Lifetime Consumption and Investment: Retirement and Constrained Borrowing*

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Extended Abstract

Saving for retirement is a primary end purpose of many parts of the financial sector, including pension plans, life insurance, and indeed much of retail banking and brokerage. This paper and a companion paper develop a group of workhorse models that can be adapted to answer many questions of personal finance. This paper focuses on retirement (modeled as irreversible) and constrained borrowing. This paper has three simple models that can be solved explicitly, at least parametrically up to the determination of a few parameters. All three models have realistic features such as a hazard rate of mortality, preference for working or not, a possible bequest motive, and wage income that can vary stochastically over time. A companion piece has models with time-dependent features, permitting the hazard rate of mortality, average wage level, and preference for working to be exogenous functions of time.

The paper only scratches the surface of possible applications, but does include several. One application is a possible resolution of a puzzle from the labor literature why consumption jumps at retirement. In our model, consumption jumps at retirement because preferences are different after retirement. This could be due to household production (time to cook instead of buying more expensive prepared food), reduced work-related expenses (for clothes or commuting), or just different preference for consumption when more leisure is available.

A second application looks at the traditional rule-of-thumb used by some practitioners that dictates a safer portfolio at older age than at earlier age. Our model is flexible enough to accommodate several of the potential effects in one model. When human capital contains little market risk, the financial investments should have decreasing proportional risk over time, while when human capital contains too much market risk, the financial investments should have increasing market risk over time. The non-negative wealth constraint limits the size of the position that can be maintained and leads to a moderation of positions, especially when wealth is low. Flexibility of retirement makes it possible to self-insure somewhat by working longer when portfolio returns are weak, and indeed we see that the present value of human capital is larger when wealth is small than when wealth is large.

The hope is that this tractable life-cycle model of consumption and investment with realistic mortality and time-dependent preferences will provide a workhorse model for analyzing problems in pensions, life-cycle consumption, and life insurance.
I. Introduction

Retirement is one of the most important economic events in a worker’s life. Not surprisingly, retirement is connected to a number of important personal decisions such as consumption and investment and also to policy issues such as those on insurance and pensions, as well as mandatory versus voluntary retirement.\(^1\) In this paper, we extend recent advances in finance to build a tractable optimal consumption and investment model with voluntary or mandatory retirement, and with or without a constraint on borrowing against future wages. This paper solves three models for which more or less complete solutions are available, either in the primal or the dual, up to determination of a constant in one case. A companion piece studies explicit dependence of the mortality rate, wage, and preference for working on the stage of life. It is hoped that these models and extensions will be useful for studying policy questions in insurance and retirement.

We consider three models in our analysis. The three models vary in the treatment of retirement and borrowing against future labor income. All three models share a number of common features: a constant hazard rate of mortality, different preferences for consumption before and after retirement, possibly stochastic labor income, bequest, and actuarially fair life insurance. Keeping these features the same makes it easy to perform a parallel comparison of the three models. All three models consider retirement to be irreversible, emphasizing that a worker may be much more valuable to a firm working full-time than when working part-time. This is an extreme alternative to models with a continuously variable labor-leisure choice, as in Liu and Neis [2002]. We have also looked at models with both types of choice, allowing the possibility of a return to part-time work at a lower wage after retirement, but not in this paper.

The first model is a benchmark case with a fixed retirement date, which we interpret as mandatory retirement.\(^2\) Our first model is a close relative of the Merton model with i.i.d. returns and constant relative risk aversion, and it can be solved explicitly.

The second model has voluntary retirement in a model in which the agent is free to borrow against future labor income. The second model is solved explicitly in the dual (as a function of the dual variable which is the marginal utility of wealth in the value function). This is a explicit parametric solution of the original problem which means that we know everything about the solution once we have conducted a one-dimensional numerical search for the value of the dual variable that corresponds with the current wealth level. Before retirement, there is a critical wealth-to-wage ratio at which it is optimal to retire. The expected time to retirement depends a lot on what the wage is today: the agent self-insures against risk in the security market by working more when the security market returns are

\(^{1}\)In the UK, mandatory retirement is still widespread, and this is still an active policy issue (see Meadows [2003]). In the US, the Age Discrimination in Employment Act of 1967 (ADEA) generally prohibits mandatory retirement. One exception is for a qualifying “bona fide executive” or person in a “high policymaking position” who can face mandatory retirement at an age of 65 or above. There used to be an exception for tenured academics who could face mandatory retirement at an age of 70 or above, but that exception expired on January 1, 1994.

\(^{2}\)A more realistic model of mandatory retirement is given by Panageas and Farhi [2003], who permit retirement at or before mandatory retirement date. Our simpler assumption is better for our benchmarking because we can solve it model exactly and it is easier to compare with the other models.
poor. By working longer in expensive states, the agent generates more income (some of which is transferred to other states) for the average time worked than if the worker worked for the same amount of time in each state.

The third model has voluntary retirement in a model in which the agent cannot borrow against future labor income. This restriction reduces the usefulness of investing in stocks because any significant negative return would wipe out the financial wealth and bring the agent against the borrowing constraint. The borrowing constraint prevents the agent from transferring income across states to the extent that would be optimal and reduces the attractiveness of working longer in expensive states of nature.

The paper contains technical innovations that permit solution up to determination of a few parameters. In particular, we combine the dual approach of He and Pagès (1993) with an analysis of the boundary to obtain a problem we can solve in parametric form even if no known solution exists in the primal problem.

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A second application looks at the traditional rule-of-thumb used by some practitioners that dictates a safer portfolio at older age than at earlier age. Our model is flexible enough to accommodate several of the potential effects in one model. When human capital contains little market risk, the financial investments should have decreasing proportional risk over time, while when human capital contains too much market risk, the financial investments should have increasing market risk over time. The non-negative wealth constraint limits the size of the position that can be maintained and leads to a moderation of positions, especially when wealth is low. Flexibility of retirement makes it possible to self-insure somewhat by working longer when portfolio returns are weak.

Liu and Neis (2002) consider the optimal consumption and investment problem with endogenous working hours. In contrast to our model, they allow an investor to borrow against future labor income. In addition, they assume that the stock price can never fall below a fixed positive level. Bodie, Merton, and Samuelson (1992) consider the effect of labor choice on optimal investment policy and Basak (1999) develops a continuous-time general equilibrium model to adapt dynamic asset pricing theory to include labor income. Similar to Liu and Neis (2002), both of these papers assume that working hours are infinitely divisible. Sundaresan and Zapatero (1997) examine how pension plans affect the retirement policies with an emphasis on the valuation of pension obligations. They abstract from modelling the disutility of working and the investor’s investment opportunities outside the pension.

The rest of the paper is organized as follows. In Section II, we describe and solve the simple model with constant wage. In Section III, we conduct an analysis of the optimal consumption, optimal investment and optimal retirement policies. In Section IV, we allow the spanned labor income to be stochastic. In Section V, we further extend the analysis to the case with unspanned stochastic labor income. Section VI concludes. Appendix contains
II. Simple Model with Constant Wage Income

Our general goal is to provide a tractable workhorse model that can be used to analyze various issues related to life cycle consumption and investment, retirement, and insurance. In this section, we present a simple stationary version of the model that illustrates our approach. Later sections contain models that are more complex and more realistic.

Many of the assumptions are common in continuous-time financial models, for example the constant riskfree rate and lognormal stock returns. Other assumptions are not standard but seem particularly appropriate for analysis of life-cycle consumption and investment. For example, our model includes mortality as well as preference for not working. In the general models in later sections, the wage is stochastic, and the mortality rate, average wage, and preference for not working can all vary as the agent moves through life. In this section, the wage, mortality rate, wage, and preference for not working are all constant. The later sections include a non-negative wealth constraint, reflecting the fact that younger people may be liquidity constrained and cannot necessarily do all the borrowing they would like to do. In the simple model of this section, we do not impose the non-negative wealth constraint; not having to deal with the boundary makes the solution and analysis more transparent.

All the models in this paper consider pure problems retirement without flexible hours, return to full-time work in retirement, or part-time work in retirement. There is no reason why these other features cannot be added to the model, but we choose to focus instead on the essential nonconvexity that says half-time work is much less valuable than full-time work in some positions. We do not have anything against more general models, but there is a limit of what can be included in one paper.

Here is the choice problem for this section.

**Problem 1** Choose adapted nonnegative consumption \( \{c_t\} \), adapted portfolio \( \{\theta_t\} \), adapted nonnegative bequest \( \{B_t\} \), and the retirement date (a stopping time) \( \tau \), to maximize expected utility of lifetime consumption and bequest

\[
E \left[ \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left( (1 - R_t) \frac{c_t^{1-\gamma}}{1-\gamma} + R_t \frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta (kB_t)^{1-\gamma} \right) dt \right]
\]

subject to

\[
W_0 = W^0, \quad R_0 \geq R^0, \quad R_t = \mathbf{1}\{t \geq \tau\},
\]

\[
dW_t = rW_t dt + \theta_t((\mu - r\mathbf{1})dt + \sigma dZ_t) + \delta(W_t - B_t) dt - c_t dt + (1 - R_t) w dt
\]

and

\[
W_t \geq (1 - R_t)W
\]

where \( W = -\frac{w}{r+\delta} \) (borrowing against labor income) or \( W = 0 \) (no borrowing at all), \( \tau = T \) (fixed retirement date) or \( \tau \) unconstrained (voluntary retirement).
The state variables are the initial wealth $W^0$ and the retirement status just before the start $R^0$. It is feasible to retire immediately, which is why we have a weak equality in (3). There is nothing to be done to change wealth initially, which is why we have an equality in (2).

The uncertainty in the model comes from two sources: the standard Wiener process $Z_t$ and the Poisson arrival of mortality at a fixed hazard rate $\delta$. These are drawn independently. The Wiener process $Z_t$ has dimensionality equal to the number of linearly independent risky returns, and maps into security returns through the constant mean vector $\mu$ and the constant standard deviation vector $\sigma$. The objective function in Problem 1 has already integrated out the impact of mortality risk: utility is discounted at the rate $\rho + \delta$ where $\rho$ is the pure rate of time discount and $\delta$ is the hazard rate of mortality. Insurance is assumed to be fairly priced at the rate $\delta$ per unit of coverage, both long and short. When $W - B > 0$, this is term life insurance purchased for a premium of $\delta(W - B)$ per unit time. If $W - B < 0$, then this is a short position in term life insurance, which is like a term version of an annuity since it trades wealth in the event of death for more consumption when living.

The utility function is a standard time-separable von Neumann-Morgenstern utility function with modified bequest. The utility function features the same constant level $\gamma > 0$ of risk aversion for consumption before retirement, after retirement, and for bequests. Felicity of consumption or bequest is discounted using a pure rate of time discount $\rho$ plus the mortality rate $\delta$. The indicator function $R_t$ is 1 after retirement and 0 before retirement. The constant $K > 1$ indicates how much better off the agent is when working than when not working, at the same consumption level, and the constant $k > 0$ gives the intensity of preference for leaving a large bequest.

The terms of the wealth equation (5) are mostly familiar. The first term says that if all wealth invested in the riskfree asset, the rate return is $r$. For the dollar investment $\theta_t$ in the risky asset, there is risk exposure $\theta_t \sigma dZ$ the mean return $\theta_t \mu$ is substituted for the corresponding riskfree return $\theta_t \mathbf{1} r$ (where $\mathbf{1}$ is a vector of 1's with dimension equal to the number of risky assets). The term $\delta(B_t - W_t)dt$ is the insurance premium we have already discussed, $c_t dt$ is payment for consumption, and $w(1 - R_t)dt$ is wage income. The factor $(1 - R_t)$ multiplying wage income says wages are received only before retirement.

The limited borrowing equation (6) says that either (a) it is not possible to borrow any more than the potential future income ($W = -\frac{w}{r+\delta}$) or (b) it is not possible to borrow at all ($W = 0$). In general, it is a subtle question what kind of constraint to add in an infinite-horizon portfolio problem to rule out borrowing without repayment and doubling strategies. Fortunately, either form of this simple and reasonable constraint suffices in this case. Later in the paper, a non-negative wealth constraint ($W = 0$) serves the same purpose. The non-negative wealth constraint is probably more realistic, but also introduces a new boundary and complicates the analysis somewhat.

Here is our characterization of the solution of Problem 1 for the two forms of borrowing constraints:

**Theorem 1** (with borrowing against wages and fixed retirement date)

Suppose $\rho + \delta > (1 - \gamma)(r + \delta + \frac{|\kappa|}{\gamma})$. The solution to Problem 1 with fixed retirement date $T$ can be written in terms of the dual variable $x_t$ (a normalized marginal utility of

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3As will be shown in Appendix, this condition is a necessary and sufficient condition for the existence of a solution to the problem without borrowing constraint (i.e., the Merton problem).
consumption). Specifically, let the dual variable be defined by
\[ x_t = x_0 e^{(\rho - r - \frac{1}{2}|\kappa|^2)t - \kappa Z_t}, \]
where \( x_0 \) solves
\[ -\varphi_x(x_0, 0) = W^0, \]
\[ \varphi(x, t) = \begin{cases} -\hat{\eta}\frac{x^b}{b} - (\hat{\eta} - \eta) \exp(-\frac{1+\delta k^{-b}}{\eta}(T - t)) + \eta \frac{x^b}{b} + \frac{w}{r+\delta} (1 - e^{-(r+\delta)(T-t)}x) & \text{if } t \geq T \\ b = 1 - 1/\gamma, \quad \kappa = \sigma^{-1}(\mu - r \bar{1}), \quad \eta = \frac{\gamma(1 + \delta k^{-b})}{\rho + \delta - (1 - \gamma)(r + \delta + |\kappa|^2/2\gamma)}, \quad \hat{\eta} = \frac{\gamma(K^{-b} + \delta k^{-b})}{\rho + \delta - (1 - \gamma)(r + \delta + |\kappa|^2/2\gamma)} & \text{otherwise}, \end{cases} \]
\[ (7) \]
Then the optimal consumption policy is
\[ c_t^* = (K^{-b})^\iota \{t \geq T\} x_t^{-1/\gamma}, \]
the optimal trading strategy is
\[ \theta_t^* = \sigma^{-2}(\mu - r)x_t \varphi_xx(x_t, t), \]
the optimal bequest policy is
\[ B_t^* = \delta k^{-b}x_t^{-1/\gamma}, \]
and the optimal wealth is
\[ W_t^* = -\varphi_x(x_t, t), \]
In addition, the value function is
\[ v(W^0, 0) = \varphi(x, 0) - x\varphi_x(x, 0) \]
where \( x \) solves
\[ -\varphi_x(x_0, \max(t\{W^0 \geq \tilde{W}\}, R^0)) = W^0, \]
\[ (8) \]
\[ (9) \]
\[ \text{Theorem 2 (with borrowing against wages)} \]
Suppose \( \rho + \delta > (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma}) \).
\[ 4 \]
The solution to Problem 1 can be written in terms of the dual variable \( x_t \) (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by
\[ x_t = x_0 e^{(\rho - r - \frac{1}{2}|\kappa|^2)t - \kappa Z_t}, \]
where \( x_0 \) solves
\[ -\varphi_x(x_0, \max(t\{W^0 \geq \tilde{W}\}, R^0)) = W^0, \]
\[ 4 \] As will be shown in Appendix, this condition is a necessary and sufficient condition for the existence of a solution to the problem without borrowing constraint (i.e., the Merton problem).
\[ W = -\varphi(x, 0), \]

\[ \varphi(x, R) = \begin{cases} 
-\hat{\eta}^{\frac{b}{b}} & \text{if } R = 1 \text{ or } x \leq x \\
A_x^{\alpha} - \eta^{\frac{b}{b}} + \frac{w}{r + \delta}x & \text{otherwise},
\end{cases} \]

\[ x = \left(\frac{(\eta - \hat{\eta})(b - \alpha)(r + \delta)}{b(1 - \alpha)w}\right)^{\gamma}, A_x = \frac{\eta - \hat{\eta}}{b}x^{b - \alpha} - \frac{w}{r + \delta}x^{1 - \alpha}, \]

\[ b = 1 - \frac{1}{\gamma}, \quad \kappa = \sigma^{-1}(\mu - r), \]

\[ \eta = \frac{\gamma(1 + \delta k^{-b})}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma})}, \] (11)

\[ \hat{\eta} = \frac{\gamma(K^{-b} + \delta k^{-b})}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma})}, \] (12)

\[ \alpha = \frac{r - \rho + \frac{1}{2}|\kappa|^2 - \sqrt{(r - \rho + \frac{1}{2}|\kappa|^2)^2 + 2(\rho + \delta)|\kappa|^2}}{|\kappa|^2}. \] (13)

Then the optimal consumption policy is

\[ c^*_t = (K^{-b})R_t^{-1/\gamma}, \]

the optimal trading strategy is

\[ \theta^*_t = \sigma^{-2}(\mu - r)x_t\varphi_{xx}(x_t, R_t^*), \]

the optimal bequest policy is

\[ B^*_t = \delta k^{-b}x_t^{-1/\gamma}, \]

the optimal retirement policy is

\[ R^*_t = \iota\{t \geq \tau^*\}, \]

and the optimal wealth is

\[ W^*_t = -\varphi_x(x_t, R_t^*), \]

where

\[ \tau^* = (1 - R^0)\inf\{t \geq 0 : x_t \leq x\}. \]

In addition, the value function is

\[ v(W^0, R^0) = \varphi(x, \max(\iota\{W^0 \geq W\}, R^0)) - x\varphi_x(x, \max(\iota\{W^0 \geq W\}, R^0)) \]

where \( x \) solves

\[ -\varphi_x(x, \max(\iota\{W^0 \geq W\}, R^0)) = W^0. \]
Theorem 3 (no borrowing against wages)

Suppose \( \rho + \delta > (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{\delta}) \). The solution to Problem 1 can be written in terms of the dual variable \( x_t \) (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

\[
x_t = \begin{cases} 
\tilde{x}_t & \text{if } t \leq \tau^* \text{ and } \tau^* \neq 0 \\
x^*_t e^{(\rho - r - \frac{1}{2}|\kappa|^2)(t - \tau^*) - \kappa(Z_t - Z_{t^*})} & \text{otherwise},
\end{cases}
\]

where

\[
\tilde{x}_t = \frac{x_0 e^{(\rho - r - \frac{1}{2}|\kappa|^2) s - \kappa Z_t}}{\max(1, \sup_{0 \leq s \leq t} x_0 e^{(\rho - r - \frac{1}{2}|\kappa|^2) s - \kappa Z_s})},
\]

\( x_0 \) solves

\[
-\varphi_x(x_0, \max(t\{W^0 \geq \bar{W}\}, R^0)) = W^0,
\]

\[
\bar{W} = -\varphi_x(\bar{x}, 0),
\]

\[
\tau^* = (1 - R^0) \inf\{t \geq 0 : \tilde{x}_t \leq \bar{x}\},
\]

\[
\varphi(x, R) = \begin{cases} 
-\frac{\eta b}{\alpha} & \text{if } R = 1 \text{ or } x \leq \bar{x} \\
A_- x^{\alpha -} + A_+ x^{\alpha +} - \frac{w}{r + \delta} x & \text{otherwise},
\end{cases}
\]

\[
A_- = \frac{\eta(b - \alpha_-)}{\alpha_+ - \alpha_-} x^{\alpha -} + \frac{(1 - \alpha_-)w}{\alpha_+ - \alpha_-}(r + \delta)x^{1 - \alpha_-},
\]

\[
A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_- - \alpha_+} x^{\alpha +} - \frac{(\alpha_+ - 1)w}{\alpha_- - \alpha_+}(r + \delta)x^{1 - \alpha_+},
\]

\[
\tau = \left(\frac{\frac{\eta}{b} x^{\alpha -} - \frac{\eta}{\alpha_-} (\alpha_+ - b)(r + \delta)}{(\gamma^{1 - \alpha_-} - 1)(\alpha_+ - 1)w}\right)^\gamma, \quad \bar{x} = \zeta \tau,
\]

\( \zeta \) solves \( q(\zeta) = 0, \) with

\[
q(\zeta) = \left(1 - \frac{K-b}{b(1 + \delta k^{-b})}\right) x^{\alpha -} - \frac{1}{\alpha_-} \left(\gamma^{1 - \alpha_-} - 1\right) (\alpha_+ - b)(\alpha_- - 1)
- \left(1 - \frac{K-b}{b(1 + \delta k^{-b})}\right) x^{\alpha +} - \frac{1}{\alpha_+} \left(\gamma^{1 - \alpha_+} - 1\right) (\alpha_+ - b)(\alpha_- - 1),
\]

and

\[
\alpha_+ = \frac{r - \rho + \frac{1}{2}|\kappa|^2}{|\kappa|^2} + \sqrt{\left(r - \rho + \frac{1}{2}|\kappa|^2\right)^2 + 2(\rho + \delta)|\kappa|^2}. \tag{16}
\]

Then the optimal consumption policy is

\[
c^*_t = (K^{-b} R^*_t)^{1/\gamma} x_t^{1/\gamma},
\]

the optimal trading strategy is

\[
\theta^*_t = \sigma^{-2}(\mu - r) x_t \varphi_x(x_t, R^*_t),
\]

\[ \]
the optimal bequest policy is
\[ B^*_t = \delta k^{-b} x^{-1/\gamma}_t, \]
the optimal retirement policy is
\[ R^*_t = \iota \{ t \geq \tau^* \}, \]
and the optimal wealth is
\[ W^*_t = -\varphi_2(x_t, R^*_t). \]

In addition, the value function is
\[ v(W^0, R^0) = \varphi(x, \max(\iota \{ W^0 \geq W \}, R^0)) - x \varphi_2(x, \max(\iota \{ W^0 \geq W \}, R^0)) \]
where \( x \) solves
\[ -\varphi_2(x, \max(\iota \{ W^0 \geq W \}, R^0)) = W^0. \]

III. Graphical Solution

Before proceeding to the more general model with stochastic wages, we explore graphically the solution to the simple model. We present many of the results normalized by total wealth, equal to financial wealth plus human capital, where human capital is the market value of future labor income in the optimal solution. The formulas for human capital are given at the end of Section IV., specialized to the simple case in which the wage is constant. While the market’s valuation of the individual’s human capital may be different from the individual’s own valuation (due to the borrowing constraint), this is still a useful normalization for interpreting the results.

The results consider three cases:

**benchmark** fixed retirement date and free borrowing against wages (Theorem 1)

**NBC ("No Borrowing Constraint")** free choice of retirement date and free borrowing against wages (Theorem 2)

**BC ("Borrowing Constraint")** free choice of retirement date but no borrowing against wages (Theorem 3)

The benchmark case is a close relative of the Merton model with i.i.d. returns and constant relative risk aversion. Moving to the NBC case isolates the impact of making retirement flexible. Subsequently moving to the BC case isolates the impact of the borrowing constraint.

Figure 1 shows the optimal stock position in the three cases, per unit of total wealth, as a function of financial wealth. The horizontal line shows the optimal portfolio choice for the benchmark case. In this case, it is as if all future wage income is capitalized and then there is fixed-proportions investment as in the Merton model. The time to retirement in the plot is 20 years, but the portfolio proportion would be the same constant whatever the time to retirement (and even after retirement). Moving to the NBC case, permitting flexibility in the retirement date permits a larger equity position because working longer can insure against variation in the stock market. The agent works longer in expensive states (when the
market is down) and the wealth from these states is transferred to other states by taking a significant position in equities. Adding the borrowing constraint in the BC case, transfer of wealth across states is restricted and indeed there is not much point of taking a significant position in equities when financial wealth is low, since that would just imply bumping into the borrowing constraint much of the time.

Figure 2 shows the consumption rate, normalized by total wealth, as a function of financial wealth. An interesting point not visible in the picture is that consumption jumps at retirement, in a direction that depends on whether risk aversion is larger or smaller than 1 (log utility). In the benchmark case, the consumption rate does not depend on financial wealth, but it does depend on time to maturity (not shown). Moving to the NBC case, adding flexible retirement leads to a higher consumption rate when wealth is higher (due to risk aversion greater than 1 in the example), since high wealth implies retirement is expected soon and demand for consumption is less after retirement. Moving to the BC case, consumption is significantly less at low wealth levels, which represents precautionary savings against market declines.

The critical wealth level at which the agent chooses to retire as a function of relative risk aversion is plotted in Figure 3. There is no curve for the benchmark case, since in that case retirement is at a fixed date, not at a freely chosen wealth boundary. The critical wealth is lower when there is a borrowing constraint than when there is none: while retirement is equally desirable in both cases, continuing to work is less desirable when borrowing is limited. The critical wealth level decreases in risk aversion because it is valuable to earn more money to take advantage of market returns when risk aversion is small.

The value of human capital can vary inversely with wealth as a form of insurance. Figure 4 shows the dependence of human capital on financial wealth. This is a constant over wealth in the benchmark case, but would vary from zero to the maximum on the NBC curve as maturity increases. In the BC and NBC cases, we see the insurance effect of how the agent hedges financial risk by working longer (and thereby increasing the value of human capital) when financial wealth is low.

Having flexible retirement and borrowing (the NBC case) is the least constrained of the three cases. It is interesting to measure the loss in value in the other two cases. Figure 5 gives the value loss from losing retirement flexibility, as a function of the fixed time to retirement. The value loss is measured as a certainty-equivalent fraction of total wealth corresponding to moving from the NBC case to the benchmark case. When wealth is high (=20), retiring soon is optimal and the loss is least when forced retirement comes soon. As wealth decreases, it becomes optimal to work longer and longer and the minimum loss is at larger and larger fixed times to retirement. Figure 6 shows the value of being able to borrow measured in certainty-equivalent units as a function of total wealth. The value is small when wealth is large and large when the borrowing constraint is nearly binding. The value of being able to borrow is greatest when the mean return on the stock is high, since being able to borrow makes it possible to make full use of equity.

Stock brokers have traditionally advised customers that young people should take on more risk than older people who are close to retirement. Our analysis can be used to generalize and confirm analysis of Jagannathan and Kocherlakota [1996] that calls into question the traditional rule. We have already seen in Figure 1 that in both the NBC case and BC case, the risky asset holding increases as a function of wealth, and higher wealth
corresponds to being nearer to retirement. Arguably this result is not a fair criticism of the traditional advice because of the normalization by total wealth. Figure 7 shows the portfolio choice normalized by financial wealth (both curves NBC), which may be closer to the intent of the traditional advice. When wages are riskless (top curve), the proportion of financial wealth put in stock does indeed fall as financial wealth increases. However, drawing on results from Section IV. below, this result can be reversed if wages are stochastic and move in the same direction as stocks, as illustrated by the bottom curve.

Part of the problem with the traditional advice can be illustrated by the benchmark case in which risk exposure is chosen to be a fixed proportion of total wealth. When young, most of the agent’s total wealth is in human capital, and if the wage is stochastic this may already represent too much exposure to risk, implying a desire for to take a short position in stocks to hedge the excess risk. When retirement is discretionary, the stock choice is less for the direct reason we have been discussing that wages are riskier, and also because it is less attractive to work more in expensive states (since those are also states of low wage when wages move in the same direction as stocks. Together with result that a borrowing constraint reduces stock demand at low levels even more, these results question the usefulness of the traditional advice, at least in the absence of a lot of qualifications of when the advice should be applied.

Table 1 contains a number of solutions illustrating the sensitivity of the equilibrium to various parameters. In particular, it shows the significant drop of the consumption and the stock investment at retirement date. In addition, as the mortality rate decreases, the investor requires a higher critical wealth-to-income ratio to retire and consumes less to save for after-retirement.

| Parameter | $\bar{W}$ | $\bar{c}(t)$ | $\bar{c}(t^+)$ | $\bar{\theta}(t)$ | $\theta(t)^+$ | $\bar{W}_{NBC}$ | $\theta_{NBC}(\tau|W_{NBC})$ |
|-----------|----------|--------------|---------------|----------------|--------------|----------------|-----------------|
| Base Case | 33.84    | 0.051        | 0.030         | 1.10           | 0.41         | 35.44          | 1.17            |
| $\gamma = 2.5$ | 32.37    | 0.053        | 0.028         | 0.93           | 0.33         | 33.31          | 0.97            |
| $\gamma = 1.5$ | 38.32    | 0.045        | 0.031         | 1.32           | 0.55         | 41.31          | 1.41            |
| $\mu = 0.04$ | 32.66    | 0.048        | 0.028         | 1.00           | 0.31         | 33.35          | 1.04            |
| $\mu = 0.06$ | 34.39    | 0.055        | 0.032         | 1.20           | 0.52         | 37.36          | 1.29            |
| $\sigma = 0.15$ | 34.39    | 0.058        | 0.034         | 1.90           | 0.89         | 38.87          | 2.07            |
| $\sigma = 0.30$ | 32.56    | 0.050        | 0.030         | 0.73           | 0.22         | 33.20          | 0.76            |
| $\delta = 0.015$ | 39.48    | 0.045        | 0.026         | 1.06           | 0.41         | 41.98          | 1.13            |
| $\delta = 0.025$ | 29.77    | 0.057        | 0.033         | 1.13           | 0.41         | 30.86          | 1.19            |
| $\rho = 0.008$ | 34.52    | 0.050        | 0.029         | 1.07           | 0.41         | 35.99          | 1.12            |
| $\rho = 0.012$ | 33.19    | 0.053        | 0.030         | 1.13           | 0.41         | 34.95          | 1.21            |
| $K = 2.5$ | 42.60    | 0.047        | 0.030         | 0.98           | 0.41         | 44.13          | 1.02            |
| $K = 3.5$ | 28.46    | 0.055        | 0.029         | 1.20           | 0.41         | 30.11          | 1.30            |
| $k = 0.025$ | 35.82    | 0.048        | 0.028         | 1.07           | 0.41         | 37.41          | 1.13            |
| $k = 0.075$ | 32.96    | 0.052        | 0.030         | 1.11           | 0.41         | 34.57          | 1.18            |

Table 1: Comparative Statics
Figure 1: Equity as a fraction of total wealth (= financial wealth plus human capital), as a function of the financial wealth $W$ for parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, $w = 1$. The horizontal line is for the benchmark case with a fixed retirement date 20 years from now and free borrowing against future wages. The NBC ("No Borrowing Constraint") case adds free choice of retirement date but still allows free borrowing. The BC ("Borrowing Constraint") case adds a nonnegative wealth constraint that restricts borrowing against future wages.
Figure 2: Consumption rate as a fraction of total wealth, as a function of financial wealth for parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$.

Figure 3: Financial wealth threshold for retirement as a function of relative risk aversion $\gamma$ for the NBC and BC cases with parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$. 

Figure 4: Human capital given financial wealth $W$ for parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$.

Figure 5: Value of voluntary retirement as a fraction of the total wealth, as a function of retirement horizon $T$ for parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$. 
Figure 6: Value of borrowing as a fraction of the total wealth, as a function of financial wealth $W$ for parameters $\sigma = 0.22$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$.

Figure 7: Impact of risky human capital on equity holdings for parameters $\mu = 0.05$, $\sigma = 0.22$, $\mu_y = 0$, $r = 0.01$, $\delta = 0.02$, $\rho = 0.01$, $\gamma = 2$, $K = 3$, $k = 0.05$, and $w = 1$. 
Base case parameters: \( \mu = 0.05, \sigma = 0.22, r = 0.01, \delta = 0.02, \rho = 0.01, \gamma = 2, K = 3, k = 0.05, w = 1. \)

IV. Richer Model with Stochastic Wage Income

This section considers the same models in the previous sections but now with stochastic wage income. The wage income is assumed to be spanned locally by security returns, without which we seem to lose the analytical solution. With this assumption of local spanning, the solution is not really any more complicated than in the simpler case, although the notation is more complex.

Problem 2 Choose adapted nonnegative consumption \( \{c_t\} \), adapted portfolio \( \{\theta_t\} \), adapted nonnegative bequest \( \{B_t\} \), and the retirement date (a stopping time) \( \tau \), to maximize expected utility of lifetime consumption and bequest

\[
E \left[ \int_{t=0}^{\infty} e^{-(\rho+\delta)t} \left( (1 - R_t) \frac{c_{t+1}^{-\gamma}}{1-\gamma} + R_t \frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt \right]
\]

subject to

\[
W_0 = W^0,
\]

\[
R_0 \geq R^0,
\]

\[
y_0 = y^0,
\]

\[
dW_t = rW_t dt + \theta_t((\mu - r)1 dt + \sigma dZ_t) + \delta(W_t - B_t)dt - c_t dt
\]

\[
+ (1 - R_t)wy_t dt,
\]

\[
\frac{dy_t}{y_t} = \mu_y dt + \sigma_y dZ_t,
\]

\[
W_t \geq W_t^0,
\]

where \( W_t = -y_t + r+\mu_y+\sigma_y\kappa_t \) (borrowing against labor income) or \( W_t = 0 \) (no borrowing at all), \( \tau = T \) (fixed retirement date) or \( \tau \) unconstrained (voluntary retirement).

Theorem 4 (with borrowing against random wages and fixed retirement date)

Suppose \( \rho + \delta > (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2r}) \). The solution to Problem 2 with fixed retirement date \( T \) can be written in terms of the dual variable \( x_t \) (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

\[
x_t = x_0 e^{(\rho - r - \frac{1}{2}|\kappa|^2)t - \kappa Z_t},
\]

where \( x_0 \) solves

\[
-y_0 \varphi_x(x_0,0) = W^0,
\]
\[ \varphi(x, t) = \begin{cases} -\hat{\eta} x^b_b & \text{if } t \geq T \\ -((\hat{\eta} - \eta) \exp(-\frac{1+\hat{\delta}k-b}{\eta}(T-t)) + \eta) x^b_b + \frac{w(1-e^{-(r+\delta-\mu y + \sigma^\top y)(T-t)})}{r+\delta-\mu y + \sigma^\top y} x & \text{otherwise} \end{cases} \]

Then the optimal consumption policy is

\[ c_t^* = (K^{-b}) t \{ t \geq T \} y_t x_t^{-1/\gamma}, \]

the optimal trading strategy is

\[ \theta_t^* = y_t \sigma^{-2}(\mu - r) x_t \varphi_{xx}(x_t, t) - \sigma^{-1} \sigma_y^\top (\gamma x_t \varphi_{xx}(x_t, t) + \varphi_x(x_t, t)), \]

the optimal bequest policy is

\[ B_t^* = \delta k^{-b} y_t x_t^{-1/\gamma}, \]

and the optimal wealth is

\[ W_t^* = -y_t \varphi_x(x_t, t). \]

In addition, the value function is

\[ v(W^0, 0) = (y^0)^{1-\gamma}(\varphi(x, 0) - x \varphi_x(x, 0)) \]

where \( x \) solves

\[ -y^0 \varphi_x(x, 0) = W^0. \]

**Theorem 5** (borrowing and random income)

Suppose \( \rho + \delta > (1-\gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma}) \), \( \beta_3 > 0 \) and \( \beta_2 + \beta_3 > 0 \). The solution to Problem 2 can be written in terms of the dual variable \( x_t \) (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

\[ x_t = x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^2)t + \sigma_x z_t}, \quad (25) \]

where

\[ \mu_x = -(r - \rho) - \frac{1}{2}\gamma(1-\gamma)|\sigma_y|^2 + \gamma \mu_y - \gamma \sigma_y^\top \kappa, \]

\[ \sigma_x = \gamma \sigma_y - \kappa, \]

\( x_0 \) solves

\[ -\varphi_x(x_0, \max(t\{W^0 \geq W\}, R^0)) = W^0, \]

\[ W = -y^0 \varphi_x(x, 0), \]

\[ \varphi(x, R) = \begin{cases} -\hat{\eta} x^b_b & \text{if } R = 1 \text{ or } x \leq x \\ A_+ x^\alpha - \eta x^b_b + \frac{w}{\beta_2 + \beta_3} x & \text{otherwise} \end{cases} \]

\[ x = \left( \frac{(\eta - \hat{\eta})(b - \alpha_-)(\beta_2 + \beta_3)}{b(1-\alpha_-)w} \right)^\gamma, \quad A_+ = \frac{\eta - \hat{\eta}}{b} x^{\beta_2 - \alpha_-} - \frac{w}{\beta_2 + \beta_3} x^{1-\alpha_-}, \]

\[ \text{18} \]
$$\alpha_- = \frac{\beta_2 + \frac{1}{2} \beta_1 - \sqrt{(\beta_2 + \frac{1}{2} \beta_1)^2 + 2 \beta_3 \beta_1}}{\beta_1},$$  
(26)$$

$$\beta_1 = |\kappa|^2 + \gamma^2 |\sigma_y|^2 - 2 \gamma \sigma_y \kappa^\top,$$

(27)$$

$$\beta_2 = r - \rho - \frac{1}{2} \gamma (1 - \gamma) |\sigma_y|^2 - \gamma \mu_y + \sigma_y \kappa^\top,$$

(28)$$

and

$$\beta_3 = \rho + \delta + \frac{1}{2} \gamma (1 - \gamma) |\sigma_y|^2 - (1 - \gamma) \mu_y.$$  
(29)$$

Then the optimal consumption policy is

$$c_t^* = (K^{-b}) R_t^* y_t x_t^{-1/\gamma},$$

the optimal trading strategy is

$$\theta_t^* = y_t [\sigma^{-2} (\mu - \rho) x_t \varphi_x (x_t, R_t^*) - \sigma^{-1} \sigma_y^\top (\gamma x_t \varphi_x (x_t, R_t^*) + \varphi_x (x_t, R_t^*))]$$

the optimal bequest policy is

$$B_t^* = \delta k^{-b} y_t x_t^{-1/\gamma},$$

the optimal retirement policy is

$$R_t^* = \iota \{ t \geq \tau^* \},$$

and the optimal wealth is

$$W_t^* = -y_t \varphi_x (x_t, R_t^*),$$

where

$$\tau^* = (1 - R^0) \inf \{ t \geq 0 : x_t \leq \bar{x} \}.$$  
In addition, the value function is

$$v(W^0, y^0, R^0) = (y^0)^{1-\gamma} \left( \varphi(x, \max(\iota \{ W^0 \geq W \}, R^0)) - x \varphi_x(x, \max(\iota \{ W^0 \geq W \}, R^0)) \right)$$

where $x$ solves

$$-y^0 \varphi_x(x, \max(\iota \{ W^0 \geq W \}, R^0)) = W^0.$$  

**Theorem 6** (no borrowing and random income)

Suppose $\rho + \delta > (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma})$, $\beta_3 > 0$ and $\beta_2 + \beta_3 > 0$. The solution to Problem 1 can be written in terms of the dual variable $x_t$ (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by

$$x_t = \begin{cases} 
\hat{x}_t & \text{if } t \leq \tau^* \text{ and } \tau^* \neq 0 \\
\hat{x}_t e^{(\mu_x - \frac{1}{2} \sigma_x^2)(t - \tau^*) - \sigma_x (Z_t - Z_{\tau^*})} & \text{otherwise},
\end{cases}$$  
(30)$$

where

$$\hat{x}_t = \frac{x_{00} e^{(\mu_x - \frac{1}{2} \sigma_x^2)t - \sigma_x Z_t}}{\max(1, \sup_{0 \leq s \leq t} x_{00} e^{(\mu_x - \frac{1}{2} \sigma_x^2)s - \sigma_x Z_s / \bar{x}})},$$
the optimal retirement policy is
\[ R_t^* = \nu \{ t \geq \tau^* \}, \]
the optimal bequest policy is
\[ B_t^* = \delta k^{-b} y_t x_t^{-1/\gamma}, \]
the optimal trading strategy is
\[ \theta_t^* = y_t \sigma^{-2} (\mu - \tau) x_t \varphi_{xx}(x_t, R_t^*) - \sigma^{-1} \sigma_y (\gamma x_t \varphi_{xx}(x_t, R_t^*) + \varphi(x_t, R_t^*)], \]
the optimal consumption policy is
\[ c_t^* = (K^{-b}) R_t^* y_t x_t^{-1/\gamma}, \]
and the optimal wealth is 

\[ W_t^* = -y_t \varphi(x_t, R_t^*). \]

In addition, the value function is

\[ v(W_0, y_0, R_0) = (y_0)^{1-\gamma} \left( \varphi(x, \max\{t\{W_0 \geq W\}, R_0\}) - x \varphi(x, \max\{t\{W_0 \geq W\}, R_0\}) \right) \]

where \( x \) solves

\[-y_0 \varphi(x, \max\{t\{W_0 \geq W\}, R_0\}) = W_0.\]

**Proposition 1** The present value of the human capital corresponding to cases in Theorems 4, 5 and 6 are respectively

\[ H(y_t, t) = \frac{w y_t}{r + \delta - \mu_y + \sigma_y^2 \kappa} \left( 1 - e^{-(r+\delta-\mu_y+\sigma_y^2 \kappa)(T-t)} \right) \mathbb{I}\{t \leq T\}, \]

\[ H(x_t, y_t) = \frac{w y_t}{r + \delta - \mu_y + \sigma_y^2 \kappa} (-x_t^{\alpha-1} + 1) \mathbb{I}\{x_t \geq \bar{x}\}, \]

\[ H(x_t, y_t) = \frac{w y_t}{r + \delta - \mu_y + \sigma_y^2 \kappa} (Ax_t^{\alpha-1} + Bx_t^{\alpha+1} + 1) \mathbb{I}\{x_t \geq \bar{x}\}, \]

where

\[ A = \frac{(1 - \alpha_+) \bar{x}^{1-\alpha_-} - (\alpha_- - 1) \bar{x}^{\alpha_+ - \alpha_-}}{\alpha_+ - 1 \bar{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1) \bar{x}^{\alpha_+ - \alpha_-}}, \]

\[ B = \frac{(\alpha_- - 1) \bar{x}^{1-\alpha_-}}{\alpha_+ - 1 \bar{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1) \bar{x}^{\alpha_+ - \alpha_-}}. \]

**Proposition 2** If \( \mu_x < \frac{1}{2} \sigma_x^2 \), then the expected time to retirement corresponding to cases for Theorems 5 and 6 are respectively

\[ E[\tau^*|x_t = x] = \frac{\log(x/\bar{x})}{\frac{1}{2} \sigma_x^2 - \mu_x}, \forall x_t > \bar{x} \]

and

\[ E[\tau^*|x_t = x] = \frac{x^m - x^m}{(\frac{1}{2} \sigma_x^2 - \mu_x)m} + \frac{\log(x/\bar{x})}{\frac{1}{2} \sigma_x^2 - \mu_x}, \forall x_t \in [\underline{x}, \bar{x}], \]

where

\[ m = 1 - \frac{2 \mu_x}{\sigma_x^2}. \]
V. Unspanned Labor Income

Now we consider the case with unspanned labor income, i.e.,

\[
(\forall t \geq 0) \quad \frac{dy_t}{y_t} = \mu_y dt + \sigma_y dZ_t + \sigma_y d\tilde{Z}_t,
\]

(34)

where \( \tilde{Z}_t \) is a one-dimensional Brownian motion independent of \( Z_t \). To avoid repetition, we present the results for the case with constant parameters. The case with time-varying parameters can be solved using the same procedure as in the previous section. The primal problem is difficult to solve due to a singular boundary condition at \( W_t = 0 \). We therefore solve this case also using the dual approach.

The dual of this problem can be written as:

\[
\min_{\psi > 0} \max_{\tau, \nu} E\left[ \int_{0}^{\tau} e^{-\gamma s} \left( -\frac{z^b_s}{b} + w y s \nu_s \right) ds + e^{-\gamma \tau} \phi^R(z_{\nu \tau}, y_{\tau}) \right] + \psi W_0,
\]

where \( \psi = z_{\nu^*0} \) and \( z_{\nu s} \) satisfies

\[
dz_{\nu s} = -(\gamma + \rho - \delta)dz - \kappa^T dZ - \nu d\tilde{Z},
\]

for some process \( \nu \in \mathcal{L}^2 \) to be determined. Define

\[
\phi(z, y) = \max_{\tau, \nu} E\left[ \int_{0}^{\tau} e^{-\gamma s} \left( -\frac{z^b_s}{b} + w y_s \nu_s \right) ds + e^{-\gamma \tau} \phi^R(z_{\nu \tau}, y_{\tau}) \right].
\]

Then we have The HJB for \( \phi \) is

\[
\frac{1}{2} |\kappa|^2 \phi_{zz} - (\gamma + \rho - \delta)z \phi_z + \frac{1}{2}(|\sigma|^2 + \sigma^2_y) y^2 \phi_{yy} + \mu_y y \phi_y - \sigma_y \kappa^T y z \phi_{yz} + \min_{\nu} \left( \frac{1}{2} \nu^2 z^2 \phi_{zz} - \sigma_y \nu y z \phi_{yz} \right) - (\rho + \delta) \phi - \frac{z^b}{b} + w y z = 0.
\]

(35)

Using the same transformation as before, we obtain

\[
\frac{1}{2} \beta_1 x^2 \varphi_{xx} - \beta_2 x \varphi_x - \beta_3 \varphi - \frac{1}{2} \sigma^2_y \varphi^2_{xx} - x^b \nu + w x = 0,
\]

(36)

where

\[
\beta_1 = |\kappa|^2 + \gamma^2 (|\sigma|^2 + \sigma^2_y) - 2 \gamma |\sigma| \kappa^T,
\]

\[
\beta_2 = \gamma + \rho - \delta - \frac{1}{2} \gamma (1 - \gamma) (|\sigma|^2 + \sigma^2_y) - \gamma \mu_y + \sigma_y \kappa^T
\]

(37)

and

\[
\beta_3 = \rho + \delta + \frac{1}{2} \gamma (1 - \gamma) (|\sigma|^2 + \sigma^2_y) - (1 - \gamma) \mu_y.
\]

(38)

(39)
Note that if the labor income is spanned, i.e., $\dot{\sigma}_y = 0$, then this ODE reduces to (53) with spanned labor income and constant parameters in Section IV.C.

We need to solve ODE (36) subject to (54)-(57), with the value function after retirement solved similarly. Different from the case with spanned labor income, the HJB ODE (36) is fully nonlinear and an explicit form for the value function seems unavailable. However, this nonlinear ODE with free boundaries can be easily solved numerically.

VI. Conclusion

In this paper, we consider how the concern about the living standard after retirement affects optimal consumption and investment policy throughout an investor’s life. We consider both the case with mandatory retirement date and the case with voluntary retirement. We show that in the case of voluntary retirement an investor retires once the ratio of financial wealth to labor income reaches a critical value. This critical wealth-to-income retirement ratio depends on financial market characteristics such as riskfree rate and risk premium on the market and investor personal characteristics such as wage rate, wealth, mortality rate, and risk aversion. We also find that the flexibility for retirement date is valuable and changes significantly the optimal investment and consumption policy, especially if we can borrow against future labor income. Generally speaking, an investor invests more in the stock market. However, the investor may save less or more than the mandatory retirement case depending on the time to the mandatory retirement date.
Appendix

In this Appendix, we collect all the proofs.

We first derive the solution to the investor’s problem after retirement. Note that the after retirement case is equivalent to the case with $R_0 = 1$ and $W_0$ equal to the current wealth. We have

**Lemma 1** If $R_0 = 1$, then the proposed strategy $(c^*, \theta^*, B^*, R^*)$ is optimal and $v$ is the value function at retirement.

At retirement, the investor’s problem is to choose admissible trading strategy $\theta$ in the stock, the consumption rate $c > 0$ and the bequest $B_t > 0$ to maximize

$$v(W^0, 1) = \max_{(c, \theta, B)} E[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt], \quad (40)$$

subject to the budget constraint:

$$dW_t = rW_t dt + \theta_t^\top (\mu - r) dt + \theta_t^\top \sigma dZ_t - c_t dt + \delta(W_t - B_t) dt \quad (41)$$

and the borrowing constraint

$$W_t \geq 0.$$

The state price density process $\xi$ can be written as follows:

$$\xi(t) = e^{-(r+\delta+\frac{1}{2}|\kappa|^2)t-\kappa^\top z_t}. \quad (42)$$

Since $W_t$ is uniformly bounded below, $c_t \geq 0$ and $B_t \geq 0$, by Dybvig and Huang (1989), $e^{-(r+\delta)t}W_t + \int_0^t e^{-(r+\delta)s}(c_s + \delta B_s) ds$ is a supermartingale under the equivalent martingale measure $Q$. Therefore for any admissible strategy $(c, \theta, B)$, we must have

$$E[\int_0^\infty \xi(c_t + \delta B_t) dt] \leq W^0. \quad (43)$$

Therefore for any admissible strategy $(c, \theta, B)$,

$$E[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(Kc_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt]$$

$$= E[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(Kc_t^{*})^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^{*})^{1-\gamma}}{1-\gamma} - x_0 e^{(\rho+\delta)t} \xi(c_t + \delta B_t) dt + x_0 \int_0^\infty \xi(c_t + \delta B_t) dt]$$

$$\leq E[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(Kc_t^{*})^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^{*})^{1-\gamma}}{1-\gamma} \right) dt + x_0 \int_0^\infty \xi(c_t + \delta B_t - c_t^{*} - \delta B_t^{*}) dt]$$

$$\leq E[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(Kc_t^{*})^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^{*})^{1-\gamma}}{1-\gamma} \right) dt]$$

where

$$x_0 = \left( \frac{\tilde{\eta}}{W^0} \right)^\gamma$$

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is the Lagrangian multiplier such that
\[ E\int_0^\infty \xi(c_t^* + \delta B_t^*)dt = W^0, \quad (44) \]

the first inequality follows from the state-by-state and time-by-time optimality of \( c_t^* \) and \( B_t^* \) and the second inequality follows from (43) and (44). In addition, it can be verified that \( (c_t^*, \theta_t^*, B_t^*) \) is admissible, satisfies the budget constraint, and \( W_t = \hat{\eta}x_t^{b-1} > 0 \). This shows the optimality of \( (c^*, \theta^*, B^*) \).

Now we show that \( \nu > 0 \) is also a necessary condition for the wellposedness of the Merton problem. Suppose \( \nu \leq 0 \). Then \( \gamma < 1 \). Consider the following strategy
\[
\hat{c}_t = \varepsilon W_t, \quad \hat{B}_t = 0, \quad \hat{\theta}_t = \frac{\mu - r}{\gamma \sigma^2} W_t.
\]

Then it is straightforward to verify that
\[
v(W^0, 1) = E\int_0^\infty e^{-(\rho + \delta)t} \left( \frac{(K\hat{c}_t)^{1-\gamma}}{1 - \gamma} + \frac{(k\hat{B}_t)^{1-\gamma}}{1 - \gamma} \right) dt = \frac{(W^0)^{1-\gamma}}{1 - \gamma} K^{1-\gamma} e^{1-\gamma} \int_0^\infty e^{-\frac{\nu(1-\gamma)}{\gamma}t} dt.
\]

If \( \nu < 0 \), then setting \( \varepsilon = -\frac{\nu}{2(1-\gamma)} > 0 \) yields \( v(W^0, 1) = +\infty \). If \( \nu = 0 \), then
\[
v(W^0, 1) = \frac{(W^0)^{1-\gamma}}{1 - \gamma} K^{1-\gamma} e^{1-\gamma} \int_0^\infty e^{(1-\gamma)t} dt = \frac{(W^0)^{1-\gamma}}{1 - \gamma} K^{1-\gamma} e^{\gamma} \int_0^\infty e^{-\gamma} dt,
\]

which converges to \( \infty \) as \( \varepsilon \) decreases to 0.

Thus \( \nu > 0 \) is also a necessary condition for the wellposedness of the Merton problem.

**Proof of Theorem 4.**

Before retirement at \( T \), the investor’s problem is to choose admissible trading strategy \( \theta \) in the stock, the consumption rate \( c > 0 \) and the bequest \( B_t > 0 \) to maximize
\[
v(W^0, 0) = \max_{(c, \theta, B)} E\int_0^T e^{-(\rho + \delta)t} \left( \frac{c_t^{1-\gamma}}{1 - \gamma} + \frac{(kB_t)^{1-\gamma}}{1 - \gamma} \right) dt + e^{-\rho t} v(W_T, 1), \quad (45)
\]

subject to the budget constraint:
\[
dW_t = rW_t dt + \theta_t^T(\mu - r) dt + \theta_t^T\sigma dZ_t - c_t dt + \delta(W_t - B_t) dt + wy_t dt, \quad (46)
\]

and the dynamics (22) for \( y \).

Since \( W_t \) is uniformly bounded below, \( c_t \geq 0 \) and \( B_t \geq 0 \), \( e^{-(\rho + \delta)t}W_t + \int_0^t e^{-(\rho + \delta)s}(c_s + \delta B_s) ds \) is a supermartingale under the equivalent martingale measure \( Q \). Therefore for any admissible strategy \( (c, \theta, B) \), we must have
\[
E[\int_0^T \xi_t(c_t + \delta B_t) dt + \xi_TW_T] \leq W^0. \quad (47)
\]
Therefore for any admissible strategy \((c, \theta, B)\),

\[
E \left[ \int_0^T e^{-(\rho+\delta)t} \left( \frac{c^{1-\gamma}_t - \delta (k B_{1-t}^{1-\gamma})}{1-\gamma} \right) dt + e^{-\delta T} T v(W_T, 1) \right]
\]

\[
= E \left[ \int_0^T e^{-(\rho+\delta)t} \left( \frac{c^{1-\gamma}_t - \delta (k B_{1-t}^{1-\gamma})}{1-\gamma} \right) dt + e^{-\delta T} T v(W_T, 1) \right]
\]

\[-x_0 \xi_T W_T + x_0 \int_0^T \xi_t (c_t + \delta B_t) dt + x_0 \xi_T W_T
\]

\[
\leq E \left[ \int_0^T e^{-(\rho+\delta)t} \left( \frac{(c^*_t)^{1-\gamma} - \delta (k B^*_t)^{1-\gamma}}{1-\gamma} \right) dt + e^{-\delta T} T v(W_T^*, 1) \right]
\]

\[+x_0 E \left[ \int_0^T \xi_t (c_t + \delta B_t - c^*_t - \delta B^*_t) dt + \xi_T (W_T - W_T^*) \right]
\]

\[
\leq E \left[ \int_0^T e^{-(\rho+\delta)t} \left( \frac{(c^*_t)^{1-\gamma} - \delta (k B^*_t)^{1-\gamma}}{1-\gamma} \right) dt + e^{-\delta T} T v(W_T^*, 1) \right],
\]

where \(x_0\) solves

\[
\varphi_\varepsilon (x_0, 0) = W^0
\]

is the Lagrangian multiplier such that

\[
E \left[ \int_0^T \xi_t (c^*_t + \delta B^*_t) dt + \xi_T W^*_T \right] = W^0,
\]

(48)

the first inequality follows from the state-by-state and time-by-time optimality of \(c^*_t\) and \(B^*_t\) and the second inequality follows from (47) and (48). In addition, it can be verified that \((c^*_t, \theta^*_t, B^*_t)\) is admissible, satisfies the budget constraint, and

\[
W_t = -\varphi_\varepsilon (x_t, t) > \frac{w(1 - e^{-(r+\delta-\mu_\gamma+\sigma^\gamma_\delta)(T-t)})}{r + \delta - \mu_\gamma + \sigma^\gamma_\delta}.
\]

This shows the optimality of \((c^*, \theta^*, B^*)\).

To prove Theorem 6, we first prove the following lemma, which contains results that will be used for the proof of Theorem 2. Let

\[
\nu = \rho + \delta - (1 - \gamma)(r + \delta + \frac{|\kappa|^2}{2\gamma}).
\]

(49)

**Lemma 2** Suppose \(\nu > 0\), \(\beta_3 > 0\), and \(\beta_2 + \beta_3 > 0\). Suppose there exists a solution \(\xi \in (0, 1)\) to equation (32) and let \(\varphi\) be as defined in Theorem 2. Then

(i). \(\varphi(x, 1)\) is strictly convex and strictly decreasing for \(x \geq 0\);

(ii). \(\forall x \leq \overline{x}, \text{ we have } \varphi(x, 0) \geq \varphi(x, 1), \forall x \in [\underline{x}, \overline{x}], \text{ we have } \varphi_\varepsilon (x, 0) \geq \varphi_\varepsilon (x, 1)\) and

\[
x < \left( \frac{1 - K^{-b}}{bw} \right)^c.
\]

(50)
(iii). \[ A_- < 0, \quad A_+ > 0, \quad \pi > \left( \frac{(1 - \alpha_-)(1 + \delta k - b)}{b - \alpha_-w} \right)^\gamma. \]

(iv). \( \varphi(x, 0) \) is strictly convex and strictly decreasing for all \( x < \pi \).

**Proof of Lemma 1:**

(i). \( \gamma > 0 \) implies that \( b = 1 - 1/\gamma < 1 \). Then since \( \nu > 0 \), direct differentiation shows that \( \varphi^R \) is strictly convex and strictly decreasing for \( x > 0 \).

(ii). First, since \( \nu > 0 \), \( \beta_3 > 0 \), and \( \beta_2 + \beta_3 > 0 \), it is straightforward to show that \( \alpha_+ > 1 > b > \alpha_- \), \( \alpha_- < 0 \).

Define \[ h(x) \equiv \varphi(x, 0) - \varphi(x, 1). \]

It can be easily verified that

\[
\frac{1}{2} \beta_1 x^2 \varphi_{xx}(x, 1) - \beta_2 x \varphi_x(x, 1) - \beta_3 \varphi(x, 1) - (K^{-b} + \delta k^{-b}) \frac{x^b}{b} = 0, \tag{52}
\]

and

\[
\frac{1}{2} \beta_1 x^2 \varphi_{xx}(x, 0) - \beta_2 x \varphi_x(x, 0) - \beta_3 \varphi(x, 0) - (1 + \delta k^{-b}) \frac{x^b}{b} + wx = 0, \tag{53}
\]

with

\[
\varphi(x, 0) = \varphi(x, 1), \tag{54}
\]

\[
\varphi_x(x, 0) = \varphi_x(x, 1), \tag{55}
\]

\[
\varphi_x(x, 0) = 0 \tag{56}
\]

and

\[
\varphi_{xx}(x, 0) = 0. \tag{57}
\]

Then by (52) and (53), \( h(x) \) must satisfy

\[
\frac{1}{2} \beta_1 x^2 h'' - \beta_2 x h' - \beta_3 h = \frac{1 - K^{-b}}{b} x^b - w x. \tag{58}
\]

By (54)-(56) and the fact that \( \varphi(x, 1) \) is monotonically decreasing for \( x > 0 \), we have

\[
h(x) = 0, \quad h'(x) = 0, \quad h'(\pi) > 0. \tag{59}
\]

Differentiating (58) once, we obtain

\[
\frac{1}{2} \beta_1 x^2 h''' + (\beta_1 - \beta_2)x h'' - (\beta_2 + \beta_3) h' = (1 - K^{-b}) x^{b-1} - w. \tag{60}
\]

We consider two possible cases.
Case 1: \((1 - K^{-b})x^{b-1} - w < 0\). In this case, the RHS of equation (60) is negative. Since \(\beta_2 + \beta_3 > 0\), \(h'(x)\) cannot have any interior nonpositive minimum. To see this, suppose \(\hat{x} \in (\underline{x}, \overline{x})\) achieves an interior minimum with \(h'(\hat{x}) \leq 0\). Then we would have \(h'''(\hat{x}) > 0\) and \(h''(\hat{x}) = 0\), which implies that the LHS is positive. A contradiction. Since \(h'(\underline{x}) = 0\), \(h'(\overline{x}) > 0\), we must have \(h'(x) > 0\) for all \(x \in (\underline{x}, \overline{x})\) because otherwise there would be an interior nonnegative minimum. Then the fact that \(h(\underline{x}) = 0\) implies that \(h(x) > 0\) for all \(x \in (\underline{x}, \overline{x})\). Since \(h'(x) > 0\) for all \(x \in (\underline{x}, \overline{x})\), we must have \(h''(x) > 0\) by (59). Then (58), (59) and \(h''(x) > 0\) imply that

\[
\overline{x} < \left( \frac{1 - K^{-b}}{b w} \right)^{\gamma},
\]

Case 2: \((1 - K^{-b})x^{b-1} - w \geq 0\). In this case, we must have \(0 < b < 1\) because \(K > 1\) and therefore \(\overline{x} \leq \left( \frac{1 - K^{-b}}{b w} \right)^{\gamma} < \left( \frac{1 - K^{-b}}{b w} \right)^{\gamma}\). This implies that \(h''(x) > 0\) by (58) and (59). Therefore there exists a \(\epsilon > 0\) such that \(h'(x) > 0\) for all \(x \in (\underline{x}, \underline{x} + \epsilon]\) because \(h'(\underline{x}) = 0\). The RHS of equation (60) is monotonically decreasing in \(x\). Let \(x^*\) be such that the RHS is 0. Then for all \(x \leq x^*\), the RHS is nonnegative and thus \(h'(x)\) cannot have any interior nonpositive (local) maximum in \([\underline{x}, x^*]\) for similar reasons to those in Case 1. Thus there cannot exist any \(\hat{x} \in (\underline{x} + \epsilon, x^*)\) such that \(h'(\hat{x}) \leq 0\). If \(x^* < \overline{x}\), then for all \(x \in (x^*, \overline{x})\), the RHS is nonpositive and thus \(h'(x)\) cannot have any interior nonpositive (local) minimum in \([x^*, \overline{x}]\). Thus there cannot exist any \(\hat{x} \in (x^*, \overline{x}]\) such that \(h'(\hat{x}) \leq 0\). Therefore, there cannot exist any \(\hat{x} \in (\underline{x}, \overline{x})\) such that \(h'(\hat{x}) \leq 0\) and thus we have \(h'(x) > 0\) and \(h(x) > 0\) for all \(x \in (\underline{x}, \overline{x})\).

Now we show for both cases, \(h(x) > 0\) for all \(x < \underline{x}\). (50) implies that the RHS of (58) is positive for \(x < \underline{x}\) and \(h(x)\) cannot achieve an interior positive maximum for \(x < \underline{x}\). On the other hand, \(h''(x) > 0\) and \(h'(x) = 0\) imply that there exists an \(\epsilon > 0\) such that

\[
\forall x \in [\underline{x} - \epsilon, \underline{x}], \quad h'(x) < 0.
\]

Thus \(\forall x \in [\underline{x} - \epsilon, \underline{x}], h(x) > 0\). Therefore \(\forall x < \underline{x}, h(x) > 0\). Otherwise \(h\) would achieve an interior positive maximum in \((0, \underline{x})\).

(iii). It can be shown that

\[
A_+ = \frac{(\eta - \tilde{\eta})(\alpha_+ - b)}{b(\alpha_+ - \alpha_-)} x^{\alpha_- - \alpha_+ - 1} - \frac{(\alpha_+ - 1)w}{(\alpha_+ - \alpha_-)(\beta_2 + \beta_3)} x^{\alpha_- - \alpha_+},
\]

and

\[
\eta = \frac{(\alpha_+ - 1)(1 - \alpha_-)(1 + \delta k^{-b})}{(\alpha_+ - b)(\beta_2 + \beta_3)}. \tag{51}
\]

(50) then implies that \(B > 0\). Since we also have

\[
A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_- (\alpha_+ - \alpha_-)} x^{\alpha_- - \alpha_+} - \frac{(\alpha_+ - 1)w}{\alpha_- (\alpha_+ - \alpha_-)(\beta_2 + \beta_3)} x^{\alpha_- - \alpha_+} > 0,
\]

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\( \pi \) must satisfy
\[
\pi > \left( \frac{\eta(\alpha - b)(\beta_2 + \beta_3)}{(\alpha+1)w} \right)^{\gamma} = \left( \frac{(1 - \alpha)(1 + \delta k - b)}{(b - \alpha)w} \right)^{\gamma}.
\]
Since
\[
\frac{\alpha - b}{\alpha+1} > \frac{b - \alpha}{1 - \alpha},
\]
we have
\[
\pi > \left( \frac{\eta(b - \alpha)(\beta_2 + \beta_3)}{(1 - \alpha)w} \right)^{\gamma},
\]
which implies that \( A_\leq 0 \). (iv). \( A_\leq 0 \), \( A_+ > 0 \) and (51) imply that
\[
A_\leq \alpha_+(\alpha_+ - 1)(\alpha - b)x^{\alpha_+ - b - 1} + A_+\alpha_-(\alpha_--1)(\alpha - b)x^{\alpha_--b - 1} < 0,
\]
which in turn implies that \( \forall x < \pi \),
\[
\varphi_{xx}(x, 0) = (A_\leq \alpha_+(\alpha_+ - 1)x^{\alpha_+ - b} + A_+\alpha_-(\alpha_--1)x^{\alpha_--b} - \eta(b - 1)x^{b-2}
\]
\[
> (A_\leq \alpha_+(\alpha_+ - 1)x^{\alpha_+ - b} + A_+\alpha_-(\alpha_--1)x^{\alpha_--b} - \eta(b - 1)x^{b-2} = 0,
\]
where the last equality follows from \( \varphi_{xx}(\pi, 0) = 0 \). Thus \( \varphi(x, 0) \) is strictly convex \( \forall x < \pi \). Since \( \varphi(x, 0) = 0 \) and \( \forall x < \pi \), \( \varphi_{xx}(x, 0) > 0 \), we must also have \( \forall x < \pi \), \( \varphi_x(x, 0) < 0 \).

**Proof of Theorem 6:**

(1) Case 1: \( R^0 = 1 \), then the investor will always remain retired since retirement is irreversible. The optimality of the proposed strategy is shown in the Lemma 1.

(2) Case 2: \( R^0 = 0 \). Before retirement, by standard argument, the investor’s problem in this case is equivalent to:

\[
v(W^0, y^0, R^0) = \max_{c, \theta, B, \tau} E[\int_0^\tau e^{-(\rho + \delta)t}(c_t^{1-\gamma} + \delta (k B_t)^{1-\gamma})dt + e^{-(\rho + \delta)\tau}v(W_\tau, 1)] \quad (62)
\]
subject to
\[
E[\int_0^\tau \xi(c_t + \delta B_t - wy_t)dt + \xi_\tau W_\tau] \leq W^0. \quad (63)
\]
and
\[
E[\int_s^\tau \xi(c_t + \delta B_t - wy_t)dt + \xi_\tau W_\tau] \geq 0, \quad \forall s > 0, \quad (64)
\]
with the income process
\[
dy_t = \mu y_t dt + \sigma y_t dZ_t,
\]
where \( \xi \) is the state price density defined in (42).
By Lemma 1, \( \varphi(x, 0) \) is strictly decreasing and strictly convex for \( x \in [\bar{x}, \tilde{x}] \), which implies that (31) has a unique solution. In addition, the existence and uniqueness (as a process) of \( x \) and \( \Lambda \) follow from Proposition 2.2.3 in Harrison (1984) (applied to (30) in logs).

It is straightforward to verify that \( W^*_t, c^*_t, \theta^*_t, R^*_t \) satisfy the budget constraint (21). In addition, since \( \varphi(x, 0) \) and \( \varphi(x, 1) \) are strictly decreasing, \( W^*_t \geq 0 \) for all \( t \geq 0 \). In addition, \( (c^*, \theta^*, B^*, R^*) \) is admissible.

Define \( W = -y \varphi_x(x, 0) \). Let \( \bar{x} \) be such that \( -y \varphi_x(\bar{x}, 1) = W \). Since \( \varphi_x(x, 0) = \varphi_x(\bar{x}, 1) \) and \( \varphi(x, 1) \) is strictly decreasing, we must have \( \bar{x} = x \), which implies that

\[
 v(W, y, 0) = y^{1-\gamma}(\varphi(x, 0) - \bar{x} \varphi_x(x, 0)) = y^{1-\gamma}(\varphi(x, 1) - \bar{x} \varphi_x(x, 1)) = y^{1-\gamma}(\varphi(x^R, 1) - x^R \varphi_x(x^R, 1)) = v(W, y, 1). \tag{65}
\]

By Lemma 1, for all \( x < \bar{x} \), we have \( \varphi_{xx}(x, 0) > 0 \), \( \varphi_{xx}(x, 1) > 0 \), and \( \varphi(x, 0) \geq \varphi(x, 1) \), with equality at \( \bar{x} \). Since \( -y \varphi_x(x, 0) = W \) and \( -y \varphi_x(x^R, 1) = W \), we have

\[
 \varphi(x, 0) - \varphi(x^R, 1) \geq \varphi(x, 1) - \varphi(x^R, 1) \geq \varphi(x^R, 1)(x - x^R) = x \varphi_x(x, 0) - x^R \varphi_x(x, 1)
\]

where the first inequality follows from \( \varphi(x, 0) \geq \varphi(x, 1) \) and the second inequality from the convexity of \( \varphi(x, 1) \). After rearranging, we obtain that \( v(W, y, 0) \geq v(W, y, 1) \) for all \( W \geq 0 \).

Let

\[
 \Lambda_t = \frac{1}{\max(1, \sup_{0 \leq s \leq t} x_0 e^{(\mu_s - \frac{1}{2} \sigma_s^2) s - \sigma_s Z_s} / \bar{x})}.
\]

Then \( \Lambda_t \) is an adapted, nonincreasing process (with \( \Lambda_0 = 1 \)) that increases only when \( x_t \) reaches \( \bar{x} \). Since \( v(W^*_t, y, 0) = y^{1-\gamma} (\varphi(x_t, 0) - x_t \varphi_x(x_t, 0)) \) with \( W^*_t = -y_t \varphi_x(x_t, 0) \), by Itô’s Lemma, we have

\[
 e^{-(\rho+\delta)t} v(W^*_t, y, 0) = v(W_0, 0, y) + \int_0^t \mathcal{L} (e^{-(\rho+\delta)s} v(W^*_s, y, 0)) ds - \int_0^t e^{-(\rho+\delta)s} y_s^{1-\gamma} x_s \varphi_{xx}(x, 0) \frac{d\Lambda_s}{\Lambda_s}
\]

\[
 + \int_0^t e^{-(\rho+\delta)s} y_s^{1-\gamma} ((1 - \gamma)(\varphi(x, 0) - x_s \varphi_x(x, 0))\sigma_y - x_s^2 \varphi_{xx}(x, 0) s) dZ_s,
\]

where \( \mathcal{L} \) is the differential operator. Since \( y_t \) is a geometric Brownian motion, \( E[y_t^{2(1-\gamma)}] < \infty \). For any time \( t \) such that \( x_t \in [x, \bar{x}] \), we have that \( \varphi(x, 0), \varphi_x(x, 0), \) and \( \varphi_{xx}(x, 0) \) are all bounded. Thus the stochastic integral is a martingale. If \( x_t < \bar{x} \), then \( x_t \) become a geometric Brownian motion, which implies that the stochastic integral is again a martingale because in this case \( \varphi(x, 0) = -\hat{\gamma}(x) \frac{b}{b} \). In addition, since \( \Lambda_t \) changes only when \( x_t = \bar{x} \) and \( \varphi_{xx}(\bar{x}, 0) = 0 \), the third term is always zero. By the construction of \( \varphi(x, 0) \), we have

\[
 \mathcal{L} (e^{-(\rho+\delta)s} v(W^*_s, y, 0)) = -e^{-(\rho+\delta)s} y_s^{1-\gamma} ((K^{-\delta} - \hat{\gamma}(W^*_s \geq \bar{W})) x_s^b 1 - \gamma + \delta k^{-\delta} x_s^b 1 - \gamma).
\]
Let \( z_t = e^{(\rho+\delta)t}\Lambda_t\xi \), then it can be shown that \( z_t = y_t^\gamma x_t \). Therefore, for any admissible strategy \((c, B, \tau)\) and the implied wealth process \( W \), by Doob’s Optional Sampling Theorem,

\[
v(W^0, y_0, 0) = E\left[ \int_0^\tau e^{-\rho t} y_s^{1-\gamma}(\frac{x_t^b}{1-\gamma} + \delta k^{-b} \frac{x_t^b}{1-\gamma}) dt + e^{-(\rho+\delta)\tau}v(W^*_\tau, y_\tau, 0) \right]
\]

\[
\geq E\left[ \int_0^\tau e^{-\rho t} y_s^{1-\gamma}(\frac{x_t^b}{1-\gamma} + \delta k^{-b} \frac{x_t^b}{1-\gamma}) dt + e^{-(\rho+\delta)\tau}v(W^*_\tau, y_\tau, 0) \right]
\]

\[
\geq E\left[ \int_0^\tau e^{-\rho t} y_s^{1-\gamma}(\frac{k B_t^{1-\gamma}}{1-\gamma}) dt + e^{-(\rho+\delta)\tau}v(W^*_\tau, y_\tau, 1) \right]
\]

\[
+ E\left[ \int_0^\tau e^{-\rho t} z_t(c_t^1 + \delta B_t - wy_t) dt + e^{-(\rho+\delta)\tau}z_\tau W^*_\tau \right]
\]

\[
- E\left[ \int_0^\tau e^{-\rho t} z_t(c_t^1 + \delta B_t - wy_t) dt + e^{-(\rho+\delta)\tau}z_\tau W^*_\tau \right]
\]

\[
\geq E\left[ \int_0^\tau e^{-\rho t} y_s^{1-\gamma}(\frac{k B^1}{1-\gamma}) ds + e^{-(\rho+\delta)\tau}v(W^*_\tau, y_\tau, 1) \right],
\]

with equality for the stopping time

\[
\tau^* = \inf\{ t \geq 0 : x_t \leq \underline{x} \},
\]

where the second inequality follows from the fact that \( v(W, y, 0) \geq v(W, y, 1) \) for all \( W \geq 0 \) as shown above, the third inequality follows from the pointwise optimality of \((c^*_t, B^*_t, W^*_\tau)\), and the fourth inequality follows from the fact that all admissible strategy must satisfy conditions (63) and (64) with inequality and the proposed strategy \((c^*_t, B^*_t, W^*_\tau)\) satisfies them with equality.

This completes the proof of Theorem 6.

**Lemma 3** Suppose \( \nu > 0, \beta_3 > 0, \) and \( \beta_2 + \beta_3 > 0 \). Then there exists a unique solution \( \zeta^* \in [0, 1] \) to equation (32) and

\[
\zeta^* < \bar{\zeta} = \text{Min}(\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}, 1).
\]

**Proof of Lemma 3:** Since \( \nu > 0 \) and \( \beta_2 + \beta_3 > 0 \),

\[
\alpha_+ > 1 > b > \alpha_-, \quad \alpha_- < 0.
\]

Next, since \( \zeta^{b-a+} \) dominates \( \zeta^{1-a+} \) as \( \zeta \to 0 \), we have

\[
\lim_{\zeta \to 0} q(\zeta) = \lim_{\zeta \to 0} -\frac{1 - K^{-b}}{b(1 + \delta k^{-b})} (\alpha_- - b)(\alpha_+ - 1)\zeta^{b-a+} = +\infty.
\]
Next, it is easy to verify that
\[ q(1) = -\frac{(\alpha_+ - 1)(\alpha_+ - 1)(\alpha_+ - \alpha_-)(K^{1-b} + \delta k^{1-b})}{\alpha_+ \alpha_-(1 + \delta k^{1-b})} < 0. \]

Now suppose \( \hat{\zeta} = \left( \frac{1-K^{1-b}}{b(1+\delta k^{1-b})} \right)^\gamma < 1. \) Then we have \( \frac{1}{b(1+\delta k^{1-b})} \hat{\zeta}^{b-\alpha_+} - \frac{1}{\alpha_+} = \hat{\zeta}^{\gamma-\alpha_+} - \frac{1}{\alpha_+}, \)
\[ \frac{1}{b(1+\delta k^{1-b})} \hat{\zeta}^{b-\alpha_-} = \hat{\zeta}^{\gamma-\alpha_-} - \frac{1}{\alpha_-} \text{ and } \hat{\zeta}^{\gamma-\alpha_+} > 1 > \frac{1}{\alpha_+}. \]
It follows that
\[ q(\hat{\zeta}) = -\frac{1}{\gamma} (\hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+})(\hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-})(\alpha_+ - \alpha_-) < 0. \]

Then by continuity of \( q, \) there exists a solution \( \zeta^* \in (0, \hat{\zeta}) \) such that \( q(\zeta^*) = 0. \) Suppose there exists another solution \( \zeta \in [0, 1] \) such that \( q(\zeta) = 0. \) Let \( v(W, y, 0) \) and \( \overline{W} \) be the value function and boundary respectively corresponding to \( \zeta^* \) and \( \hat{v}(W, y, 0) \) and \( \hat{W} \) be the value function and boundary respectively corresponding to \( \hat{\zeta}. \) Without loss of generality, suppose \( W > \hat{W}. \) Since \( \overline{W} \) is the retirement boundary, the value function corresponding to \( \hat{\zeta} \) for \( W > \overline{W} \) is equal to \( v(W, y, 1). \) However, Lemma 1 implies that \( v(W, y, 0) > v(W, y, 1) \) for all \( W < \overline{W}. \) This implies that \( \overline{W} \) cannot be the optimal retirement boundary which contradicts Theorem 4. Therefore the solution to equation (32) is unique.

**Proof of Proposition 1.** Given the optimal policies of the investor, the present value of the investor’s human capital at \( t \) is
\[ H(x_t, y_t, t) = \xi_t^{-1} E\left[ \int_t^{T_R} \xi_s wy_s ds \right], \]
where \( T_R = T \) for the fixed retirement date case and \( T_R = \tau^* \) for the discretionary retirement case.

(i) for the fixed retirement date we have the HJB equation
\[ H_t + \frac{1}{2} \sigma_x^2 x^2 H_{x^2} + \mu_x x H_x + \frac{1}{2} \sigma_y^2 y^2 H_{y^2} + \mu_y y H_y + \sigma_y \sigma_x x y H_{xy} - (x \sigma_x^T H_x + y \sigma_y^T H_y) \kappa - (r + \delta) H + w y = 0, \]
subject to
\[ H(x_T, y_T, T) = 0. \]
It can be verified that the first \( H \) function solves this PDE. In addition, the diffusion term \( \int_0^T (x_t H_x \sigma_x + y_t H_y \sigma_y - \xi_t \kappa^T) dB_s \) is a martingale since both \( y_t \) and \( x_t \) are geometric Brownian motions.

(ii) for the cases with discretionary retirement, we have the same HJB equation (66). It can be shown that the general solution for \( h \) is
\[ H(x_t, y_t, t) = \frac{w}{r + \delta - \mu_y + \sigma_y \kappa}(C_1 x_t^{\alpha_-} + C_2 x_t^{\alpha_+} + 1) \delta t \{ x_t \geq \overline{x} \}, \]
where \( C_1 \) and \( C_2 \) are integration constants to be determined. Since at the retirement date the present value of future human capital is zero, we have the boundary condition

\[
H(x, y, t) = 0. \tag{67}
\]

If there is no borrowing constraint, then we must have \( C_2 = 0 \). The boundary condition (67) implies the second function for \( H \). If there is a borrowing constraint, then by applying Itô’s lemma to \( H \), we have another boundary condition

\[
H_x(x, y, t) = 0. \tag{68}
\]

Conditions (67) and (68) then implies \( C_1 = A \) and \( C_1 = B \).

**Proof of Proposition 2.** Recall that

\[
\frac{dx_t}{t} = \mu_x x_t + \sigma x_t dB_t.
\]

Let

\[
g(x) = E[\tau^* | x_t = x].
\]

Then by Itô’s lemma, \( g \) must satisfy

\[
\frac{1}{2} \sigma^2 x^2 g_{xx} + \mu_x x f_x + 1 = 0,
\]

subject to

\[
g(x) = 0. \tag{69}
\]

The solution is

\[
g(x) = C_1 \left( x^{1 - 2 \mu_x / \sigma_x^2} - \frac{\log(x/x)}{\mu_x - \frac{1}{2} \sigma_x^2} \right).
\]

In the absence of borrowing constraint, we must have \( C_1 = 0 \) for the diffusion term in the dynamics of \( g(x_t) \) to be a martingale, which gives the first expression in the proposition. In the presence of borrowing constraint, we have a second boundary condition

\[
g'(x) = 0,
\]

which yields the second expression.
References


