t_j} Y(j)]. \tag{8}$$

We assume $0 < \delta_i < 1$. The effect of discounting is similar to the effect due to the costs $c_i(A)$ and $c_i(B)$ in that both decrease the value of payoffs in the future relative to the same payoffs today.

Proposition 2. *Given Axioms* N1–N4 *and preferences given by* (8), *there is a unique solution to the embedded Nash bargaining problem. The solution is computed via a rolling-back procedure in the joint decision tree. At each decision node, the payoff is the Nash bargaining solution to the game with disagreement payoffs given by the expected utility of the discounted payoffs of the subsequent node net of costs.*

For example, suppose event $A = a$ has occurred (hence, $c_1(A)$ and $c_2(A)$ have already been expended). If the DMs disagree, DM1 pays $c_1(B)$ to observe the outcome of B and earn $Eu_1[\delta_1^{t_B} x_1(a, B) - \delta_1^{t_A} c_1(B) - c_1(A)]$. The disagreement payoff is given by

$$\begin{aligned} d_1(a) = {} & P(b \mid a) u_1[\delta_1^{t_B} x_1(a, b) - \delta_1^{t_A} c_1(B) - c_1(A)] \\ & + P(\tilde{b} \mid a) u_1[\delta_1^{t_B} x_1(a, \tilde{b}) - \delta_1^{t_A} c_1(B) - c_1(A)]. \end{aligned} \tag{9}$$

Defining the disagreement payoff to DM2 similarly, the split determined by the Nash bargaining solution at $A = a$ is the pair $(s_1(a), v - s_1(a))$ that maximizes $[u_1(\delta_1^{t_A} s_1(a) - c_1(A)) - d_1(a)][u_2(\delta_2^{t_A}(v - s_1(a) - c_2(A)) - d_2(a)]$. A similar process leads to the split determined by the Nash bargaining solution at $A = \tilde{a}$.

Working backward in the tree, the disagreement payoff to DM1 prior to A being resolved is given by

$$\begin{aligned} d_1 = {} & P(a) u_1[\delta_1^{t_A} s_1(a) - c_1(A)] \\ & + P(\tilde{a}) u_1[\delta_1^{t_A} s_1(\tilde{a}) - c_1(A)]. \end{aligned} \tag{10}$$

The split $(s_1, v - s_1)$ determined by the embedded Nash bargaining payoff maximizes $[u_1(s_1) - d_1] \cdot [u_2(s_2) - d_2]$.

4. Comparative Statics

With embedded Nash bargaining, disagreement leads to a random payoff. To establish comparative statics for embedded Nash bargaining requires understanding the effects of increasing risk aversion on the Nash bargaining solution.

4.1. Comparative Statics for the Nash Bargaining Solution

Roth and Rothblum (1982) present results on the effects of risk aversion when the disagreement point is deterministic. They show that an increase in the risk aversion of DM2 can either increase, decrease, or leave unaffected the utility to DM1. The direction of the effect depends, roughly, on the relationship between the payoff at the disagreement point and the possible payoffs in the support of the Nash bargaining solution. Safra et al. (1990) provide similar results when the disagreement payoffs are random but the set of possible negotiation points is finite. Our next result differs from the results in these two papers in three ways: first, because the DMs seek to divide an infinitely divisible asset, the bargaining set is a continuum rather than finite; second, each payoff pair in the bargaining set is deterministic; and finally, because the final node reached is random, the disagreement payoff pair is random.

For the moment, ignore discounting and the cost of resolving uncertainties, and for simplicity assume A is the only chance event. The decision makers can divide the asset v, or they can fail to agree and DM1 will receive $x_1(a)$ with probability p and $x_1(\tilde{a})$ with probability $1 - p$. DM2 receives the remainder in each case. Thus, the disagreement payoff pair is

A follow-up paper by Prakash (1977) shows that a von Neumann and Morgenstern (1947) consistent preference relation has the property that time discounting and risk adjustment are interchangable; we make use of this result. Fishburn and Rubinstein (1982) further develop the model for the special case of discounted expected utility with constant discounting and instantaneous risk aversion. They provide a set of axioms that guarantees the existence of a concave instantaneous utility function for some discount factor.

random. In disagreement, DM1's expected utility is $d_1 = pu_1(x_1(a)) + (1-p)u_1(x_1(\tilde{a}))$. The Nash bargaining solution $(s_1, v-s_1)$ is such that s_1 solves $\max_s(u_1(s)-d_1)(u_2(v-s)-d_2)$. Per Axiom N1, there is no loss in renormalizing so that $d_1 = d_2 = 0$. Thus, the first-order condition shows that s_1 solves

$$\frac{u_1(s_1)}{u_1'(s_1)} = \frac{u_2(v-s_1)}{u_2'(v-s_1)}. \tag{11}$$

We are interested in the effect on s_1 of an increase in the risk aversion of DM1. Let $u(x)$ be a twice-differentiable, strictly increasing, concave utility function. The absolute risk aversion of $u(x)$ is defined by $r_u(x) = -u''(x)/u'(x)$. We say that the utility function $\hat{u}$ is more risk averse than u if $r_{\hat{u}}(x) \geq r_u(x)$ for all x. Let $\hat{u}_1(x)$ represent a utility function that is more risk averse than $u_1(x)$, and let $\hat{s}_1$ be the Nash bargaining payoff to DM1 with the more risk-averse utility function $\hat{u}_1$.

Proposition 3. *If the DM's utility functions $u_1(x)$ and $u_2(x)$ are twice-differentiable, strictly increasing, and concave, the Nash bargaining payoff to DM1 is decreasing in DM1's aversion to risk, i.e., $\hat{s}_1 \leq s_1$.*

Proof. Because $\hat{u}_1$ is more risk averse than u_1, $\hat{u}_1 = w(u_1)$ for some increasing concave w. We can renormalize w so that $w(0) = 0$; i.e., $\hat{d}_1 = 0$. Because w is concave and $w(0) = 0$, it follows that $w(y) = w(y) - w(0) = \int_0^y w'(t)\,dt \geq w'(y)\cdot y$ provided $y \geq 0$. In particular, set $y = u_1(s_1) \geq d_1 = 0$. Then, $\hat{u}_1(s_1) = w(u_1(s_1)) \geq w'(u_1(s_1))u_1(s_1)$. Hence,

$$\frac{\hat{u}_1(s_1)}{\hat{u}_1'(s_1)} = \frac{w(u_1(s_1))}{w'(u_1(s_1))u_1'(s_1)} \geq \frac{u_1(s_1)}{u_1'(s_1)}. \tag{12}$$

Because u_1 is strictly increasing and concave, it follows that

$$[u_1(x)/u_1'(x)] \quad \text{is strictly increasing in } x. \tag{13}$$

Suppose $\hat{s_1} > s_1$. Using Equations (11), (13), (12), (11), and (13) in order produces

$$\frac{u_2(v-\hat{s}_1)}{u_2'(v-\hat{s}_1)} = \frac{\hat{u}_1(\hat{s}_1)}{\hat{u}_1'(\hat{s}_1)} > \frac{\hat{u}_1(s_1)}{\hat{u}_1'(s_1)} \geq \frac{u_1(s_1)}{u_1'(s_1)} \tag{14}$$

$$= \frac{u_2(v-s_1)}{u_2'(v-s_1)} > \frac{u_2(v-\hat{s}_1)}{u_2'(v-\hat{s}_1)}, \tag{15}$$

a contradiction. Hence, $\hat{s}_1 \leq s_1$. □

In the special case wherein each of the DMs has the same exponential utility, $u_i(x) = 1 - e^{-x/R}$, with risk tolerance R, the same discount factor δ, and zero costs, it is possible to derive a closed form solution for the Nash bargaining payoff. Let δX be the discounted random payoff to DM1 in disagreement, and let $\delta(v - X)$ be the discounted random payoff to DM2. Straightforward algebraic manipulation results in the Nash bargaining payoff s_1 to DM1 given by

$$s_1 = \frac{(1-\delta)v}{2} - \frac{R}{2}\ln\left(\frac{Ee^{-\delta X/R}}{Ee^{\delta X/R}}\right). \tag{16}$$

We are interested in the effects of changes in the random disagreement distribution on the Nash bargaining payoff to DM1. Because $X \geq 0$ in our setup, the argument of the logarithm in (16) is less than one; hence, the second term is positive and acts to increase s_1. Let X and Y be two random variables distributed on $[0, v]$, representing two different possible distributions of the share received by DM1 in disagreement. Let s_X and s_Y, respectively, be the Nash bargaining solution shares received by DM1 when δX and δY are the discounted payoffs to DM1 in disagreement. Suppose Y is stochastically larger than X. Because it is well known that this stochastic ordering implies that there exists a random variable $Z \geq 0$ such that $Y = X + Z$, it follows immediately from (16) that $s_Y \geq s_X$: a stochastic increase in DM1's disagreement payoff increases his Nash bargaining payoff.

Next, suppose that Y is more variable than X in the sense of second-order stochastic dominance. A standard result is that there exists a random variable Z independent of X, with $E(Z) \geq 0$, such that Y has the representation $Y = X + Z$. Because of this independence, it follows that

$$\ln\left(\frac{Ee^{-\delta Y/R}}{Ee^{\delta Y/R}}\right) = \ln\left(\frac{Ee^{-\delta X/R}}{Ee^{\delta X/R}}\right) + \ln\left(\frac{Ee^{-\delta Z/R}}{Ee^{\delta Z/R}}\right). \tag{17}$$

It follows from (16) that $s_Y > s_X$ if $E(e^{-\delta Z/R}) < E(e^{\delta Z/R})$. This last condition does not hold if $Z = -Z$, but there are examples when it does hold; for example, it holds if $P(Z=3) = P(Z=-1) = P(Z=-2) = 1/3$.

4.2. Numerical Example Part 2

We return to the numerical example of §2.3 wherein the decision makers have v =\$1,000,000 to divide; the expected final share to DM1 is $Ex_1(A, B) = 405{,}000$;

costs are $c_1(A) = 1{,}000$, $c_2(A) = 1{,}200$, $c_1(B) = 10{,}000$, and $c_2(B) = 15{,}000$; and the DMs have instantaneous exponential utility with risk tolerances 25,000 and 5,000,000, respectively. Assume that each uncertainty takes one period to be resolved.

If the DMs are time insensitive, i.e., $\delta_1 = \delta_2 = 1$, following the process laid out just after Proposition 2, the certainty equivalent of the Nash bargaining payoff at $A = a$ is (785,564; 214,436), and at $A = \tilde{a}$ is (157,868; 842,132). Discounting those certainty equivalents, subtracting the cost of resolving A, and taking the expected utility yields the disagreement payoff prior to the resolution of A. The certainty equivalent of the embedded Nash bargaining solution prior to the resolution of A is (200,282; 799,172).

On the other hand, if the DMs have the same discount factor $\delta_1 = \delta_2 = 0.9$, the certainty equivalent of the embedded Nash bargaining prior to the resolution of A is (201,183; 798,817). We now consider several parameter changes that offer suggestive comparative statics, focusing on the embedded Nash bargaining payoff to DM1. All comparisons are to the case just described, where both DMs have the same discount factor of 0.9, and the embedded Nash bargaining payoff to DM1 is 201,183. We know from Proposition 3 that if DM1 becomes more risk averse, that is, has a smaller risk tolerance, then DM1's embedded Nash bargaining payoff decreases; for example, if R_1 decreases from 25,000 to 22,500, then the embedded Nash bargaining payoff decreases from 201,183 to 194,028.

- If DM1 becomes less patient, that is, has a smaller discount factor $\delta_1 = 0.8$, then DM1's embedded Nash bargaining payoff decreases to 194,790.
- If DM1 faces a higher cost of resolving uncertainty, that is, $c_1(B)$ increases to 20,000, then DM1's embedded Nash bargaining payoff decreases to 191,187.
- If $x_1(a, b)$, the payoff to DM1 when (a, b) occurs, decreases from 900,000 to 800,000, then the DM1's embedded Nash bargaining payoff decreases to 200,660.

4.3. Comparative Statics for Embedded Nash Bargaining

Armed with Proposition 3 on the impact of increasing risk aversion on the Nash bargaining solution, we can now establish the impact of increasing risk aversion on the embedded Nash bargaining solution.

Proposition 4. *Given Axioms* N1–N4 *and preferences given by* (8), *increasing the risk aversion of DM*1 *reduces the share of the asset awarded to DM*1 *under the embedded Nash bargaining solution.*

Proof. The Nash bargaining solution at the last stage becomes the disagreement payoff at the previous stage. By Proposition 3, the Nash bargaining solution payoff to DM1 is decreasing in his risk aversion; hence, the disagreement payoff to DM1 in the previous stage is decreasing in his risk aversion. Furthermore, the Nash bargaining solution is monotonic with respect to changes in the disagreement point: a decrease in the disagreement payoff to DM1 leads to a decrease in the Nash bargaining payoff to DM1 in the previous stage. □

The suggestive comparative statics from Part 2 of the numerical example are formalized and shown below to always hold.

Proposition 5. *The payoff to DM*1 *in the embedded Nash bargaining solution with risk aversion and time preferences characterized by time-invariant discount factors* δ_i *for DMi is* (i) *increasing in DM*1*'s payoff in any state* $x_1(\cdot, \cdot)$, (ii) *decreasing in DM*1*'s own costs* $c_1(A)$ *and* $c_1(B)$, *and* (iii) *increasing in his discount factor* δ_1.

Proof. As with Proposition 3, the proof follows from the monotonicity of the Nash bargaining solution with respect to the disagreement point. To establish (i), note that an increase in $x_1(a, b)$ leads to an increase in $d_1(A)$, which in turn leads to an increase in $s_1(A)$. An increase in $s_1(A)$ leads to an increase in d_1, which leads to an increase in s_1. Items (ii) and (iii) are established similarly. □

Because the DMs are sharing the asset v, an increase in the share (and hence, utility) to one DM implies a decrease in the share to the other; that is, not only does DM1 benefit from a decrease in his own risk aversion, impatience, and costs, but DM1 also benefits from an increase in the risk aversion, impatience, and costs of DM2. For example, small firms that developed an internet presence sometimes were sued for patent infringement regarding the look and functionality of their online purchasing software. These firms often found it less costly to settle (by paying a royalty to the alleged copyright holder) than to face the vagaries of a trial. The alleged copyright holder, by bringing suit against a large number of small firms, was able to allocate its potential costs of discovery and trial into smaller pieces. In this example, the alleged copyright holder is DM1 and $c_1(A) + c_1(B) < c_2(A) + c_2(B)$.

5. Conclusion and Further Research

This paper provides analytic grounding for embedded Nash bargaining, a method originally proposed by Lippman and McCardle (2004). As is the case with both decision analysis and Nash bargaining, the proposed embedded Nash bargaining solution is arrived at axiomatically. Through the Nash bargaining process, the embedded Nash bargaining solution inherits equilibrium properties: Pareto optimality and an equitable sharing of the potential surplus. Through the folding back process of decision analysis, the solution inherits dynamic consistency, i.e., subgame perfection in the bargaining procedure. We also provide comparative statics on the solution: an increase in the cost, impatience for positive outcomes, or risk aversion of either decision maker benefits the other.

This paper focuses on the Nash solution to the bargaining problem, but alternative solutions satisfying various other axiom systems exist. For example, one could imagine that the DMs have differing "power," in which case symmetry is violated. The weighted Nash bargaining solution, which allocates weights (α_1, α_2) to the DMs, would then obtain. See Thompson (1994) for this and other examples.

With Christopher S. Tang, we are developing an application of the embedded Nash approach to a project-management contracting model. Other applications are also possible. For example, the presentation and examples used throughout this paper treat the shares and costs as dollar figures. It is easy to imagine, however, an application where the bargaining is over a non-dollar-denominated asset, such as water rights. In that case, a multiattribute utility analysis would likely be required.

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