Does Private Information about Inventories Matter?

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We study the impact of multilateral private information about inventory levels on the overall performance of two-echelon supply chains. We focus on two polar cases: i) multiple suppliers serving a single retailer; and ii) a single supplier facing multiple retailers. In both cases the single party has all the bargaining power, i.e. it can commit to take-it-or-leave-it offers. In the former case, we find that the inventory policy that solves the centralized cost minimization problem remains optimal for the retailer, even in the presence of multilateral private information. Moreover, the retailer can appropriate the cost reduction of the entire supply chain. In the latter case, the contract that is optimal for the supplier induces a simple ranking allocation rule, based on critical fractiles of adjusted demand distributions that account for the incentives generated by private information. This rule also reverses the order of all served retailers’ inventory positions; and, for intermediate values of the supplier’s initial inventory level, “overshoots” the positions of some retailers, i.e. pushes them beyond their optimal levels without private information. Our results suggest that supply chains in which parties that are more exposed to demand uncertainty have stronger bargaining positions perform better in terms of operating cost.

Key words: decentralized supply chain, mechanism design, informed principal, private information

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1. Introduction

Consider a two-echelon supply chain in which retailers face stochastic demand. The centralized problem of minimizing the total operating cost in this setting has been extensively studied, and the main features of the optimal allocation are well understood (Zipkin 2000). Recently, several decentralized supply chains models have highlighted various aspects of the fundamental tension between the unilateral incentives of suppliers and retailers versus the benefits of system coordi-

In this paper, we study the impact of multilateral private information regarding inventory levels on both the performance of the entire supply chain and the main features of the optimal allocation. We consider a single period model in which all parameters except initial inventory levels are common knowledge. Thus we isolate the effect of private information about inventory levels on the performance of the supply chain. More specifically, we address the following questions:

- Can a decentralized supply chain be as efficient as its centralized counterpart?
- What are the main properties of the optimal contract?

We find that the answers to these questions depend upon the structure of the supply chain. Our analysis focuses on two scenarios: i) multiple suppliers serving a single retailer (MSSR), and ii) a single supplier facing multiple retailers (SSMR). In both scenarios retailers face stochastic demands, all parties have privately known inventory levels, and the single party (the retailer in the MSSR case and the supplier in the SSMR case) has all the bargaining power, i.e. can commit to a “take-it-or-leave-it” contract offer. The optimal mechanisms minimize the overall cost (operating plus monetary) for the single party in the presence of strategic agents and multilateral private information about initial inventory levels.

The MSSR case applies to situations in which a large retail company procures its products from many suppliers on a regular basis. Typically, the retailer’s inventory can hardly be monitored by any individual supplier. On the other hand, it may be costly for the retailer to gather accurate and timely information about all its suppliers’ current inventory levels.

To motivate the SSMR case, consider a new hot-selling product during a holiday shopping season. For example, *Zhu Zhu Pet* became the hottest toy during the Christmas season of 2009 in the US (Mabrey and Janik 2009, Wernau 2009). The overnight frenzy created a drastic shortage, and the manufacturer Cepia LLC had to deal with the problem of restocking its retailers, having limited knowledge about their current inventory positions and local demand conditions.
Our first main result is that private information about inventories matters in the SSMR case, and is irrelevant in the MSSR case. The irrelevance result hinges on two reasons. First, the retailer’s private information does not matter because its cost function is not directly affected by the suppliers’ inventory levels. In the literature on mechanism design by an informed principal, this property is labeled “private values” (Maskin and Tirole 1990). Second, each supplier’s private information is inconsequential because the benefit of selling any additional unit is constant up to its inventory level. In our static environment, where production has already taken place, this is equivalent to the standard assumption of constant marginal handling cost. In turn, this is also due to the fact that suppliers are not directly exposed to any demand uncertainty. Thus in the MSSR case the presence of multilateral private inventory information does not hinder the implementability of the inventory policy that minimizes the total operating cost of the system (without private information). Moreover, the retailer can appropriate the full cost reduction of the entire supply chain using a straightforward fixed price contract.

In contrast, in the SSMR case the centralized optimal inventory policy is no longer optimal for the supplier. This is because the exposure to demand uncertainty makes each retailer’s cost function non-linear – the cost reduction from an additional unit is decreasing.

Contracts that are optimal for the supplier are characterized by a simple ranking allocation rule, based on critical fractiles of an appropriately adjusted demand distribution, which accounts for the incentives generated by the presence of private information. Even in symmetric environments, where retailers differ only with respect to their privately known inventory levels, it is optimal for the supplier to reverse the order of the inventory positions of all served retailers. As it is well-known, the solution to the centralized problem (without private information) aims at balancing all retailers’ inventory positions. Thus the “reverse ranking” of final inventory position is entirely due to the presence of private information. We also show that the optimal allocation is more selective, i.e. it serves a weakly smaller number of retailers relative to the centralized allocation.
Confirming a well known feature of environments with private information, the total number of units sold by the supplier cannot exceed the one prescribed by the centralized solution. However, in our model, even with symmetric retailers, the quantities allocated to some retailers may “overshoot,” i.e. go beyond the levels prescribed by the centralized solution. We provide necessary and sufficient conditions for the occurrence of overshooting, which, to the best of our knowledge, is unique in the literature.

The rest of the paper is organized as follows. Sections 2 and 3 are devoted to the analyses of the MSSR model and the SSMR model respectively. In Section 4, we restrict our attention to the symmetric case of the SSMR model and establish that the optimal allocation entails reverse ranking of inventory positions and overshooting distortions. Section 5 contains numerical examples to illustrate our findings. All proofs and technical comments are relegated to the appendices.

1.1. Literature Review

Centralized supply chains has been extensively studied. The main properties of the allocations that minimize inventory operating costs in many different situations are well understood (see Allen (1958) and Zipkin (2000)). More recently, there has been a wealth of research on the tension between competition and coordination in decentralized supply chains. We defer to Cachon (2003) for an extensive literature review, and a general discussion on how various types of contracts can be used to manage incentive conflicts. In particular, Section 10 of Cachon (2003) focuses on issues generated by private information in supply chain environments. In this work we consider models that contribute to the decentralized supply chain literature by incorporating multilateral private information about inventory levels and examining different bargaining power configurations under capacity constraints.

In their pioneering work, Cachon and Lariviere (1999) study a model with a capacity constrained supplier serving multiple retailers that are privately informed about their optimal stocking level. They show that several empirically relevant contracts with a fixed wholesale price are vulnerable to manipulation. Deshpande and Schwarz (2005) characterize optimal contracts in an abstract model
similar to Cachon and Lariviere (1999). Both papers also consider the capacity choice problem under the optimal contract.


Chen (2007) considers the optimal procurement auction for a buyer facing multiple suppliers with private information about their cost structures. Iyengar and Kumar (2008) study a similar model to Chen (2007), where suppliers have private information about cost parameters as well as capacity levels.

Lee and Whang (2000) point out that informational decentralization and incentives are key factors for the overall supply chain performance. Lee and Whang (1999) investigate performance measurement schemes to improve the decentralized decisions, Porteus and Whang (1991) develop mechanisms which include an internal futures market to account for different incentives in the supply chain.

Zhang et al. (2009) investigate a model in which a supplier interacts over time with a retailer who has private information about its current inventory level. Their single period model is essentially identical to the SSMR case we consider, with a single retailer, no capacity constraint and no private information on the supplier’s side. In the dynamic model, they show that under certain conditions on the parameter values and demand distributions it is optimal for the supplier to offer a batch contract in every period.

2. Multiple Suppliers and a Single Retailer

In this section we study a two-echelon supply chain system with $M$ suppliers serving a single retailer.

The retailer is modeled as a newsvendor, with a privately known inventory level $x_r$, facing the standard trade-off between holding excess inventory, which entails a unit cost $h_r$, and not being
able to meet its stochastic demand $D_r$ which generates a per-unit penalty cost $b_r$. After receiving a total quantity $Q$ from all suppliers, its expected operating cost can be written as

$$C_r(x_r, Q) = h_r \mathbb{E}_{D_r} [(x_r + Q - D_r)^+] + b_r \mathbb{E}_{D_r} [(x_r + Q - D_r)^-],$$

(1)

where the expectation is taken with respect to the demand $D_r$.

Similarly, supplier $i \in \{1, \ldots, M\}$ is privately informed about its inventory $x_i$ and incurs unit holding and shipping/transaction costs $h_i$ and $c_i$, respectively. Thus its total cost of supplying $q_i$ units is given by

$$C_i(x_i, q_i) = h_i(x_i - q_i) + c_i q_i, \quad 0 \leq q_i \leq x_i. \quad (2)$$

The last two inequalities capture the facts that inventory cannot flow back from any retailer to the supplier and production has already taken place, hence the quantity sold $q_i$ cannot exceed the inventory.

Relabeling suppliers if necessary, we can assume that $c_1 - h_1 \leq \cdots \leq c_M - h_M$. To avoid uninteresting cases we also assume that $h_i < h_r$ and $c_i < b_r$ for all $i = 1, \ldots, M$.

We assume that the retailer has all the bargaining power, i.e. can to commit to a take-it-or-leave-it contract offer. Its problem is to design a procurement contract that minimizes its total expected cost given by the sum of its monetary payments to the suppliers plus its operating cost. By the revelation principle (Myerson 1979, Dasgupta et al. 1979), without loss of generality, we can restrict attention to the set of incentive compatible and individually rational revelation mechanisms. Formally, a revelation mechanism specifies the quantity $q_i(x_r, x)$ sold by supplier $i$ and the payment $m_i(x_r, x)$ received from the retailer, for each inventory profile $x_r, x := (x_1, \ldots, x_M)$, and each $i = 1, \ldots, M$.

The incentive compatibility and individual rationality constraints can be stated as: for all $x_r$ and $x'_r$

$$\mathbb{E}_x \left[ C_r \left( x_r, \sum_{i=1}^M q_i(x_r, x) \right) + \sum_{i=1}^M m_i(x_r, x) \right] \leq \mathbb{E}_x \left[ C_r \left( x_r, \sum_{i=1}^M q_i(x'_r, x) \right) + \sum_{i=1}^M m_i(x'_r, x) \right], \quad (IC_r)$$
and
\[ \mathbb{E}_x \left[ C_r \left( x_r, \sum_{i=1}^{M} q_i(x_r, x) \right) + \sum_{i=1}^{M} m_i(x_r, x) \right] \leq C_r(x_r, 0), \quad (\text{IR}_r) \]
where the expectation is taken with respect to the retailer’s belief about the suppliers’ inventory profile \( x \). Similarly, and for each \( i = 1, \ldots, M \), and for all \( x_i, x'_i \),
\[ \mathbb{E}_{x_r, x_{-i}} \left[ C_i(x_i, q_i(x_r, x)) - m_i(x_r, x) \right] \leq \mathbb{E}_{x_r, x_{-i}} \left[ C_i(x_i, q_i(x_r, (x_{-i}, x'_i))) - m_i(x_r, (x_{-i}, x'_i)) \right], \quad (\text{IC}_i) \]
where \( x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M) \), and
\[ \mathbb{E}_{x_r, x_{-i}} \left[ C_i(x_i, q_i(x_r, x)) - m_i(x_r, x) \right] \leq C_i(x_i, 0) \quad (\text{IR}_i) \]
where the expectation is taken with respect to supplier’s \( i \) belief about all other parties’ inventory levels.

The retailer’s problem, given its privately known inventory level \( x_r \), can be stated as
\[ \min_{q, m} \mathbb{E}_x \left[ C_r \left( x_r, \sum_{i=1}^{M} q_i(x_r, x) \right) + \sum_{i=1}^{M} m_i(x_r, x) \right] \]
\[ \text{s.t. (IC}_r), (\text{IR}_r), (\text{IC}_i), (\text{IR}_i) \text{ and } 0 \leq q_i(x_r, x) \leq x_i, \quad i = 1, \ldots, M. \quad (3) \]

Our first theorem establishes that the retailer’s problem (3) is solved by the allocation function \( q^o \) which also solves the centralized problem: for each inventory profile \( x_r, x \)
\[ q^o(x_r, x) = \arg \min_q \quad C_r \left( x_r, \sum_{i=1}^{M} q_i \right) + \sum_{i=1}^{M} C_i(x_i, q_i) \]
\[ 0 \leq q_i \leq x_i \quad \text{for} \quad i = 1, \ldots, M. \quad (4) \]

**Theorem 1.** In the MSSR model, the following contract is optimal for the retailer’s problem. For any inventory profile \( x_r, x \), there exists an integer \( i^o \) such that, for each \( i = 1, 2, \ldots, M \)
\[ q^o_i(x_r, x) = \begin{cases} x_i, & \text{if} \quad i < i^o, \\ G_r^{-1} \left( \frac{h_i + h_{i-1} - x_{i-1}}{h_{i-1} + b_r} \right) - x_r - \sum_{j=1}^{i-1} x_j, & \text{if} \quad i = i^o, \\ 0, & \text{if} \quad i > i^o, \end{cases} \]
where \( G_r \) denotes the cumulative distribution function of the retailer’s demand \( D_r \), and
\[ m^o_i(x_r, x) = (c_i - h_i) q^o_i(x_r, x). \quad (5) \]
Since the allocation rule \( q^o \) defined in (4) already minimizes the entire system cost for any given inventory profile \( x_r, x \) without any incentive constraints, the proof of Theorem 1 consists in showing that the payments \( m^o \) defined in (5) together with \( q^o \) satisfy all incentive constraints and at the same time generate no cost reduction for any supplier. This immediately implies that the retailer is able to appropriate the full cost reduction of the entire supply chain. We record this observation for future reference in the following corollary.

**Corollary 1.** In the MSSR model, the retailer appropriates the full cost reduction of the entire supply chain.

The contract characterized in Theorem 1 would remain optimal even if all inventory levels were publicly known. Thus the presence of private information has no impact on the efficiency of entire supply chain in the MSSR model. Intuitively, there are two reasons for this result. First, the irrelevance of the retailer’s private information is due to the fact that its cost function is not directly affected by the suppliers’ private information.\(^2\) Second, the irrelevance of each supplier’s private information hinges on the property that its benefit of selling any additional unit is constant, which in turn follows from the fact that they are not directly exposed to any demand uncertainty.

**Remark 1.** The contract characterized in Theorem 1 satisfies all incentive constraints “ex post”: the retailer’s constraints \( (IC_r) \) and \( (IR_r) \) for each realization of the suppliers’ inventory profile \( x \), and supplier \( i \)’s satisfies \( (IC_i) \) and \( (IR_i) \), \( i = 1, \ldots, M \), for all realizations of \( x_{-i} \) and \( x_r \). In particular the implementation of the optimal contract does not require any knowledge by the retailer of the suppliers’ beliefs about the others’ inventory levels.\(^3\)

### 3. A Single Supplier and Multiple Retailers

We now turn to the case with a single supplier, indexed by \( s \), serving \( N \) retailers. As in the previous model, each party is privately informed about its own inventory level.

The supplier’s cost of delivering \( q_i \) units to retailer \( i = 1, \ldots, N \) is given by

\[
C_s(x_s, q) = h_s \left( x_s - \sum_{i=1}^{N} q_i \right) + \sum_{i=1}^{N} c_i q_i, \quad \text{for } \sum_{i=1}^{N} q_i \leq x_s, \quad q \geq 0, \quad (6)
\]
where $x_s$ denotes its inventory level, $h_s$ denotes the unit holding cost for unsold inventory, and $c_i$ the shipping/transaction cost for any unit sold to retailer $i$. The supplier cannot allocate more than its inventory/capacity $x_s$, cannot discard its inventory, nor can receive inventory from retailers.

Retailer $i$ is modeled as a newsvendor with unit holding cost $h_i \geq h_s$, unit penalty cost $b_i \geq c_i$, facing stochastic demand $D_i$ with cdf and pdf denoted by $G_i$ and $g_i$. Its expected cost function given by

$$C_i (x_i, q_i) = h_i \mathbb{E}_{D_i} [(x_i + q_i - D_i)^+] + b_i \mathbb{E}_{D_i} [(x_i + q_i - D_i)^-],$$

(7)

where $x_i$ denotes the retailer’s (privately known) inventory level.

The supplier’s problem is to design a contract that minimizes its expected total cost given by its operating cost minus the payments made by the retailers. The supplier’s belief about the profile $x := (x_1, \ldots, x_N)$ is represented by the joint probability distribution $F^N$ on the rectangular support $\mathcal{X} = \prod_{i=1}^N \mathcal{X}_i = \prod_{i=1}^N [\bar{x}_i, \bar{x}_i]$ with marginals $F_1, \ldots, F_N$. Similarly, retailer $i$ believes that retailer $j$’s inventory is distributed accordingly to the cdf $F_j$ and the supplier’s inventory level is distributed accordingly to $F_s$ with support $\mathcal{X}_s$.

The incentive constraints that determine the feasible set for the supplier’s problem are: for each retailer $i = 1, \ldots, N$, and all $x_i \in \mathcal{X}_i$, $x'_i \in \mathcal{X}_i$,

$$\mathbb{E}_{x_{-i}, x_{-i}} [C_i (x_i, q_i (x_i, x)) + m_i (x_s, x)] \leq \mathbb{E}_{x_{-i}, x_{-i}} [C_i (x_i, q_i (x'_i, x_{-i})) + m_i (x_s, (x'_i, x_{-i}))], \quad (\text{IC}_i)$$

for all $x_i \in \mathcal{X}_i$

$$\mathbb{E}_{x_{-i}, x_{-i}} [C_i (x_i, q_i (x_i, x)) + m_i (x_s, x)] \leq C_i (x_i, 0), \quad (\text{IR}_i)$$

for any $x_s, x'_s \in \mathcal{X}_s$,

$$\mathbb{E}_{x} \left[ C_s (x_s, q(x_s, x)) - \sum_{i=1}^N m_i (x_s, x) \right] \leq \mathbb{E}_{x} \left[ C_s (x_s, q(x'_s, x)) - \sum_{i=1}^N m_i (x'_s, x) \right], \quad (\text{IC}_s)$$

and for all $x_s \in \mathcal{X}_s$

$$\mathbb{E}_{x} \left[ C_s (x_s, q(x_s, x)) - \sum_{i=1}^N m_i (x_s, x) \right] \leq C_s (x_s, 0), \quad (\text{IR}_s)$$

where all expectations are taken to their corresponding distributions, and $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M)$. 
Formally, the supplier’s problem can be stated as
\[
\min_{q,m} \mathbb{E}_x \left[ C_s(x_s, q(x_s, x)) - \sum_{i=1}^{N} m_i(x_s, x) \right]
\]
s.t. \((\text{IC}_s), (\text{IR}_s), (\text{IC}_i), (\text{IR}_i), q_i(x_s, x) \geq 0 \quad i = 1, \ldots, N \)
\[
\sum_{i=1}^{N} q_i(x_s, x) \leq x_s.
\] (8)

The results for the SSMR model will be established under the following regularity conditions.

**Assumption 1.** The survival function of the demand \(\bar{G}_i(\cdot) = 1 - G_i(\cdot)\) is log-concave for \(i = 1, 2, \ldots, N\).

**Assumption 2.** The retailers’ inventory levels \(x_1, \ldots, x_N\) are distributed independently, with marginal densities \(f_1, \ldots, f_N\), and cumulative distributions \(F_1, \ldots, F_N\). For each \(i = 1, \ldots, N\) the reversed hazard rate \(\frac{f_i}{F_i}\) is non-increasing.

Assumption 1 allows for a variety of demand distributions. It is weaker than the log-concavity of the probability density functions \(g_i\) which is imposed in Zhang et al. (2009). Assumption 2 is standard in the literature (e.g. Myerson (1979), Corbett (2001) and Zhang et al. (2009)). We are now ready to state the main result of this section.

**Theorem 2.** In the SSMR model, under Assumptions 1 and 2, the following contract is optimal for the supplier’s problem (8): for each \(x \in \mathcal{X}\) and \(x_s \in \mathcal{X}_s\)
\[
q^*(x_s, x) \in \arg \min_{q \geq 0} \quad \pi^*(x, q) \quad \sum_{i=1}^{N} q_i \leq x_s
\] (9)

where \(\pi^*(x, q) := \sum_{i=1}^{N} \left[ (c_i - h_i)q_i + C_i(x_i, q_i) + (h_i + b_i) \frac{f_i(x_i)}{f_i(x_i)} G_i(x_i + q_i) \right]\), and each retailer payment is given by
\[
m^*_i(x_s, x) = C_i(x_i, 0) - C_i(x_i, q^*_i(x_s, x)) - (h_i + b_i) \int_{x_i}^{x_i} \left[ G_i(z + q^*_i(x_s, (z, x_{-i}))) - G_i(z) \right] dz.
\]

The function \(\pi^*(x, q)\) is given by the sum of all retailers’ “virtual costs”. It adjusts the supply chain costs by taking into account the incentives generated by the presence of private information.

The allocation \(q^*\) characterized in Theorem 2 may differ significantly from the centralized solution, due to the presence of the retailers’ private information in the SSMR model. Indeed, the
supplier’s private information is irrelevant – the contract of Theorem 2 would remain optimal even if \( x_s \) were publicly known. This is because the supplier’s cost function is not directly affected by the retailers’ private information. The retailers’ private information limits the supplier’s gains because each retailer’s cost reduction of receiving additional units is decreasing. This feature in turn is a consequence of the retailers bearing the entire supply chain risk.

**Remark 2.** The contract characterized in Theorem 2 satisfies (IC\(_i\)) and (IR\(_i\)) for each \( i = 1, \ldots, N \), for all realizations of \( x_{-i} \) and \( x_s \). Therefore, the implementation of the optimal contract does not require any knowledge by the supplier of the retailers’ beliefs about the others’ inventory levels.

In the remainder of this section and in Section 4 we derive additional structural properties of the optimal contract.

### 3.1. Fractile-Based Interpretation of the Optimal Allocation

This section provides a characterization of the optimal contract derived in Theorem 2 based on critical fractiles of appropriately adjusted demand distributions. This allows for a direct comparison with the centralized solution. We begin by defining the following family of adjusted cumulative distribution functions, parameterized by the inventory level \( x_i \)

\[
\tilde{G}_i(y|x_i) := G_i(y) + \frac{F_i(x_i)}{f_i(x_i)} g_i(y), \quad i = 1, \ldots, N. \tag{10}
\]

In general, \( \tilde{G}_i(\cdot|x_i) \) is not a cumulative probability distribution as it can be both non-monotone and larger than one. It turns out however that, under Assumption 1, \( \min\{1, \tilde{G}_i(\cdot|x_i)\} \) is a cumulative distribution function, and its fractiles agree with the inverse of \( \tilde{G}_i(\cdot|x_i) \) restricted to \((0, 1)\). This allows us to speak meaningfully, albeit informally, of the “fractiles” of \( \tilde{G}_i(\cdot|x_i) \).

To formally state our results let

\[
\alpha\text{-fractile of } \tilde{G}_i(\cdot|x_i) := \inf\{y \in [x_i, \bar{x}_i] : \tilde{G}_i(y|x_i) \geq \alpha\}, \quad i = 1, \ldots, N, \tag{11}
\]

and

\[
\mu_i(x_i) := (h_i + b_i) \tilde{G}_i(x_i|x_i) - (h_s + b_i - c_i). \tag{12}
\]
Note that the index $\mu_i$ coincides with the derivative of $\pi^*(x, q)$ with respect to $q$, evaluated at $x_i$.

The next theorem characterizes the optimal allocation defined in (9) by first ranking the retailers based on indices $\mu_i(x_i)$, and then expressing the allocated quantity to each retailer as a fractile of the adjusted distribution $\tilde{G}_i$.

**Theorem 3 (Allocation Rule via Ranking and Fractiles).** In the SSMR model, fix an inventory profile $x_s \in X_s$, $x \in X$, and relabel retailers, if necessary, so that

$$\mu_1(x_1) \leq \mu_2(x_2) \leq \cdots \leq \mu_N(x_N). \quad (13)$$

Under Assumption 1, there exist a Lagrange multiplier $U^* \in \left[ \min_{i=1,\ldots,N} \{ c_i - h_s - b_i \}, 0 \right]$ and an integer $n^*$ such that the solution $q^*$ to (9) is determined by:

(i) for all $i \leq n^*$, we have $q^*_i > 0$ and $x_i + q^*_i = (\frac{U^* + h_s + b_i - c_i}{h_i + b_i})$-fractile of $\tilde{G}_i(\cdot | x_i)$;

(ii) for all $j > n^*$, we have $q^*_j = 0$ and $x_j \geq (\frac{U^* + h_s + b_j - c_j}{h_j + b_j})$-fractile of $\tilde{G}_j(\cdot | x_j)$; and

(iii) $U^* \cdot (\sum_{i=1}^{N} q^*_i - x_s) = 0$.

The idea of ranking retailers to characterize optimal allocations under a capacity constraint is already present in Zipkin (1980). Theorem 3 shows that this characterization extends to the problem in (9) where the presence of private information induces non-convexities in the objective function. The expressions in (i)–(iii) generalize the solution of the classic newsvendor problem to our setting with limited capacity, multiple retailers, and private information. The Lagrange multiplier $U^*$ and the adjusted demand distribution $\tilde{G}_i$ account for the capacity constraint and the presence of private information, respectively.

### 3.2. Distortions of the Optimal Allocations

This section examines distortions due to the presence of private information under limited capacity with multiple retailers. In particular, we compare the optimal allocations derived in Theorem 2 with the solutions of two standard benchmarks: the classic newsvendor problem without capacity constraint,

$$\bar{q}_i(x_i) = \left[ G_i^{-1} \left( \frac{h_s + b_i - c_i}{b_i + h_i} \right) - x_i \right]^+ \quad \text{for} \quad x_i \in X_i, \quad i = 1, \ldots, N; \quad (14)$$
and the centralized distribution system in which a limited quantity \( x_s \) is allocated to minimize the overall cost of the supply chain

\[
q^c_i(x_s, x) = \left[ G_i^{-1} \left( \frac{U^o + h_s + b_i - c_i}{b_i + h_i} \right) - x_i \right]^+ \quad \text{for} \quad x_s \in \mathcal{X}, \ x \in \mathcal{X}, \ i = 1, \ldots, N, \tag{15}
\]

where the Lagrange multiplier \( U^o = U^o(x_s, x) \leq 0 \) ensures that \( \sum_{i=1}^N q^c_i(x_s, x) \leq x_s \).

The case in which the supplier has no private information and unlimited capacity collapses to the single retailer case studied in Zhang et al. (2009), under slightly different assumptions. Indeed, once the capacity constraint is removed, the problem becomes separable across retailers and all results in Zhang et al. (2009) extend immediately to the multiple-retailer case.

We begin by comparing the total quantities allocated in (9), (14) and (15).

**Proposition 1.** In the SSMR model, under Assumptions 1 and 2, we have, for all \( x_s \in \mathcal{X}_s \) and \( x \in \mathcal{X} \),

\[
\sum_{i=1}^N q^c_i(x_s, x) \leq \sum_{i=1}^N q^o_i(x_s, x) \leq \sum_{i=1}^N \bar{q}_i(x_i). \tag{16}
\]

The first inequality is due to the presence of private information. This “friction” effect is familiar from standard mechanism design results. The second inequality follows directly from removing the capacity constraint in the centralized problem. The next proposition pertains to individual quantities.

**Proposition 2.** In the SSMR model, under Assumptions 1 and 2, for all \( x_s \in \mathcal{X}_s, \ x \in \mathcal{X}, \) and \( i = 1, \ldots, N, \) we have

\[
q^c_i(x_s, x) \leq \bar{q}_i(x_i) \quad \text{and} \quad q^o_i(x_s, x) \leq \bar{q}_i(x_i).
\]

As in Proposition 1, the first inequality is due to private information and the second to the removal of the capacity constraint.

The individual allocations \( q^c_i \) and \( q^o_i \), however, cannot be ranked, in general. This is because the simultaneous presence of private information and limited supplier’s inventory creates two conflicting effects: (i) the “friction” effect described above, and (ii) a “redistribution” effect. Due to friction effect the supplier restricts quantities allocated to high inventory retailers. This allows the supplier
to redistribute the released units to low inventory retailers. The redistribution effect is formalized in the next proposition.

**Proposition 3.** In the SSMR model, under Assumptions 1 and 2, we have, for all $x_s \in X_s, x \in X$,

$$U^*(x_s, x) \leq U^o(x_s, x) \leq 0.$$  \hspace{1cm} (17)

Proposition 3 implies that the adjusted fractiles are larger under private information, hence the capacity released by the friction effect is reallocated more profitably for the supplier. The lack of exact knowledge regarding the retailers inventory levels lowers the marginal value of additional capacity units for the supplier relative to the centralized distribution system.

### 4. Single Supplier with Multiple Symmetric Retailers

In this section, we restrict attention to the case where all the retailers are *ex ante* symmetric. This allows us to isolate the distortions arising purely due to private information about inventories.\textsuperscript{11} By ex ante symmetric retailers we mean that the shipping/transaction costs for all retailers are the same $c = c_i$, and they have the same inventory cost function $C(\cdot) = C_i(\cdot)$ for $i = 1, \ldots, N$. In particular the holding and penalty costs are the same $h = h_i$, $b = b_i$ and all retailers face a stochastic demand with the same distribution $G = G_i$ and have the same belief $F_i = F$ about the others’ inventory levels over the support $[\underline{x}_i, \bar{x}_i], i = 1, \ldots, N$.

In this symmetric environment, it is well known that, without private information, the optimal allocation $q^o(x_s, x)$ for the centralized distribution system balances the final positions of the $n^o(x_s, x)$ retailers that are served, see Allen (1958). Relabeling retailers if necessary so that $x_1 \leq x_2 \leq \cdots \leq x_N$, we have

$$x_i + q^o_i = \frac{x_s + \sum_{i=1}^{n^o} x_i}{n^o} \quad \text{for all} \quad i \leq n^o := \min_{1 \leq n \leq N} \left\{ n : \frac{x_s + \sum_{i=1}^{n} x_i}{n} \leq x_{n+1}, \quad x_{N+1} = \infty \right\},$$ \hspace{1cm} (18)

if the capacity constraint is binding, and $q^o_i = \left[ G^{-1} \left( \frac{h_s + b - c}{b + h} \right) - x_i \right]^+$ as in the newsvendor problem, otherwise. This yields the following balanced final positions among served retailers

$$x_1 + q^o_1 = x_2 + q^o_2 = \cdots = x_{n^o} + q^o_{n^o} \leq x_{n^o+1} \leq \cdots \leq x_N.$$ \hspace{1cm} (19)
In general, the balancing policy defined in (18) and (19) is no longer optimal under private information. For example, consider the unlimited capacity case with two retailers and $F$ uniform. When $x_1 < x_2 < G^{-1}\left(\frac{h_s + b - c}{b + h}\right)$, we have $\tilde{G}(y|x_1) < \tilde{G}(y|x_2)$ which leads to the unbalanced final inventory positions $x_1 + q_1^* > x_2 + q_2^*$ by Theorem 3.

The next theorem generalizes this example and establishes that, for all served retailers, the order of their inventory levels is completely reversed by the optimal allocation to the problem (9).

**Theorem 4 (Reverse Inventory Positions).** In the SSMR model with symmetric retailers, for any $x_s \in X_s$, $x \in X$, relabel retailers, if necessary, so that $x_1 \leq x_2 \leq \cdots \leq x_N$, and let $q^* = q^*(x_s, x)$ denote the solution to (9). If Assumptions 1 and 2 hold, there exists an integer $n^* = n^*(x_s, x)$ such that $q_i^* > 0$ if and only if $i \leq n^*$. Furthermore

1. the final positions of all served retailers are reversed, i.e.
   \[ x_1 + q_1^* \geq x_2 + q_2^* \geq \cdots \geq x_{n^*} + q_{n^*}^* . \]  
2. fewer retailers are served relative to the centralized distribution system, i.e.
   \[ n^* \leq n^* . \]

Under our assumptions, ranking retailers with respect to the indices $\mu_i$ is equivalent to ranking them by their inventory levels $x_i$. This allows us to use the characterization derived in Theorem 3.

In both allocations $q^*$ and $q^o$, retailers with lower inventory levels receive larger shares of the available quantity $x_s$. The inequalities in (20) however imply that, under private information, this imbalance is larger. Nonetheless, as remarked in the previous section, the individual allocations $q_i^*$ and $q_i^o$ cannot be ranked in general. The next section will shed some light on this comparison.

The next result provides a characterization of a new type of distortion induced by the presence of private information in our setting, which is a consequence of the interplay between the “friction” effect and “redistribution” effect. We provide necessary and sufficient conditions for a retailer to
be allocated strictly more under private information than it would be in the optimal allocation of the centralized distribution system (i.e., $q^*_i > q^0_i$ for some $i$). We refer to this type of distortion as “overshooting”.

**Theorem 5 (Overshooting).** In the SSMR model with symmetric retailers, let $N \geq 2$. Suppose that Assumptions 1 and 2 hold, and $F(x)/f(x)$ is strictly increasing over $[\underline{x}, \bar{x}]$. Fix $x_s \in \mathcal{X}_s$ and $x \in \mathcal{X}$, relabeling retailers, if necessary, by their inventory levels, with $x_1 < x_2$. Then, overshooting occurs

$$q^*_i(x_s, x) > q^0_i(x_s, x) \quad \text{for some } i = 1, \ldots, N$$

if and only if

1. $x_s > x_2 - x_1$
2. $x_s < \sum_{i=1}^{n^0} \left \lceil \tilde{G}^{-1}\left( \frac{h_s + b - c}{h + b} \bigg| x_1 \right) - x_i \right \rceil$.

The proof also shows that overshooting occurs if and only if it occurs for the retailer with the lowest initial inventory, i.e. $q^*_1(x_s, x) > q^0_1(x_s, x)$.

Condition (i) in Theorem 5 guarantees that it is optimal for the supplier to serve at least two retailers in the centralized distribution system, i.e. $n^0 \geq 2$. Otherwise, by (16) and (21) we would have that $q^*_1 \leq q^0_1$, hence no overshooting.

Condition (ii) can be rewritten as $\tilde{G}([x_s + \sum_{i=1}^{n^0} x_i]/n^0|x_1) < (h_s + b - c)/(h + b)$, in light of (18). Since $[x_s + \sum_{i=1}^{n^0} x_i]/n^0$ is non-decreasing in $x_s$, this shows that $x_s$ cannot be too large which in turn implies that capacity is binding at the centralized allocation $q^0$. With unlimited capacity, the problem would collapse to the single retailer case, studied in Zhang et al. (2009), in which overshooting cannot occur. However, overshooting can also occur when capacity is not binding for $q^*$.

Deviations from the centralized solution similar to the ones discussed Proposition 1 and 2, and in particular the friction effect generated by private information that tends to reduce the total quantity allocated in the system, have been observed in many other supply chain environments, e.g. Cachon and Lariviere (1999), Corbett (2001), Deshpande and Schwarz (2005), and Zhang et al.
(2009). However, in many of these models without any capacity constraint the problem decomposes into a family of independent single-agent problems. The overshooting distortion can arise only in the presence of multiple retailers and limited capacity. To the best of our knowledge, Theorem 5 is the first result that demonstrates that an agent (a retailer) can be allocated in the optimal mechanism more than it would in the absence of private information even within symmetric environments.

5. Numerical Examples

This section presents numerical simulations that illustrate the impact of private information in the SSMR model. To isolate the distortion effects due to the presence of private information about inventory, we focus on the symmetric case.

Consider the following numerical example: cost parameters $h_s = 0.3$, $c = 0.1$, $h = 0.5$, $b = 0.5$; each retailer’s demand distribution is $D \sim \text{Unif}(0,1)$; and identical independent probability assessment for the inventory level $x_i$ given by $F_i \sim \text{Unif}(0,1)$.

For each supplier’s capacity level $x_s = 0.1, 0.2, \ldots, 3$ and number of retailers $N = 1, 5, 10, 50$ and each $x_i$ drawn according to $F$, we compute the centralized solution $q^\circ(x_s, x)$ and the decentralized solution $\{q^\star(x_s, x), m^\star(x_s, x)\}$, based on which we computed several measures to illustrate the main results of Sections 3 and 4.

Letting $C(x_s, x, q) = C_s(x_s, 0) - C_s(x_s, q) + \sum_{i=1}^N [C_i(x_i, 0) - C_i(x_i, q_i)]$ denote the overall cost reduction of the supply chain, we define the expected efficiency of the decentralized solution (relative to the centralized solution) as $E_x [C(x_s, x, q^\star)] / E_x [C(x_s, x, q^\circ)]$. This is a measure of how detrimental private information about inventory is to the overall performance of the supply chain. As can be seen from Figure 1, the expected efficiency is always below 1 due to the “friction effect”. The expected efficiency decreases with $x_s$ and increases with $N$ increases. Intuitively, as the capacity $x_s$ becomes more stringent while the number of retailers becomes larger, the competition for the limited capacity becomes more intense. As a result, the “redistribution effect”, which promotes efficiency, becomes larger relative to the “friction effect”.

We define the “service ratio” between the decentralized solution and the centralized solution.
Figure 1  Expected efficiency ratios.

Figure 2  Service ratios.

as $\mathbb{E}_x [n^*(x_s, x)] / \mathbb{E}_x [n^c(x_s, x)]$. As can be seen from Figure 2, the service ratio is always below 1, which illustrates part (ii) of Theorem 4.

In order to illustrate the presence of overshooting distortions, we compute the probability $\mathbb{P}_x [q^*_1(x_s, x) > q^c_1(x_s, x)]$, and the expected number of overshooting retailers
\[ \mathbb{E}_x [\{ i : q_i^*(x, x) > q_i(x, x) \}] \]. Figures 3 and 4 illustrate these two measurements. The u-shape patterns in both figures indicate that Conditions (i) and (ii) in Theorem 5 are more likely to hold for intermediate values of capacity.

**Figure 3**  Probabilities of overshooting.

**Figure 4**  Expected number of overshooting retailers.
6. Concluding Remarks

In this paper we investigated a single period two-echelon supply chain in which all parties have private information regarding their inventory level. Thus this work is complementary to most literature on decentralized supply chains with private information, which focuses on unilateral private information on costs.

Two bargaining power configurations were studied. In the MSSR case, where the retailer bears the entire supply chain risk and has all the bargaining power, we find that the centralized allocation is optimal and allows for the full extraction of all cost reductions by the retailer. This results hinges critically on the assumption that the supplier’s cost function is linear up to its initial inventory level. In our static environment, where production has already taken place, this is equivalent to the standard assumption of constant marginal handling cost.

In the SSMR case, where the supplier has all the bargaining power, the optimal allocation can be significantly different from the centralized solution. In particular, the total quantity sold tends to be lower. We characterize the contract via a ranking rule based on critical fractiles of a virtual demand distributions, which is adjusted to account for the incentives created by private information. Under the optimal allocation for symmetric environments, retailers with smaller initial inventory levels are brought to larger final inventory positions (reverse ranking), and in some cases final allocations exceed the ones prescribed by the centralized solution.

Several interesting questions remain beyond the scope of this paper. Zhang et al. (2009) have studied a dynamic extension of our SSMR with one retailer and unlimited supplier’s inventory. Extending both SSMR and MSSR models to dynamic settings appears to be the natural next step in this line of research. It would also be interesting to develop practical procedures for implementing the optimal contract in the SSMR case. We note however that even without a practical implementation, the characterization of the optimal mechanism provides useful insights, as well as a benchmark for evaluating the performance of any alternative mechanism. Finally, intermediate bargaining power structures should also be investigated.
Endnotes

1. Technically, the retailer faces an “informed principal problem.” Appendix A provides the details of the formal approach leading to (3).

2. This result was first established by Maskin and Tirole (1990). Mylovanov and Tröger (2008) provide sufficient conditions in more general environment for the irrelevance result. See Appendix A for a more detailed discussion on this issue, which also arises in the SSMR model studied in the next sections.

3. Formally, this contract satisfies ex post incentive compatibility and individual rationality for all parties.

4. As in the previous section there is a potential signaling issue due to the fact that the contract designer also has private information. Potentially, the other participants may make inferences about the designer’s private information through the offered contract and exploit this information. Appendix A provides a more detailed discussion of this issue.

5. The expression “virtual utility” was introduced by Myerson (1979).

6. This result was first established by Maskin and Tirole (1990). Mylovanov and Tröger (2008) provide sufficient conditions in more general environment for the irrelevance result.

7. Under Assumption 1, the non-monotonicity can only occur if \( \tilde{G}_i(y|x_i) \geq 1 \).

8. Zipkin (1980) also provides an efficient algorithm to find the index \( n^\star \) and the multiplier \( U^\star \).

9. Both standard benchmarks have no private information, i.e. the inventory levels of retailers are known by the supplier.

10. Zhang et al. (2009) also considered an interesting dynamic mechanism design problem which we do not consider here.

11. Asymmetry among retailers introduce additional distortions.

Appendix A: Mechanism Design by an Informed Principal

Following the mechanism design nomenclature, we refer to the party with all the bargaining power (the retailer in the MSSR case, and the supplier in the SSMR case) as the principal. In both models
the principal has private information about its inventory level. This changes the mechanism design problem into an “informed-principal problem.” On one hand the principal could exploit its private information to achieve higher payoffs, on the other hand the proposed contract could signal its own private information to the other parties.

In order to deal with the incentives faced by the informed principal, Myerson (1983) introduces the concept of “strong solution” for environments with finite type spaces and finite outcome spaces. An extension to non-finite environments can be found in Mylovanov and Tröger (2008).

**Definition 1 (Safe, Dominated, and Strong Solution).** A mechanism is said to be **safe** if it would remain incentive feasible (incentive compatible and individually rational) for the principal and all agents, even if all agents knew the principal’s private information. An incentive feasible mechanism is said to be **dominated** if there exists another incentive feasible mechanism such that all types of the principal are at least as well off and a positive mass of types of the principal is strictly better off. A mechanism is said to be a **strong solution** if it is both safe and not dominated.

Myerson (1983) (Theorem 2) proves that in any environment with finite type spaces and a finite outcome space, any strong solution is an equilibrium outcome of an informed-principal game where any finite simultaneous-move game form is a feasible mechanism. Moreover, Myerson (1983) also points out that any two strong solutions yield the same expected payoff for all types of the principal.

Mylovanov and Tröger (2008) provide an extension of this result to non-finite environments, which can be adopted in our setting. Formally Mylovanov and Tröger show that no type of the principal has an incentive to deviate by offering any finite (simultaneous-move or multi-stage) game. The restriction to finite deviating mechanisms is due to the need to guarantee that any feasible mechanism has an equilibrium.

In Appendices B and C, we construct an incentive feasible mechanism that (i) solves the principal problem when its type is publicly known (full-information-optimal), and (ii) minimizes the principal’s expected cost (ex-ante optimal). By Lemma 1 of Mylovanov and Tröger (2008), since we are in a private values model, this mechanism is a strong solution for the informed principal problem.
A.1. Specializing to SSMR

In this section we specialize the analysis of the informed principal problem to the SSMR scenario (the MSSR scenario is analogous so it is omitted).

The full information optimal for each inventory level $x_s$ of the supplier is given by
\[
\min_{q,m} \mathbb{E}_x \left[ C_s(x_s, q(x_s, x)) - \sum_{i=1}^N m_i(x_s, x) \right]
\] (22)
\[
\text{s.t. } (\text{IC}_i)(x_s), \ (\text{IR}_i)(x_s), \ q_i(x_s, x) \geq 0 \quad i = 1, \ldots, N \\
\sum_{i=1}^N q_i(x_s, x) \leq x_s
\]
where $(\text{IC}_i)(x_s)$ and $(\text{IR}_i)(x_s)$ represent the incentive constraints for the retailers knowing the supplier’s inventory level $x_s$.

The ex-ante optimal is given by
\[
\min_{q,m} \mathbb{E}_{x_s, x} \left[ C_s(x_s, q(x_s, x)) - \sum_{i=1}^N m_i(x_s, x) \right]
\] (23)
\[
\text{s.t. } (\text{IC}_s), \ (\text{IR}_s), \ (\text{IC}_i), \ (\text{IR}_i), \ q_i(x_s, x) \geq 0 \quad i = 1, \ldots, N \\
\sum_{i=1}^N q_i(x_s, x) \leq x_s
\]
The supplier’s problem (8) has the objective function of (22) and the constraints of (23). Under our assumptions, the optimal contract $q^*, m^*$ characterized in Theorem 2 solves (8), (22), and (23) simultaneously. In particular, by solving (22) and (23), since we have private values environment, Lemma 1 of Mylovanov and Tröger (2008) establishes that the optimal contract $q^*, m^*$ is a strong solution. Thus, the mechanism $q^*, m^*$ is also an equilibrium of the associated game, see Myerson (1983) Theorem 2.

Appendix B: Proofs in Section 2

Proof of Theorem 1  We divide the proof in two steps: i) the main arguments, and ii) technical results on the allocation $q^o$ used in i).

Step 1. Main Arguments. First observe that setting $m_i^o(x) = (c_i - h_i) q_i^o(x)$ implies that $(\text{IR}_i)$ holds with equality for all $i = 1, \ldots, M$, thus the retailer extracts the entire cost reduction given by (4). It remains to show that the contract $\{q_i^o(x_r, x), m_i^o(x_r, x)\}_{i=1}^M$ also satisfies $(\text{IC}_i)$, $(\text{IC}_r)$ and $(\text{IR}_r)$. For any $x_r, x$, we have
\[
C_i(x_i, q_i^o(x_r, x)) - m_i^o(x_r, x) = C_i(x_i, 0) = h_i x_i, \quad \text{for } i = 1, \ldots, M,
\]
which implies that (IC) holds with equality. Moreover, by (4) we have

\[ C_r \left( x_r, \sum_{i=1}^{M} q_i^* (x_r, x) \right) + \sum_{i=1}^{M} C_i (x_i, q_i^* (x_r, x)) \leq C_r (x_r, 0) + \sum_{i=1}^{M} C_i (x_i, 0) \]

and by the previous equality

\[ C_r \left( x_r, \sum_{i=1}^{M} q_i^* (x_r, x) \right) + \sum_{i=1}^{M} m_i^* (x_r, x) = C_r \left( x_r, \sum_{i=1}^{M} q_i^* (x_r, x) \right) + \sum_{i=1}^{M} \left[ C_i (x_i, q_i^* (x_r, x)) - C_i (x_i, 0) \right] \]

\[ \leq C_r (x_r, 0). \]

The last inequality is (IR). Finally, to see that (IC) also holds, for each \( x_r, x_r', x \), define

\[ \Delta := C_r \left( x_r, \sum_{i=1}^{M} q_i^* (x_r, x) \right) + \sum_{i=1}^{M} m_i^* (x_r, x) - \left[ C_r \left( x_r, \sum_{i=1}^{M} q_i^* (x_r', x) \right) + \sum_{i=1}^{M} m_i^* (x_r', x) \right] \]

\[ = (h_r + b_r) \int_{x_r + \sum_{i=1}^{N} q_i^* (x_r', x)}^{x_r + \sum_{i=1}^{N} q_i^* (x_r', x)} G_r (z) dz - \sum_{i=1}^{M} \left( b_r + h_i - c_i \right) (q_i^* (x_r, x) - q_i^* (x_r', x)). \]

We will show that \( \Delta \leq 0 \) hence (IC) holds by integrating \( x \) out.

Let \( \eta_i = G_r^{-1} ((h_i + b_r - c_i) / (h_r + b_r)) \) for \( i = 1, \ldots, M \). Note that \( \eta_i \) is decreasing in \( i = 1, \ldots, M \) since \( c_i - h_i \) is increasing. The index \( i^v (x_r, x) \) is defined in Step 2 below.

If \( \eta_1 \leq x_r < x_r' \), then \( q_i^* (x_r, x) = q_i^* (x_r', x) = 0 \) by Step 2 (***), and hence \( \Delta = 0 \).

If \( x_r < x_r' \) and \( x_r \leq \eta_i \leq x_r + \sum_{i=1}^{M} x_i \) for some \( i = 1, \ldots, M \), then by Step 2 (**),

\[ G_r \left( x_r + \sum_{j=1}^{M} q_j^* (x_r, x) \right) = \frac{b_r + h_i (x_r, x) - c_i (x_r, x)}{h_r + b_r} \]

Furthermore, we have \( q_i^* (x_r, x) \geq q_i^* (x_r', x) \) by Step 2 (*). Hence,

\[ \Delta \leq \sum_{i=1}^{i^v (x_r, x)} \left[ (h_r + b_r) G_r \left( x_r + \sum_{i=1}^{N} q_i^* (x_r, x) \right) - (b_r + h_i - c_i) \right] (q_i^* (x_r, x) - q_i^* (x_r', x)) \]

\[ = \sum_{i=1}^{i^v (x_r, x)} \left[ (h_i (x_r, x) - c_i (x_r, x)) - (h_i - c_i) \right] (q_i^* (x_r, x) - q_i^* (x_r', x)) \leq 0. \]

If \( x_r < x_r' \) and \( \eta_M \geq x_r + \sum_{i=1}^{M} x_i \), then by Step 2 (***),

\[ G_r \left( x_r + \sum_{j=1}^{M} q_j^* (x_r, x) \right) \leq \frac{b_r + h_M - c_M}{h_r + b_r}. \]

Furthermore, we have \( q_i^* (x_r, x) \geq q_i^* (x_r', x) \) by Step 2 (*). Hence,

\[ \Delta \leq \sum_{i=1}^{M} \left[ (h_r + b_r) G_r \left( x_r + \sum_{i=1}^{N} x_i \right) - (b_r + h_i - c_i) \right] (q_i^* (x_r, x) - q_i^* (x_r', x)) \]
\[ \sum_{i=1}^{M} \left[ (h_M - c_M) - (h_i - c_i) \right] \left( q_i^\circ(x_r, x) - q_i^\circ(x'_r, x) \right) \leq 0. \]

If \( x_r > x'_r \geq \eta_i \), then \( q_i^\circ(x_r, x) = q_i^\circ(x'_r, x) = 0 \) by Step 2 \((***)\) and hence \( \Delta = 0 \).

If \( x_r > \eta_i \geq x'_r \), we have \( 0 = i^\circ(x_r, x) < i^\circ(x'_r, x) \) and \( 0 = q_i^\circ(x_r, x) \leq q_i^\circ(x'_r, x) \) for all \( i = 1, \ldots, M \) by Step 2 \((***)\)

\[
\Delta \leq \sum_{i=i^\circ(x_r, x)}^{i^\circ(x'_r, x)} \left[ (h_r + b_r)G_r(x_r) - (b_r + h_i - c_i) \right] \left( q_i^\circ(x_r, x) - q_i^\circ(x'_r, x) \right) 
\leq \sum_{i=i^\circ(x_r, x)}^{i^\circ(x'_r, x)} \left[ (h_i - c_i) \right] \left( q_i^\circ(x_r, x) - q_i^\circ(x'_r, x) \right) \leq 0.
\]

If \( x_r > x'_r \), which implies \( i^\circ(x_r, x) \leq i^\circ(x'_r, x) \) and \( q_i^\circ(x_r, x) \leq q_i^\circ(x'_r, x) \) by Step 2 \((*)\), and \( x_r \leq \eta_i \leq x_r + \sum_{i=1}^{M} x_i \) for some \( i = 1, \ldots, M \), which implies, by Step 2 \((***)\),

\[
G_r \left( x_r + \sum_{j=1}^{M} q_i^\circ(x_r, x) \right) = \frac{b_r + h_i^\circ(x_r, x) - c_i^\circ(x_r, x)}{h_r + b_r},
\]

then we have

\[
\Delta \leq \sum_{i=i^\circ(x_r, x)}^{i^\circ(x'_r, x)} \left[ (h_r + b_r)G_r \left( x_r + \sum_{i=1}^{N} q_i^\circ(x_r, x) \right) - (b_r + h_i - c_i) \right] \left( q_i^\circ(x_r, x) - q_i^\circ(x'_r, x) \right) 
\leq \sum_{i=i^\circ(x_r, x)}^{i^\circ(x'_r, x)} \left[ (h_i - c_i) \right] \left( q_i^\circ(x_r, x) - q_i^\circ(x'_r, x) \right) \leq 0.
\]

If \( x_r > x'_r \) but \( \eta_M \geq x_r + \sum_{i=1}^{M} x_i \), we have \( q_i^\circ(x_r, x) = q_i^\circ(x'_r, x) = x_i \) for all \( i = 1, \ldots, M \). Hence, \( \Delta = 0 \).

Step 2. Properties of the allocation \( q^\circ \).

Recall the definition of \( \eta_i = G_r^{-1}\left( (h_i + b_r - c_i) / (h_r + b_r) \right) \) for \( i = 1, \ldots, M \). Note that \( \eta_i \) is decreasing in \( i = 1, \ldots, M \) since \( c_i - h_i \) is increasing. Because of the convexity of objective function in (4), KKT condition is the necessary and sufficient for the global optimality of \( q^\circ \), namely there exists \( \nu_i, \lambda_i \geq 0, i = 1, \ldots, M \) such that the following (in)equalities are satisfied for \( i = 1, \ldots, M \):

\[
(h_r + b_r)G_r \left( x_r + \sum_{j=1}^{M} q_j \right) - b_r + c_i - h_i + \nu_i - \lambda_i = 0 \\
0 \leq q_i \leq x_i, \quad \nu_i (x_i - q_i) = 0, \quad \lambda_i q_i = 0
\]
Hence, for given \( x \), we consider the following greedy algorithm:

Set \( i = 0 \) and \( q^o = 0 \). If \( x_r \geq \eta_1 \), **terminate**.

**Iteration Step:** if \( i \geq M \) **terminate**;

\[ i \leftarrow i + 1; \]

if \( x_r + \sum_{j=1}^{i} x_j < \eta_i \), set \( q^o_i = x_i \) and return to **Iteration Step**;

else set \( q^o_i = \eta_i - x_r - \sum_{j=1}^{i-1} x_j \) and **terminate**.

Let \( i^o \) denote the exiting index \( i \) and note that \( q^o_i = 0 \) for all \( i > i^o \). Hence \( q^o_i(x_r, x), i = 1, \ldots, M \), is the one given in the statement of the theorem. Moreover, define

\[


\nu_i = \begin{cases} 
(c_i - h_i) - (c_{i^o} - h_{i^o}), & \text{if } i < i^o, \\
0, & \text{if } i \geq i^o,
\end{cases}

\lambda_i = \begin{cases} 
(c_i - h_i), & \text{if } i \leq i^o, \\
(c_{i^o} - h_{i^o}), & \text{if } i > i^o.
\end{cases}

\]

Then \( \nu_i, \lambda_i \geq 0 \) for all \( i \), together with \( q^o_i(x_r, x), i = 1, \ldots, M \) satisfy (24). Therefore, \( q^o_i(x_r, x), i = 1, \ldots, M \), indeed solves (4).

It follows that:

(*) \( i^o = i^o(x_r, x) \) and \( q^o_i(x_r, x) \) is non-increasing in \( x_r \) for \( i = 1, 2, \ldots, M \);

(**) \( x_r + \sum_{j=1}^{M} q^o_j(x_r, x) = \eta_i \) if \( x_r \leq \eta_i \), \( x_r + \sum_{i=1}^{M} x_i \) for some \( i = 1, \ldots, M \);

(***) \( q^o_i(x_r, x) = 0 \) for \( i = 1, \ldots, M \) if \( x_r \geq \eta_i \) (and hence \( x_r \geq \eta_i \) for all \( i = 1, \ldots, M \)).

(****) \( q^o_i(x_r, x) = x_i \) for \( i = 1, \ldots, M \) if \( \eta_i \geq x_r + \sum_{i=1}^{M} x_i \) (and hence \( \eta_i \geq x_r + \sum_{i=1}^{M} x_i \) for all \( i = 1, \ldots, M \)).

\[ \square \]

**Appendix C: Proofs of Section 3**

**Proof of Theorem 2**  The proof consists in showing that the contract \( q^*, m^* \) constitutes a strong solution (see Appendix A).

By Lemma EC.4, we note that for any given \( x_s \), \( \{q^*(x_s, x), m^*(x_s, x)\} \) satisfies (IC\(_i\))(\( x_s \)) and (IR\(_i\))(\( x_s \)) and hence is incentive feasible for all the retailers even if they knew \( x_s \). By Definition 1,
in order to show \( \{q^*(x_s, x), m^*(x_s, x)\} \) is safe for the supplier, we need to verify (IC\(_s\)) and (IR\(_s\)). Since the contract \( q^*, m^* \) minimizes \( \mathbb{E}_x \left[ C_s(x_s, q(x_s, x)) - \sum_{i=1}^N m_i(x_s, x) \right] \) over a set that contains the contract \( q \equiv 0 \) and \( m \equiv 0 \), we have

\[
\mathbb{E}_x \left[ C_s(x_s, q^*(x_s, x)) - \sum_{i=1}^N m^*_i(x_s, x) \right] \leq C_s(x_s, 0)
\]

and (IR\(_s\)) holds. In order to verify (IC\(_s\)), by the supplier’s cost structure and Lemma EC.4, for any \( x_s, x'_s \in X_s \),

\[
\mathbb{E}_x \left[ C_s(x_s, q^*(x'_s, x)) - \sum_{i=1}^N m^*_i(x'_s, x) \right] = \mathbb{E}_x [\pi^*(x, q^*(x', x))] + h_s x_s \tag{25}
\]

where \( \sum_{i=1}^N q^*_i(x'_s, x) \leq x_s \) for all \( x \in X \).

Thus, for any \( x_s < x'_s \), we cannot have \( \sum_{i=1}^N q^*_i(x'_s, x) > x_s \) for any \( x \). If \( \sum_{i=1}^N q^*_i(x'_s, x) \leq x_s < x'_s \) for all \( x \), then \( x_s \) is enough to bring retailer \( i \)'s final inventory position to the minimum \( y^*_i(x_i) \) of \( V_i(\cdot|x_i) \) as defined in (EC.13) for all \( i \), i.e. for all \( x \),

\[
q^*_i(x_s, x) = q^*_i(x'_s, x) = (y^*_i(x_i) - x_i)^+.
\]

Thus, \( \pi^*(x, q^*(x_s, x)) = \pi^*(x, q^*(x'_s, x)) \) for all \( x \). Therefore, (IC\(_s\)) holds (with equality) by (25).

For any \( x_s > x'_s \), we must have \( \sum_{i=1}^N q^*_i(x'_s, x) \leq x'_s < x_s \) and \( \sum_{i=1}^N q^*_i(x_s, x) \leq x_s \). Thus by Lemma EC.4, since \( q^* \) solves (EC.10), for any \( x \),

\[
\pi^*(x, q^*(x_s, x)) \leq \pi^*(x, q^*(x'_s, x)),
\]

which immediately implies (IC\(_s\)) by (25).

On the other hand, because (EC.1) minimizes the supplier’s cost for any given \( x_s \) subject to only the retailers’ incentive constraints (IC\(_i\)) and (IR\(_i\)) by ignoring (IC\(_s\)) and (IR\(_s\)). Hence, we must have for any other incentive feasible mechanism \( \{\tilde{q}(x_s, x), \tilde{m}(x_s, x)\} \) such that, for all \( x_s \),

\[
\mathbb{E}_x \left[ C_s(x_s, \tilde{q}(x_s, x)) - \sum_{i=1}^N m_i(x_s, x) \right] \leq \mathbb{E}_x \left[ C_s(x_s, \tilde{q}(x_s, x)) - \sum_{i=1}^N \tilde{m}_i(x_s, x) \right] .
\]

Thus, by Definition 1, \( \{q^*(x_s, x), m^*(x_s, x)\} \) is not dominated by any other incentive feasible mechanism. Therefore, by Definition 1, \( \{q^*(x_s, x), m^*(x_s, x)\} \) is a strong solution for the supplier’s problem. □
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**Proof of Theorem 3** The result follows from Lemma EC.7 and the observation that the fractile representation arises from \( \mu_i(x_i + q_i^*(x_i,x)|x_i) = U^* \) being equivalent to \( \tilde{G}_i(x_i + q_i^*(x_i,x)|x_i) = (U^* + h_s + b_i - c_i)/(h_i + b_i) \). □

The classic centralized problem where there is no private information is:

\[
\Pi^c_N(x_s, x) := \min_{q \geq 0} C_s(x_s, q) + \sum_{i=1}^{N} \{ C_i(x_i, q_i) - C_i(x_i, 0) \} - C_s(x_s, 0)
\]

(26)

As a corollary of Theorem 3, we can immediately characterize the solution \( \{q^o, m^o\} \) to (26) in the next result, where \( \phi_i(x_i) := (h_i + b_i)G_i(x_i) - (h_s + b_s - c_i) \) for \( i = 1, \ldots, N \) and \( \pi^o(x,q) := \sum_{i=1}^{N} \{ C_i(x_i, q_i) - (h_s - c_i)q_i \} \):

**Corollary 2.** Fix an inventory profile \( x_s \in \mathcal{X}_s, x \in \mathcal{X} \), and relabel retailers, if necessary, so that

\[ \phi_1(x_1) \leq \phi_2(x_2) \leq \cdots \leq \phi_N(x_N). \]

Then there exists a Lagrange multiplier \( \min_{i=1, \ldots, N} \{ c_i - h_s - b_i \} \leq U^o \leq 0 \) and an integer \( n^o \) such that the allocation rule \( q^o \) to (26) is determined by

\[
\begin{align*}
(2.1) & \text{ for all } i \leq n^o, \ q_i^o > 0 \text{ and } x_i + q_i^o = \left( \frac{U^o + h_s + b_i - c_i}{h_i + b_i} \right) \text{-fractile of } G_i(\cdot); \\
(2.2) & \text{ for all } j > n^o, \ q_j^o = 0 \text{ and } x_j \geq \left( \frac{U^o + h_s + b_i - c_i}{b_j + h_i} \right) \text{-fractile of } G_j(\cdot); \text{ and} \\
(2.3) & \sum_{i=1}^{N} q_i^o \leq x_s \text{ and } U^o \cdot (\sum_{i=1}^{N} q_i^o - x_s) = 0.
\end{align*}
\]

**Proof.** Following the same argument as in the proof of Lemma EC.4 and Theorem 3 by substituting \( \frac{F_i(x_i)}{f_i(x_i)} \) with 0, the relaxed problem (EC.10) is reduced to the centralized problem (26). Notice that this is a convex program even without Assumption 1. Furthermore, there is no monotonicity constraints and hence we do not need Assumption 2. □

**Proof of Proposition 2** By Theorem 3, if \( q_i^* > 0 \) we have \( q_i^* = \tilde{G}_i^{-1} \left( \frac{U^o + h_s + b_i - c_i}{b_i + h_i} \right) |x_i \) \leq G_i^{-1} \left( \frac{U^o + h_s + b_i - c_i}{b_i + h_i} \right) \leq G_i^{-1} \left( \frac{h_s + b_i - c_i}{b_i + h_i} \right) \) since \( G_i(y) \leq \tilde{G}_i(y|x_i) \) and \( U^* \leq 0 \). □

**Proof of Proposition 1** The second inequality is straightforward. Let us focus on the first inequality. Since \( y_i^*(x_i) \) is the unrestricted minimum for each function \( V_i(\cdot|x_i) \) defined in (EC.13).
Note also that the unrestricted centralized allocation corresponds to $y^*(x)$, since the factor $F_i(x) = f_i(x)$ if $x_i > 0$. Thus, we have

$$
\sum_{i=1}^{N} q_i^*(x) = \min \left\{ x_s, \sum_{i=1}^{N} (y_i^*(x_i) - x_i)^+ \right\}, \quad \text{and} \quad \sum_{i=1}^{N} q_i^*(x) = \min \left\{ x_s, \sum_{i=1}^{N} (y^*(x) - x_i)^+ \right\}.
$$

Since $y_i^*(\cdot)$ is non-increasing by Lemma EC.6, we have

$$
\min \left\{ x_s, \sum_{i=1}^{N} (y_i^*(x_i) - x_i)^+ \right\} \leq \min \left\{ x_s, \sum_{i=1}^{N} (y_i^*(x) - x_i)^+ \right\},
$$

which immediately yields the result. \(\square\)

**Proof of Proposition 3.** If $x_s > \sum_{i=1}^{N} (y_i^*(x_i) - x_i)^+$ the lagrangian multiplier $U^*(x_s, x) = 0$ by Lemma 3 and the result follows. Otherwise, $x_s \leq \sum_{i=1}^{N} (y_i^*(x_i) - x_i)^+$ and we have $\sum_{i=1}^{N} q_i^* = \sum_{i=1}^{N} q_i^0 = x_s$ by Proposition 1. Since $\tilde{G}_i^{-1}(\alpha | x_i) \leq \tilde{G}_i^{-1}(\alpha | x_i) = G_i^{-1}(\alpha)$ for $\alpha \in [0, 1]$ and $i = 1, \ldots, N$, if $U^* < U^0$ it follows by Lemma 3 that $x_i + q_i^* < x_i + q_i^0$ for all $i$ such that $q_i^* > 0$. That yields $\sum_{i=1}^{N} q_i^0 > x_s$ and therefore $U^* < U^0$ cannot hold. \(\square\)

**Appendix D: Proofs of Section 4.**

**Proof of Theorem 4.** Under Assumption 2 for $i < j$ that $F(x_i) / f(x_i) \leq F(x_j) / f(x_j)$. Hence by definition of $\mu_i(\cdot | x_i)$ in (EC.12), $\mu_i(y|x_i) \leq \mu_j(y|x_j)$ for all $y$. Moreover, for any $i$, $\mu_i(y|x_i) \leq 0$ and non-decreasing for $y \in [0, y_i^*(x_i)]$, and $\mu_i(y|x_i) \geq 0$ otherwise by Lemma EC.5. Thus, if $\mu_i(x_i|x_i) > 0$, we have $x_i > y_i^*(x_i)$ and we have correspondingly $q_i^* = 0$ and the $i$th retailer is not served; so, without loss of generality, we may assume $\mu_i(x_i|x_i) \leq 0$, i.e. $x_i \leq y_i^*(x_i)$. Thus, the condition $x_1 \leq x_2 \leq \cdots \leq x_N$ implies that $\mu_i(x_i|x_i) \leq \mu_2(x_2|x_2) \leq \cdots \leq \mu_N(x_N|x_N)$ since $\mu_i(x_i|x_i) \leq \mu_{i+1}(x_i|x_{i+1}) \leq \mu_{i+1}(x_{i+1}|x_{i+1})$.

By Lemma 3 we have two cases. Either $q^* = 0$, in which case $n^* = 0$ and both results (1) and (2) hold. Otherwise, for $i \leq n^*$ we have $q_i^* > 0$, $\mu_i(x_i + q_i^*|x_i) = U^*$, and $x_i + q_i^* \leq y_i^*(x_i)$. Thus, $\mu_{i+1}(x_i + q_i^*|x_{i+1}) \geq U^*$ which implies that $x_i + q_i^* \geq x_{i+1} + q_{i+1}^*$ by the monotonicity of $\mu_{i+1}(|x_{i+1})$ up to $y_{i+1}^*(x_{i+1}) \geq x_{i+1} + q_{i+1}$.

To establish the second result we can assume $q^* \neq 0$. Note that if for some $i_s \leq n^*$ we have $q_{i_s}^* \geq q_{i_s}^0$, for any $i \leq i_s$ we have $x_i + q_i^* \geq x_i + q_i^0 = x_{i_s} + q_{i_s}^0$ by the result (1) and the balancing
property of the centralized supply chain solution. Thus, if \( n^* > n^\circ \), \( q^*_i > 0 = q^\circ_i \), which yields \( \sum_{i=1}^{N} q^*_i > \sum_{i=1}^{N} q^\circ_i \) which contradicts Proposition 1. Thus, \( n^* \leq n^\circ \) and (2) follows. □

Proof of Theorem 5  Fix \( x \in \mathcal{X} \) with \( x_i \leq x_{i+1} \) and \( x_1 < x_2 \), and let \( q^* = q^*(x_s,x) \) and \( q^\circ = q^\circ(x_s,x) \) be the optimal allocations for the asymmetric information and the centralized supply chain.

It suffices to show that \( q^*_i > q^\circ_i \). Indeed, if \( q^*_i > q^\circ_i \) we have \( x_j + q^*_j \geq x_i + q^*_i > x_i + q^\circ_j = x_j + q^\circ_j \), hence \( q^*_j > q^\circ_j \) for all \( j \leq i \). Thus, if overshooting happens for retailer \( i \), it must also happen for all retailers \( j \leq i \).

We first establish that Conditions (i) and (ii) are sufficient for overshooting.

First assume that \( x_s \leq \sum_{i=1}^{N} (y^*_i(x_i) - x_i)^+ \), so that \( \sum_{i=1}^{N} q^*_i = \sum_{i=1}^{N} q^\circ_i = x_s \). Therefore either \( q^* = q^\circ \) or there is an index \( i \) such that \( q^*_i > q^\circ_i \) since they are non-negative. Since \( x_s > x_2 - x_1 \), we have \( q^\circ_2 > 0 \) by (19). Thus, if \( q^\circ_2 = 0 \) we are done. Otherwise, by Lemma 3, \( \mu_1(x_1 + q^\circ_1|x_1) = \mu_2(x_2 + q^\circ_2|x_2) \) and since \( F(x_1)/f(x_1) < F(x_2)/f(x_2) \) we have \( x_1 + q^\circ_1 > x_2 + q^\circ_2 \). Therefore \( q^* \neq q^\circ \).

Second, if \( \sum_{i=1}^{N} (y^*_i(x_i) - x_i)^+ < x_s \), we have \( q^*_i = (y^*_i(x_i) - x_i)^+ \). Also, note that \( x_1 + q^\circ_1 = (x_s + \sum_{i=1}^{n^\circ} x_i)/n^\circ \) so that Condition (ii) is equivalent to \( \tilde{G}(x_1 + q^\circ_1|x_1) < (h_s + b - c)/(h + b) \) which implies that \( x_1 + q^\circ_1 < y^*_i(x_1) \) by Lemma EC.5. Hence we have \( q^\circ_i < q^*_i \), i.e. overshooting occurs.

Next we show the necessity of Conditions (i) and (ii). If \( x_s \leq x_2 - x_1 \) we have \( q^\circ_i = (G^{-1}(\frac{h_s + b - c}{h + b}) - x_1)^+ \land x_s \) and \( q^\circ_i = 0 \) for all \( i = 2,\ldots,N \). By Theorem 4 part (ii), we also have \( q^\circ_i = 0 \) for \( i = 2,\ldots,N \), and, by Proposition 1, \( q^*_i \leq q^\circ_i \). Thus, overshooting does not occur.

If \( \tilde{G}(x_1 + q^\circ_1|x_1) \geq \frac{h_s + b - c}{h + b} \), we have \( y^*_1(x_1) \leq x_1 + q^\circ_1 \) by Lemma EC.5. Therefore, \( q^*_1 \leq (y^*_1(x_1) - x_1)^+ \leq q^\circ_1 \). Using the fact that \( q^\circ \) balances the final inventory position of served retailers and Theorem 4, we have that \( x_i + q^\circ_i \geq x_1 + q^\circ_1 \geq x_1 + q^*_i \) so that \( q^*_i \leq q^\circ_i \), i.e. overshooting does not occur. □

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Characterization of \((\text{IC}_i)\) and \((\text{IR}_i)\) in the SSMR case

The following is the supplier’s problem in Section 3 if its inventory/capacity level were publicly known:

\[
\Pi_N(x_s) := \min_{q, m} \mathbb{E}_x \left[ C_s(x_s, q(x_s, x)) - \sum_{i=1}^N m_i(x_s, x) \right]
\]

subject to \(\sum_{i=1}^N q_i(x_s, x) \leq x_s\), \((\text{IC}_i)(x_s), (\text{IR}_i)(x_s), q_i(x_s, x) \geq 0\) for \(i = 1, \ldots, N\) \hspace{1cm} (EC.1)

where \((\text{IC}_i)(x_s)\) and \((\text{IR}_i)(x_s)\) are the incentive constraints for the retailers when \(x_s\) is publicly known.

In this section, we abbreviate \(m_i(x_s, x)\) as \(m_i(x), q_i(x_s, x)\) as \(q_i(x)\). For any \(q\) and \(m\) let

\[
W_i(x_i) := \mathbb{E}_{x_{-i}} \left[ C_i(x_i, q_i(x)) + m_i(x) \right], \quad i = 1, \ldots, N. \hspace{1cm} (EC.2)
\]

**Lemma EC.1.** Fix any \(i \in \{1, \ldots, N\}\), suppose that \(q_i(x)\) is non-increasing in \(x_i \in \mathcal{X}_i\) for all \(x_{-i} \in \mathcal{X}_{-i}\). Then the pair \(q_i(x), m_i(x)\), where

\[
m_i(x) = C_i(x_i, 0) - C_i(x_i, q_i(x)) - (h_i + b_i) \int_{x_i}^{x_i'} \{G_i(z + q_i(z, x_{-i})) - G_i(z)\} dz, \quad x \in \mathcal{X}.
\]

satisfies the following inequality

\[
C_i(x_i, q_i(x)) + m_i(x) \leq C_i(x_i, q_i(x', x_{-i})) + m_i(x', x_{-i}), \quad \forall x_i' \in \mathcal{X}_i, x \in \mathcal{X}, \hspace{1cm} (EC.3)
\]

and hence satisfies \((\text{IC}_i)\).

**Proof.** Notice that

\[
C_i(x_i, q_i) - C_i(x_i', q_i) = (h_i + b_i) \int_{x_i}^{x_i'} G_i(z + q_i) dz - b_i(x_i - x_i').
\]

Thus, we have

\[
[C_i(x_i, q_i(x_i', x_{-i})) + m_i(x_i', x_{-i})] - [C_i(x_i, q_i(x)) + m_i(x)]
\]

\[
= [C_i(x_i, q_i(x_i', x_{-i})) - C_i(x_i', q_i(x_i', x_{-i}))] + [C_i(x_i', q_i(x_i', x_{-i})) + m_i(x_i', x_{-i})] - [C_i(x_i, q_i(x)) + m_i(x)]
\]

\[
= [C_i(x_i, q_i(x_i', x_{-i})) - C_i(x_i', q_i(x_i', x_{-i}))] + [C_i(x_i', 0) - C_i(x_i, 0)] - (h_i + b_i) \int_{x_i'}^{x_i} [G_i(z + q_i(z, x_{-i})) - G_i(z)] dz
\]

\[
= (h_i + b_i) \int_{x_i'}^{x_i} [G_i(z + q_i(x_i', x_{-i})) - G_i(z + q_i(z, x_{-i}))] dz \geq 0,
\]

where \(\int_{x_i'}^{x_i} [G_i(z + q_i(x_i', x_{-i})) - G_i(z + q_i(z, x_{-i}))] dz \geq 0\).
where the last inequality is due to the fact that if \( x_i > x'_i \), \( q_i(x'_i, x_{-i}) \geq q_i(z, x_{-i}) \) for all \( z \in [x_i, x'_i] \) because of monotonicity of \( q_i(x) \) in \( x_i \) for any given \( x_{-i} \). (Similarly if \( x_i < x'_i \) we have \( q_i(x'_i, x_{-i}) \leq q_i(z, x_{-i}) \) for all \( z \in [x_i, x'_i] \).) This shows (EC.3) and taking the expectation of (EC.3) with respect to \( x_{-i} \) yields (IC_i).

**Lemma EC.2.** For any \( i = 1, \ldots, N \), if \( q_i \) and \( m_i \) satisfy (IC_i), then for any \( x_i \in [\underline{x}_i, \bar{x}_i] \), we have

\[
W_i(x_i) = W_i(\bar{x}_i) - \mathbb{E}_{x_{-i}} \left[ \int_{x_i}^{\bar{x}_i} \{ (h_i + b_i)G_i(z + q_i(z, x_{-i})) - b_i \} \, dz \right],
\]

or equivalently,

\[
\mathbb{E}_{x_{-i}} [m_i(x)] = W_i(\bar{x}_i) - \mathbb{E}_{x_{-i}} \left[ C_i(x_i, q_i(x)) + \int_{x_i}^{\bar{x}_i} \{ (h_i + b_i)G_i(z + q_i(z, x_{-i})) - b_i \} \, dz \right].
\]

**Proof.** Define

\[
\tilde{W}_i(x_i, x'_i) := \mathbb{E}_{x_{-i}} [C_i(x_i, q_i(x'_i, x_{-i})) + m_i(x'_i, x_{-i})].
\]

Then (IC_i) can be formulated as

\[
W_i(x) = \min_{x'_i \in [\underline{x}_i, \bar{x}_i]} \tilde{W}_i(x_i, x'_i).
\]

We have

\[
\frac{\partial}{\partial x_i} \tilde{W}_i(x_i, x'_i) = \mathbb{E}_{x_{-i}} \left[ \frac{\partial}{\partial x_i} C_i(x_i, q_i(x'_i, x_{-i})) \right] = \mathbb{E}_{x_{-i}} [(h_i + b_i) G_i(x_i + q_i(x'_i, x_{-i})) - b_i],
\]

because the differentiation and expectation operators commute (Rosenthal 2000, Proposition 9.2.1).

Thus \( \left| \frac{\partial}{\partial x_i} \tilde{W}_i(x_i, x'_i) \right| \leq \max \{ h_i, b_i \} \) for any \( (x_i, x'_i) \), and (EC.4) follows from the *envelope theorem* of Milgrom and Segal (2002). The equality in (EC.5) follows from the definition of \( W_i \) in (EC.2). 

**Corollary EC.1.** For \( i = 1, \ldots, N \), if \( q_i \) and \( m_i \) satisfies (IC_i), then (IR_i) reduces to

\[
W_i(\bar{x}_i) \leq C_i(\bar{x}_i, 0).
\]
Proof. By Lemma EC.2, we have

\[
W_i(x_i) - C_i(x_i, 0) = W_i(\bar{x}_i) - C_i(\bar{x}_i, 0) - (h_i + b_i)\mathbb{E}_{\bar{x}_i} \left[ \int_{x_i}^{\bar{x}_i} \left\{ G_i(z + q_i(z, x_i)) - G_i(z) \right\} dz \right] \\
\leq W_i(\bar{x}_i) - C_i(\bar{x}_i, 0).
\]

Thus the (IR_i) constraint, \( W_i(x) - C_i(x, 0) \leq 0 \) for all \( x_i \), is equivalent to imposing only (EC.6).

\( \square \)

Lemma EC.3. Let \( \{q_i^*(\cdot)\}_{i=1}^N \) denote the solution to the following program:

\[
\min_{q(\cdot)} \mathbb{E}_x \left[ \sum_{i=1}^N \left\{ (c_i - h_i)q_i(x) + C_i(x_i, q_i(x)) + (h_i + b_i)\frac{E(x)}{f_i(x_i)} G_i(x_i + q_i(x)) \right\} \right]
\]

s.t. \( \sum_{i=1}^N q_i(x) \leq x_s, \ q_i(x) \geq 0, \ x \in \mathcal{X} \).

If \( q_i^*(x) \) is also non-increasing in \( x_i \) for all \( x_i \in \mathcal{X}_i \) and \( i = 1, \ldots, N \), then the allocation rule \( \{q_i^*(\cdot)\}_{i=1}^N \) together with the payment rule

\[
m_i^*(x) = C_i(x_i, 0) - C_i(x_i, q_i^*(x)) - (h_i + b_i)\int_{x_i}^{\bar{x}_i} \left\{ G_i(z + q_i^*(z, x_i)) - G_i(z) \right\} dz, \quad i = 1, \ldots, N, \ x \in \mathcal{X}.
\]

solves (EC.1).

Proof. Replace the term \( \{\mathbb{E}_{\bar{x}_i}[m_i(x)]\}_{i=1}^N \) in (EC.1) with the expression in (EC.5). In light of Corollary EC.1 and Lemma EC.1, we can focus on the following relaxed minimization problem:

\[
\min_{q(\cdot), \ldots, q_N(\cdot)} \mathbb{E}_x \left[ C_s(x_s, q(x)) \right] + \sum_{i=1}^N \mathbb{E}_x \left[ C_i(x_i, q_i(x)) \right] \\
+ \sum_{i=1}^N (h_i + b_i)\mathbb{E}_x \left[ \int_{x_i}^{\bar{x}_i} G_i(z + q_i(z, x_i)) dz \right] - \sum_{i=1}^N W_i(\bar{x}_i)
\]

s.t. \( \sum_{i=1}^N q_i(x) \leq x_s, \ q_i(x) \geq 0, \) and \( W_i(\bar{x}_i) \leq C_i(\bar{x}_i, 0) \) for all \( x \in \mathcal{X}, i = 1, \ldots, N \).

Since the objective function is decreasing in \( W_i(\bar{x}_i) \) for \( i = 1, \ldots, N \), it is optimal to set,

\[
W_i^*(\bar{x}_i) = C_i(\bar{x}_i, 0), \quad i = 1, \ldots, N.
\]
Furthermore, we have,

\[
\mathbb{E}_x \left[ \int_{x_i}^{x_i} G_i(z + q_i(z, x_{-i})) dz \right] = \mathbb{E}_{x_{-i}} \left[ \mathbb{E}_{x_i} \left[ \int_{x_i}^{x_i} G_i(z + q_i(z, x_{-i})) dz \right] \right]
\]

\[
= \mathbb{E}_{x_{-i}} \left[ \mathbb{E}_{x_i} \left[ \frac{F_i(x_i)}{f_i(x_i)} G_i(x_i + q_i(x)) \right] \right]
\]

\[
= \mathbb{E}_x \left[ \frac{F_i(x_i)}{f_i(x_i)} G_i(x_i + q_i(x)) \right], \quad i = 1, \ldots, N
\]

where the second equality follows from integration by parts. Substituting the last expression into (EC.9) yields (EC.7). The result follows from Lemma EC.1. 

Our approach of solving the problem reformulated in Lemma EC.3 is to solve the following pointwise optimization problem by ignoring the monotonicity constraints.

\[
\min_{q \geq 0} \pi^*(x, q) \quad \sum_{i=1}^{N} q_i \leq x_s,
\]

where \( \pi^*(x, q) := \sum_{i=1}^{N} \left[ (c_i - h_s)q_i + C_i(x_i, q_i) + (h_i + b_i) \frac{F_i(x_i)}{f_i(x_i)} G_i(x_i + q_i) \right] \).

In particular, one needs to account for the lack of global convexity in the objective function. We circumvent by establishing that the function is (globally) quasi-convex and convexity holds on the relevant region so that KKT conditions can still be used to characterize the global solution, which turns out to satisfy the neglected monotonicity conditions under our mild conditions on the distribution functions.

**Lemma EC.4.** Under Assumptions 1 and 2, the solution \( q^*(x_s, x) \) given in (EC.10) and the payment \( m^*(x_s, x) \) defined by (EC.8) solves (EC.1).

**Proof.** Combining Lemma EC.3, Lemma EC.7 and Lemma EC.8 we see that the relaxed solution to the pointwise optimization problem (EC.10) indeed solves the problem (EC.1) under Assumptions 1 and 2. 

**Quasi-convexity and Monotonicity in the SSMR Case**

Lemma EC.5 below will be applied throughout the paper with \( \alpha = (h_s + b_i - c_i)/(h_i + b_i) \), \( G = G_i, \quad g = g_i, \quad \zeta = [F_i(x_i)/f_i(x_i)] \geq 0. \)
LEMMA EC.5. Let $G$ be a probability distribution with a upper-semi continuous probability density function $g$ such that $1 - G$ is log-concave, $\zeta \geq 0$, and $\alpha \in (0, 1)$. Define $y^* = \inf\{y \in [0, +\infty)\} - \alpha + G(y) + \zeta g(y) \geq 0\}$. Then

(i) $-\alpha + G(y) + \zeta g(y)$ is non-decreasing for $y \in [0, y^*]$;

(ii) $-\alpha + G(y) + \zeta g(y) \geq 0$, for all $y \in [y^*, +\infty)$.

Proof. It is equivalent to show that $1 - G(y) - \zeta g(y)$ is non-increasing in $y \in [0, y^*]$ and for $y \in [y^*, \infty)$

$$1 - G(y) - \zeta g(y) \leq 1 - \alpha. \quad \text{(EC.11)}$$

By definition of $y^*$, if $1 - G(y) - \zeta g(y) \leq 0$, we must have $y \in [y^*, \infty)$ and (EC.11) holds for this case. Otherwise note that

$$\log (1 - G(y) - \zeta g(y)) = \log(1 - G(y)) + \log \left(1 - \zeta \frac{g(y)}{1 - G(y)}\right).$$

By the monotonicity of $G$ we have $\log(1 - G(y))$ is non-increasing and by log-concavity of $1 - G$ we have that $-g(y)/[1 - G(y)]$ is non-increasing in $y$. Therefore both terms are non-increasing in $y$, establishing (i) and (ii) since $1 - G(y^*) - \zeta g(y^*) \leq 1 - \alpha$. \qed

Lemma EC.6 below will typically be applied to $\alpha_i = (h_s + b_i - c_i)/(h_i + b_i)$, $a = 1$, and $\zeta_i(x_i) = F_i(x_i)/f_i(x_i)$ as a non-decreasing function of $x_i$ for fixed $x_{-i}$.

LEMMA EC.6. Assume that $\zeta_i$ is non-decreasing (non-increasing) in $x_i$, and let $\alpha_i \in (0, 1)$, $a > 0$, and $G_i$ be a probability distribution with density function $g_i$. Then $y_i^*(x_i) = \inf\{y \in [0, +\infty) : -\alpha_i + G_i(y) + a\zeta_i(x_i)g_i(y) \geq 0\}$ is non-increasing (non-decreasing) in $x_i$.

Proof. Pick $x_i < x_i'$ so that $\zeta_i(x_i) \leq \zeta_i(x_i')$. For any $y$

$$0 \leq -\alpha_i + G_i(y) + a\zeta_i(x_i)g_i(y) \leq -\alpha_i + G_i(y) + a\zeta_i(x_i')g(y)$$

so that any $y$ considered in the infimum problem for $y_i^*(x_i)$ is also considered in the infimum problem for $y_i^*(x_i')$. Thus, $y_i^*(x_i) \geq y_i^*(x_i')$. Similar arguments apply when $\zeta_i(x_i)$ is non-increasing in $x_i$. \qed
In what follows we need the definition
\[ \mu_i(y|x_i) := (h_i + b_i)\tilde{G}_i(y|x_i) - (h_s + b_i - c_i) \tag{EC.12} \]
which generalizes (12).

**Lemma EC.7.** Under Assumption 1, fix \( x \in X \), relabeling retailers if necessary so that
\[ \mu_1(x_1|x_1) \leq \mu_2(x_2|x_2) \leq \cdots \leq \mu_N(x_N|x_N), \]
the solution \( q^*(\cdot) \) to (EC.10) can be characterized as follows: there is \( U^* = U^*(x_s, x) \in \{ \min_{j=1,...,N} \{ c_j - h_s - b_j \}, 0 \} \) and index \( n^* \) such that

1. \( \mu_i(x_i + q^*_i(x_s, x)|x_i) = U^* \) for those \( i \) such that \( q^*_i(x_s, x) > 0 \), \( i \leq n^* \).
2. \( \mu_j(x_j|x_j) \geq U^* \) for those \( j \) such that \( q^*_j(x_s, x) = 0 \), \( i > n^* \).
3. \( \sum_{i=1}^{N} q^*_i(x_s, x) \leq x_s \) and \( U^* \cdot \left( \sum_{i=1}^{N} q^*_i(x_s, x) - x_s \right) = 0. \)

**Proof.** Let
\[ V_i(y|x_i) := (c_i - h_s)y + C_i(y, 0) + (h_i + b_i)\frac{F_i(x_i)}{f_i(x_i)}G_i(y). \tag{EC.13} \]

The objective function in problem (EC.10) can be written as \( \sum_{i=1}^{N} V_i(x_i + q_i|x_i) \). Straightforward computation reveals that \( \frac{\partial}{\partial y} V_i(y|x_i) = \mu_i(y|x_i) \) defined in (EC.12). Lemma EC.5 establishes that \( V_i(\cdot|x_i) \) is quasi-convex over the whole line, reach its minimum at \( y^*_i(x_i) \) and convex over \( [-\infty, y^*_i(x_i)] \). Thus it entails that \( q^*_i(x_s, x) = 0 \) for all those \( i \) such that \( \mu_i(x_i|x_i) \geq 0 \). For those \( i \) such that \( \mu_i(x_i|x_i) < 0 \), KKT conditions are necessary and sufficient to characterize the solution. Let \( U^* = U^*(x_s, x) \) be the Lagrangian multiplier for the capacity constraint \( \sum_{i=1}^{N} q_i \leq x_s \). Thus we have \( \mu_i(x_i + q^*_i(x_s, x)|x_i) = U^* \) for those \( i \) such that \( q^*_i(x_s, x) > 0 \) and \( \mu_j(x_j|x_j) \geq U^* \) for those \( j \) such that \( q^*_j(x_s, x) = 0 \).

If for all \( j = 1, \ldots, N \), \( q^*_j(x_s, x) = 0 \), set \( U^* = \min \{ \mu_j(x_j|x_j) : j = 1, \ldots, N \} \leq 0 \) and hence \( U^* \geq \min \{ -(h_s + b_j - c_j) : j = 1, \ldots, N \} \). If, for some \( i \), \( q^*_i(x_s, x) > 0 \), we have \( U^* = \mu_i(x_i + q^*_i(x_s, x)|x_i) \geq -(h_s + b_i - c_i) \geq \min \{ -(h_s + b_j - c_j) : j = 1, \ldots, N \} \) and \( U^* = \mu_i(x_i + q^*_i(x_s, x)|x_i) \leq 0 \) because \( x_i + q^*_i(x_s, x) \in [-\infty, y^*_i(x_i)] \), over which \( \mu_i(\cdot|x_i) \leq 0 \) by quasi-convexity (Lemma EC.5). The complementarity condition follows from the fact that if \( \sum_{i=1}^{N} q^*_i(x_s, x) < x_s \), then we must have \( \mu_i(x_i + q^*_i(x_s, x)|x_i) \geq \mu_i(y^*_i(x_i)|x_i) = 0 \) and it suffices to set \( U^* = 0 \).
Finally, since $\mu_i(x_i|x_i)$ are increasing, $n^* = \max\{i \geq 0 : q_i^*(x_s, x) > 0\}$.

□

**Lemma EC.8.** Under Assumptions 1 and 2, let $q^*$ be the solution to (EC.10) characterized in Lemma EC.7. Then for $i = 1, \ldots, N$ we have that $q_i^*(x_s, x)$ is non-increasing in $x_i$. Moreover, $q_i^*(x_s, x)$ is non-decreasing in $x_i$ for any $j \neq i$.

**Proof.** For fixed $x_{-i}$ and $x_i \leq x_i'$ consider $q = q^*(x_s, x_{-i}, x_i)$ and $q' = q^*(x_s, x_i', x_{-i})$. We can assume $q' > 0$ otherwise we are done. Moreover, since $\mu_i(x_i|x_i) \leq \mu_i(x_i'|x_i')$, we have that $q_i' > 0$ implies that $q_i > 0$. By Lemma EC.7, $q$ and $q'$ are characterized by the existence of multipliers $U$ and $U'$, and integers $n$ and $n'$ such that

\[
\begin{align*}
    j \leq n, j \neq i, & \quad \mu_j(x_j + q_j|x_j) = U, q_j > 0 & \quad j \leq n', j \neq i, & \quad \mu_j(x_j + q_j'|x_j) = U', q_j' > 0 \\
    j > n, j \neq i, & \quad \mu_j(x_j|x_j) \geq U, q_j = 0 & \quad j > n', j \neq i, & \quad \mu_j(x_j|x_j) \geq U', q_j' = 0 \\
    U \cdot \left(\sum_{j=1}^N q_j - x_s\right) = 0 & \quad U' \cdot \left(\sum_{j=1}^N q_j' - x_s\right) = 0.
\end{align*}
\]

Since $\mu_i(y|x_i) \leq \mu_i(y|x_i')$ for all $y$, it follows that $U' \geq U$. Moreover, note that $x_j + q_j \leq y_j^*(x_j)$ if $j \leq n$ and $x_j + q_j' \leq y_j^*(x_j)$ if $j \leq n'$. Therefore, since $\mu_j(\cdot|x_j)$ is non-decreasing for $y \in [0, y_j^*(x_j)]$ by Lemma EC.5 (i), it follows that $q_j' \geq q_j$ for $j \leq \max\{n, n'\}, j \neq i$.

Thus, note that if $U < 0$ the conditions above imply $\sum_{i=1}^N q_i = x_s$ and we have

\[
q_i' \leq x_s - \sum_{j \neq i} q_j' \leq x_s - \sum_{j \neq i} q_j = q_i
\]

and the result follows. On the other hand, if $U = 0$, we also have $U' = 0$ and $q_i' = \max\{0, y_i^*(x_i') - x_i'\} \leq \max\{0, y_i^*(x_i) - x_i\} = q_i$, since $y_i^*(x_i) - x_i$ is non-increasing in $x_i$ by Lemma EC.6. □

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