Dynamic Assortment Customization with Limited Inventories

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We consider a retailer with limited inventory of identically priced, substitutable products. Customers arrive sequentially and the firm decides which subset of products to offer to each arriving customer depending on the customer’s preferences, the inventory levels, and the remaining time in the season. We show that it is optimal to limit the choice set of some customers (even when the products are in stock), reserving products with low inventory levels for future customers who may have a stronger preference for those products. In certain settings, we prove that it is optimal to follow a threshold policy under which a product is offered to a customer segment if its inventory level is higher than a threshold value. The thresholds are decreasing in time and increasing in the inventory levels of other products. We introduce two heuristics derived by approximating the future marginal expected revenue by the marginal value of a newsvendor function that captures the substitution dynamics between products. We test the impact of assortment customization using data from a large fashion retailer. We find that the revenue impact of dynamic assortment customization can be significant, especially when customer heterogeneity is high and when the products’ inventory-to-demand ratios are asymmetric. Our findings suggest that assortment customization can be used as another lever for revenue maximization in addition to pricing.

1. Introduction

Motivation and Objective: Online retailers have been investing heavily in diverse technologies, such as sophisticated analytics and personalization tools. In contrast to traditional bricks-and-mortar stores, online retailers can not only easily collect personal information, but they can also record their customers’ purchase histories. As a result, online retailers are able to send personalized
promotions or display customized recommendations to individual customers based on their personal information and browsing and purchasing history. This can effectively enhance the customers’ shopping experience and influence their purchasing activities. For example, Amazon.com’s website, like many other online retailers’ websites, requires a customer to log in before making a purchase. This allows the company to track the customer’s personal information and purchase history. Based on this data, the company can estimate the customer’s preferences on styles, colors, and other features of a product. An online retailer can use this information to customize the set of products made available to customers. While retailers typically display a selection of product options over multiple pages, the products on the first few pages tend to receive more attention. Therefore, the company may be able to control the selection of products offered to a customer by restricting the set of product variants displayed in the first few pages of the search result. In addition, the company can more directly exercise that control by selecting a product’s “in-stock” or “out-of-stock” status information displayed to customers. This idea also applies, for example, to hotel consolidators such as www.hotel-savers.com that purchase blocks of inventory (e.g., hotel rooms) from 25,000 hotels in 5,000 cities to sell through their websites. The travel websites collect information about their customers’ purchase histories and preferences, and have access to many products (e.g., hotels) with different inventory levels in the customer’s desired price range.\footnote{In fact, Expedia.com is currently exploring the design of customized displays for its website (Expedia 2009).}

The goal of this paper is to explore the revenue impact of dynamic assortment optimization. To that end, we develop a stylized model of a multi-product retailer and show that the optimal dynamic assortment policy involves offering customized assortments to individual customers (thus, rationing products to some customers) based on inventory conditions and the distribution of preferences of future customers. Furthermore, using a case study based on a large retailer’s data-set, we show that dynamic assortment customization has the potential to significantly increase revenue.

**Model and Results:** We consider a firm that sells multiple products in a retail category. There are limited inventories of the products to sell over a finite selling season. The selling prices of the products are all equal (as in the case of different colors and sizes of the same style of shirt). The customer base is heterogeneous and characterized by multiple segments with different product preference distributions. These segments are essentially customer clusters with similar purchase histories as described in Linden et al. (2003). We use the Multinomial Logit framework to model
the choice process of each customer. The retailer can identify the segment of an arriving customer and customize the assortment to that particular customer without incurring additional cost. The customer then selects a product among those in the offered assortment or selects the no-purchase option. We formulate this as a dynamic assortment optimization problem in which the assortment decisions depend on the inventory levels, the current customer’s segment, and the distribution of preferences of future customers.

Deriving the optimal dynamic assortment policy for settings with multiple products is complex due to the combinatorial nature of assortment problems. Indeed, research on choice-based network revenue management has generally focused on heuristic approaches to solve assortment problems. In this paper, we analytically characterize the optimal policy in specific settings. We find that if the retailer is not able to identify the types of arriving customers, it is optimal to offer every product to all customers. However, when such identification is possible, we show that the retailer may benefit from rationing products to some customer segments. In a setting with two products, we prove that it is optimal to follow a threshold-type policy in which a product is offered only if its inventory level is above a certain value. Hence, the firm has the potential to increase revenues by strategically restricting the set of product options it makes available to customers, even when all products are in stock and identically priced. In other words, the company may conceal a product short on inventory in anticipation of future sales to other customers who may have a stronger preference for this product (and who may therefore walk away if that product is not available).

Contributions of the paper

1. **Theoretical Contribution:** Our paper is the first to propose the idea of assortment customization in the presence of heterogeneous customer segments. We analytically characterize the optimal policy for the case of two products. The optimal policy involves rationing even when products’ prices are equal. Note that in the revenue management literature, discussed in detail in Section 2, rationing has been shown to be optimal in various settings – including single-period models and models with unlimited inventory – when product costs/prices or customers’ willingness to pay are different. In contrast, the results in this paper hold if the following features are present: a dynamic (multi-period) model, multiple heterogeneous customer segments, and supply restrictions (limited inventory). Moreover, as noted earlier, rationing requires that the retailer have the ability to identify individual customers’ types.
2. Practical Implications: Ideas similar to those presented in this paper are gaining more traction in practice. Retailers are becoming more sophisticated in the way they utilize both supply conditions and customer-specific information to customize decisions. For example, in conversations with a large U.S.-based retailer of women apparel, it was noted that the company customizes delivery lead times when inventory is limited. The company segments its online customers based on a variety of factors and bounds some customers to longer delivery lead times (7 days) to reserve products for other potentially more valuable customers. In the context of our model, we demonstrate the potential revenue impact of assortment customization with a case study based on data from a high-end Turkish fashion retailer that operates online and offline channels. This company buys high-end fashion products well in advance of the selling season and sells them at full price until the clearance season. We use weekly item-level sales (units and prices) and inventory data for the category of women’s shoes from the Fall-Winter 2011-2012 season. The data suggests that there are significant differences in the preferences of customers from three distinct regions in Turkey. We demonstrate the potential impact of assortment customization by setting the parameters of our model based on the actual demand and inventory data, and by comparing the expected revenue from assortment customization with the benchmark policy that involves offering all available products to any arriving customer in every period. We find that inventory-demand imbalances tend to emerge early in the season and employing assortment customization can lead to significant benefits in those cases. Specifically, in this study, the revenue benefit from assortment customization can be as high as 5% relative to the benchmark policy.

3. Heuristics: The analytical results in this paper suggest that rationing as a result of customer heterogeneity and supply conditions may be optimal in more general settings. Hence, we introduce two heuristics that are derived from the dynamic program used to characterize the optimal policy for the more stylized settings. The heuristics are derived by approximating the marginal expected revenue generated by each product and in each period in the dynamic program by the marginal value of a newsvendor (single-period) function that captures the substitution dynamics between products (if a product that is low on inventory has a high substitution rate into another product, then the inventory of the latter product is evaluated against its total effective demand). These heuristics capture the demand-supply conditions for each product (and in each period) based on newsvendor-type approximations. Applying these heuristics, we demonstrate the revenue impact
of assortment customization in cases with larger numbers of products.

Organization of the paper: Section 2 provides a review of the relevant literature. Section 3 describes the model. Section 4 analyzes the optimal dynamic assortment policy. Section 5 introduces heuristics for more general cases. Section 6 discusses the impact of assortment customization in a case study. Section 7 concludes the paper. All proofs are provided in an online appendix.

2. Literature Review

Our work is related to three streams of research. The first one is the literature on retail assortment planning, with papers focusing on assortment and inventory decisions for a single customer segment. Kök et al. (2008) provide a review of this literature. van Ryzin and Mahajan (1999) derive the optimal assortment policy for a category with homogenous products. Cachon et al. (2005) incorporate consumer search costs in a similar context. Kök and Xu (2010) study assortment and pricing decisions in retail categories with multiple subgroups of products. Besbes and Saure (2010) study assortment and price competition between two retailers. Smith and Agrawal (2000) discuss an optimization approach for the assortment selection and inventory management problems in a multi-item setting with demand substitution. Kök and Fisher (2007) describe a methodology for estimation of demand and substitution rates and for assortment optimization using data from a supermarket chain. Mahajan and van Ryzin (2001) and Honhon et al. (2008) optimize starting inventory levels for a model with dynamic customer substitution (i.e., customers choose from those products that are available at the time of their arrival). In our model, customers also engage in dynamic substitution, but the set of products displayed to each customer is a decision variable in our case. Caro and Gallien (2007), Rusmevichientong et al. (2010), Saure and Zeevi (2009), and Ulu et al. (2010), study dynamic assortment selection with demand learning during a single selling season. Caro and Martinez-de-Albeniz (2009) find that renewing the assortment frequently can allow a firm to charge higher prices. The models in this stream of research do not consider multiple customer segments with different preferences, and therefore the assortment policy does not involve any form of customization. Kim et al. (2002) develop a methodology for estimating the product preferences of households and propose that web retailers such as Net Grocer and Peapod could offer customized assortments to each household – rather than the full assortment – to reduce the search cost of customers, which has been shown to negatively influence sales.
The second related stream of research includes work on choice-based network revenue management. Zhang and Cooper (2005) study revenue maximization in a setting where customers choose from a set of parallel flights. Talluri and van Ryzin (2004a) describe a framework for choice-based network revenue management models with multiple products (itineraries) and components (flight-legs), where product prices can be different. The authors characterize the optimal assortment policy for settings in which all products share the same resource (aircraft capacity for one flight-leg). As in other revenue management papers with a single flight-leg, the optimal policy is a booking-limit based policy, under which some products with lower fare prices are not offered if the remaining capacity is low. In these models, the products command different prices but share the same resources. In contrast, each product variant has its own dedicated inventory in our setting.

For general choice-based network revenue management models, Gallego et al. (2004) and Liu and van Ryzin (2008) use a choice-based linear programming (CDLP) model to approximate the dynamic control problem. In addition, Liu and van Ryzin (2008) propose a dynamic programming decomposition heuristic and characterize the efficient sets which are used in the optimal policy. Zhang and Adelman (2009) use an affine function to approximate the value function of the dynamic program and Chen and Homem de Mello (2009) develop an approximation that consists of a sequence of two-stage stochastic programs with simple recourse. van Ryzin and Vulcano (2008a, b) study virtual nesting policies in a similar context where the demand process consists of a stochastic sequence of heterogeneous customers. Miranda Bront et al. (2009) show that the CDLP model of the assortment problem with multiple segments is NP-hard and propose a column generation algorithm. Rusmevichientong and Topaloglu (2011) propose a robust formulation of the assortment optimization problem. The above papers consider multiple segments characterized by different choice probabilities for the different products, and these products are sold at different prices (i.e., fare-route options). However, in these papers, the decision maker cannot observe the type of an arriving customer and therefore, at any given time, all customers are offered the same assortment (which is a list of fare class and route combinations). Hence, customization is not possible in those settings and the optimal assortment decisions are based on aggregate choice probabilities across segments, inventory levels, and price differences between products.

Assortment customization has begun to attract attention in academic research and in industry. Golrezai et al. (2012) study an assortment customization problem similar to ours and develop an
inventory-balancing algorithm that minimizes the asymptotic worst-case gap with an upper bound. Lederman and Saure (2013) consider a hierarchical customization problem in which a customer is offered product assortments in an incremental fashion.

Cross-selling is commonly used to maximize revenues by online retailers. Netessine et al. (2006) study a retailer with limited inventories that dynamically selects product bundles to offer to each arriving customer. The authors develop several heuristics and identify those that are effective in their setting. In a single product setting, Aydin and Ziya (2009) consider personalized dynamic pricing after receiving a signal about each customer’s willingness-to-pay. Fudenberg and Villas-Boas (2006) provide a review of the literature on personalized pricing. Personalized pricing is also closely related to price discrimination, which has been studied extensively in the marketing literature. Price discrimination is achieved by offering a vertically differentiated product line (Mussa and Rosen 1978) and by offering product bundles (e.g., Fay and Xie 2008). Usually, in these settings, a static assortment is offered to all customers.

The ability of a company to limit the assortment to its customers is a form of inventory rationing. Therefore, our paper is also related to research on this topic. Ha (1997a) considers a single-item, make-to-stock production system with several demand classes (characterized by the different prices they are willing to pay) and lost sales, and demonstrates that the optimal policy is characterized by rationing levels for each demand class. Ha (1997b) studies a similar system with two demand classes and backordering. de Vericourt et al. (2002) extend this model to a setting with multiple demand classes characterized by different backorder penalty costs. Shumsky and Zhang (2009) analyze a dynamic capacity allocation problem with upgrading. The paper evaluates the value of an optimal upgrading policy.

3. Model Formulation

We consider a retailer that sells a set of identically-priced product variants within a retail category over a finite selling season. The retailer decides an assortment to offer to each arriving customer from a set of horizontally differentiated products, denoted by \( \mathcal{N} = \{1, \cdots, N\} \). Each product variant may represent a combination of features that does not change the functionality of the product, such as a color/size/design combination in a clothing category. Other examples include music CDs, DVDs, variations of similar products from different brands, and online travel services.
The selling season has $T$ periods. There is no replenishment during the season. The initial amount of stock for all products is denoted by an $N$-dimensional vector $y_0 = (y_{01}, \ldots, y_{0N})$ and $y$ denotes a generic vector of inventory levels. This model is applicable, for example, to short-life-cycle products with long procurement lead times and to end-of-season sales for seasonal products.

There are $M$ customer segments characterized by different product preferences. We model customer preferences using a Mixed Logit model. That is, each arriving customer belongs to a segment with a certain probability and the choice process of all customers in a segment follows a specific Multinomial Logit (MNL) model. Given an assortment $S \in \mathcal{N}$, the utility derived from choosing product $i \in S$ for a customer in segment $m$ is $u_{mi} + \xi_{mi}$, where $u_{mi}$ is the expected utility derived from product $i$ and $\xi_{mi}$ is a random variable representing the heterogeneity of utilities across customers in the same segment. In addition, customers can always choose not to purchase any product, receiving a utility $u_{m0} + \xi_{m0}$. Each customer chooses the product that offers the maximum utility. We assume that $\xi_{mi}$ are i.i.d. random variables following a Gumbel distribution with mean zero and variance $\pi^2/6$. The probability of a customer choosing product $i$ that arises from this utility maximization problem is given by

$$q_{mi}(S) = \frac{\theta_{mi}}{\sum_{j \in S} \theta_{mj} + \theta_{m0}}, i \in S \cup \{0\},$$

where $\theta_{mi} = e^{u_{mi}}$. See Anderson et al. (1992) for more details on the MNL model and Kök et al. (2008) for a comparison of the MNL model with other demand models. We refer to $\theta_{mi}$ as a segment $m$ customer’s preference for product $i$ and we let $\Theta_m = (\theta_{m1}, \theta_{m2}, \ldots, \theta_{mN})$ be the preference vector of segment $m$. Without loss of generality, we set $\theta_{m0} = \theta_0 > 0$ for all $m$, i.e., all segments have the same preference for the no-purchase option. We assume that the retailer knows the preference vector for each customer segment. The retailer is able to estimate these preferences based on the customers’ purchase history. Mixed Logit models are commonly used by retailers and marketing firms to identify multiple latent customer segments in the customer base and to estimate purchasing behavior for each segment (see, e.g., Gupta and Chintagunta 1994 and Wedel and Kamakura 1998). A common approach in online retailing is collaborative filtering (Linden et al. 2003), which measures similarity of customers to each other based on past history and infers preferences of an arriving customer for the category of interest.

\[2\] Mixed Logit models are quite flexible in modeling demand elasticities and do not suffer from some of the main criticisms of the Multinomial Logit Model, such as the IIA (independence of irrelevant alternatives) assumption.
We consider a Poisson arrival process and assume that at most one customer arrives in each period. The sequence of events in each period is as follows: At the beginning of the period, a consumer arrives with probability $\lambda$ and the arriving customer belongs to segment $m$ with probability $\rho_m$, with $\sum_{i=1}^{M} \rho_i = 1$. Because of the identification process that takes place upon arrival, the retailer has perfect information on the customer’s segment. The retailer offers an assortment (subset of the available products) to the customer. Next, the customer makes a purchasing decision according to the choice process and the revenue is received if a product is sold. One can allow for an additional generic segment with product preferences matching those of the general population. If there is no sales history or information on the identity of a particular customer, this customer would be assumed to belong to this generic segment.

To isolate the effects of customer heterogeneity and limited inventories, we focus on a model with identical prices for all products, denoted by $p$. This is usually the case in apparel, e.g., for items of different colors and sizes or for similar product styles. In a setting with non-identical prices, there is a clear incentive for rationing (e.g., not offering a lower-priced product at certain levels of inventory) even with a homogeneous customer base. (Section 5 discusses two heuristics that also apply to product categories in which prices are not necessarily equal.) We also assume that a product’s price is the same for all customers – that is, we do not consider dynamic customized pricing of individual products. It is a matter of debate whether customized pricing is legal with respect to antitrust laws (Ramasasty 2005). As an example, in 2000, Amazon.com acknowledged that it had presented different prices to different customers in its DVD store for experimentation purposes, but denied that it did so on the basis of any past purchasing behavior at Amazon.

Finally, without loss of generality, we assume that the salvage value for unsold units at the end of the season is zero. We use the following notation throughout the paper. We let $e_i$ denote the $i^{th}$ unit vector. In addition, we let $S(y)$ be the set of products with positive inventory and denote the cardinality of set $S$ as $|S|$.

4. The Dynamic Assortment Optimization Problem

In this section, we examine the optimal dynamic assortment policy and provide results for specific settings. These results suggest that it is optimal to ration inventory when the customer base is heterogeneous and there is limited inventory of the available products. We build on the ideas
generated in this section to construct two heuristics to solve the general system. These heuristics
are presented in Section 5.

Define the value function in period \( t \) as \( V_t(y|m) \), given the vector of inventory levels \( y \) and
that the customer arriving in this period is of segment \( m \). Taking expectation across all customer
segments, the value function at the beginning of period \( t \) is given by

\[
V_t(y) = \sum_{m \in M} \rho_m V_t(y|m).
\]

Provided that the current arriving customer is in segment \( m \), the goal is to select an assortment
to maximize revenue through the selling season. Therefore, the optimality equation is given by

\[
V_t(y|m) = \max_{S \subseteq S(y)} \left[ \sum_{i \in S} \lambda q_{mi}(S)(p + V_{t+1}(y - e_i)) + \lambda q_{m0}(S)V_{t+1}(y) \right] + (1 - \lambda)V_{t+1}(y). \tag{1}
\]

The term inside the brackets is the value function for an arriving customer in segment \( m \) (an
arrival occurs with probability \( \lambda \)). For a given assortment \( S \), this term accounts for the probability
of selling one unit of product \( i \in S \), earning a revenue of \( p \) from the sale, plus the revenue-to-go
function in period \( t + 1 \) evaluated at the current inventory level minus the unit sold in period \( t \). The
term also accounts for the possibility that the customer does not make a purchase, in which case
the revenue is the profit-to-go function in period \( t + 1 \) evaluated at the current vector of inventory
levels. We maximize this term over all possible subsets of variants with positive inventory. The
last term is the future value function if no customer arrives (with probability \( 1 - \lambda \)). Because the
products prices are equal, it is optimal to sell as many units of any product as possible throughout
the selling season. Let \( S_{mt}(y) \) denote the optimal assortment for a segment \( m \) customer at time \( t \)
given inventory levels \( y \). The total optimal revenue over the selling season is given by \( V_1(y_0) \). The
terminal condition of this dynamic program is the value function in period \( T \). Because there are
no more customers beyond the last period, the optimal policy is to offer all products with positive
inventory to any arriving customer. Therefore, \( V_T(y|m) = \sum_{i \in S(y)} \lambda q_{mi}(S(y))p \).

This formulation leads to a dynamic program with an \( N \)-dimensional state space and a large
action space (for each segment, there are \( 2^{|S(y)|} \) possible assortments that can be offered). Thus,
the above dynamic assortment optimization problem is intractable for large \( N \). Note that the
myopic solution to this dynamic program (which maximizes the current period reward ignoring the
impact on future revenues) is to offer all products in every period, thus maximizing the probability
of sales to any arriving customer. This policy serves as a benchmark for the value of dynamic assortment customization. We next discuss general properties of the optimal policy.

For \(i \in S(y)\), define \(\Delta_{i+1}(y) = V_{t+1}(y) - V_{t+1}(y - e_i)\) as the marginal expected revenue generated by the \(y_i\)-th unit of inventory of product \(i\) in period \(t + 1\). Clearly, \(0 \leq \Delta_{i+1}(y) \leq p\).

We rewrite the optimality equation in (1) as

\[
V_t(y|m) = \max_{S \subseteq S(y)} \left\{ \sum_{i \in S} \lambda_{q_{mi}}(S)(p - \Delta_{i+1}(y)) \right\} + V_{t+1}(y). \tag{2}
\]

If product \(i\) is offered and sold in period \(t\), then the revenue consists of the price \(p\) minus the lost opportunity revenue of selling this unit after period \(t\), given by \(\Delta_{i+1}(y)\). We denote the effective marginal price of product \(i\) in period \(t\) as \(p_{i+1}^t(y) = p - \Delta_{i+1}(y)\) if \(i \in S(y)\) and note that \(p_{i+1}^t(y) = 0\) if \(i \in N \setminus S(y)\). Consider an ordering of the products in period \(t\) given inventory levels \(y\) so that \(p_{i+1}^1(y) \geq \cdots \geq p_{i+1}^N(y)\). Based on this ordering, we define a set consisting of the products with the largest effective marginal prices, given by \(A_k(y) = \{i_1, \cdots, i_k\}\). The next result shows that the optimal assortment for each customer segment is restricted to one of \(N\) possible sets \(A_i(y)\).

**Lemma 1** Given inventory levels \(y\) in period \(t\), the optimal assortment for a segment \(m\) customer is given by \(S^*_m(y) \in \{A_1(y), \cdots, A_N(y)\}\).

Lemma 1 is similar to Theorem 1 in Talluri and van Ryzin (2004b), although our setting involves an \(N\)-dimensional state-space (representing the separate inventories for each product) as opposed to a single resource in their setting. Moreover, unlike Talluri and van Ryzin (2004b), our formulation also involves customized sets offered to each customer type. Lemma 1 is helpful in the numerical study as it allows us to reduce the size of the action space from \(2^N\) to \(N\). Because the sets \(A_k(y)\) are nested, in each period there is a product \(i_1\) that is offered to any arriving customer.

In the remainder of this section, we develop structural results of the optimal policy for some specific settings. We first consider the case in which the firm is not able to segment its customer base or to observe the segment of an incoming customer.

**Proposition 1** Consider a setting in which it is not possible to segment customers or to identify the segment of arriving customers. Then, in every period, it is optimal to offer any arriving customer all products with positive inventory.
Because the firm is unable to identify the preference profile of arriving customers, all future customers have the same preference in expectation, so the firm cannot benefit from reserving a product in anticipation of future sales. Thus, it is optimal to offer all products. This result emphasizes the requirement that the retailer have the ability to identify different customer types upon arrival. We next provide two properties that apply in general settings.

**Lemma 2** If the inventory level of product $i$ in period $t$ is large relative to the remaining time-horizon, i.e., $y_i \geq T - t + 1$, then it is optimal to offer this product to all customers. Moreover, if a single product is available, then it is optimal to offer this product to all arriving customers.

Assortment customization is more relevant when there is heterogeneity in the customer base and when there is a limited amount of inventory of some or all of the available products. To elaborate on the factors that induce rationing, we next present two results in stylized settings that highlight different aspects of rationing in our context. Namely, Proposition 2 and the subsequent example study the effects of limited inventory and overlapping customer preferences on the optimal policy. Proposition 3 examines which products are more likely to be rationed and which customer segments are more likely to face rationing. We first consider a situation in which customer demands are symmetric (this condition is formalized in the statement of Proposition 2 below).

**Proposition 2** Consider a setting with $N$ products and $M$ customer segments, but with symmetric product demands. Specifically, assume that

$$\sum_{m \in \mathcal{M}} \rho_m q_{ml}(S) = \sum_{m \in \mathcal{M}} \rho_m q_{ml'}(S')$$

for any two subsets $S, S'$ and any two products $l, l'$, with $l \in S, l' \in S'$ (or $l = l' = 0$), and $|S| = |S'|$.

Then, for an arriving customer of segment $m$ in period $t$, there exists a threshold $y^*_{ml}(y)$ such that product $i$ is in the assortment offered to this customer if $y_i \geq y^*_{ml}(y)$ – i.e., it is optimal to offer each arriving customer those products with inventory levels higher than the threshold value. The same policy is optimal if $\theta_{mi} = \theta$ for all $m \in \mathcal{M}$ and all $i \neq 0$, but the preference for the no-purchase option is different across segments. Moreover, in this case, if $\theta_{m_10} \geq \theta_{m_20}$, then the optimal assortment offered to segment $m_2$ is a subset of that offered to segment $m_1$.

This result indicates that, when product demands are symmetric, the optimal dynamic assortment policy is a threshold policy. Products low in inventory tend to be rationed to more customer
segments. In that way, the firm reserves the more inventory-constrained products for potential future customers that would not make a purchase if those products were not available. The condition in Proposition 2 ensures that aggregate demand for each product is the same, regardless of the specific set of products in the assortment (i.e., all products in any two equal-sized assortments attract the same demand). For example, consider a setting with $N$ products and $N$ customer segments with $\rho_m = \frac{1}{N}$ for all $m$. Customer segment $m = 1, ..., N$, has a preference vector with $\theta_{mm} = \theta$ and $\theta_{mj} = \theta'$ for $j \neq m$. One can show (using similar arguments as in Proposition 1) that the thresholds $y_{mt}(\theta') = 0$ when $\theta' = 0$ or $\theta' = \theta$, i.e., when there is either no overlap (each segment only likes a single product) or full overlap (all customers like all products equally). In those cases, assortment customization has no impact on revenue. That is, rationing occurs due to customer heterogeneity, but it has a meaningful revenue impact when there is some overlap in the customer preferences. Otherwise, customers segments do not interact with each other and assortment customization does not create much value. As exhibited in Figure 1 below, the benefit associated with assortment customization relative to the myopic policy is not monotone in the preference values $\theta'$. The impact is highest when there is some overlap between preferences, but the segments are still quite distinct regarding their product preferences. Proposition 2 also shows that a threshold policy is optimal if segments only differ in terms of the value they assign to the outside option. In that case, segments with a relatively higher value for the outside option (indicating a higher probability of no purchase) are offered a relatively larger assortment.

![Figure 1: Revenue impact of assortment customization as a function of $\theta'$. Parameters: $T = 50$, $N = 4$, $I_1 = 17$, $I_2 = 11$, $I_3 = 3$, $I_4 = 2$, and symmetric demands with $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.25$, $\theta = 10$, $0 \leq \theta' \leq 10$.](image)
We next focus on characteristics of the segments’ preferences that tend to induce rationing. To that end, we focus on a simple scenario in which inventory levels are all equal and set to \( y_i = 1 \) for all \( i \in \mathcal{N} \), to reflect an environment with limited inventory – if inventory was not scarce, rationing would not be optimal. Customer segments are nested in the sense that they increasingly incorporate more products in their consideration sets.

**Proposition 3** Consider a setting with \( N \) products and \( M \) customer segments. Suppose that product preferences for segment \( m \) are given by \( \theta_{mj} = \theta \) for \( j \leq m \) and \( \theta_{mj} = 0 \) for \( j > m \) (that is, a segment \( m \) customer has equal preference for products \( \{1, \cdots, m\} \) and does not consider any product outside this set), and arbitrary \( \theta_0 \). Suppose that the inventory levels in period \( t \) are \( y_1 = y_2 = \cdots = y_N = 1 \). Then, an arriving customer of segment \( m \) is offered an assortment of the form \( \{k_m, \cdots, m\} \) with \( k_m \leq m \). Moreover, \( k_m \) is non-decreasing in \( m \).

Proposition 3 shows that customers with larger consideration sets are rationed more products than customers with a narrower preference for products (relative to their preference sets). Moreover, it is more likely to ration the more sought-after products because customers with a larger consideration set may purchase one of the other products in the offered assortment. The result indicates that if product \( k \) is not offered to segment \( m \), then that product is not offered to segment \( m + 1 \) either. Moreover, the result suggests that the interplay of demand-supply conditions – and not just the inventory levels – is relevant to the revenue impact of assortment customization.

We next present the main result of the paper, which characterizes the optimal dynamic assortment policy in a setting with two products and two customer segments. The preference vectors for customer segments 1 and 2 are given by \( (\theta_{11}, \theta_{12}) \) and \( (\theta_{21}, \theta_{22}) \), respectively. The scenarios of interest are those in which the range of preferences for the two segments have some intersection, i.e., \( \theta_{11} \geq \theta_{21} \) and \( \theta_{22} \geq \theta_{12} \).

**Theorem 1** Consider a setting with \( N = M = 2 \) and preference vectors with \( \theta_{11} \geq \theta_{21}, \theta_{22} \geq \theta_{12}, \) and \( \theta_{11} - \theta_{12} \geq |\theta_{21} - \theta_{22}| \). If both products are available in period \( t \), then:

(i) Product \( i \) is always in the optimal assortment set for segment \( i \), i.e., \( i \in \mathcal{S}_{it}^* (y) \).

(ii) For a segment \( i \) customer, there exists a threshold level \( y_{ij}^*(y_i) \) such that:

If \( y_j \geq y_{ij}^*(y_i) \), then \( \mathcal{S}_{it}^* (y) = \{1, 2\} \); If \( y_j < y_{ij}^*(y_i) \), then \( \mathcal{S}_{it}^* (y) = \{i\} \).
Theorem 1 shows that the optimal policy involves inventory rationing. This policy is characterized by a set of thresholds $y^*_i$ and $y^*_j$ in each period $t$. A customer from segment $i$ is always offered product $i$ in the optimal policy (product $i$ is segment $i$’s more preferred product for $i = 1$, but not necessarily for $i = 2$). Furthermore, a customer from segment $i$ is also offered product $j$ if the inventory level of that product is large enough. Otherwise, the customer is only offered product $i$. If the inventory level of product $j$ is low, then it may be optimal to reserve that inventory for future arriving customers of segment $j$ as those customers are more likely to leave without purchasing any product if product $j$ is not available. If both segments have the same order of preference for the products (i.e., if $\theta_{11} > \theta_{12}$ for $i = 1, 2$), then only the segment with a relatively stronger preference for product 1 (as measured by $\theta_{11} - \theta_{12}$) is guaranteed to have that product in the optimal assortment. In this case, a customer from the other segment is offered product 1 only if this product’s inventory level is high enough. In sum, rationing is driven by the relative strength of product preferences for each customer segment (and not simply by the order of their preferences). We next present an example to further expand on the intuition behind this result.

**Example 1** Consider the second to last period before the end of the horizon, i.e., period $t = T - 1$, and let $y = (y_1 = 1, y_2 = 2)$. Suppose that a customer from segment 2 arrives in period $T - 1$ and consider the realization of her preferences for the three options – purchase product 1, purchase product 2, or no purchase. If the customer’s highest realized preference is either to purchase product 2 or the no-purchase option, then rationing product 1 has no effect on current or future revenues. Consider the case where the customer’s first (realized) choice is product 1. If the offered assortment is $S_{2,T-1} = \{1, 2\}$, then the customer buys product 1, yielding a revenue of $p + V_T (0, 2)$. If the offered assortment is $S_{2,T-1} = \{2\}$, then the customer either buys product 2 (with probability $\frac{\theta_{22}}{\theta_0 + \theta_{22}}$), yielding a revenue of $p + V_T (1, 1)$, or chooses the no-purchase option, yielding a revenue of $V_T (1, 2)$. Note that $V_T (1, 2) = V_T (1, 1)$ and $V_T (0, 2) = V_T (0, 1)$, because the firm can sell at most one unit in the last period. Hence, the assortment $S_{2,T-1} = \{1, 2\}$ yields a revenue of $p + V_T (0, 1)$ and $S_{2,T-1} = \{2\}$ yields a revenue of $p \frac{\theta_{22}}{\theta_0 + \theta_{22}} + V_T (1, 1)$. With $S_{2,T-1} = \{2\}$, the firm loses some revenue in the current period, but it gains $V_T (1, 1) - V_T (0, 1)$ in the last period. Thus, depending on the choice probabilities and the size of the segments, $S_{2,T-1} = \{2\}$ could be the optimal solution – implying rationing product 1. This is the case, for example, when $\theta_{22}$ is high relative to $\theta_0$ or
when segment 1 is relatively large ($\rho_1$ is close to one). In such cases, it is optimal to direct the segment 2 customer to buy product 2, reserving product 1 for the last period. □

To conclude the characterization of the optimal policy in this setting, we discuss monotonicity properties of the threshold levels introduced in Theorem 1.

**Proposition 4** Consider a setting with $N = M = 2$ and preference vectors as in Theorem 1. The policy thresholds satisfy the following properties.

(i) The threshold $y^*_j(y_j)$ is increasing in $y_j$.

(ii) The threshold levels are decreasing in $t$.

Part (i) of Proposition 4 implies that, given inventory levels $y_i$ and $y_j$, if product $i$ is offered to segment $j$, then it is also optimal to offer product $i$ to segment $j$ for lower inventory levels of product $j$. Thus, for a given inventory level of product $i$, it is more likely to offer that product to an arriving customer of segment $j$ when the inventory level of product $j$ is low. In this case, it may be optimal to re-direct some segment $j$ customers to purchase product $i$ in order to reduce the likelihood of running out of stock of product $j$ in the future. On the other hand, when the inventory level of product $j$ is high, it may be optimal to offer only product $j$ to an arriving customer of segment $j$, thereby increasing the demand for that product. Part (ii) suggests that towards the end of the selling season, there is more incentive to offer both variants to any arriving customer. The threshold levels decrease as time increases, indicating that less rationing occurs as time approaches the end of the selling season. This occurs because there is less concern about future sales and therefore it is optimal to sell as many units as possible.

We next explore a general setting with multiple products and multiple customer segments. Because this problem is intractable due to the combinatorial aspect of assortment customization, we propose two heuristics. These heuristics are used in the case study discussed in Section 6.

5. Heuristics

In this section, we propose two heuristics for the general problem. These heuristics build on the dynamic program studied in Section 4. Recall that the optimality equation in period $t$ is given by

$$V_t(y|m) = \max_{S \subseteq S(y)} \left\{ \sum_{i \in S} \lambda q_m(S)(p - \Delta_{i+1}(y)) \right\} + V_{t+1}(y).$$
The goal of these heuristics is to approximate $\Delta_{i+1}(y)$ to solve the maximization problem efficiently. To that end, we first consider an approximation of the marginal expected revenue generated by the $y_i$-th unit of inventory of product $i$ in period $t+1$ by the marginal value of a newsvendor function that takes into account the potential future demand for this product and captures the substitution dynamics between products (i.e., considers spillover demand when products are out of stock). In particular, when a product is sold out, a portion of the excess demand substitutes the unavailable product by one that is in stock. Let $y^t$ be the vector of inventory levels available at time $t$. We define

$$d^t_j = \lambda \sum_{m \in M} \rho_m \frac{\theta_{mj}}{\sum_{k \in S(y^t) \setminus \{i\}} \theta_{mk} + \theta_0},$$

and let $D^t_j$ be a Poisson distributed random variable with rate $d^t_j \ast (T - t)$. Following the problem studied in Netessine and Rudi (2003), we model the first-order substitution demand (or effective demand) in period $t$ as $D^t_{i|} = D^t_i + \sum_{j \in S(y^t) \setminus \{i\}} \alpha_{ij} (D^t_j - y^t_j)^+$. The coefficient $\alpha_{ij}$ approximates the fraction of customers that may substitute product $i$ by product $j$ if product $i$ is not available.

We define these fractions as:

$$\alpha_{ij} = \sum_{m \in M} \rho_m \left( \frac{\theta_{mj}}{\sum_{k \in S(y^t) \setminus \{i\}} \theta_{mk} + \theta_0} \right) \text{ for all } j \neq i.$$

The quantity in parenthesis is the proportion of customers in segment $m$ that chooses product $j$ from the set of products in $S(y^t) \setminus \{i\}$, i.e., the set of products available at time $t$ when product $i$ is excluded from the assortment. The approximate marginal expected revenue is given by the derivative of $p \ast \sum_{i \in S} E \left[ \min\{D^t_{i|}, y^t_i\} \right]$ with respect to $y^t_i$, i.e., the marginal value of the $y^t_i$-th unit of inventory in a multi-product newsvendor model with substitution. That is,

$$\Delta^i_{t+1}(y^t) = p \ast \left[ Pr \left( D^t_{i|} > y^t_i \right) - \sum_{j \neq i} \alpha_{ij} Pr \left( D^t_{j|} \leq y^t_j, D^t_i > y^t_i \right) \right]. \tag{3}$$

The first term in the squared brackets accounts for the probability that the effective demand of product $i$ (primary demand plus substitution demand from other out-of-stock products) is greater than the current inventory level. The second term accounts for the probability that demand for product $i$ in excess of the available inventory of that product will be served by inventory of other products in stock.

---

3 Similar demand substitution models have been discussed in Lippman and McCardle (1997), Netessine and Shumsky (2005), and Zhao and Atkins (2008), among others.
Using this approximation, and given a vector of inventory levels \( y \), we solve

\[
\max_{S \subseteq \mathcal{S}(y)} \left\{ \sum_{i \in S} \lambda q_{m_i}(S)(p - \Delta_{i+1}^t(y)) \right\},
\]

using the result in Lemma 1, for \( \Delta_{i+1}^t(y) \) defined as in (3). We refer to this heuristic as \( \text{Sub}_t \). As we report in Section 6, this heuristic is very effective.

The second heuristic is a variant of \( \text{Sub}_t \), in which we approximate the expected marginal revenue as follows:

\[
\Delta_{i+1}^t(y^t) = p^t \left[ Pr(D_{i}^{0S} > y_{i}^t) - \sum_{j \neq i} \alpha_{ij}^t Pr(D_{j}^{0S} \leq y_{j}^t, D_{i}^0 > y_{i}^0) \right],
\]

where \( y_{j}^0 \) is product \( j \)'s inventory level at the beginning of the selling season, \( D_{j}^0 \) is a Poisson distributed random variable with rate \( d_{j}^0 \cdot T \), and

\[
D_{j}^{0S} = D_{i}^0 + \sum_{j \neq i} \alpha_{ji}^0 (D_{j}^0 - y_{j}^0)^+, \quad D_{j}^{0S} = D_{i}^t + \left( \frac{T-t}{T} \right) \sum_{j \in \mathcal{S}(y^t) \setminus \{i\}} \alpha_{ji}^t (D_{j}^0 - y_{j}^0)^+.
\]

We refer to this heuristic as \( \text{Sub}_0 \). Unlike \( \text{Sub}_t \), under which the approximate \( \Delta_{i+1}^t(y^t) \) depends on the \textit{entire} vector of inventory levels in period \( t \), the approximate marginal expected revenue under \( \text{Sub}_0 \) depends on \( y^t \) only through the set \( \mathcal{S}(y^t) \) of products with positive inventory level in period \( t \). This enables significant pre-processing (prior to initiating the heuristic), resulting in shorter running times. This heuristic is also extremely effective, as demonstrated in Section 6. Moreover, we next show that the heuristic \( \text{Sub}_0 \) results in a threshold-type policy, like the optimal policy for two products derived in Theorem 1.

**Proposition 5** The dynamic assortment policy that emerges under the heuristic \( \text{Sub}_0 \) is a threshold-type policy, i.e., in every period \( t \) and for each product \( i \) and customer segment \( m \), there exists a threshold \( y_{it}^m(y) \) such that product \( i \) is in the assortment offered to an arriving customer of segment \( m \) if \( y_i \geq y_{it}^m(y) \). Moreover, \( y_{it}^m(y) \) is increasing in \( y_j \) for \( j \neq i \).

The proof of this result follows from a similar logic to that used in the proof of Theorem 1. In the heuristic \( \text{Sub}_t \), the approximate marginal value depends on the entire vector of inventory levels. Under this heuristic, one can show that the effective marginal price of each product \( j \) increases with the inventory level of product \( i \), i.e., \( p_{i}^t(y) \leq p_{j}^t(y + \varepsilon_i) \) for all \( j \). However, a threshold-type result requires a stronger condition that involves the joint effect of a change in the inventory level.
of product $i$ on the effective marginal prices of all products. This threshold-type result holds for the optimal policy in the case of two products as the dynamic program takes the full substitution effect into account.

6. Value of Dynamic Assortment Customization

In this section, we demonstrate the potential impact of assortment customization on revenue with a case study. We measure the impact on revenue of employing the optimal policy by calculating the percentage revenue increase due to assortment customization relative to a benchmark (myopic) policy under which all available products are offered to any arriving customer. We also assess the efficiency of the heuristics developed in Section 5.

Beymen is a privately held upscale fashion retailer in Turkey. The retailer operates 21 department stores and mono-brand stores in Turkey, and an online sales channel. Beymen places orders from the wholesale arms of high-end fashion brands with firm purchasing commitments six to eight months in advance of the beginning of each season. There are two main buying cycles every year, resulting in two distinct selling seasons: Fall-Winter and Spring-Summer. There is little to no replenishment opportunity during the season for most of the brands, and leftover inventory at the end of the season is liquidated at outlet stores.

The company’s buyers make product assortment and chain-level inventory decisions (at brand level and at style level) to minimize the occurrence of stockouts and the cost of leftover inventory. However, demand, especially at the style level, is highly uncertain and inventory imbalances start to emerge after the first few weeks of the season. Because the markdown period starts only at the end of the sixth month of the season, the merchandising team reviews the remaining inventory-to-demand ratios every week and manages demand-inventory imbalances using various methods. One method involves increasing visibility of some product groups in the online channel, either by showing them more prominently on opening pages or by assigning higher rankings in customer search results. We next provide details on the data used for this case study.

1. **Data:** We have country-wide weekly item-level sales (units and prices) and inventory data for the top seven women’s shoe brands (Christian Louboutin, Miu Miu, Prada, Valentino, YSL, Tory Burch, Tod’s) from the Fall-Winter 2011-2012 season. The season starts in July.
2011 and ends in February 2012. The markdown period starts around January 1st, 2012. Because there are significant price differences across products, we rank all styles with respect to their prices and create three price clusters that contain styles with similar prices. Average prices within clusters are $1000, $500 and $200, respectively. It is reasonable to assume that there is no substitution across products from different price classes.

2. **Segmentation:** A customer segmentation approach used by the company is based on the customers’ location. When a customer logs in to the firm’s website, the location of the customer is identified either through the customer login information or the IP address. We consider three location-based segments: the more affluent neighborhoods of Istanbul (Segment 1), other neighborhoods in Istanbul (Segment 3), and other cities in Turkey (Segment 2); see Figure 2 below. Figure 3 illustrates the preferences of the three segments for the top four selling shoe styles (referred to by their SKU numbers) for price group 3 (the highest-priced styles). We estimate the segment probabilities from the relative volume of sales for each location, based on historical data. In this example, we have $\rho_1 = 0.57, \rho_2 = 0.28, \rho_3 = 0.15$.

![Figure 2: Three customer segments identified by their location](image)

3. **Demand estimation:** To simulate a setting in which our model could be applied, we suppose that we are in a given week $w$ into the selling season and estimate demand for the remainder of the season. Demand from early season can be used to compute high-quality estimates for full-season demand, as demonstrated by Fisher and Raman (1994). We estimate seasonality from the total shoe sales across the chain. Let $x(w)$ denote the share of week $w$’s demand to full-season demand, which can be estimated from the previous year’s same season data. We
denote demand of item $k$ from segment $j$ up to period $w$ by $d_{jk}(w)$ and the estimate of demand from week $w$ until the end-of-season $W$ as $\bar{d}_{jk}(w,W)$, or simply as $\bar{d}_{jk}$ when the relevant time periods are clear from the context. At time point $w$, $d_{jk}(w)$ can be observed from data and $\bar{d}_{jk}(w,W) = \alpha(w)d_{jk}(w)$, where $\alpha(w)$ is the ratio of demand after week $w$ to demand up to week $w$ and is given by $(\sum_{i=w}^{W} x(i)) / (\sum_{i=1}^{w-1} x(i))$. The season starts on week 25 of 2011 and all products are offered at regular prices until week 53. We set $w$ equal to the 44th week of the year. Given the demand estimates for each product-segment combination, we estimate the segment’s choice probabilities (preferences) following the analysis in Anderson et al. (1992). For further discussion on the identification of latent customer segments and estimation of their preferences, we refer to Gupta and Chintagunta (1994) and Wedel and Kamakura (1998).

4. **Inventory:** Inventory of product $k$ at time $w$ is denoted by $I_k(w)$ and can be observed from the data provided by Beymen. Figure 3 shows the remaining inventory over time for four styles of shoes (the curves are scaled by dividing the inventory of each product in each week by the initial inventory level for that product). Note the marked imbalance of inventory levels throughout the selling season – one product stocks out early in the season while others experience a relatively constant rate of sales.

![Figure 3: Preference distribution by location and inventory depletion curves for four styles of shoes.](image)

Two key ideas emerge from this dataset, which are more broadly applicable to fashion retailing in general: (1) Some form of segmentation is possible and segments’ preferences can be easily estimated from the sales data (as shown in Figure 3, these preferences can be markedly different
across segments); (2) Significant demand-inventory imbalances across products tend to develop within the season due to the uncertainty of product preferences (as illustrated in Figure 3). In what follows, we demonstrate the potential impact of assortment customization by estimating the parameters of our model based on the company’s data and by comparing the expected profit from assortment customization with the benchmark policy that involves offering all available products to any arriving customer. We begin with an example with two products and three customer segments that illustrates the magnitude of the benefits associated with assortment customization.

**Example 1**  We consider an example with two products. Their SKUs are 100486728 and 100484123 (see Figure 3). We can calculate reliable estimates of popularity of all items on week 44. We set $q_0 = 0.2$, $\theta_0 = 1$, and solve for the set of preferences $\theta_{jk}$ for all $j, k$, to satisfy the demand ratio equations and the given no purchase probability, i.e.,

$$\frac{\theta_{jk}}{\sum_{i=1}^{2} \theta_{ji}} = \frac{d_{jk}}{\sum_{i=1}^{2} d_{ji}}, \quad j = 1, 2, 3, \quad k = 1, 2,$$

and

$$\frac{\theta_0}{\theta_0 + \sum_{i=1}^{2} \theta_{ji}} = q_0 = 0.2, \quad j = 1, 2, 3,$$

where the $d_{jk}$ estimates are obtained from the sales data. We have $\sum_{j=1}^{3} d_{j1} = 44$, $\sum_{j=1}^{3} d_{j2} = 34$, and $I_1 = 70$, $I_2 = 13$. The former represent estimates of demand for each product across segments from week 44 to week 53 (as explained earlier, we use the data during the initial weeks to construct these demand estimates). The latter represents the available inventory of products 1 and 2, respectively, at the beginning of week 44. We set $T = 100$ and $\lambda = 1$ for the dynamic program, which means that $T(1 - q_0) = 80$ customers are expected to make a purchase, roughly matching the total estimated demand for the remaining weeks in the season.

Based on the data from Example 1, Figure 4 below shows the percentage increase in revenue due to assortment customization as a function of the inventory levels. For the specific inventory combination on week 44 from the company’s data, i.e., $I_1 = 70$, $I_2 = 13$, adopting assortment customization would result in a 2.78% increase in revenue relative to the myopic policy. It can also be seen from the graph that when the inventory levels of both products are large, the percentage revenue impact is small. In that case, the retailer is less likely to run out of stock, so there is no need for rationing inventory. Similarly, when inventory levels of both products are low relative to the time horizon (low values of $y_1$ and $y_2$ in the graph), there is enough time to sell all units, so both products are likely to be offered in the optimal solution, and again the revenue impact is
small. The revenue impact is more significant in the areas where the inventory level of one product is relatively high while the inventory level of the other product is relatively low (a maximum of 3.04% in this example). Because retailers usually operate with low profit margins, even a small percentage increase in revenue can have a significant impact on profit.

Figure 4: Percentage revenue increase due to assortment customization relative to the benchmark policy that involves offering all available products to all customers. Parameters: $T = 100, p = 1, \lambda = 1, \rho = (0.57, 0.15, 0.28), \Theta_1 = (2.3, 1.9), \Theta_2 = (3.1, 1.2), \Theta_3 = (0.8, 3.4), \theta_0 = 1.$

To focus on a richer data set, we next discuss a case study that considers all possible combinations of four products chosen among the top 20 selling shoe styles across the chain (the top 20 products are determined based on sales until week 44 of the year 2011).

**Case Study 1** The case study consists of a total of 4,845 combinations. Each combination represents an experiment with four products and three segments. Total estimated future demand (from week 44) is given by $D = \sum_{j=1}^{3} \rho_j \sum_{k=1}^{4} \bar{d}_{jk}$ and inventory levels at week 44 are given by $I_k$ for $k = 1, \ldots, 4$. We define the load factor as the ratio of total demand to total inventory (available on week 44), given by $D/\sum_{k=1}^{4} I_k$. We focus on the 1,974 cases in which the load factor is between 0.8 and 1.3. We set $q_0 = 0.1, \theta_0 = 10,$ and solve for the set of preferences $\theta_{jk}$ for all $j, k,$ to satisfy the demand ratio equations and the given no purchase probability,

$$\frac{\theta_{jk}}{\sum_{i=1}^{4} \theta_{ji}} = \frac{\bar{d}_{jk}}{\sum_{i=1}^{4} \bar{d}_{ji}}, \quad j = 1, 2, 3, \quad k = 1, 2, 3, 4$$

and

$$\frac{\theta_0}{\theta_0 + \sum_{i=1}^{4} \theta_{ji}} = q_0 = 0.1, \quad j = 1, 2, 3,$$
where the $\bar{d}_{jk}$ estimates are obtained based on sales data up to week 44. Note that $q_0$ is difficult to estimate as the no-purchase alternative is not observed (see for example, Kök and Fisher 2007 and Vulcano et al. 2012). In general, $1 - q_0$ determines the probability to accept a substitute, which has been shown to be around 40% for supermarket categories (Gruen et al. 2002) and is suspected to be much lower for online retail settings. We set $T = 50$ for the dynamic program, implying an average of 45 potential buyers in the remaining weeks of the season. Since $D$ may well exceed 45, we re-scale inventories to match the scaling of demand, i.e., $\hat{I}_k = \frac{T(1-q_0)}{D}I_k$ for $k = 1, ..., 4$. (If $D$ is equal to 45, then inventory levels remain the same; otherwise, inventory and demand are both scaled down in the same proportion.) Note that the load factor remains the same before and after the re-scaling described above.

For the experiments in this case study, we compute the optimal assortment policy by solving the dynamic program and making use of the result in Lemma 1. Figure 5 below shows the percentage increase in revenue due to assortment customization as a function of the load factor. The impact of assortment customization is non-negligible when the load factor is close to 1. The average improvement over the benchmark policy across the 1,974 experiments is 0.21%, with a maximum of 4.92%. In particular, 231 cases yield an impact higher than 0.5% and 102 cases yield an impact higher than 1%. The experiment yielding the maximum improvement has the following parameters. $\rho = (0.74, 0.2, 0.06), \theta_1 = (43, 18, 17, 12), \theta_2 = (70, 14, 3, 3), \theta_3 = (57, 8, 16, 8)$, and $\hat{I} = (12, 4, 28, 1)$. In this example, while the first product is the most popular in all three segments, the segments differ in terms of the strength of their preference for the first product and their choice of the second most preferred product.

To measure the effect of segment sizes on the benefits of assortment customization, we define the following segment distance metric: $\rho^{\text{dist}} = \sum_{j=1}^3 |\rho_j - 0.333|$. A lower $\rho^{\text{dist}}$ value indicates a more heterogeneous population with three relatively equal-sized segments. A higher value of $\rho^{\text{dist}}$ implies a less heterogeneous population concentrated primarily in one large segment (e.g., an extreme scenario would be $\rho = (1, 0, 0)$). We then group the 1,974 experiments into three equal sized groups with low, medium, and high values of the metric $\rho^{\text{dist}}$. In this data set, segment 1 is generally larger than the other two segments combined (with a minimum value of $\rho_1 = 0.54$). The minimum, average, and maximum $\rho^{\text{dist}}$ values are 0.41, 0.86, and 1.25, respectively. Figure
6 reports the average percentage gain from assortment customization for each group, controlling for the load factor. Although the net impact depends on the relationship between the sizes of the segments and the preference vectors of each segment, Figure 6 clearly suggests that the impact of assortment customization is highest in settings with more heterogeneous populations. This finding makes sense, as there would be no benefit to assortment customization if all customers belonged to the same segment (as in the extreme scenario with $\rho = (1, 0, 0)$).

The actual scale of demand and inventory levels is generally higher than that which can be handled in a dynamic program. In the next case study, we demonstrate the impact of assortment customization in settings with larger $T$ by evaluating the benchmark (myopic) policy and the
heuristics introduced in Section 5 using simulation.

**Case Study 2** We select a subsample of 100 experiments from Case Study 1 (in which \( N = 4 \)) with load factors between 1 and 1.3. We have run two sets of experiments with this subsample (for a total of 200 experiments), namely: (i) \( T = 90 \) and \( q_0 = 0.5 \); (ii) \( T = 180 \) and \( q_0 = 0.75 \). In addition to these experiments with \( N = 4 \), we consider all possible combinations of 10 products chosen among the top 20 selling shoe styles and select a subsample of 100 experiments with load factors between 1 and 1.1. We have also run two sets of experiments with this subsample (for another 200 experiments), with: (i) \( T = 90 \) and \( q_0 = 0.5 \); (ii) \( T = 180 \) and \( q_0 = 0.75 \).

For these 400 experiments, we run the myopic policy, the two heuristics introduced in Section 5, i.e., Sub, and Sub, and a heuristic proposed in Golrezaei et al. (2012), which we denote by GNR. The GNR heuristic is based on a penalty function that uses supply information (i.e., information about the prevailing inventory levels) to determine the set of products to offer to each arriving customer. The heuristic achieves the best worst-case asymptotic performance (as \( T \to \infty \)) within the class of single-variable decreasing penalty functions. The advantage of the GNR heuristic is that it does not require demand forecasting. On the other hand, unlike the optimal policy and the heuristics Sub and Sub, the GNR heuristic ignores demand-supply imbalances across products and the substitution dynamics between products. Table 1 reports the results corresponding to the two heuristics developed in Section 5 and to the GNR heuristic, for the experiments described in Case Study 2. The numbers reported in Table 1 represent the percentage revenue improvement over the myopic benchmark policy \( M \) that offers all products to any arriving customer.

The results in Table 1 suggest that the heuristics introduced in Section 5 lead to significant revenue improvement (including settings with a large number of products and long time-horizon), provided that the load factors are around 1. Moreover, the heuristic Sub dominates the other heuristics both in terms of the average and maximum revenue improvement.

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<th>N</th>
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<td>0.23%</td>
<td>1.34%</td>
<td>0.21%</td>
</tr>
<tr>
<td>10</td>
<td>90</td>
<td>0.5</td>
<td>-0.04%</td>
<td>0.66%</td>
<td>-0.55%</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>0.75</td>
<td>0.15%</td>
<td>0.98%</td>
<td>0.13%</td>
</tr>
</tbody>
</table>

Table 1: Impact of Assortment Customization for Case Study 2
7. Conclusion

We consider a retailer with limited inventory of substitutable products in a category with equal selling prices. The retailer faces a heterogeneous customer base, consisting of multiple segments characterized by different preferences for the products. The retailer can identify the segment of each arriving customer and therefore offer a customized assortment based on that segment’s preferences. We formulate this problem as a dynamic assortment customization problem where the assortment decision depends on the inventory levels of the products, the time remaining in the selling season, the segment of the arriving customer, and the distribution of preferences of future customers.

If the retailer is not able to identify the segment (type) of an arriving customer, then it is optimal to offer all available products to any arriving customer. In contrast, when the retailer has the ability to identify customer types, it may be optimal to ration products to some customers – even if the products’ prices are equal. This is a novel result that applies when the following factors are present: heterogeneous segments, limited inventory, and multi-period dynamics. For a setting with two products and two segments, we show that it is optimal to follow a threshold policy under which it is optimal to offer a product to a customer segment only if the inventory level of that product is higher than a threshold level. Otherwise, the product is not offered to that segment in order to reserve those units for future customers who may have a stronger preference for that product. We show that the threshold levels are increasing in the other product’s inventory level and decreasing in time, making customization (and rationing) less likely as the system runs out of inventory or time to sell the products.

We introduce two heuristics based on an approximation of the marginal expected revenue generated by each product and in each period in the dynamic program by the marginal value of a newsvendor (single-period) function that captures the substitution dynamics between products. These heuristics capture the interplay of demand-supply conditions for each product (and in each period) based on newsvendor-type approximations.

We demonstrate the potential revenue impact of assortment customization with a case study based on a high-end fashion retailer. This study reveals the potential impact of assortment customization by comparing the revenue derived from the optimal policy and the heuristics with that
of the myopic policy that involves offering all available products to any arriving customer. Specifically, in this case study, the revenue benefit from assortment customization can be as high as 5% relative to the benchmark policy – such revenue increases can have a significant impact on profit.

Our analytical results and the numerical demonstration of the value of assortment customization in the case study suggest that dynamic assortment optimization is another lever (in addition to pricing) for revenue maximization in online retailing and in other environments in which the identification and customization processes are feasible.

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References


Dynamic Assortment Customization with Limited Inventories

Online Appendix: Proofs

This appendix contains the proofs of all results in the paper.

Proof of Lemma 1. We prove that \( S_{mt}^* \in \{ A_1(y), \ldots, A_N(y) \} \). The optimality equation (2) can be written as

\[
V_t(y|m) = \max_{S \subseteq S(y)} \left\{ \sum_{i \in S} \lambda q_{mi}(S)p_i^t(y) \right\} + V_{t+1}(y).
\]

For a given vector of inventory levels \( y \), assume, without loss of generality, that \( p_i^t(y) \geq \cdots \geq p_i^{tN}(y) \) and \( S(y) = N \). Note that these effective marginal prices are the same for all customer segments. We prove the result by contradiction. Suppose that the arriving customer in period \( t \) belongs to segment \( m \). If \( S_{mt}^* \) is not one of the sets in \( \{ A_1(y), \ldots, A_N(y) \} \), then there exist \( i_k \) and \( i_l \) such that \( i_k \in S_{mt}^* \), \( i_l \in \N \setminus S_{mt}^* \), and \( p_i^t(y) > p_i^t(y) \). Because \( i_k \in S_{mt}^* \), we have \( p_i^t(y) \geq \sum_{j \in S_{mt}^*} q_{mj}(S_{mt}^*)p_i^t(y) \). (If this inequality did not hold, then \( 1 - q_{mk}(S_{mt}^*)p_i^t(y) < \sum_{j \in S_{mt}^* \setminus \{i_k\}} q_{mj}(S_{mt}^*)p_i^t(y) \), which implies \( p_{i_k}^t(y) < \sum_{j \in S_{mt}^* \setminus \{i_k\}} q_{mj}(S_{mt}^* \setminus \{i_k\})p_i^t(y) \). This, in turn, implies that the assortment \( S_{mt}^* \setminus \{i_k\} \) leads to a higher profit than that of \( S_{mt}^* \), resulting in a contradiction.) Thus, \( p_i^t(y) > p_i^t(y) \geq \sum_{j \in S_{mt}^*} q_{mj}(S_{mt}^*)p_i^t(y) = a/b \), where \( a = \sum_{j \in S_{mt}^*} \theta_{mj}p_i^t(y) \) and \( b = \sum_{j \in S_{mt}^*} \theta_{mj} + \theta_{m0} \). denote \( \theta \) and \( x \) as the preference and effective marginal price of product \( i_l \), respectively. Because \( (a + \theta x)/(b + \theta) \geq a/b \) for any positive \( \theta \) if and only if \( x \geq a/b \), and \( p_i^t(y) > \sum_{j \in S_{mt}^*} q_{mj}(S_{mt}^*)p_i^t(y) \), we have

\[
\sum_{j \in S_{mt}^* \cup \{i_l\}} q_{mj}(S_{mt}^* \cup \{i_l\})p_i^t(y) > \sum_{j \in S_{mt}^*} q_{mj}(S_{mt}^*)p_i^t(y).
\]

This implies that assortment \( S_{mt}^* \cup \{i_l\} \) is strictly better than \( S_{mt}^* \), which is a contradiction. ■

Proof of Proposition 1. For simplicity, we show the result for the case \( N = 2 \). The same arguments apply to the case of an arbitrary number of products. We prove the result by induction, using a sample-path argument. In period \( T \), it is clearly optimal to offer all products with positive inventory. Suppose the same is true for all periods \( t+1, \ldots, T \). Consider now period \( t \) with starting
inventory levels \((y_1, y_2)\). We compare the expected revenue obtained by offering the assortment \(\{1, 2\}\) versus the assortment \(\{1\}\) (without loss of generality). Let us denote by \(A_t\) the assortment offered in period \(t\). If the realized preference of the customer arriving in period \(t\) (if any) is either \([1, 2, 0]\), \([1, 0, 2]\), \([0, 1, 2]\), or \([0, 2, 1]\), then both assortments result in identical expected revenue. The notation \([#1, #2, #3]\) indicates that the customer first prefers product \(#1\), then product \(#2\), and then \(#3\), where 0 denotes the outside option. If 0 is preferred over another product (e.g., \([1,0,2]\)), then the customer does not purchase anything if a more preferred product (product 1 in the example) is not available. We next consider simultaneously the two remaining cases in which the customer’s realized preference is either \([2, 0, 1]\) or \([2, 1, 0]\). If \(A_t = \{1, 2\}\), then the resulting expected revenue under either realized preference is \(p + V_t(y_1, y_2 - 1)\). If \(A_t = \{1\}\), then the resulting revenue is \(V_t(y_1, y_2)\) if the realized preference is \([2, 0, 1]\) and \(p + V_t(y_1 - 1, y_2)\) if the realized preference is \([2, 1, 0]\). We now consider two scenarios that differ in the realization of preferences after period \(t\). Suppose first that \(y_1 - \tilde{y}_1\) units of product 1 and \(y_2 - 1\) units of product 2 are sold through period \(t' - 1 > t\), with \(1 \leq \tilde{y}_1 \leq y_1\). Then, if \(A_t = \{1, 2\}\), the revenue in period \(t' - 1\) is \(pB + V_{t'}(\tilde{y}_1, 0)\) under either realized preference of the customer arriving in period \(t\), where \(B = y_1 - \tilde{y}_1 + y_2\). If \(A_t = \{1\}\), then the revenue in period \(t' - 1\) is \(V_{t'}(\tilde{y}_1, 1)\) if the customer’s realized preference in period \(t\) is \([2, 0, 1]\), and \(pB + V_{t'}(\tilde{y}_1 - 1, 1)\) if the customer’s realized preference in period \(t\) is \([2, 1, 0]\). The next table shows the revenue in period \(t'\) for given realizations of preferences in periods \(t\) and \(t'\) and for a given assortment offered in period \(t\). (Note that we do not consider realizations in which the customer’s first preference is product 1, as there is still inventory of that product. We later analyze the case in which product 1 runs out of inventory first.)

<table>
<thead>
<tr>
<th>Realized preference in (t)</th>
<th>Realized preference in (t')</th>
<th>Revenue in (t') if (A_t = {1, 2})</th>
<th>Revenue in (t') if (A_t = {1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>([2, 0, 1])</td>
<td>([2, 0, 1])</td>
<td>(pB + V_{t'+1}(\tilde{y}_1, 0))</td>
<td>(pB + V_{t'+1}(\tilde{y}_1, 0))</td>
</tr>
<tr>
<td></td>
<td>([2, 1, 0])</td>
<td>(p(B + 1) + V_{t'+1}(\tilde{y}_1 - 1, 0))</td>
<td>(pB + V_{t'+1}(\tilde{y}_1, 0))</td>
</tr>
<tr>
<td>([2, 1, 0])</td>
<td>([2, 0, 1])</td>
<td>(pB + V_{t'+1}(\tilde{y}_1, 0))</td>
<td>(p(B + 1) + V_{t'+1}(\tilde{y}_1 - 1, 0))</td>
</tr>
<tr>
<td></td>
<td>([2, 1, 0])</td>
<td>(p(B + 1) + V_{t'+1}(\tilde{y}_1 - 1, 0))</td>
<td>(p(B + 1) + V_{t'+1}(\tilde{y}_1 - 1, 0))</td>
</tr>
</tbody>
</table>

If the realized preferences in periods \(t\) and \(t'\) are \([2, 0, 1]\) and \([2, 0, 1]\), respectively, or \([2, 1, 0]\) and \([2, 1, 0]\), respectively, then the revenues are the same under both options for \(A_t\). If the realized preferences in periods \(t\) and \(t'\) are \([2, 0, 1]\) and \([2, 1, 0]\), respectively, then \(A_t = \{1, 2\}\) leads to a higher revenue than \(A_t = \{1\}\) and the difference is \(p - \Delta_{t+1}^L > 0\). Conversely, if the realized preferences in periods \(t\) and \(t'\) are \([2, 1, 0]\) and \([2, 0, 1]\), respectively, then \(A_t = \{1, 2\}\) leads to a
lower revenue than $A_t = \{1\}$ and the difference is again $p - \Delta^{1}_{\bar{t} + 1} > 0$. Because preferences are constant over time, the probability of observing either sample path is the same and therefore these differences cancel out when expectation is taken over all sample paths.

<table>
<thead>
<tr>
<th>Realized preference in $t$</th>
<th>Realized preference in $t'$</th>
<th>Revenue in $t'$ if $A_t = {1, 2}$</th>
<th>Revenue in $t'$ if $A_t = {1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, 0, 1]</td>
<td>[1, 0, 2]</td>
<td>$p(C + 1) + V_{t+1}(0, \bar{y}_2 - 1)$</td>
<td>$pC + V_{t+1}(0, \bar{y}_2)$</td>
</tr>
<tr>
<td></td>
<td>[1, 2, 0]</td>
<td>$p(C + 1) + V_{t+1}(0, \bar{y}_2 - 1)$</td>
<td>$pC + V_{t+1}(0, \bar{y}_2)$</td>
</tr>
<tr>
<td>[2, 1, 0]</td>
<td>[1, 0, 2]</td>
<td>$p(C + 1) + V_{t+1}(0, \bar{y}_2 - 1)$</td>
<td>$pC + V_{t+1}(0, \bar{y}_2)$</td>
</tr>
<tr>
<td></td>
<td>[1, 2, 0]</td>
<td>$p(C + 1) + V_{t+1}(0, \bar{y}_2 - 1)$</td>
<td>$p(C + 1) + V_{t+1}(0, \bar{y}_2 - 1)$</td>
</tr>
</tbody>
</table>

We now consider the second possible scenario. Suppose that the realization of preferences over time is such that $y_1 - 1$ units of product 1 and $y_2 - \bar{y}_2$ units of product 2 are sold through period $t' - 1$, with $1 \leq \bar{y}_2 \leq y_2$. Then, if $A_t = \{1, 2\}$, the revenue in period $t' - 1$ is $pC + V_{t'}(1, \bar{y}_2 - 1)$ under either realized preference of the customer arriving in period $t$, where $C = y_1 + y_2 - \bar{y}_2$. If $A_t = \{1\}$, then the revenue in period $t' - 1$ is $V_{t'}(1, \bar{y}_2)$ if the customer’s realized preference in period $t$ is $[2, 0, 1]$ and $pC + V_{t'}(0, \bar{y}_2)$ if the customer’s realized preference in period $t$ is $[2, 1, 0]$. The table above shows the revenue in period $t'$ for given realizations of preferences in periods $t$ and $t'$ and for a given assortment offered in period $t$. (Note that we do not consider realizations in which the customer’s first preference is product 2, as there is still inventory of that product. The case in which product 2 runs out of inventory first was analyzed earlier.)

Because $p - \Delta^{2}_{\bar{t} + 1} > 0$, we have that under the first three possible combinations of realized preferences shown in the table, the revenue under $A_t = \{1, 2\}$ strictly dominates that under $A_t = \{1\}$. The last combination of preferences leads to equal revenue in both cases. We conclude that, taking expectation over all possible sample paths, choosing $A_t = \{1, 2\}$ strictly dominates choosing $A_t = \{1\}$. Therefore, the seller is better off offering both products in period $t$, concluding the induction argument. Finally note that this proof holds for any choice model (in particular, a mixed MNL model) as long as the probability of making a sale increases with a broader assortment.

**Proof of Lemma 2.** If $y_i \geq T - t + 1$, then $p_i^t(y) = p \geq p_i^t(y)$ for all other products $j$. Therefore, it is optimal to offer product $i$ to any arriving customer.

**Proof of Proposition 2.** We prove the result by showing that $V_t(y - e_i) \geq V_t(y - e_j)$ for any vector of inventory levels with $y_1 \geq y_2 \geq \cdots \geq y_N$ and any $i < j$, as this implies that $p_i^t \geq p_i^t$, leading to the desired result. We first focus on the last period $T$. We know that in the last period,
it is optimal to offer all products with positive inventory. Let $i < j$, i.e., $y_i \geq y_j$. Let's prove that $V_T(y - e_i) \geq V_T(y - e_j)$ for all inventory vectors with $y_1 \geq y_2 \geq \cdots \geq y_N \geq 1$. Suppose first that $y_l > 1$ for all $l = 1, \ldots, N$. Then, $V_T(y - e_i) = V_T(y - e_j)$ since $T$ is the last period. If $y_i > 1$ but $y_j = 1$, then

$$V_T(y - e_i) = \lambda p \sum_{m \in M} \rho_m \left( \sum_{l=1}^{k} q_{ml}(S) \right) \quad \text{and} \quad V_T(y - e_j) = \lambda p \sum_{m \in M} \rho_m \left( \sum_{l=1,l \neq j}^{k} q_{ml}(S \setminus \{j\}) \right),$$

with $k = |S(y)| > j$ and $S = \{1, \ldots, k\}$. Clearly, $V_T(y - e_i) > V_T(y - e_j)$. Now suppose that $y_i = 1$ so that $y_j = 1$ as well. Let again $k = |S(y)| > j$, the number of items with positive inventory under the vector $y$. Then,

$$V_T(y - e_i) = \lambda p \sum_{m \in M} \rho_m \left( \sum_{l=1,l \neq i}^{k} q_{ml}(S \setminus \{i\}) \right) = \lambda p \sum_{l=1,l \neq i}^{k} \left( \sum_{m \in M} \rho_{ml}(S \setminus \{i\}) \right), \quad \text{and}$$

$$V_T(y - e_j) = \lambda p \sum_{m \in M} \rho_m \left( \sum_{l=1,l \neq j}^{k} q_{ml}(S \setminus \{j\}) \right) = \lambda p \sum_{l=1,l \neq j}^{k} \left( \sum_{m \in M} \rho_{ml}(S \setminus \{j\}) \right).$$

The assumption of symmetric demands implies that $\sum_{m \in M} \rho_{ml}(S \setminus \{i\}) = \sum_{m \in M} \rho_{ml}(S \setminus \{j\})$ for any two products $l \in S \setminus \{i\}, l' \in S \setminus \{j\}$ since $|S \setminus \{i\}| = |S \setminus \{j\}|$. Therefore, $V_T(y - e_i) = V_T(y - e_j)$. This proves the result for period $T$. Let us now assume that the result holds for period $t+1$, i.e., $V_{t+1}(y - e_i) \geq V_{t+1}(y - e_j)$ for any vector of inventory levels $y$ with $y_1 \geq y_2 \geq \cdots \geq y_N \geq 1$ and any $i < j$. We want to prove that $V_t(y - e_i) \geq V_t(y - e_j)$. From (1), we have

$$V_t(y|m) = \max_{S \subset S(y)} \left[ \sum_{l \in S} \lambda_{qml}(S)(p + V_{t+1}(y - e_l)) + \lambda_{qml}(S)V_{t+1}(y) \right] + (1 - \lambda)V_{t+1}(y).$$

Let us first assume that $y_i > 1$. Let $S^*$ be the set of products that achieves the maximum in the above optimization when the vector of inventory levels is $y - e_j$, $S^* \subset S(y - e_j)$. Then,

$$V_t(y - e_j|m) = \left[ \sum_{l \in S^*} \lambda_{qml}(S^*)(p + V_{t+1}(y - e_i - e_l)) + \lambda_{qml}(S^*)V_{t+1}(y - e_j) \right] + (1 - \lambda)V_{t+1}(y - e_j) \leq \left[ \sum_{l \in S^*} \lambda_{qml}(S^*)(p + V_{t+1}(y - e_i - e_l)) + \lambda_{qml}(S^*)V_{t+1}(y - e_i - e_j) \right] + (1 - \lambda)V_{t+1}(y - e_i).$$

Using the inductive hypothesis for the vectors $y$ and $y - e_i$. We only need to verify that $S^* \subset S(y - e_i)$. This holds because $S(y - e_j) \subset S(y - e_i)$ since $y_i \geq y_j$ and $y_i > 1$. Then, $V_t(y - e_i|m) \leq V_t(y - e_i|m)$ for each segment $m$, so that $\sum_{m \in M} \rho_m V_t(y - e_j|m) \leq \sum_{m \in M} \rho_m V_t(y - e_i|m)$. Consider
now a vector of inventory levels with $y_i = 1$ (and therefore $y_j = 1$ as well). Following the same reasoning as above, let $S^*$ be the set of products that achieves the maximum when the vector of inventory levels is $y - e_j$. Then, $S^*$ does not contain product $j$. If $i \not\in S^*$ either, then the same argument as above applies. Suppose that $i \in S^*$. As before,

$$V_t(y - e_j | m) = \left[ \sum_{l \in S^*} \lambda q_m l(S^*)(p + V_{t+1}(y - e_j - e_i)) + \lambda q_{m0}(S^*)V_{t+1}(y - e_j - e_i) \right] + (1 - \lambda)V_{t+1}(y - e_j) =$$

$$= \left[ \sum_{l \in S^*, l \not= i} \lambda q_m l(S^*)(p + V_{t+1}(y - e_l - e_j)) + \lambda q_{mi}(S^*)(p + V_{t+1}(y - e_j - e_i)) + \lambda q_{m0}(S^*)V_{t+1}(y - e_j - e_i) \right]$$

$$+ (1 - \lambda)V_{t+1}(y - e_j) \leq$$

$$\leq \left[ \sum_{l \in S^*, l \not= i} \lambda q_m l(S^*)(p + V_{t+1}(y - e_l - e_i)) + \lambda q_{mi}(S^*)(p + V_{t+1}(y - e_j - e_i)) + \lambda q_{m0}(S^*)V_{t+1}(y - e_i) \right]$$

$$+ (1 - \lambda)V_{t+1}(y - e_i).$$

Let $\tilde{S}^* = S^* \setminus \{i\} \cup \{j\}$. Summing up the inequality over all $m$, we have that $V_t(y - e_j) =$

$$\sum_{m \in M} \rho_m V_t(y - e_j | m) \leq \sum_{l \in S^*, l \not= i} \lambda \sum_{m \in M} \rho_m q_m l(S^*)(p + V_{t+1}(y - e_l - e_i)) + \lambda \sum_{m \in M} \rho_m q_{mi}(S^*)(p + V_{t+1}(y - e_j - e_i))$$

$$+ \lambda \sum_{m \in M} \rho_m q_{m0}(S^*)V_{t+1}(y - e_i) + (1 - \lambda)V_{t+1}(y - e_i) = \sum_{l \in S^*, l \not= j} \lambda \sum_{m \in M} \rho_m q_m l(\tilde{S}^*)(p + V_{t+1}(y - e_l - e_i))$$

$$+ \lambda \sum_{m \in M} \rho_m q_{mj}(\tilde{S}^*)(p + V_{t+1}(y - e_j - e_i)) + \lambda \sum_{m \in M} \rho_m q_{m0}(\tilde{S}^*)V_{t+1}(y - e_i) + (1 - \lambda)V_{t+1}(y - e_i)$$

$$= \sum_{m \in M} \rho_m \left[ \sum_{l \in \tilde{S}^*} \lambda q_m l(\tilde{S}^*)(p + V_{t+1}(y - e_l - e_i)) + \lambda q_{m0}(\tilde{S}^*)V_{t+1}(y - e_i) \right] + (1 - \lambda)V_{t+1}(y - e_i) \leq$$

$$\leq \sum_{m \in M} \rho_m V_t(y - e_i | m) = V_t(y - e_i),$$

establishing the result.

**Proof of Proposition 3.** We first show that $V_t(y - e_i) < V_t(y - e_j)$ for $i < j$ and any period $t$, using induction. First, for period $T$, we have

$$V_T(y - e_i | m) = \sum_{k \in S_T(y - e_i)} \lambda p q_{mk} = \left\{ \begin{array}{ll} \lambda p \frac{m \theta}{m \theta + \theta_0} & \text{if } m < i, \\ \lambda p \frac{(m - 1) \theta}{(m - 1) \theta + \theta_0} & \text{if } m \geq i \end{array} \right\},$$

and similarly for $V_T(y - e_j | m)$. For $m < i$ or $m \geq j$, we have that $V_T(y - e_j | m) \leq V_T(y - e_i | m)$, while $V_T(y - e_j | m) > V_T(y - e_i | m)$ for $i \leq m < j$. This establishes the result for the last period $T$. Suppose now that the result holds for $t + 1, \ldots, T$ and consider period $t$. Let $S^*$ be the optimal
assortment offered to a customer from segment $m$ in period $t$ under the vector $y - e_i$. Clearly, $i \not\in S^*$. Let $s = |S^* \cap \{1, \ldots, m\}|$. We have

$$V_t(y - e_i|m) = \sum_{k \in S^*} \lambda \frac{\theta}{s\theta + \theta_0} (p + V_{t+1}(y - e_i - e_k)) + \lambda \frac{\theta_0}{s\theta + \theta_0} V_{t+1}(y - e_i) + (1 - \lambda)V_{t+1}(y - e_i).$$

If $j \not\in S^*$, then the result is immediate by induction using the fact that $S^*$ is a feasible set for the maximization in $V_t(y - e_j|m)$. Suppose that $j \in S^*$. Then, by induction

$$V_t(y - e_i|m) = \sum_{k \in S^*} \lambda \frac{\theta}{s\theta + \theta_0} (p + V_{t+1}(y - e_i - e_k)) + \lambda \frac{\theta_0}{s\theta + \theta_0} V_{t+1}(y - e_i) + (1 - \lambda)V_{t+1}(y - e_i)$$

$$< \sum_{k \in S^* \cup \{i\} \setminus \{j\}} \lambda \frac{\theta}{(s-1)\theta + \theta_0} (p + V_{t+1}(y - e_i - e_k))^+ \lambda \frac{\theta_0}{(s-1)\theta + \theta_0} V_{t+1}(y - e_i) + (1 - \lambda)V_{t+1}(y - e_i),$$

and this quantity is itself no larger than $V_t(y - e_j|m)$, establishing that $V_t(y - e_i) < V_t(y - e_j)$.

This implies that $p^1_t(y) < p^2_t(y) < \cdots < p^n_t(y)$ and, therefore, for an arriving customer of segment $m$ there exists $k_m$ such that the optimal assortment for that customer is of the form $\{k_m, \ldots, m\}$.

We finally show that $k_m$ is non-decreasing in $m$, i.e., if $k_m \in S^*_m$ then $k_m \in S^*_m - 1, t$, where $S^*_m$ and $S^*_m - 1, t$ are the optimal sets offered to arriving customers of segments $m$ and $m - 1$, respectively, in period $t$. Because $k_m \in S^*_m - 1, t$, we have that $p^k_t(y) \geq \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m}^m p^i_t(y)$. This implies that $p^k_t(m) \geq \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m+1}^m p^i_t(y)$. Also, $p^m_t(y) \geq \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m+1}^m p^i_t(y)$ which implies that

$$\frac{\theta}{(m-k_m)\theta + \theta_0} \sum_{i=k_m+1}^m p^i_t(y) \geq \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m+1}^m p^i_t(y).$$

We then have $p^k_t(m) \geq \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m+1}^m p^i_t(y)$, which implies that

$$\frac{\theta}{(m-k_m)\theta + \theta_0} \sum_{i=k_m}^{m-1} p^i_t(y) = \frac{\theta}{(m-k_m)\theta + \theta_0} \sum_{i=k_m+1}^{m-1} p^i_t(y) + \frac{\theta}{(m-k_m)\theta + \theta_0} p^k_t(m)$$

$$> \frac{\theta}{(m-k_m)\theta + \theta_0} \sum_{i=k_m+1}^{m-1} p^i_t(y) + \frac{\theta}{(m-k_m)\theta + \theta_0} (m-k_m+1)\theta + \theta_0 \sum_{i=k_m+1}^{m-1} p^i_t(y)$$

$$= \frac{\theta}{(m-k_m+1)\theta + \theta_0} \sum_{i=k_m+1}^{m-1} p^i_t(y).$$

Therefore, product $k_m \in S^*_m - 1, t$, which implies that $k_m - 1 \leq k_m$. □

**Proof of Theorem 1 and Proposition 4.** To prove these two results, we show that the following five properties hold for each period $t$ using an induction argument that makes use of these five properties in period $t + 1$. 


P1(\(t, i, y_1, y_2\)): \(\Delta_j^i(y_1, y_2)\) is decreasing in \(y_j\), \(j \in \{1, 2\}\).

P2(\(t, i, y_1, y_2\)): \(\Delta_i^j(y_1, y_2)\) is decreasing in \(t\).

P3(\(t, i, y_1, y_2\)): \(p - \Delta_i^j(y_1, y_2) \geq \frac{\theta_i}{\theta_j + \theta_0} (p - \Delta_j^i(y_1, y_2))\) for \(j \neq i\). For \(i = 2\), this equivalent to
\[
\theta_0 p + (\theta_{2} + \theta_{0})V_{t+1}(y_1, y_2 - 1) \geq \theta_{0} V_{t+1}(y_1, y_2) + \theta_{21} V_{t+1}(y_1 - 1, y_2).
\]

P4(\(t, i, y_1, y_2\)): \(\Delta_i^j(y) - \Delta_i^j(y + e_i) \geq \frac{\theta_i}{\theta_j + \theta_0} (\Delta_j^j(y) - \Delta_j^j(y + e_i))\) for \(j \neq i\). For \(i = 1\), this equivalent to
\[
(\theta_{22} + 2\theta_{0})V_{t+1}(y) + \theta_{22} V_{t+1}(y - e_2) \geq (\theta_{22} + \theta_{0}) V_{t+1}(y - e_1) + \theta_{22} V_{t+1}(y + 1, y_2 - 1) + \theta_{0} V_{t+1}(y + e_1).
\]

P5(\(t, i, y_1, y_2\)): If \(p - \Delta_i^j(y) \geq \frac{\theta_i}{\theta_j + \theta_0} (p - \Delta_i^j(y))\), then \(p - \Delta_i^j(y - e_j) \geq \frac{\theta_i}{\theta_j + \theta_0} (p - \Delta_i^j(y - e_j))\) for \(j \neq i\). For \(i = 2\), this equivalent to having that if \(\theta_0 p + (\theta_2 + \theta_0) V_{t+1}(y_1 - 1, y_2) \geq \theta_0 V_{t+1}(y_1, y_2) + \theta_{22} V_{t+1}(y_1, y_2 - 1)\), then \(\theta_0 p + (\theta_{22} + \theta_0) V_{t+1}(y_1 - 1, y_2 - 1) \geq \theta_0 V_{t+1}(y_1, y_2 - 1) + \theta_{22} V_{t+1}(y_1, y_2 - 2)\).

P1 implies that the expected marginal revenue of a unit of inventory is decreasing in any product’s inventory level, implying concavity in \(y_i\) and submodularity in \(y\). P2 implies that the expected marginal revenues are decreasing in time. P3 implies part \((i)\) of the theorem, i.e., that the assortment offered to segment \(j\) always contains product \(j\). P4 is an inequality that implies the threshold policy in part \((ii)\), which can be expressed as follows: If it is optimal to offer product \(j\) to segment \(i\) at inventory level \(y\), then it is also optimal to offer product \(j\) to segment \(i\) when there is one more unit of inventory of product \(j\), i.e., if \(p - \Delta_j^j(y) \geq \frac{\theta_i}{\theta_j + \theta_0} (p - \Delta_j^j(y))\) then \(p - \Delta_j^j(y + e_j) \geq \frac{\theta_i}{\theta_j + \theta_0} (p - \Delta_j^j(y + e_j))\). P5 implies that if it is optimal to offer product \(i\) to segment \(j\) at inventory level \(y\), then it is also optimal to offer this product to segment \(j\) when the vector of inventory levels is \(y - e_j\), that is, \(y_i \geq y_i^*(y_j - 1)\). This property implies part \((i)\) of Proposition 4. The proof of each property requires comparing the alternative assortments in the current period given the customer choice probabilities and the continuation value functions based on the implementation of the optimal assortment policy in future periods. This results in a large number of cases to be checked (corresponding to multiple future outcomes). We provide the proof for many of the cases in this appendix. The proof of the remaining cases follows using similar arguments (details are available from the authors).
We now proceed with the proof. We provide the proof of the results for \( i = 2 \). We can similarly prove the results for \( i = 1 \). For notational simplicity, define \( \Lambda_{\{ij0\}}^m = \theta_{mi} + \theta_{mj} + \theta_0 \) and \( \Lambda_{\{i0\}}^m = \theta_{mi} + \theta_0 \), where \( m \) denotes the customer segment, and \( i \) and \( j \) represent the products.

Throughout the proof, we will frequently use the fact that, for a given vector of inventory levels \( (y_1, y_2) \), the set \( \{1, 2\} \) is preferred over the set \( \{2\} \) [resp., \( \{1\} \)] for a customer of segment 2 [resp., 1] at time \( t \) if and only if \( p - \Delta_{t+1}^1(y_1, y_2) \geq \frac{\theta_m}{\Lambda_{\{i0\}}^m} (p - \Delta_{t+1}^2(y_1, y_2)) \) [resp., \( p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_m}{\Lambda_{\{i0\}}^m} (p - \Delta_{t+1}^1(y_1, y_2)) \)].

First, because the optimal assortment in period \( T \) is to offer all available products, it is easy to verify that the results hold for period \( T \). We then assume that these five properties hold in period \( t + 1 \), and show that they also hold in period \( t \). Note that if P3 and P4 hold in period \( t + 1 \), then this implies that the optimal assortment policy in period \( t \) is a threshold policy. Note also that because \( V_t(y_1, y_2) \) is a linear combination of \( V_t(y_1, y_2|m) \) for \( m = 1 \) and \( m = 2 \), we show that the results hold for \( V_t(y_1, y_2|m) \) for a given \( m = 1, 2 \), which then implies that they hold for \( V_t(y_1, y_2) \).

The proof focuses on the case with \( \lambda = 1 \) (the results can be similarly proved for \( \lambda < 1 \)). Finally, we focus on the case in which \( \mathcal{S}(\mathbf{y}) = \{1, 2\} \). If only one product is available, then it is always optimal to offer that product to any arriving customer.

We begin with the proof of P1\((t, 2, y_1, y_2)\), which involves two results. One is that \( \Delta_{t+1}^2(y_1, y_2+1) \) is decreasing in \( y_1 \), which is equivalent to \( V_t(y_1 - 1, y_2 + 1) - V_t(y_1 - 1, y_2) \geq V_t(y_1, y_2 + 1) - V_t(y_1, y_2) \); the other result is that \( \Delta_{t+1}^2(y_1, y_2) \) is decreasing in \( y_2 \), which is equivalent to \( V_t(y_1, y_2) - V_t(y_1, y_2 - 1) \geq V_t(y_1, y_2 + 1) - V_t(y_1, y_2) \). We provide the proof for the first result and the second result follows with a similar argument.

We further consider two cases: (i) \( p - \Delta_{t+1}^1(y_1, y_2 + 1) \geq p - \Delta_{t+1}^2(y_1, y_2 + 1) \) and (ii) \( p - \Delta_{t+1}^1(y_1, y_2 + 1) < p - \Delta_{t+1}^2(y_1, y_2 + 1) \). We prove the result for case (i). This case implies that \( S_{2t}^*(y_1, y_2 + 1) = \{1, 2\} \). In addition, because by induction P5 holds in period \( t + 1 \), we have that \( S_{2t}^*(y_1, y_2) = \{1, 2\} \). The optimal assortment sets under the various inventory levels are given by one of the combinations shown in the table below:

<table>
<thead>
<tr>
<th>( S_t^* )</th>
<th>( (y_1, y_2) )</th>
<th>( (y_1 - 1, y_2) )</th>
<th>( (y_1, y_2 + 1) )</th>
<th>( (y_1 - 1, y_2 + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{1t}^* )</td>
<td>{1, 2} or {1}</td>
<td>{1, 2} or {1}</td>
<td>{1, 2} or {1}</td>
<td>{1, 2} or {1}</td>
</tr>
<tr>
<td>( S_{2t}^* )</td>
<td>{1, 2}</td>
<td>{1, 2} or {2}</td>
<td>{1, 2}</td>
<td>{1, 2} or {2}</td>
</tr>
</tbody>
</table>

In addition to the scenarios described in the table, we further consider two possible cases: (a) \( p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_m}{\Lambda_{\{i0\}}^m} [p - \Delta_{t+1}^1(y_1, y_2)] \) and (b) \( p - \Delta_{t+1}^2(y_1, y_2) < \frac{\theta_m}{\Lambda_{\{i0\}}^m} [p - \Delta_{t+1}^1(y_1, y_2)] \). We
provide the proof for case (a). In this case, because \( p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_{10}}{\Lambda_{10}(t)}[p - \Delta_{t+1}^1(y_1, y_2)] \), we have \( S^*_{1t}(y_1, y_2) = \{1, 2\} \). Because of P4\((t + 1, 1, y_1, y_2)\), we also have \( p - \Delta_{t+1}^2(y_1, y_2 + 1) \geq \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^1(y_1, y_2 + 1)] \), which implies that \( S^*_{1t}(y_1, y_2 + 1) = \{1, 2\} \). Similarly, because of P5\((t + 1, 1, y_1, y_2)\) and P4\((t + 1, 1, y_1 - 1, y_2)\), \( S^*_{1t}(y_1 - 1, y_2) = \{1, 2\} \) and \( S^*_{1t}(y_1 - 1, y_2 + 1) = \{1, 2\} \). Therefore, when \( y_1 > 1 \), the optimal assortment sets under the relevant inventory levels are:

| Inventory Level | \( S^*_{1t} \) | \( S^*_{2t} \) |
|-----------------|-----------------|
| \( y_1, y_2 \)  | \{1, 2\}        |
| \( y_1 - 1, y_2 \)   | \{1, 2\}        |
| \( y_1, y_2 + 1 \)  | \{1, 2\}        |
| \( y_1 - 1, y_2 + 1 \) | \{1, 2\} or \{2\} |

Because the optimal assortment sets for all the relevant inventory levels are the same for segment 1 (all equal to \( \{1, 2\} \)), it is easy to prove the result for that segment. We then focus on the proof for segment 2. If \( p - \Delta_{t+1}^1(y_1 - 1, y_2) \geq \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^2(y_1 - 1, y_2)] \) and \( p - \Delta_{t+1}^1(y_1 - 1, y_2 + 1) \geq \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^2(y_1 - 1, y_2 + 1)] \), then \( S^*_{2t}(y_1 - 1, y_2) = \{1, 2\} \) and \( S^*_{2t}(y_1 - 1, y_2 + 1) = \{1, 2\} \). Hence, the optimal assortment set for segment 2 is \( \{1, 2\} \) in all cases, so the result follows for segment 2 as well. If \( p - \Delta_{t+1}^1(y_1 - 1, y_2) < \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^2(y_1 - 1, y_2)] \) and \( p - \Delta_{t+1}^1(y_1 - 1, y_2 + 1) < \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^2(y_1 - 1, y_2 + 1)] \), then \( S^*_{2t}(y_1 - 1, y_2) = \{1, 2\} \) and \( S^*_{2t}(y_1 - 1, y_2 + 1) = \{2\} \). Because \( S^*_{2t}(y_1 - 1, y_2 + 1) = \{2\} \), we have that

\[
V_t(y_1 - 1, y_2 + 1| 2) = \frac{1}{\Lambda_{120}(2)}[\theta_{21}p + \theta_{22}V_{t+1}(y_1 - 1, y_2) + \theta_0V_{t+1}(y_1 - 1, y_2 + 1)] \\
+ \frac{1}{\Lambda_{120}(2)}[\theta_{21}V_{t+1}(y_1 - 2, y_2 + 1) + \theta_{22}V_{t+1}(y_1 - 1, y_2) + \theta_0V_{t+1}(y_1 - 1, y_2 + 1)] \\
- \frac{1}{\Lambda_{120}(2)}[\theta_{21}V_{t+1}(y_1 - 2, y_2 + 1) + \theta_{22}V_{t+1}(y_1 - 1, y_2) + \theta_0V_{t+1}(y_1 - 1, y_2 + 1)].
\]

where we are adding and subtracting the same term. Therefore,

\[
V_t(y_1, y_2| 2) - V_t(y_1 - 1, y_2| 2) - V_t(y_1, y_2 + 1| 2) + V_t(y_1 - 1, y_2 + 1| 2) \geq \\
- \frac{1}{\Lambda_{120}(2)}[\theta_{21}V_{t+1}(y_1 - 2, y_2 + 1) + \theta_{22}V_{t+1}(y_1 - 1, y_2) + \theta_0V_{t+1}(y_1 - 1, y_2 + 1)] \\
- \frac{\theta_{21}\theta_0}{(\Lambda_{120}(2))(\Lambda_{20}(2))} + \frac{1}{\Lambda_{120}(2)}[\theta_{22}V_{t+1}(y_1 - 1, y_2) + \theta_0V_{t+1}(y_1 - 1, y_2 + 1)] \\
= - \frac{\theta_{21}\theta_0}{(\Lambda_{120}(2))(\Lambda_{20}(2))} \theta_{21}p - \frac{\theta_{21}}{\Lambda_{20}(2)}V_{t+1}(y_1 - 2, y_2 + 1) + \frac{\theta_{21}\theta_{22}}{\Lambda_{20}(2)^2}V_{t+1}(y_1 - 1, y_2) \\
+ \frac{\theta_{21}\theta_0}{(\Lambda_{120}(2))(\Lambda_{20}(2))}V_{t+1}(y_1 - 1, y_2 + 1) > 0.
\]

The first follows inductively from P1\((t + 1, 2, y_1 - 1, y_2 + 1)\), P1\((t + 1, 2, y_1, y_2)\), and P1\((t + 1, 2, y_1, y_2 + 1)\), and the last inequality holds because \( p - \Delta_{t+1}^1(y_1 - 1, y_2 + 1) \leq \frac{\theta_{20}}{\Lambda_{20}(t)}[p - \Delta_{t+1}^2(y_1 - 1, y_2 + 1)] \), which
is equivalent to \( \theta p + (\Lambda^2_{(20)}) V_{t+1} (y_1 - 2, y_2 + 1) < \theta V_{t+1} (y_1 - 1, y_2 + 1) + \theta_2 V_{t+1} (y_1 - 1, y_2) \). A similar argument follows if \( p - \Delta^2_{(t+1)} (y_1 - 1, y_2) < \frac{\theta_2}{\Lambda^2_{(20)}} [p - \Delta^2_{(t+1)} (y_1 - 1, y_2)] \) and \( p - \Delta^1_{(t+1)} (y_1 - 1, y_2 + 1) < \frac{\theta}{\Lambda^1_{(20)}} [p - \Delta^1_{(t+1)} (y_1 - 1, y_2 + 1)] \).

We now study the cases where \( y_1 = 1 \) and \( y_2 > 0 \). The optimal assortment sets for segment 1 under all relevant inventory levels are \( S^*_t(1, y_2) = \{1\} \), \( S^*_t(0, y_2) = \{2\} \), \( S^*_t(1, y_2 + 1) = \{1, 2\} \) or \( \{1\} \), and \( S^*_t(0, y_2 + 1) = \{2\} \). If \( p - \Delta^2_{(t+1)} (1, y_2 + 1) \geq \frac{\theta}{\Lambda^1_{(10)}} [p - \Delta^1_{(t+1)} (1, y_2 + 1)] \), then \( S^*_t(1, y_2 + 1) = \{1, 2\} \) and \( S^*_t(1, y_2) = \{1\} \). Thus,

\[
V_t(1, y_2 | 1) - V_t(0, y_2 | 1) = V_t(1, y_2 + 1 | 1) + V_t(0, y_2 + 1 | 1)
\]

\[
\geq - \frac{\theta_1}{\Lambda^1_{(120)}} [\theta_1 V_t(0, y_2) + \theta_0 V_{t+1} (1, y_2)]
\]

\[
= \frac{1}{\Lambda^1_{(120)}} [\theta_1 (V_t(0, y_2) - V_{t+1} (1, y_2 + 1)) + \theta_0 (V_{t+1} (1, y_2 + 1) - V_{t+1} (0, y_2))]
\]

The above term is positive because \( \Delta^2_{(t+1)} (0, y_2) \geq \Delta^2_{(t+1)} (0, y_2 + 1) \geq \Delta^2_{(t+1)} (1, y_2 + 1) \) and \( \Delta^2_{(t+1)} (0, y_2) \geq \Delta^2_{(t+1)} (1, y_2) \), leading to the desired result. A similar argument follows if \( p - \Delta^2_{(t+1)} (1, y_2 + 1) < \frac{\theta}{\Lambda^1_{(10)}} [p - \Delta^1_{(t+1)} (1, y_2 + 1)] \). This completes the proof of all cases for \( P1(t, 2, y_1, y_2) \).

We now turn attention to \( P2(t, 2, y_1, y_2) \). To prove the result, it is sufficient to show that \( V_t(y_1, y_2 | m) - V_t(y_1, y_2 - 1 | m) \geq V_t(y_1, y_2) - V_{t+1} (y_1, y_2 - 1) \) for \( m = 1, 2 \). Consider two cases:

(i) \( p - \Delta^1_{(t+1)} (y_1, y_2) \geq p - \Delta^2_{(t+1)} (y_1, y_2) \) and (ii) \( p - \Delta^1_{(t+1)} (y_1, y_2) < p - \Delta^2_{(t+1)} (y_1, y_2) \). We provide the proof for the first case. The first inequality implies that product 1 is offered to both customer segments. This inequality and the inductive argument imply the possible optimal assortment sets for both customer segments in period \( t \):

| \( S^*_t | \) | \( (y_1, y_2) \) | \( (y_1, y_2 - 1) \) |
|---|---|---|
| \( S^*_t^1 \) | \( \{1, 2\} \) or \( \{1\} \) | \( \{1, 2\} \) or \( \{1\} \) |
| \( S^*_t^2 \) | \( \{1, 2\} \) | \( \{1, 2\} \) |

For \( m = 2 \), because it is optimal to offer both products at time \( t \), the proof follows by noting that

\[
V_t(y_1, y_2 | 2) - V_t(y_1, y_2 - 1 | 2) - V_{t+1} (y_1, y_2) + V_{t+1} (y_1, y_2 - 1) = \frac{\theta_1}{\Lambda^1_{(120)}} (\Delta^2_{(t+1)} (y_1 - 1, y_2) - \Delta^2_{(t+1)} (y_1, y_2) \right) + \frac{\theta}{\Lambda^1_{(120)}} (\Delta^2_{(t+1)} (y_1, y_2 - 1) - \Delta^2_{(t+1)} (y_1, y_2))
\]

The last term is non-negative using \( P1(t + 1, 2, y_1, y_2) \). Following a similar argument, the inequality can be shown for \( m = 1 \) when \( S^*_t(y_1, y_2) = \{1, 2\} \) and
$S^*_t(y_1, y_2 - 1) = \{1, 2\}$, or when $S^*_t(y_1, y_2) = \{1\}$ and $S^*_t(y_1, y_2 - 1) = \{1\}$. The cases in which the optimal sets are different for $(y_1, y_2)$ and $(y_1, y_2 - 1)$ require a different argument. We focus on the case $S^*_t(y_1, y_2) = \{1, 2\}$ and $S^*_t(y_1, y_2 - 1) = \{1\}$. For this case, we have

$$V_t(y_1, y_2)[1] - V_t(y_1, y_2 - 1)[1] - V_{t+1}(y_1, y_2) + V_{t+1}(y_1, y_2 - 1) \geq \frac{\theta_{11}}{N_{\{10\}}} \left[ V_{t+1}(y_1 - 1, y_2) + V_{t+1}(y_1, y_2 - 1) - V_{t+1}(y_1 - 1, y_2 - 1) - V_{t+1}(y_1, y_2) \right]$$

$$= \frac{\theta_{11}}{N_{\{10\}}} \left[ \Delta_{t+1}^2(y_1 - 1, y_2) - \Delta_{t+1}^2(y_1, y_2) \right] \geq 0.$$

The first inequality follows from the fact that $\theta_0 p + (\theta_{11} + \theta_0) V_{t+1}(y_1, y_2 - 1) \geq \theta_{11} V_{t+1}(y_1 - 1, y_2) + \theta_0 V_{t+1}(y_1, y_2)$ in the first term, which in turn follows from $p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_{11}}{\theta_{11} + \theta_0} [p - \Delta_{t+1}^1(y_1, y_2)]$ (this inequality holds because $S^*_t(y_1, y_2) = \{1, 2\}$). The last inequality is due to $P1(t + 1, 2, y_1, y_2)$.

This completes the proof of $P2(t, 2, y_1, y_2)$.

We next prove $P3(t, 2, y_1, y_2)$, which is stated as

$$\theta_0 p + (\theta_{21} + \theta_0) V_t(y_1, y_2 - 1) \geq \theta_0 V_t(y_1, y_2) + \theta_{21} V_t(y_1 - 1, y_2).$$

If $V_t(y_1, y_2 - 2) \geq V_t(y_1 - 1, y_2)$, then $P3(t, 2, y_1, y_2)$ is true because $p + V_t(y_1 - 1, y_2) \geq V_t(y_1, y_2)$.

We then focus on the case in which $V_t(y_1, y_2 - 2) < V_t(y_1 - 1, y_2)$. We prove the result by considering two cases: (i) $p - \Delta_{t+1}^1(y_1, y_2) \geq p - \Delta_{t+1}^2(y_1, y_2)$ and (ii) $p - \Delta_{t+1}^1(y_1, y_2) < p - \Delta_{t+1}^2(y_1, y_2)$. We provide the proof for the first case. For the case under consideration, the optimal assortment sets under the various inventory levels are given by one of the combinations shown in the table below:

<table>
<thead>
<tr>
<th>$S^*_t$</th>
<th>$(y_1, y_2)$</th>
<th>$(y_1 - 1, y_2)$</th>
<th>$(y_1, y_2 - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^*_t$</td>
<td>${1, 2}$</td>
<td>${1, 2}$</td>
<td>${1, 2}$</td>
</tr>
<tr>
<td>$S^*_t$</td>
<td>${1}$</td>
<td>${1}$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>

We further consider two cases: (a) $p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_{11}}{N_{\{10\}}} [p - \Delta_{t+1}^1(y_1, y_2)]$ and (b) $p - \Delta_{t+1}^2(y_1, y_2) < \frac{\theta_{11}}{N_{\{10\}}} [p - \Delta_{t+1}^1(y_1, y_2)]$. We provide the proof for case (a). In this case, the set $\{1, 2\}$ is preferred over the set $\{1\}$ for segment 1 in period $t$ under the vector of inventory levels $(y_1, y_2)$. The same holds for the inventory vector $(y_1 - 1, y_2)$ from $P5(t + 1, 1, y_1, y_2)$. Then, when $y_1, y_2 \geq 2$, the optimal assortment sets under the relevant inventory levels are:

<table>
<thead>
<tr>
<th>$S^*_t$</th>
<th>$(y_1, y_2)$</th>
<th>$(y_1 - 1, y_2)$</th>
<th>$(y_1, y_2 - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^*_t$</td>
<td>${1, 2}$</td>
<td>${1, 2}$</td>
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</tr>
<tr>
<td>$S^*_t$</td>
<td>${1, 2}$</td>
<td>${1}$</td>
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</tr>
<tr>
<td>$S^*_t$</td>
<td>${1}$</td>
<td>${1}$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>
If \( p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2)] \) and \( p - \Delta_{t+1}^1(y_1, y_2) \geq \frac{\theta_{22}}{\Lambda_{(20)}^2}[p - \Delta_{t+1}^2(y_1 - 1, y_2)] \), then we have that \( S^*_t(y_1, y_2 - 1) = \{1, 2\} \) and \( S^*_t(y_1, y_2) = \{1, 2\} \). The result follows by induction in this case. If \( p - \Delta_{t+1}^2(y_1, y_2 - 1) < \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2 - 1)] \) and \( p - \Delta_{t+1}^1(y_1 - 1, y_2) < \frac{\theta_{22}}{\Lambda_{(20)}^2}[p - \Delta_{t+1}^2(y_1 - 1, y_2)] \), then we have \( S^*_t(y_1, y_2 - 1) = \{1\} \) and \( S^*_t(y_1, y_2) = \{2\} \). Because \( S^*_t(y_1, y_2 - 1) = \{1\} \) for segment 1, we have

\[
\theta_0 p + (\theta_0 + \theta_2) V_i(y_1, y_2 - 1) - \theta_0 V_i(y_1, y_2) - \theta_2 V_i(y_1 - 1, y_2) - \theta_2 V_i(y_1, y_2)
\]

where the inequality follows from P3(\( t+1, 2, y_1 - 1, y_2 \)), P3(\( t+1, 2, y_1, y_2 - 1, y_2 \)), and P3(\( t+1, 2, y_1, y_2 \)), and from \( p - \Delta_{t+1}^2(y_1, y_2 - 1) < \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2 - 1)] \). A similar analysis follows for segment 2. Similar arguments apply to the cases with \( y_1 = 1 \) and \( y_2 \geq 2 \), with \( y_1 \geq 2 \) and \( y_2 = 1 \), and with \( y_1 = 1 \) and \( y_2 = 1 \).

We now proceed with P4(\( t, 2, y_1, y_2 \)). Here again, we consider two cases: (i) \( p - \Delta_{t+1}^1(y_1, y_2) \geq p - \Delta_{t+1}^2(y_1, y_2) \) and (ii) \( p - \Delta_{t+1}^1(y_1, y_2) < p - \Delta_{t+1}^2(y_1, y_2) \). We provide the proof for the first case. In this case, using induction, the optimal assortment sets under the relevant inventory levels are:

<table>
<thead>
<tr>
<th>( S^*_t )</th>
<th>( (y_1, y_2) )</th>
<th>( (y_1, y_2 - 1) )</th>
<th>( (y_1 - 1, y_2) )</th>
<th>( (y_1 + 1, y_2 - 1) )</th>
<th>( (y_1 + 1, y_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^*_t )</td>
<td>{1, 2} or {1}</td>
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</tr>
<tr>
<td>( S^*_t )</td>
<td>{1, 2}</td>
<td>{1, 2}</td>
<td>{1, 2}</td>
<td>{1, 2}</td>
<td>{1, 2}</td>
</tr>
</tbody>
</table>

We further consider two cases: (a) \( p - \Delta_{t+1}^2(y_1, y_2) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2)] \) and (b) \( p - \Delta_{t+1}^2(y_1, y_2) < \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2)] \). We focus on the proof of case (a). When \( y_1, y_2 > 1 \), the possible optimal assortment sets are:

<table>
<thead>
<tr>
<th>( S^*_t )</th>
<th>( (y_1, y_2) )</th>
<th>( (y_1, y_2 - 1) )</th>
<th>( (y_1 - 1, y_2) )</th>
<th>( (y_1 + 1, y_2 - 1) )</th>
<th>( (y_1 + 1, y_2) )</th>
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</thead>
<tbody>
<tr>
<td>( S^*_t )</td>
<td>{1, 2}</td>
<td>{1, 2} or {1}</td>
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</tr>
<tr>
<td>( S^*_t )</td>
<td>{1, 2}</td>
<td>{1, 2}</td>
<td>{1, 2} or {2}</td>
<td>{1, 2} or {2}</td>
<td>{1, 2}</td>
</tr>
</tbody>
</table>

We start by verifying the result for segment 1. If \( p - \Delta_{t+1}^2(y_1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2 - 1)] \) and \( p - \Delta_{t+1}^2(y_1 + 1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1 + 1, y_2 - 1)] \), then \( S^*_t(y_1, y_2 - 1) = \{1, 2\} \) and \( S^*_t(y_1 + 1, y_2 - 1) = \{1, 2\} \), so the proof of this case is straightforward. If \( p - \Delta_{t+1}^2(y_1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1, y_2 - 1)] \), \( p - \Delta_{t+1}^2(y_1 + 1, y_2 - 1) < \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1 + 1, y_2 - 1)] \), and \( p - \Delta_{t+1}^2(y_1 + 1, y_2) \geq \frac{\theta_{11}}{\Lambda_{(10)}^1}[p - \Delta_{t+1}^1(y_1 + 1, y_2)] \), then \( S^*_t(y_1, y_2) = \{1, 2\} \), \( S^*_t(y_1, y_2 - 1) = \{1, 2\} \),
$S_{1t}^*(y_1 - 1, y_2) = \{1, 2\}$, $S_{1t}^*(y_1 + 1, y_2 - 1) = \{1\}$, and $S_{1t}^*(y_1 + 1, y_2) = \{1, 2\}$. For segment 1, because $S_{1t}^*(y_1 + 1, y_2 - 1) = \{1\}$, we have

$$-\theta_{22}V_t(y_1 + 1, y_2 - 1|1) = -\frac{\theta_{22}}{\Lambda_{1}^{10}}[\theta_{11}p + \theta_{11}V_{t+1}(y_1, y_2 - 1) + \theta_0V_{t+1}(y_1 + 1, y_2 - 1)]$$

$$+ \frac{\theta_{22}}{\Lambda_{1}^{120}}[\theta_{11}V_{t+1}(y_1, y_2 - 1) + \theta_0V_{t+1}(y_1 + 1, y_2 - 1)]$$

$$- \frac{\theta_{22}}{\Lambda_{1}^{120}}[\theta_{11}V_{t+1}(y_1, y_2 - 1) + \theta_0V_{t+1}(y_1 + 1, y_2 - 1)]$$

(6)

We denote the equation in (6) as $A(y_1 + 1, y_2 - 1|1)$, as this equation will be used in the other cases. Following (6) and the optimal assortment sets under the relevant inventory levels, we have

$$(\theta_{22} + 2\theta_0)V_t(y_1 + 1, y_2|1) + \theta_{22}V_t(y_1, y_2 - 1|1) - (\Lambda_{1}^{2})V_t(y_1 - 1, y_2|1) - \theta_{22}V_t(y_1 + 1, y_2 - 1|1) - \theta_0V_t(y_1 + 1, y_2|1)$$

$$\geq \frac{\theta_{12}\theta_0\theta_{22}p}{(\Lambda_{1}^{120})(\Lambda_{1}^{10})} + \frac{\theta_{12}}{\Lambda_{1}^{120}}[(\theta_{22} + 2\theta_0)V_t(y_1, y_2 - 1) + \theta_{22}V_{t+1}(y_1, y_2 - 2)$$

$$- (\Lambda_{1}^{2})V_{t+1}(y_1 - 1, y_2 - 1) - \theta_0V_{t+1}(y_1 + 1, y_2 - 1)] + \frac{\theta_{22}}{\Lambda_{1}^{120}}[\theta_{11}V_{t+1}(y_1, y_2 - 1)$$

$$+ \theta_0V_{t+1}(y_1 + 1, y_2 - 1)] - \frac{\theta_{22}}{\Lambda_{1}^{120}}[\theta_{11}V_{t+1}(y_1, y_2 - 1) + \theta_0V_{t+1}(y_1 + 1, y_2 - 1)]$$

$$\geq \frac{\theta_{12}\theta_0(\theta_{22} + \theta_{11} + \theta_0)}{(\Lambda_{1}^{120})(\Lambda_{1}^{10})}[2V_{t+1}(y_1, y_2 - 1) - V_{t+1}(y_1 - 1, y_2 - 1) - V_{t+1}(y_1 + 1, y_2 - 1)] \geq 0.$$

The first inequality follows inductively from P4($t + 1, 2, y_1 - 1, y_2$) and P4($t + 1, 2, y_1, y_2$), and the second inequality follows from $p - \Delta_{1t+1}^2(y_1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1, y_2 - 1)]$ and P1($t + 1, 1, y_1, y_2 - 1$). A similar argument follows if $p - \Delta_{1t+1}^2(y_1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1, y_2 - 1)]$, $p - \Delta_{1t+1}^2(y_1 + 1, y_2 - 1) \leq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1 + 1, y_2 - 1)]$, and similarly if $p - \Delta_{1t+1}^2(y_1, y_2 - 1) \leq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1, y_2 - 1)]$, $p - \Delta_{1t+1}^2(y_1 + 1, y_2 - 1) \leq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1 + 1, y_2 - 1)]$, and $p - \Delta_{1t+1}^2(y_1 + 1, y_2) \leq \frac{\theta_{11}}{\Lambda_{1}^{10}}[p - \Delta_{1t+1}^1(y_1 + 1, y_2)]$. A similar proof also follows for segment 2. The proof for the cases with $y_1 = 1$ and $y_2 > 1$, $y_1 > 1$ and $y_2 = 1$, and $y_1 = 1$ and $y_2 = 1$ follows similarly.

We now proceed to the proof of P5($t, y_1, y_2$). If $V_t(y_1 - 1, y_2 - 1) \geq V_t(y_1, y_2 - 2)$, then $\theta_0 V_t + (\Lambda_{2})V_t(y_1 - 1, y_2 - 1) \geq \theta_0 V_t(y_1, y_2 - 1) + \theta_{22}V_t(y_1, y_2 - 2)$ because $p + V_t(y_1 - 1, y_2 - 1) \geq V_t(y_1, y_2 - 1)$. Therefore, we focus on the case in which $V_t(y_1 - 1, y_2 - 1) \leq V_t(y_1, y_2 - 2)$. We further consider two cases: (a) $p - \Delta_{1t+1}^1(y_1 - 1, y_2 - 1) \geq \frac{\theta_{22}}{\Lambda_{2}}[p - \Delta_{1t+1}^2(y_1 - 1, y_2 - 1)]$ and (b)
\( p - \Delta^1_{t+1}(y_1 - 1, y_2 - 1) < \frac{\theta_{22}}{\Lambda^{12}_{10}} [p - \Delta^2_{t+1}(y_1 - 1, y_2 - 1)] \). We provide the proof for case (a). When \( y_1 > 1 \) and \( y_2 > 2 \), the possible optimal assortment sets under the relevant inventory levels are:

<table>
<thead>
<tr>
<th>( S^*_1 )</th>
<th>( (y_1 - 1, y_2 - 1) )</th>
<th>( (y_1, y_2 - 2) )</th>
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The proof for segment 2 is straightforward. For segment 1, if \( p - \Delta^2_{t+1}(y_1 - 1, y_2 - 1) \geq \frac{\theta_{12}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1 - 1, y_2 - 1)] \) and \( p - \Delta^2_{t+1}(y_1, y_2 - 2) \geq \frac{\theta_{22}}{\Lambda^{12}_{10}} [p - \Delta^1_{t+1}(y_1, y_2 - 2)] \), then the optimal assortment for any customer type under either of the inventory levels \((y_1 - 1, y_2 - 1), (y_1, y_2 - 2), \) or \((y_1, y_2 - 1)\), is \{1, 2\}. In this case, it is easy to prove that \( P5(t, 2, y_1, y_2) \) holds for each customer segment. If \( p - \Delta^2_{t+1}(y_1 - 1, y_2 - 1) \geq \frac{\theta_{12}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1 - 1, y_2 - 1)] \), \( p - \Delta^2_{t+1}(y_1, y_2 - 2) \geq \frac{\theta_{22}}{\Lambda^{12}_{10}} [p - \Delta^1_{t+1}(y_1, y_2 - 2)] \), and \( p - \Delta^2_{t+1}(y_1, y_2 - 1) \geq \frac{\theta_{11}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1, y_2 - 1)] \), then \( S^*_1(y_1 - 1, y_2 - 1) = \{1, 2\}, S^*_1(y_1, y_2 - 2) = \{1\}, \) and the optimal assortment for segment 2 is always \{1, 2\}. Then,

\[
\theta_0p + (\Lambda^2_{120})V_1(y_1 - 1, y_2 - 1) - \theta_0 V_1(y_1, y_2 - 1) - \theta_{22} V_1(y_1, y_2 - 2) = \theta_0p + (\Lambda^2_{120})[\theta_{11} V_1(y_1 - 1, y_2 - 1) + \theta_{12} V_1(y_1 - 1, y_2 - 2) + \theta_0 V_1(y_1, y_2 - 1)]
\]

The first term is positive because \( S^*_1(y_1 - 1, y_2 - 1) = \{1\} \). The second and third terms are positive because \( S^*_2(y_1 - 1, y_2 - 1) = \{1, 2\} \) and \( S^*_2(y_1, y_2 - 1) = \{1, 2\} \), respectively.

If \( p - \Delta^2_{t+1}(y_1 - 1, y_2 - 1) \leq \frac{\theta_{12}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1 - 1, y_2 - 1)] \), \( p - \Delta^2_{t+1}(y_1, y_2 - 2) \leq \frac{\theta_{22}}{\Lambda^{12}_{10}} [p - \Delta^1_{t+1}(y_1, y_2 - 2)] \), and \( p - \Delta^2_{t+1}(y_1, y_2 - 1) \leq \frac{\theta_{11}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1, y_2 - 1)] \), then \( S^*_1(y_1 - 1, y_2 - 1) = \{1, 2\}, S^*_1(y_1, y_2 - 2) = \{1\}, \) and \( S^*_1(y_1, y_2 - 1) = \{1, 2\} \). If \( p - \Delta^2_{t+1}(y_1 - 1, y_2 - 1) < \frac{\theta_{11}}{\Lambda^{11}_{10}} [p - \Delta^1_{t+1}(y_1 - 1, y_2 - 1)] \), then \( S^*_1(y_1 - 1, y_2 - 1) = \{1\} \). The proofs of these two cases are similar to that of the previous case.

We next prove part (ii) of Proposition 4, which states that the threshold values are decreasing in time \( t \). We again provide the proof for the case \( i = 2 \). Assume that the optimal assortment is given
by \{1, 2\} in period \(t\) under the inventory level \((y_1, y_2)\), i.e., \(\theta_0(p - \Delta_{t+1}^1(y_1, y_2)) + \theta_2(\Delta_{t+1}^2(y_1, y_2) - \Delta_{t+1}^1(y_1, y_2)) \geq 0\). It suffices to prove that the optimal assortment is still given by \{1, 2\} in period \(t+1\) under the same inventory level, i.e., \(\theta_0(p - \Delta_{t+2}^1(y_1, y_2)) + \theta_2(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2)) \geq 0\). If \(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2) \geq 0\), then \(\theta_0(p - \Delta_{t+2}^1(y_1, y_2)) + \theta_2(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2)) \geq 0\) and the proof is complete. If \(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2) < 0\), then the marginal expected revenue of product 1 is larger than that of product 2 in period \(t+2\), i.e., \(V_{t+2}(y_1, y_2 - 1) - V_{t+2}(y_1 - 1, y_2) > 0\). Similar to the proof of P2, we can show that \(V_{t+1}(y_1, y_2 - 1) - V_{t+1}(y_1 - 1, y_2) \geq V_{t+2}(y_1, y_2 - 1) - V_{t+2}(y_1 - 1, y_2)\) by induction, which implies that \(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2) \geq \Delta_{t+1}^2(y_1, y_2) - \Delta_{t+1}^1(y_1, y_2)\). Moreover, from P2, we have that \(\Delta_{t+1}^1(y_1, y_2) > \Delta_{t+2}^2(y_1, y_2)\). Therefore, \(\theta_0(p - \Delta_{t+2}^1(y_1, y_2)) + \theta_2(\Delta_{t+2}^2(y_1, y_2) - \Delta_{t+2}^1(y_1, y_2)) \geq \theta_0(p - \Delta_{t+1}^1(y_1, y_2)) + \theta_2(\Delta_{t+1}^2(y_1, y_2) - \Delta_{t+1}^1(y_1, y_2)) \geq 0\), concluding the proof. \(\blacksquare\)

**Proof of Proposition 5.** Consider a given period \(t\) and an arriving customer of segment \(m\). We show that if product \(i\) is offered to this customer under a vector of inventory levels \(y\), then it is also offered under the vector of inventory levels \(y + e_i\). Suppose, without loss of generality, that under the vector \(y\), the products’ effective prices are ordered as \(p_i^1(y) \geq p_i^2(y) \geq \cdots \geq p_i^N(y)\) and that the assortment dictated by the heuristic is \(A_k(y) = \{1, \ldots, k\}\). We first show that \(p_i^1(y + e_i) \geq p_i^1(y)\) and \(p_j^1(y + e_i) = p_j^1(y)\) for all \(j \neq i\) (these properties imply properties P1 and P4 in the proof of Theorem 1). In the Sub heuristic, \(p_i^1(y + e_i) - p_i^1(y) = p * Pr(y_i^k < D_i^{OS} \leq y_i^1 + 1) > 0\) and it is immediate that \(p_j^1(y + e_i) = p_j^1(y)\) for all \(j \neq i\).

We will use the following result in this proof: If \(A\) is the optimal assortment for an arriving customer of segment \(m\), then \(p_i^1(y) \geq \sum_{j \in A} q_{mj}(A) p_i^1(y)\) for all \(i \in A\). If this inequality did not hold, we would have \((1 - q_{mi}(A)) p_i^1(y) < \sum_{j \in A \setminus \{i\}} q_{mj}(A) p_j^1(y)\), which implies \(p_i^1(y) < \sum_{j \in A \setminus \{i\}} q_{mj}(A) p_j^1(y)\). If that is the case, let \(a = \sum_{j \in A \setminus \{i\}} \theta_{mj} p_j^1(y)\) and \(b = \sum_{j \in A \setminus \{i\}} \theta_{mj} + \theta_{m0}\). Denote \(\theta\) and \(x\) as the preference and effective marginal price of product \(i\), respectively. Note that \((a + \theta x)/(b + \theta) < a/b\) for any positive \(\theta\) if and only if \(x < a/b\). Therefore, \(p_i^1(y) < \sum_{j \in A \setminus \{i\}} q_{mj}(A \setminus \{i\}) p_j^1(y)\) would imply that \(\sum_{j \in A} q_{mj}(A) p_i^1(y) < \sum_{j \in A \setminus \{i\}} q_{mj}(A \setminus \{i\}) p_j^1(y)\), contradicting the optimality of \(A\).

Suppose now that \(i\) is in the optimal assortment \(A_k\) under the vector \(y\) (i.e., \(i < k\)). We want to show that \(i\) is also in the optimal assortment under the vector \(y + e_i\). Let \(\hat{A}\) be the optimal
assortment under $\mathbf{y} + e_i$. Because $i \in A_k$, we have that
\[
p_i^t(\mathbf{y}) \geq \sum_{j \in A_k} q_{m_j(A_k)} p_i^j(\mathbf{y}) \geq \sum_{j \in \tilde{A}} q_{m_j(\tilde{A})} p_i^j(\mathbf{y}),
\]
where the last inequality follows because $A_k$ is optimal under the vector $\mathbf{y}$. Suppose that $i \not\in \tilde{A}$. Then,
\[
p_i^t(\mathbf{y} + e_i) \geq p_i^t(\mathbf{y}) \geq \sum_{j \in \tilde{A}} q_{m_j(\tilde{A})} p_i^j(\mathbf{y}) = \sum_{j \in \tilde{A}} q_{m_j(\tilde{A})} p_i^j(\mathbf{y} + e_i),
\]
which is a contradiction. This implies that for each product $i$, segment $m$, and time period $t$, there exists a threshold $y_{it}^m(\mathbf{y})$ such that product $i$ is offered to an arriving customer of segment $m$ if the current inventory level $y_i^t \geq y_{it}^m(\mathbf{y})$.

We now show that $y_{it}^m(\mathbf{y})$ is increasing in $y_j$ for $j \neq i$. To that end, we show that if product $i$ is offered under the vector of inventory levels $\mathbf{y}$, then it is also offered under the vector $\mathbf{y} - e_j$ for $j \neq i$. Let $A^*$ and $A_j^*$ be the optimal assortments under the vectors $\mathbf{y}$ and $\mathbf{y} - e_j$, respectively. Then,
\[
\sum_{k \in A_j^*} q_{mk}(A_j^*) p_k^j(\mathbf{y} - e_j) \leq \sum_{k \in A_j^*} q_{mk}(A_j^*) p_k^j(\mathbf{y}) \leq \sum_{k \in A^*} q_{mk}(A^*) p_k^j(\mathbf{y}) \leq p_i^t(\mathbf{y}) = p_i^t(\mathbf{y} - e_j),
\]
where the first inequality follows because $p_k^j(\mathbf{y} - e_j) = p_k^j(\mathbf{y})$ for $k \neq j$ and $p_i^t(\mathbf{y} - e_j) \leq p_i^t(\mathbf{y})$, the second inequality follows from the optimality of $A^*$ under the vector $\mathbf{y}$, and the last inequality follows because $i \in A^*$. We thus have that $i \in A_j^*$, concluding the proof. ■