Magical Thinking: A Representation Result*

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Abstract

This paper suggests a novel way to import the approach of axiomatic theories of individual choice into strategic settings and demonstrates the benefits of this approach. We propose both a tractable behavioral model as well as axioms applied to the behavior of the collection of players, focusing first on Prisoners’ Dilemma games. A representation theorem establishes these axioms as the precise behavioral content of the model, and that the model’s parameters are (essentially) uniquely identified from behavior. The behavioral model features magical thinking: players behave as if their expectations about their opponents’ behavior vary with their own choices. The model provides a unified view of documented behavior in a range of often-studied games, such as the Prisoners’ Dilemma, the Battle of the Sexes, Hawk-Dove, and the Stag Hunt, and also generates novel predictions across games.

Key words: Magical thinking, axioms/representation theorem, prisoners’ dilemma, coordination games

JEL Codes: C7, D8

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1 Introduction

This paper suggests a novel way to import the approach of axiomatic theories of individual choice into game-theoretic settings. We propose a behavioral model of play in symmetric $2 \times 2$ games, which features *magical thinking*: players behave as if they expect that choosing an action $a$ increases the likelihood that their opponents also select action $a$. We then provide axioms and a representation result that establishes the equivalence between the axioms and the equilibrium play of the behavioral model, focusing first on behavior in Prisoners’ Dilemma (PD) games. Further, the model’s parameters are (essentially) uniquely identified from behavior.

The novelty lies in the behavioral data to which our axioms apply. The axioms concern players’ preferences over *actions* contingent on the payoffs of the (one-shot) game, rather than preferences over outcomes. In addition, they restrict not only individual behavior, but also place a joint restriction on the behavior of a finite collection of players. We motivate our axioms as simple and intuitive behavioral regularities across games and individuals, without reference to any particular strategic model.

The contribution of the paper is therefore threefold. First, we provide a tractable and empirically plausible theory of magical thinking, a phenomenon that has received attention in psychology and philosophy (discussed below), applied to strategic games. Most importantly here, we demonstrate that our model provides a unified view of observed behavior in a range of often-studied games including the Battle of the Sexes, Hawk-Dove (also known as Chicken), and the Stag Hunt, in addition to the PD.

Second, distinct from typical work in applied or behavioral game theory, we present a representation result that establishes equivalence between the model’s predictions and a set of empirically plausible axioms. This result allows for the evaluation and empirical testing of the model, and facilitates its comparison to alternative theories. Further, the model’s parameters can be identified from behavior, which is both useful for comparative statics and allows the analyst to traverse between the model and the axioms whenever convenient. For example, observed behavior satisfying the axioms on PD games can be used to identify the parameters of the model, which can be used in turn to generate predictions for (not-yet-observed) behavior in a different set of games. All of this is important for applied work.

Third, a key component of our approach is that the axioms apply to players’ preferences over actions (rather than outcomes). Axiomatizing this type of data confers the following benefits. (1) The primitive of our axiomatic analysis is exactly the type of data we aim to address, namely players’ preferences over their own actions, across games and across players. (2) This type of data is straightforward and common to collect in experiments. (3) We do not have to rely on auxiliary assumptions about an equilibrium concept or on commonality.
of beliefs. Instead, as we discuss below, we can derive these, as well as individual value functions, from the data. We hope that our approach will prove useful in future research beyond this one application.

The Domain of Games: For several reasons, we begin our analysis on the set of PD games. PD games constitute perhaps the most important class of games in applications, and cooperation in the (one-shot) PD is a much discussed behavioral puzzle.¹ We demonstrate that our model makes behavioral predictions distinct from other explanations of cooperative behavior in PD games. Further, focusing on PD games helps build intuition for the workings of the model. Most importantly though, we demonstrate that behavior in PD games provides sufficient data to precisely characterize the behavioral model via axioms and identify its parameters. Isolating such a small, yet economically interesting, domain for both the representation and identification results has the same advantages as it does in theories of individual choice.²

We then apply the behavioral model to all symmetric 2×2 games, where its predictions continue to align with experimental evidence.³ Hence, the model’s ability to explain behavior is not tailored to PD games at the expense of descriptive accuracy in other games in the class, but instead it provides a single account of observed play. Correspondingly, the model generates novel predictions for how behavioral patterns should correlate across games. Finally, we extend the model to allow for larger action sets and investigate the manner in which the connection between magical thinking and cooperative behavior likewise extends.

Further extensions of the model are possible, but would require additional modeling choices. At a very general level, the two key components are that players believe their action choices have stochastic influence over the decisions of others and that equilibrium beliefs are biased as a result (evidence for each is discussed in Section 4). In principle, players could have arbitrary (magical) beliefs about how their choices affect others. However, given the formulation of magical thinking (and related concepts) in psychology and philosophy, we

¹Of course, cooperation is easier to explain in the repeated PD, provided players are patient enough. For finitely repeated versions, reputation models starting with Kreps et al. (1982) offer a potential explanation. For infinitely repeated versions, cooperation models starting with Kreps et al. (1982) offer a potential explanation. For infinitely repeated versions, cooperation is part of some subgame perfect Nash equilibria. Interestingly, in an infinitely repeated, noisy version, Fudenberg et al. (2012) find substantial levels of cooperation (over 30% across all rounds) under parameters for which the unique equilibrium strategy is always defect.

²First, it distills the behavioral implications of the psychological phenomenon (here magical thinking) by abstracting from as many complications as possible. Second, the less data needed for identification of the parameters, the better. Third, the model can be assessed by testing the individual axioms on the small domain of interest (for example, see our comparison to alternative explanations of cooperation in PD games in the Supplement, §S.1). In contrast, an axiomatization of the same model on a larger domain might involve axioms that have less bite when restricted to the small domain, and can therefore not serve as a checklist for testing the model on that domain.

³Extension of the axiomatic analysis is found in the Supplement, §S.3.
believe a natural starting point is for players to believe they influence others to select “the same action” as they do. Although it is possible that real-world context could imbue meaning into strategically irrelevant action labels, symmetric games provide a setting in which “the same action” is meaningful strategically. Next, in games with more players, there would be the added modeling choice of which other players $i$ believes he is influencing and whether or not there is correlation in his perceived influencing.\footnote{For example, in a two-party voting game, player $i$ might believe that turning out independently increases the probability that others from his own party turn out, while not affecting the turnout of the opposing party.} These modeling choices will likely need to be tailored to the setting at hand, but the two key features would remain.

**Summary of Results**: Within the model, each player $i$ in the collection of players, $I$, is endowed a type, $\alpha_i$, and there is a CDF over types, $F$, from which players perceive types to be i.i.d. draws. Given a game, player $i$ forms the following nonstandard beliefs. He assigns probability $\alpha_i$ that the action of his anonymous opponent, $j$, will correlate perfectly with his own, and probability $(1 - \alpha_i)$ that $j$’s action will be determined independently. In the latter case, $i$’s belief about $j$’s behavior is consistent with $j$’s equilibrium strategy. We refer to $\alpha_i$ as $i$’s degree of magical thinking. A player with $\alpha_i = 0$ corresponds to a standard game-theoretic agent—though, one who recognizes that he may be playing against a nonstandard opponent. We characterize the equilibria of the model, and establish a necessary and sufficient condition on $F$ for the equilibrium to be unique in all PD games.

Turning to the axioms, as one would expect, some of them describe plausible regularities of individual behavior. Specifically, we posit *Monotonicity*, which requires that a player who is willing to defect in one PD does not prefer to cooperate in another PD with greater payoffs from defection, as well as appropriate notions of *Continuity*, *Convexity*, and *Invariance to Positive Affine Payoff Transformations*. In addition, we posit a novel *Inter-Player Sensitivity Comparison* axiom. Roughly, the idea behind the axiom is that the behavior of a player who is more prone to defection is also more sensitive to changes in the gains from defecting on a cooperating opponent. We will see that this pattern is consistent with a player’s willingness to cooperate being responsive to the true cost of doing so. In surveying the experimental literature, we find that our axioms are broadly consistent with the available evidence and also offer new testable implications for future studies.

Our representation theorem establishes that the axioms are equivalent to the behavioral model with the condition on $F$ that is necessary and sufficient for uniqueness of the equilibrium in all PD games. Further, the collection of types $(\alpha_i)_{i \in I}$ and the quantiles $(F(\alpha_i))_{i \in I}$ are uniquely identified from behavior, which allows us to provide stronger comparative statics in
terms of those parameters. Finally, note that in the representation, $F$ is the common belief among players regarding the distribution that types are drawn from. In the Supplement, §S.2, we provide an axiomatic characterization of this belief being *empirically valid* when the collection of players is arbitrarily large.

In addition to generating a positive degree of cooperation in PD games that decreases monotonically with the incentives for defection, the model comports with observed behavior in other well-known games. In Hawk-Dove games, our model predicts that players will choose *dove* more often than is predicted by the symmetric (mixed-strategy) Nash equilibrium of the standard model, in line with experimental evidence. In Battle of the Sexes games, the prediction of our model matches the symmetric (mixed-strategy) Nash equilibrium of the standard model, which also aligns with experimental findings. Consider next coordination games with multiple symmetric Nash equilibria that are Pareto ranked (e.g., the Stag Hunt game). Our model uniquely predicts coordination on the payoff-dominant Nash equilibrium only if it is also not “too risky,” in a sense similar to the concept of *risk dominance* (Harsanyi and Selten, 1988), and in line with evidence. However, the prediction is more nuanced than risk dominance in that whether the payoff-dominant Nash equilibrium is too risky or not depends on the (perceived) distribution of types, $F$.

Note that the model’s ability to capture all of these findings does *not* owe to any flexibility across different games. Our results show that play in PD games alone (essentially) pins down the model, leaving no additional flexibility. Hence, the model also makes predictions across classes of games that are often studied independently. For example, collections with higher rates of cooperation in PD games also have a larger set of coordination games in which the payoff-dominant Nash equilibrium is uniquely selected in our model.

Of course, alternative explanations of non-standard behavior in games—most notably models based on other-regarding preferences—have been studied and shown to align with important experimental findings. However, in both PD games as well as other prominent games in our domain, there remains significant evidence of non-standard behavior that is not explained by these theories, but is consistent with our model of magical thinking, as we discuss in Sections 4, 5.1, and the Supplement, §S.1.

**Magical Thinking:** Psychologists have collected evidence that is consistent with individuals exhibiting magical thinking. Starting first with inanimate “opponents,” the term *illusion of control* was coined by Langer (1975) to describe subjects who acted as if their choices had influence over physical outcomes. For example, subjects placed higher bets on a coin about

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5That is, because of identification, our comparative statics (Section 3) describe not only the implication of changes in parameters for changes in behavior (as is common in applied game theory), but can establish equivalence between them (as is standard in decision theory).
to be flipped than on a coin already flipped, but whose outcome was still unknown.\textsuperscript{6}

Section 4 discusses evidence suggestive of magical thinking in strategic settings. Presenting one example here may be useful. Shafir and Tversky (1992) had subjects play a standard PD with the twist that in some treatments the game was played sequentially, such that one player knew the other’s action before choosing his own. They observe that second-movers cooperate significantly less often in the sequential PD—even following cooperation by the first-mover—than in the standard, simultaneous-move version of the game. This finding is inconsistent with standard forms of other-regarding preferences (such as reciprocity), but can be explained by players believing that their actions directly influence their opponents’ not-yet-chosen action, but cannot influence those that have already been taken.

Throughout, we refer to magical thinking as the belief that one’s action choice influences one’s opponent to choose the same action. A related notion is found in a normative debate in philosophy that concerns Newcomb’s Paradox (Nozick, 1969) and extends to the PD if one presumes a notion of \textit{self similarity}.\textsuperscript{7} \textit{Evidentiary} decision theorists argue that one’s opponent is probably similar to one’s self, and hence one \textit{should} believe that the other player will go through the same deliberations and come to a similar conclusion as one’s self (Lewis, 1979; Jeffrey, 1983). They conclude from this that cooperation is the optimal choice. Hence, while their psychological mechanism is slightly different, evidentiary decision theorists advocate for a player to behave as if his choice influences his opponent’s choice, and the notion is observationally equivalent to magically thinking on our domain. In contrast, \textit{causal} decision theorists argue that one \textit{should not} believe that one’s own action affects the other player’s action, as the simultaneous-move game leaves no room for a causal explanation (Joyce, 1999).

We mention this debate not because we will participate in it—the nature of the behavioral data we consider presupposes that magical thinking is a cognitive error—but to highlight that a number of intelligent, serious individuals have reasoned in such a manner.\textsuperscript{8} Finally, a similar idea is apparent in common causal reasoning, such as, “I contribute/recycle/volunteer because if I didn’t, then how could I \textit{believe} that others are doing it?”

The remainder of the paper is organized as follows. For PD games, Section 2 presents our model, axioms, and representation theorem. Section 3 presents comparative statics, and Section 4 compares our theory to experimental evidence and alternative theories of play. Section 5 first applies the model to all symmetric 2×2 games and then extends it to allow for larger action sets. Section 6 provides extended discussion including a comparison of

\textsuperscript{6}The interpretation that a decision-maker’s beliefs about random states of nature vary with his own choice is also common in the theory of ambiguity aversion (see, for example, Gilboa and Schmeidler (1989)).

\textsuperscript{7}Such as described by Rubinstein and Salant (2016) (and citations therein) as the belief that others are likely to make similar judgements and choices as one’s self.

\textsuperscript{8}Experimental evidence suggestive of evidentiary reasoning is found in Quattrone and Tversky (1984).
our axiomatic methodology to alternative approaches. Proofs are in the Appendix. Three supplements, §S.1–§S.3, contain extended formal results.

2 A Theory of Magical Thinking

We begin with the class of Prisoners’ Dilemma games as shown in Figure 1, where \( r > p \) and \( x, y > 0 \), which we refer to as \( PD^0 \) (the reason for the superscript will become apparent shortly).\(^9\) In each game, two players, \( i \) and \( j \), can each choose to defect (\( d \)) or to cooperate (\( c \)). Often \( r + x \) is denoted as \( t \) and \( p - y \) as \( s \), but the above parametrization will be more convenient for our purposes. Note that \( x \) captures the benefit from defecting on a cooperating opponent, while \( y \) is the benefit from defecting on a defector. We refer to an arbitrary game as \( g \in PD^0 \), or, if it is useful to be more explicit about its payoffs, as \((r, p, x, y)\).

\[
\begin{array}{c|cc|c|c}
 & c & d & & \\
\hline
r, r & p - y, r + x & & \\
\hline
r + x, p - y & & p, p & \\
\end{array}
\]

Figure 1: An Arbitrary Prisoners’ Dilemma in \( PD^0 \)

We consider a finite collection of players, indexed by \( I := \{1, \ldots, n\} \), and each player \( i \)'s preferred action for each possible game in \( PD^0 \) when played as a one-shot game against an anonymous opponent, as is typical in experimental settings.

We present the behavioral model, or representation, first and then the axioms in Section 2.2. Compared to axiomatic theories of individual choice, the most notable procedural difference is the necessity to conduct equilibrium analysis (Section 2.1.1) in order to apply our representation.\(^10\)

2.1 The Behavioral Model

For the set of atomless probability distributions each with support \([0,1]\) and differentiable CDF, let \( \mathcal{F} \) be the corresponding set of CDFs. In the behavioral model, each player \( i \in I \) is

\(^9\)Throughout, we interpret game-payoffs in monetary terms to facilitate comparison with experimental findings. However, there is no formal sense in which our theory relies on this interpretation rather than the interpretation of game-payoffs as vNM utilities, as is customary in game theory. (See the discussion of methodology in Section 6 for more.)

\(^10\)This can be viewed as a generalization of the single-agent exercise. There, the prototypical result is the equivalence between axioms and a decision-maker acting as if he maximizes a certain utility function. However, each choice problem can be interpreted as a single-player game, with the notions of optimization and equilibrium coinciding. Therefore, the standard result is identical to showing that the axioms are equivalent to the decision-maker playing an equilibrium in every (single-player) game where payoffs are defined by the utility representation.
privately endowed with a type $\alpha_i \in [0, 1]$. In addition, there is a common prior that types are drawn i.i.d. from a distribution with CDF $F \in \mathcal{F}$. For each $g \in PD^0$, player $i$ evaluates the expected payoff of action $a_i \in \{c, d\}$ as:

$$V_i(c) = \alpha_i \cdot r + (1 - \alpha_i) \left[ P_i \cdot (p - y) + (1 - P_i) \cdot r \right]$$
$$V_i(d) = \alpha_i \cdot p + (1 - \alpha_i) \left[ P_i \cdot p + (1 - P_i) \cdot (r + x) \right]$$  \hspace{1cm} (1)$$

where $P_i$ is the probability $i$ assigns to being defected on in game $g$, conditional on $a_i$ and $a_j$ being determined independently. That is, $i$ evaluates options as if he thinks that there is probability $\alpha_i$ that his opponent will match whatever action choice $i$ makes, and probability $1 - \alpha_i$ that his opponent determines $a_j$ uninfluenced by $a_i$. This is the sense in which player $i$ exhibits magical thinking, and the degree to which he does so is measured by $\alpha_i$.

Given a game $g \in PD^0$, a strategy for player $i$ (denoted $\sigma_i$) is completely characterized by the probability with which he selects $a \in \{c, d\}$ if his type is $\alpha_i$ (denoted $\sigma_i(a|\alpha_i) \in [0, 1]$), and his interim expected payoff from strategy $\sigma_i$ is $\sigma_i(c|\alpha_i)V_i(c) + \sigma_i(d|\alpha_i)V_i(d)$.\footnote{There can be measurability issues for mixed strategies with uncountable type spaces (Aumann, 1964). We use a convenient formulation that handles those issues. A strategy is a function $\sigma_i : \mathcal{A} \times [0, 1] \rightarrow [0, 1]$, where $\mathcal{A}$ is the collection of all subsets of $\{c, d\}$, that satisfies two properties: (i) for every $B \in \mathcal{A}$, the function $\sigma_i(B|\cdot) : [0, 1] \rightarrow [0, 1]$ is measurable, and (ii) for every $\alpha_i \in [0, 1]$, the function $\sigma_i(\cdot|\alpha_i) : \mathcal{A} \rightarrow [0, 1]$ is a probability measure. In a slight abuse of notation, then, we write $\sigma_i(\{a\}|\alpha_i)$ as $\sigma_i(a|\alpha_i)$, and if $\sigma_i(a|\alpha_i) = 1$ we say that player $i$ “chooses/selects/plays” action $a$ when his type is $\alpha_i$. See Milgrom and Weber (1985) for further details and equivalence between this and other notions of mixing with uncountable type spaces. Finally, while the formula for interim expected payoff is standard (taking (1) as given), it implies that the bias in a player’s beliefs depends only on his type and ultimate action choice, and not on $\sigma_i(\cdot|\alpha_i)$ directly.} Throughout, we consider only symmetric equilibria, defined as follows.

**Definition 2.1** Fix any CDF $F$ and $g \in PD^0$. An equilibrium is a pair $(\sigma, P)$, such that, for all $i \in I$, and $V_i$ as given by (1),

1. $\sigma_i = \sigma$
2. For $a, a' \in \{c, d\}$, $\sigma(a|\alpha_i) > 0 \implies V_i(a) \geq V_i(a')$
3. $P_i = P = \int_0^1 \sigma(d|\alpha)dF(\alpha)$

The first two requirements are standard: the first is the symmetry condition, and the second states that the strategy assigns positive probability only to actions that yield the highest expected payoff, given a player’s type and beliefs. The third requires that any player’s belief conditional on not influencing his opponent is consistent with his opponent’s equilibrium strategy. If $\alpha_i = 0$, player $i$ corresponds to a standard game-theoretic agent in that he assigns probability zero to directly influencing his opponent, and his belief about his opponent’s behavior is consistent with his opponent’s equilibrium strategy. If $\alpha_i > 0$, player
i’s belief is a convex combination of this belief and the belief that i’s opponent will match the action played by i.\footnote{Because players in the model seek to maximize their expected payoff (albeit, with non-standard beliefs), one could obviously employ an alternative, reduced-form assumption that a player simply receives a direct utility gain from selecting c. In Section 4 we discuss how this modeling choice would require a counterintuitive form of dependence on the payoff parameters to emulate our model (which is only exacerbated when we extend to games beyond PD\(^0\)) and be at odds with additional experimental evidence.}

2.1.1 Equilibrium Analysis

We now characterize the equilibrium properties of the behavioral model. First, we observe that the set of equilibria is invariant to positive affine transformations of the payoffs.

**Lemma 1** If \((\sigma, P)\) is an equilibrium of the game \((r, p, x, y) \in PD^0\), then it is also an equilibrium of the game \(\kappa(r + \xi, p + \xi, x, y) \in PD^0\) for all \(\kappa > 0\) and \(\xi \in \mathbb{R}\).

All proofs are located in the Appendix. From the lemma, the set of equilibria is identical in games \((r, p, x, y)\) and \((1, 0, \frac{x}{r-p}, \frac{y}{r-p})\); the latter being the positive affine transformation of the former with \(\kappa = \frac{1}{r-p} > 0\) and \(\xi = -p\). Let \(PD \subset PD^0\) denote the subset of games in which \(r\) and \(p\) are normalized to 1 and 0, respectively, with \((x, y) \in PD\) being an arbitrary element. Given Lemma 1, it is sufficient to characterize equilibrium behavior for games in \(PD\), which we focus on for the remainder of Section 2.1.

**Definition 2.2** An equilibrium \((\sigma, P)\) is a **cutoff equilibrium** if \(\sigma\) is of the form: \(\sigma(d|\alpha) = 1\) if \(\alpha < \alpha^*\), and \(\sigma(d|\alpha) = 0\) if \(\alpha > \alpha^*\), for some \(\alpha^* \in [0, 1]\).

**Proposition 1** For any \(F \in \mathcal{F}\) and \((x, y) \in PD\), i) any equilibrium is a cutoff equilibrium with \(\alpha^* \in (0, 1)\), ii) \(\alpha^*\) is an equilibrium cutoff if and only if it is a solution to (2) below, and iii) an equilibrium exists.

- Fixing any \((x, y) \in PD\), the cutoff nature of the equilibrium is immediate: for any (common) equilibrium belief \(P_i = P, V_i(c) - V_i(d)\) is strictly increasing in \(\alpha_i\). Then, in equilibrium, \(P_i = P = F(\alpha^*)\), and the cutoff type, \(\alpha^*\), is indifferent between \(c\) and \(d\). So, the set of equilibria is identical to the set of solutions to the equation:\footnote{Definition 2.2 does not specify the behavior of the cutoff type, who is indifferent between \(c\) and \(d\). We will not always distinguish equilibria that have the same cutoff, but in which the cutoff type behaves differently since this type has measure zero and the distinction has no effect on payoffs.}

\[
V_i(c|\alpha_i = \alpha^*) = \alpha^* \cdot 1 + (1 - \alpha^*) \left[F(\alpha^*) \cdot (-y) + (1 - F(\alpha^*)) \cdot 1\right]
= \alpha^* \cdot 0 + (1 - \alpha^*) \left[F(\alpha^*) \cdot 0 + (1 - F(\alpha^*)) \cdot (1 + x)\right] = V_i(d|\alpha_i = \alpha^*) \quad (2)
\]

Noting that for \(\alpha_i = 0\), \(V_i(c|\alpha_i = \alpha^*) < V_i(d|\alpha_i = \alpha^*)\) and for \(\alpha_i = 1\), \(V_i(c|\alpha_i = \alpha^*) > V_i(d|\alpha_i = \alpha^*)\), all solutions to (2) are interior and existence is guaranteed by the continuity of both the left- and right-hand sides. This leaves only the question of uniqueness.
Proposition 2 For any fixed $F \in \mathcal{F}$, there is a unique equilibrium cutoff in each $(x, y) \in PD$ if and only if $\frac{F'(\alpha)}{F(\alpha)} \leq \frac{1}{\alpha - \alpha^2}$ for all $\alpha \in (0, 1)$ (hereafter referred to as Condition $S$).

Condition $S$ restricts how steep $F$ can be, by limiting its reverse hazard rate, in a manner that depends on $\alpha$. For example, the CDF $F(\alpha) = \alpha^{1/k}$, $k \geq 1$, satisfies the condition, even though $\frac{F'(\alpha)}{F(\alpha)} \to \infty$ as $\alpha \to 0$. Note, then, that by taking $k$ arbitrarily large, we can generate arbitrarily close approximations of the standard model (in which $F(\alpha) = 1$ for all $\alpha \in [0, 1]$), while continuing to satisfy Condition $S$.

To gain intuition for the potential multiplicity of equilibria, first note that for type $\alpha_i$, defection carries the cost of $r - p = 1$ with perceived probability $\alpha_i$, while the benefit of defection is $F(\alpha_i)y + (1 - F(\alpha_i))x$ with perceived probability $1 - \alpha_i$. If $x > y$, then the benefit of defection is decreasing in $F(\alpha_i^*)$ (i.e., the probability that one’s opponent defects if his choice is made independently), and the indifference equation (2) has a unique solution.\footnote{For $x = y$, (2) has a unique solution, which is independent of $F$: $\alpha^* = \frac{x}{1 + x} = \frac{y}{1 + y}$.}

But, if $x < y$, then the benefit of defection is increasing in $F(\alpha_i^*)$. If $F$ is steep on some range this means that (in expectation) there are many players making essentially the same calculation; so each is happy to cooperate if the equilibrium calls for all of them to do so, but each prefers to defect if the equilibrium calls for them all to do so. These types face a coordination problem. This problem is ameliorated if $F$ is never too steep. Not surprisingly, the most difficult games to maintain uniqueness are those with the smallest $x$ values, which is used to derive the tightness of Condition $S$ for uniqueness (see the proof in the Appendix).

2.2 The Axioms

We now present the axioms, doing so without reliance on the model. The data we consider is each player $i$’s preferred action for each possible game in $PD^0$ when played as a one-shot game against an anonymous opponent. The behavior of player $i$ partitions $PD^0$ into three sets: the set of games $D^0_i$ for which $i$ strictly prefers $d$, the set of games $C^0_i$ for which $i$ strictly prefers $c$, and the set of games $M^0_i = PD^0 \setminus (D^0_i \cup C^0_i)$ for which $i$ is indifferent in his choice of $d$ or $c$. We denote by $\overline{D}^0_i = PD^0 \setminus C^0_i$ and $\overline{C}^0_i = PD^0 \setminus D^0_i$ the sets of games for which $i$ weakly prefers $d$ or $c$, respectively. The primitive of our analysis is the collection of pairs $(D^0_i, C^0_i)_{i \in I}$, which fully summarizes the behavior of all players in $I$.\footnote{Our primitive differentiates the games where $i$ strictly prefers $d$ or $c$ from those in which he is indifferent. This is analogous to the standard assumption in axiomatic decision theory that the primitive is a preference relation (not simply choice), which also distinguishes strict from weak preferences. Formally, for every $g \in PD^0$, $i$ ranks the actions in $\{d, c\}$. Each ranking is a complete binary relation $>_i$. Our primitive is $(D^0_i, C^0_i)_{i \in I}$, where $D^0_i$ and $C^0_i$ are the subsets of $PD^0$ for which $d>_i c$ and $c>_i d$, respectively.}

Our first four axioms consider individual behavior. It can be noted that a player who adheres to the standard prediction of always defecting, $D^0_i = PD^0$, satisfies all of these axioms (and can never generate a violation of our fifth and final axiom).
Axiom 1 (Invariance to Positive Affine Transformations)

For all $i \in I$, if $(r, p, x, y) \in D^0_i$, then $\kappa (r + \xi, p + \xi, x, y) \in D^0_i$ for all $\kappa > 0$ and $\xi \in \mathbb{R}$; and analogously for $C^0_i$.

The axiom states that positive affine transformations of all game-payoffs have no effect on individual behavior. For the dollar-stakes used in the laboratory, evidence seems to be consistent with the axiom, both in the Prisoners’ Dilemma and also in many other games (see Section 4). The axiom has a flavor of risk neutrality (which we have already seen is part of the behavioral model). One interpretation is that subjects themselves treat strategic risk differently from environmental risk, focusing on the strategic aspects of their choice rather than their attitude toward risk.\footnote{Of course, the axiom is also consistent with the alternative interpretation of game-payoffs as vNM utilities as is customary in game theory. See the discussion of methodology in Section 6 for more.}

Axiom 1 implies that any player $i$ behaves identically in games $(r, p, x, y)$ and $(1, 0, \frac{x}{r-p}, \frac{y}{r-p})$. Hence, under Axiom 1, it is sufficient to characterize behavior on the subset $PD \subset PD^0$. We pose the remainder of our axioms on $PD$, meaning that, on their own, they are weaker than their obvious counterparts applying to $PD^0$. To do so, let $D_i = D^0_i \cap PD$, and analogously for $C_i, M_i, D_i$, and $C_i$.

The remaining two payoff parameters, $x$ and $y$, correspond to the two motives for defection: the exploitative motive of gaining at the expense of a cooperating opponent and reaping an extra payoff of $x$, and the defensive motive to avoid being the “sucker” and losing $y$. Our remaining axioms describe the effects of changing $x$ and $y$ on behavior.

Axiom 2 (Continuity)

For all $i \in I$, $D_i$ and $C_i$ are open.

The axiom says that no individual has a jump from a strict preference for defection to a strict preference for cooperation as the motives for defection vary continuously.

Axiom 3 (Monotonicity)

For all $i \in I$, if $(x, y) \in \overline{D}_i$, $(x', y') \geq (x, y)$, and $(x', y') \neq (x, y)$, then $(x', y') \in D_i$.

The axiom requires that strengthening the motives for defection (at least one of them strictly) will lead a player who initially weakly prefers to defect to strictly prefer defection.

Axiom 4 (Convexity)

For all $i \in I$, $D_i$ and $C_i$ are convex.

The intuition behind the axiom is that a larger change in the motives for defection should have a weakly larger effect on behavior than does a proportionally smaller change. Suppose
that player $i$ strictly prefers to, say, defect in both $(x, y)$ and $(x', y')$. The change from $(x, y)$ to $(x', y')$ can be interpreted as trading off the two motives at a rate, $\frac{y - y'}{x - x'}$, and a scale, normalized to 1. Comparatively, the change from $(x, y)$ to $(\gamma x + (1 - \gamma)x', \gamma y + (1 - \gamma)y')$, where $\gamma \in (0, 1)$, is unambiguously smaller: it trades off the two motives at the same rate, but on a smaller scale. Axiom 4 states that if the larger change in payoffs does not alter $i$’s strict preference for $d$ (or for $c$), then neither should this smaller change in the payoffs.\(^{17}\)

While we allow different players to behave differently in a given game, we now pose a new type of axiom that compares the behavior of any two players across games. Informally, the inter-player axiom says the following: Suppose that player $i$ defects under lower incentives for defection than does $j$. Then, when $i$ is at the cusp of flipping between $d$ or $c$, his choice is more sensitive to changes in $x$ (the exploitative motive) than is $j$’s choice when $j$ is likewise at the cusp.

It seems natural that the interpretation of an inter-player axiom would be contingent on at least some basic properties of individual behavior; in our case this will be Monotonicity (Axiom 3). Intuitively, if a player cooperates in a given Prisoners’ Dilemma game, he does so at a cost to his own game-payoff. This cost depends on his opponent’s behavior: specifically, the more likely the opponent is to cooperate, the greater is the influence of $x$ on this cost. Hence, if all players satisfy Axiom 3, then in games where player $i$ (who defects under lower incentives for defection) is on the cusp of flipping his behavior it must be that the arbitrary opponent is more likely to be cooperating than in games where player $j$ (who defects only under higher incentives for defection) is similarly on the cusp. If behavior is responsive to the true cost of cooperation, then player $i$’s behavior should be more sensitive to changes in $x$ than is player $j$’s. We now present the formalisms.

**Definition 2.3** For $H, H' \subset PD$ we write $H < H'$ if, for all $(x, y) \in H$ and $(x', y') \in H'$, $x < x'$ and $y < y'$.

**Axiom 5 (Inter-player Sensitivity Comparison)**

For all $\{i, j : i \neq j\} \subset I$, and $\varepsilon, \delta \in \mathbb{R}_{++}$, if (i) $\{(x, y), (x + \varepsilon, y - \delta)\} < \{(x', y'), (x' + \varepsilon, y' - \delta)\}$, (ii) $(x, y) \in \overline{D}_i$, (iii) $(x + \varepsilon, y - \delta) \in \overline{C}_i$, and (iv) $(x', y') \in \overline{C}_j$, then (v) $(x' + \varepsilon, y' - \delta) \in C_j$.

To see that the axiom captures the pattern described above, note first that, in the context of Axiom 3, (i), (ii) and (iv) imply that player $i$ indeed defects under lower incentives for

\(^{17}\)It may be useful to note that while reminiscent of the classic two-good consumer-preference diagram, in our context the choice objects are $c$ and $d$, not “$(x, y)$-bundles”; so $M_i$ is not an indifference curve and $D_i$ and $C_i$ are not “better-than/worse-than” sets, meaning Axiom 4 is not related to the standard convexity-of-consumer-preferences assumptions (for example, Mas-Colell et al. (1995, Chap 3.B)).
defection than does player $j$ in the four games. Second, (ii) and (iii) imply that $i$ is willing to flip between choosing $d$ or $c$ when moving from $(x, y)$ to $(x + \varepsilon, y - \delta)$. Third, (iv) says that $j$ is willing to cooperate in $(x', y')$. Now, clearly, the movements from $(x, y)$ to $(x + \varepsilon, y - \delta)$ and from $(x', y')$ to $(x' + \varepsilon, y' - \delta)$ entail the same increase, $\varepsilon$, in the exploitative motive and the same reduction, $\delta$, in the defensive motive. Hence, if, contrary to (v), $j$ were willing to defect in $(x' + \varepsilon, y' - \delta)$, then $j$ would have to be more sensitive to changes in $x$ (relative to changes in $y$) than is $i$, which violates the pattern described at the outset. Hence, Axiom 5 requires that (i)-(iv) imply (v).

We note that insofar as one views both defection in more games and a greater responsiveness to the exploitative motive to be features of a more “aggressive disposition” on the part of player $i$, the axiom is consistent with the view, and the motivation based on objective incentives and Axiom 3 provides a micro-foundation for this correlation.

Finally, as this type of inter-player axiom is novel to our approach, it may be worth previewing the role it plays in the representation result. The intuition provided for the axiom above refers to behavior being “responsive to the true cost of cooperation.” In the representation, player $i$’s behavior is a response to the cost of cooperation as measured by his perception of the distribution $F$, call it $F^i$. The axiom, then, disciplines the heterogeneity in this perception. As we will see, it ensures that $F^i(\alpha_i) \leq F^j(\alpha_j)$ if and only if $\alpha_i \leq \alpha_j$, which must hold if all players perceive the same $F$.

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18 Let $H$ denote the set of four games. To see that $i$ is more prone to defection than $j$ in $H$, note that Axiom 3 implies that $\{(x', y'), (x' + \varepsilon, y' - \delta)\} \subseteq D_i$ and that $\{(x, y), (x + \varepsilon, y - \delta)\} \subseteq C_j$. Therefore, $D_j \cap H \subseteq D_i \cap H$ and $C_i \cap H \subseteq C_j \cap H$.

19 Conversely, if the behavioral model were expanded to accommodate heterogenous perceptions of $F$, and $F^i(\alpha_i) > F^j(\alpha_j)$ despite $\alpha_i \leq \alpha_j$, the implied behavior would violate Axiom 5.
2.3 The Representation Theorem

Having studied the behavioral model and the axioms, we present the representation result.

**Definition 2.4** For $I' \subset I$, the behavior of the players in $I'$, $(D^0_i, C^0_i)_{i \in I'}$, can be explained by the behavioral model $[F, (\alpha_i)_{i \in I}]$ if for all $g \in PD^0$ there exists an equilibrium such that for all $i \in I'$, with $V_i$ as defined by (1):

1. $g \in C^0_i$ if and only if $\{c\} = \operatorname{argmax}_{\{c,d\}} \{V_i(c), V_i(d)\}$
2. $g \in D^0_i$ if and only if $\{d\} = \operatorname{argmax}_{\{c,d\}} \{V_i(c), V_i(d)\}$

**Theorem 1** The primitive $(D^0_i, C^0_i)_{i \in I}$ satisfies Axioms 1-5 if and only if it can be explained by a behavioral model $[F, (\alpha_i)_{i \in I}]$, where $F \in \mathcal{F}$ satisfies Condition S. Furthermore, for all $i \in I$, $\alpha_i$ and $F(\alpha_i)$ are unique.

Before sketching the proof, it is worth noting a few interesting features. First, a central concern in representation results is the degree to which the parameters in the representation, here $F$ and $(\alpha_i)_{i \in I}$, are unique. Theorem 1 establishes that each player’s $\alpha_i$ (the degree to which he exhibits magical thinking) is uniquely determined by the primitive and will, in fact, only depend on $(D^0_i, C^0_i)$ as we sketch below. Further, the quantiles of $F$ at all $\alpha_i$ in the collection are also unique.

Second is the interpretation of the magical thinking component. Given the nature of our primitive, we have taken the position that this is an error, and the choices of each player are not directly influenced by the choices of any other player. In other words, our assumptions about the nature of human agency are the standard ones, but we allow that the players act as if they have nonstandard ones. There is also an important subtlety in understanding the $F$ in the representation: (it is as if) $F$ is the CDF of the distribution that all players perceive the $\alpha$-types to be drawn from. This suggests an interpretation in which the players conceive of a grand population of which $I$ is a random sample. In §S.2, we provide an axiomatic characterization of this belief being empirically valid when the collection is large.

Third, a common concern in game-theoretic analysis is the issue of equilibrium multiplicity.\(^{20}\) A reader might therefore object to the terminology that a model can explain behavioral data if the data is always consistent with one of the model’s equilibria (Definition 2.4) as too permissive. The definition was chosen so that equilibrium uniqueness is not

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\(^{20}\)In single-player games/decision problems, the agent may be indifferent between multiple payoff-maximizers, which can be interpreted as equilibrium multiplicity. However, in this scenario, the payoff to all agents is equivalent across all equilibria (by hypothesis). In general, the same statement does not hold for multiplayer games with multiple equilibria. This is one reason why the multiplicity issue is of perhaps greater concern in game theory than in decision theory.
forced into the very notion of representation. Nevertheless, this objection is easily addressed. Notice that Theorem 1 includes the provision that $F$ satisfies Condition $S$. Under this provision, Proposition 2 (with Lemma 1) guarantees that the equilibrium cutoff is unique for all $g \in PD^0$ (and all equilibria are cutoff equilibria—Proposition 1). It is immediate, therefore, that the representation satisfies the more stringent definition of can explain attained if the requirements of Definition 2.4 must instead hold in all equilibria.

A couple of notational definitions will simplify exposition for the remainder of the paper. Let $\mathcal{F}_S$ denote the set of CDFs in $\mathcal{F}$ that satisfy Condition $S$. Let $\alpha_{-i}$ denote an arbitrary assignment of types to players in $I \setminus \{i\}$ (i.e., $(\alpha_j)_{j \in I \setminus \{i\}}$).

**Sketch of Proof of Theorem 1**

It is clear that Lemma 1 is the precise behavioral content of Axiom 1. Hence, we need only prove that Axioms 2-5 are equivalent to the behavioral model on $PD$.

As is typical, showing that the representation implies the axioms is the easier direction. First, extreme players, $\alpha_i = 0, 1$, either always defect or always cooperate, so trivially satisfy our axioms. Next, recall that in the behavioral model, the unique equilibrium of any game $(x, y) \in PD$ is of cutoff form, where the cutoff, $\alpha^*$, is characterized by (2). To find the set of games in $PD$ for which $i$ is indifferent between $c$ and $d$, we fix $\alpha_i \in (0, 1)$ and solve (2) for $y$ as a function of $x$ to get

$$M_i = \left\{(x, y) \in PD \mid y = \frac{\alpha_i}{1 - \alpha_i} F(\alpha_i) - x \left(1 - \frac{F(\alpha_i)}{F(\alpha_i)}\right)\right\}.$$  

Note that $M_i$ is a downward sloping line in $PD$. $D_i$ and $C_i$ are the strict-upper and strict-lower contour sets of $M_i$, respectively (Figure 3(a)). Axioms 2-4 follow immediately.

In addition, see that $\frac{\alpha}{(1-\alpha)F(\alpha)}$ is weakly increasing and $\frac{1-F(\alpha)}{F(\alpha)}$ is strictly decreasing in $\alpha_i$; the former by Condition $S$, the latter by $F \in \mathcal{F}$. This implies that if $0 < \alpha_i < \alpha_j < 1$, then $M_i$ and $M_j$ do not intersect in $PD$ and, further, that they “fan out” as $x$ increases (Figure 3(b)). It is straightforward to verify that this property ensures Axiom 5.

The proof that the axioms imply the representation has two main parts. In the first part, we show that for any individual player $i$, if $(D_i, C_i)$ satisfies Axioms 2-4, then there exists a pair $(\alpha_i, F_i) \in [0, 1]^2$ such that $(D_i, C_i)$ can be explained by any model $[F, (\alpha_i, \alpha_{-i})]$ satisfying $F \in \mathcal{F}$ and $F(\alpha_i) = F_i$. Further, $\alpha_i$ and $F_i$ are unique. In other words, the axioms on individual behavior are enough to establish that each individual is playing in accordance with our behavioral model—though not necessarily with agreement among individuals about $F$. The second part of the proof establishes that there is a common $F \in \mathcal{F}_S$ that can simultaneously explain all of $(D_i, C_i)_{i \in I}$. This relies on Axiom 5.
To begin the first part suppose that \((D_i, C_i)\) satisfies Axioms 2-4: Continuity, Monotonicity, and Convexity. By Continuity, it is straightforward to show that either: i) \(D_i = PD\), ii) \(C_i = PD\), or iii) \(M_i \neq \emptyset\). If (i), then \((\alpha_i, F_i) = (0, 0)\); and if (ii), then \((\alpha_i, F_i) = (1, 1)\). Suppose now that (iii) holds. Continuity and Monotonicity imply that there is a continuous, strictly decreasing function \(\overline{y}\) such that \(M_i = \{(x, y) \in PD| y = \overline{y}(x)\}\), \(C_i = \{(x, y) \in PD| y < \overline{y}(x)\}\), and \(D_i = \{(x, y) \in PD| y > \overline{y}(x)\}\). Finally, Convexity of \(D_i\) and \(C_i\) mean \(\overline{y}\) is linear, so can be summarized by two scalars that we denote \(\text{int}_i\) and \(\text{slp}_i\): \(M_i = \{(x, y) \in PD| y = \text{int}_i - \text{slp}_i \cdot x\}\).

Having established the linearity of \(M_i\) from behavioral data, recall from the argument above that in the behavioral model

\[
M_i = \left\{ (x, y) \in PD \left| y = \frac{\alpha_i}{1 - \alpha_i} F(\alpha_i) - x \left(1 - F(\alpha_i)\right) \right. \right\}
\]

Inverting the bijection \((\text{int}_i, \text{slp}_i) = \left(\frac{\alpha_i}{1-\alpha_i} F_i, \frac{1-F_i}{F_i}\right)\) establishes the first part of the proof.

For the second part, consider two players \(i\) and \(j\), such that \(M_i, M_j \neq \emptyset\) and who satisfy Axiom 5. This means \(\text{int}_i < \text{int}_j\) implies \(\text{slp}_i > \text{slp}_j\).\(^{21}\) The translation of this condition under the bijection yields that \(0 < \alpha_i < \alpha_j < 1\) implies \(F_i < F_j \leq F_i^\alpha_j(1-\alpha_j)\). The first inequality means that there exists a strictly increasing CDF \(F\) that, together with \((\alpha_i)_{i \in I}\), can simultaneously explain the behavior of all players (the inclusion of the \(\alpha_i = 0, 1\) players is trivial). The second inequality is a discretized version of Condition \(S\). It is then

\(^{21}\)To see this, note that \(\text{int}_i < \text{int}_j\) implies that there are games \((x, y), (x + \epsilon, y - \delta), (x', y')\), and \((x' + \epsilon, y' - \delta)\) that satisfy (i) \(\{(x, y), (x + \epsilon, y - \delta)\} < \{(x', y'), (x' + \epsilon, y' - \delta)\}\), (ii) \((x, y) \in M_i\), (iii) \((x + \epsilon, y - \delta) \in M_i\), and (iv) \((x', y') \in M_j\). Axiom 5 then implies that \((x' + \epsilon, y' - \delta) \in C_j\), and consequently \(\text{slp}_i > \text{slp}_j\).
straightforward, but cumbersome, to show that it is without loss of generality to take $F$ to be differentiable and to satisfy Condition $S$.

Finally, a comment on the properties of $F$ in the representation. As made clear from the sketch above, the axioms do not require $F$ to have full support or to be differentiable, but merely allow for these properties. This is because the data of a finite number of players generates values for $F$ at only a finite number of points (§S.2 provides an analysis with a continuum of players). These features are chosen to be part of the representation because they are commonly assumed, appealing properties for applied models that facilitate a tractable analysis (recall Section 2.1). For example, they allow for a simple statement of Conditions $S$. It is not difficult to show that a larger class of behavioral models satisfies the axioms, and that any primitive that satisfies the axioms can be explained by another model $[F,(\alpha_i)_{i \in I}]$, where $F$ lacks full support and/or is not everywhere differentiable. It is worth noting, however, that the unique identification of parameters in Theorem 1 continues to hold across this larger class of models since, as outlined above, these parameters are pinned down by individual behavior that satisfies Axioms 1-4.

3 Comparative Statics

In this section we illustrate how the predictions of the model vary with the parameters. In light of Axiom 1/Lemma 1, we do so on the smaller set of games, $PD$, without loss.

**Definition 3.1** Let $|A|$ be the size of any finite set of players $A$. Consider two arbitrary sets of players $A$ and $\tilde{A}$ such that $|A| = |\tilde{A}|$.

- We say that, in $H \subset PD$, the set of players $A$ defects (weakly) more than $\tilde{A}$ if $|\{i \in A | (x,y) \in D_i\}| \geq |\{j \in \tilde{A} | (x,y) \in D_j\}|$ for each $(x,y) \in H$.

- We say that the set of players $A$ is (weakly) more influenced by $x$ relative to $y$ than is $\tilde{A}$ if $A$ defects more than $\tilde{A}$ in $\{(x,y) | x \geq y\}$ and $\tilde{A}$ defects more than $A$ in $\{(x,y) | x \leq y\}$.

The notion of defects more is straightforward. For singletons $A = \{i\}$ and $\tilde{A} = \{j\}$, it is simply: in $H \subset PD$, player $i$ defects (weakly) more than player $j$ if $D_j \cap H \subset D_i \cap H$. When convenient, we use the term cooperates (weakly) more for the obvious analog. The notion of more influenced by $x$ isolates the idea that players in set $A$ are more driven to defection than players in $\tilde{A}$ when $x$ is relatively large but without being more prone to defection overall.

We begin with comparative static results that, as is typically done in applied work, investigate the effects of varying one parameter, assuming (rather than determining from behavior) that all other parameters stay fixed. The cutoff feature of equilibria (Proposition
1) immediately gives us our first comparative static: for fixed $F \in \mathcal{F}_S$, a player of type $\alpha$ cooperates more in $PD$ than does a player of type $\tilde{\alpha}$ if and only if $\alpha \geq \tilde{\alpha}$. Intuitively, a player who believes he has more influence over his opponent’s behavior cooperates in a larger set of games.

Proposition 3 below explores how predictions change as the population becomes more inclined toward magical thinking (in the sense of first-order stochastic dominance). It shows the equivalence between a first-order stochastically ranked pair of distributions and properties of both choice behavior in the observable domain (i.e., (b) and (d)) and their manifestations in the behavioral model (i.e., (c) and (e)). This may also serve to illustrate the usefulness of the equivalence between the axioms and representation.

**Proposition 3** For any $F, \tilde{F} \in \mathcal{F}_S$, let $I$ and $\tilde{I}$ be independently drawn collections of common size $n$ from $F$ and $\tilde{F}$ respectively. For any $(x, y) \in PD$, let $\alpha^*_{x,y}$ and $\tilde{\alpha}^*_{x,y}$ be the unique equilibrium cutoffs for $F$ and $\tilde{F}$, and the random variables $k_{x,y}$ and $\tilde{k}_{x,y}$ be the number of players cooperating in their respective collections. The following statements are equivalent:

a) $F$ first-order stochastically dominates (f.o.s.d.) $\tilde{F}$ (i.e., $F(\alpha) \leq \tilde{F}(\alpha) \forall \alpha \in [0, 1]$).

b) For all $(x, y) \in PD$, the distribution of $k_{x,y}$ f.o.s.d. the distribution of $\tilde{k}_{x,y}$.

c) For all $(x, y) \in PD$, $F(\alpha^*_{x,y}) \leq \tilde{F}(\tilde{\alpha}^*_{x,y})$.

d) For any $\alpha \in [0, 1]$, a player of type $\alpha$ is more influenced by $x$ relative to $y$ when facing $F$ than when facing $\tilde{F}$.

e) For any $(x, y) \in PD$, $\alpha^*_{x,y} \leq \tilde{\alpha}^*_{x,y}$ if and only if $x \leq y$.

Interpreting the proposition, (b) and (c) show specific manners in which greater degrees of population-wide magical thinking and of cooperation are synonymous. Notice that (b) is only useful if the analyst either assumes the empirical validity of $F$ and $\tilde{F}$ (see §S.2), or if she is interested in understanding how much cooperation the players themselves predict as their common belief about the distribution of $\alpha$-types changes—which does provide some useful intuition for the final two claims.

The final two statements are perhaps a bit more surprising. They can be interpreted as answering the question, “How does the behavior of the player with magical-thinking type $\alpha$ change if (the players believe that) the magical thinking of the population increases/decreases?” The answer depends on the relative magnitudes of the two motives for defection. From (b) and (c), $F$ f.o.s.d. $\tilde{F}$ means more cooperation from the $F$ population than from the $\tilde{F}$ population. As discussed following Proposition 2, when $x < y$, players want to cooperate if enough others are cooperating, which (d) and (e) reflect. However, when $x > y$, the gain from defecting on cooperators is relatively large, and the $\alpha$-type takes
advantage of increased cooperation in the populace by defecting in more games when facing $F$ than when facing $\widetilde{F}$.

In axiomatic theories of individual choice, customarily, the aim of comparative statics results is to disentangle the behavioral content of different parameters, relying crucially on the separate identification of those parameters. Consider first the individual types $(\alpha_i)_{i \in I}$. If the analysts wishes to know if differences in the behaviors of two collections are at least partially due to differences in individual types, she can leverage the facts that in the model, equilibrium behavior is independent of $F$ when $x = y$ (Section 2.1.1), and that any player’s type can be identified from play in such games. Intuitively, when $x = y$ any player’s incentive to defect is independent of what he believes about his opponent’s decision. This is formalized in Proposition 4(a) below.

For the commonly believed distribution of types, $F$, part (b) of the proposition captures the exact behavioral content of keeping the actual types in the collection fixed and changing only these beliefs. Similar to Proposition 3(a) and (d), (the discretized analog of) a first-order stochastic shift in beliefs is equivalent to players becoming more influenced by $x$.

**Proposition 4** Consider two collections $I$ and $\bar{I}$ such that $|I| = |ar{I}| = n$, and whose behavior is described by $[F, (\alpha_j)_{j \in I}]$ and $[\widetilde{F}, (\tilde{\alpha}_j)_{j \in \bar{I}}]$, respectively, with $F, \widetilde{F} \in \mathcal{F}_S$ and each collection ordered by increasing $\alpha$-values.

a) In $\{(x, y) | x = y\}$, player $i \in I$ defects more than player $j \in \bar{I}$ if and only if $\alpha_i \leq \tilde{\alpha}_j$.

b) Collection $I$ is more influenced by $x$ relative to $y$ than is $\bar{I}$ if and only if, for all $i \leq n$, $\alpha_i = \tilde{\alpha}_i$ and $F(\alpha_i) \leq \widetilde{F}(\tilde{\alpha}_i)$.

4 Evidence and Alternative Theories

In this section, we first discuss how the available experimental evidence aligns with our axioms. We then discuss additional evidence, drawn from studies of manipulated variants of PD games, finding support for the magical-thinking interpretation of the behavioral model.

The rationale for discussing both types of evidence is as follows. The utility of our representation result is that it establishes (a) the (non-obvious) behavioral content of a model built on a documented psychological phenomenon (see the Introduction), applied to a domain of economic interest, and (b) that empirically plausible axioms on the domain of interest can be explained by a tractable model that is not obvious from mere inspection of those axioms. Hence, the first set of evidence presented speaks to the plausibility of the axioms as empirical regularities, while the second set speaks more to the relevance of the psychological decision-making process.

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Starting with Rapoport and Chammah (1965), experimentalists have investigated how the payoffs in the Prisoners’ Dilemma affect observed levels of cooperation. For the stakes typically used in experiments, a positive affine transformation of the game payoffs seems to have little effect on the level of cooperation in the Prisoners’ Dilemma (for example, Jones et al. (1968)), or on play in games more generally (Camerer and Hogarth, 1999; Kocher et al., 2008), consistent with Axiom 1. For very significant stakes, evidence from televised game shows where contestants play a one-shot Prisoners’ Dilemma (of course, without anonymity) paints a similar picture (List, 2006; Van de Assen et al., 2012). In fact, Axiom 1 is commonly assumed, and most experiments do not even test it. Also, as in more familiar contexts, continuity (Axiom 2) is hard to falsify empirically and should be thought of as a technically useful abstraction.

The main experimental finding for Prisoners’ Dilemma games is that a substantial proportion of subjects choose to cooperate (see Dawes and Thaler (1988) for a survey), and that cooperation monotonically decreases with the motives to defect: $x$ and $y$. For example, Charness et al. (2015) find that cooperation levels decrease monotonically from 60% to 23% when varying $(x, y)$ on an increasing path from $(\frac{1}{4}, \frac{1}{4})$ to $(4, 1)$ (modulo positive affine transformations). Monotonicity has also been verified within subject (Ahn et al., 2001; Engel and Zhurakhovska, 2012), giving strong support to Axiom 3. Any theory that aims to explain observed play in PD games should account for this evidence.

Axiom 4 is testable, but the available evidence on play in the PD is too incomplete to evaluate it directly. However, again starting with Rapoport and Chammah (1965), various unidimensional indices have been proposed (though with little theoretical foundation) to capture the magnitude of the incentive to defect, depending on the payoff parameters, and then used to forecast the level of cooperation across different Prisoners’ Dilemma games. Empirically, the best validated of such indices are increasing in $\frac{r-p}{r-p+x+y}$ (see Steele and Tedeschi (1967), for example). This ratio is invariant to positive affine transformations of game-payoffs, consistent with Axiom 1, and becomes $\frac{1}{1+x+y}$ in PD. Therefore, these indices predict that the level curves of constant aggregate cooperation will be thin, linear, and downwards sloping, as they are in our model—owing to Axioms 1-4 and the fact that individual $M_i$ lines do not cross, an implication of Axiom 5. The empirical support for these indices then provides indirect evidence in support of Axioms 1-4, but not of the differing slopes of level curves that are also implied by Axiom 5 (illustrated in Figure 3(b)), meaning

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22Ignoring the possible differences in behavior between the one-shot game and its finitely repeated version, early experimental works simply report aggregate behavior across rounds and subjects (for example, Rapoport and Chammah (1965); Steele and Tedeschi (1967); Jones et al. (1968)). More recent studies of the one-shot game either randomly rematch subjects after every round of play (for example, Ahn et al. (2001)), or use the “strategy method” (Engel and Zhurakhovska, 2012), or truly have each subject play just a single game one time (Charness et al., 2015).
our axioms/model provide a more nuanced prediction.\textsuperscript{23}

Axiom 5 is a novel type of assumption that is central for our theory. It describes the correlation of behavior across players and games. This correlation has not been a focus of experimental investigation. Recall that, if players are sensitive to the true cost of cooperating, Axiom 3 implies Axiom 5. The strong support in favor of Axiom 3, therefore, strengthens the empirical plausibility of Axiom 5. Ultimately, however, the validity of the axiom is an empirical question, and in that sense our theory suggests a fruitful avenue for future experiments.

Because the axioms distill the precise behavioral content of our theory, they facilitate comparison not only with the experimental evidence, but also with alternative models. In §S.1, we formally demonstrate that canonical models with the three most common forms of other-regarding preferences—altruism (Ledyard, 1995; Levine, 1998), inequity-aversion (Fehr and Schmidt, 1999), and reciprocity (Rabin, 1993)—violate our axioms, and hence make different predictions on our domain.\textsuperscript{24} In McKelvey and Palfrey’s (1995) notion of quantal-response equilibrium (QRE) each player chooses every available action with positive probability, which can be interpreted as random errors. Immediately then, QRE predicts a positive degree of cooperation in the Prisoners’ Dilemma. Further, given the distribution of opponent play, the probability of selecting an action increases with the expected payoff from doing so, as is also true in our model. However, despite the many degrees of freedom afforded QRE, its implications for aggregate behavior differ from those of our model.\textsuperscript{25} More importantly though, instead of attributing differences in observed behavior to randomness, our axioms and model speak directly to heterogeneity in individual behavior.

We now discuss evidence suggestive of magical thinking from games that are in natural extensions of our domain (which for brevity we do not formalize here):

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\textsuperscript{23}We are unaware of studies that provide detailed enough data to test the predictions of our model against the predictions based on these indices.

\textsuperscript{24}A succinct intuition is that the most altruistic players in a population always fail Axiom 3 because, in games where they (correctly) predict their opponent will defect with probability one, increasing their opponent’s payoff from doing so increases the altruistic player’s preference for cooperation. The models of inequity-aversion and reciprocity have a coordination feature to them: players are willing to cooperate if and only if they believe cooperation by their opponent is sufficiently likely. This leads to equilibrium multiplicity: for every game, all players defecting is an equilibrium, but in some games cooperation by some players occurs in other equilibria. Further, because of this coordination component, the set of games that have equilibria with some cooperation end abruptly, as coordinated cooperation unravels due to a small increase in the incentive to defect, leading to abundant violations of Axiom 2.

\textsuperscript{25}For example, even though the expected payoff from defection is always larger than from cooperation, (in expectation) the majority of individuals in our model will cooperate for small enough $x$ and $y$, in line with evidence (Charness et al., 2015), but in contrast to QRE. Also, beyond PD games, there are games for which our model predicts that some actions are never played: for instance, any game where the socially optimal action is also dominant (Proposition 5). See also Proposition 10 on games with larger action sets.
1. Most immediately, players in our model would have completely standard preferences over the domain of final game-payoff vectors (unlike altruistic or inequity-averse players). Consistent with this, when the Prisoners’ Dilemma is modified to have a passive opponent (so the unconstrained player is unilaterally selecting the payoff vector), higher rates of “defection” are found (Ellingsen et al., 2012).

2. Shafir and Tversky’s (1992) observation that the level of cooperation by second-movers is significantly lower in the sequential Prisoners’ Dilemma than in the standard, simultaneous-move version of the game—even if the first-mover cooperates—is highly suggestive of magical thinking, but inconsistent with standard forms of other-regarding preferences. Reciprocity, notably, predicts that second-movers should be more likely to cooperate following cooperation than in the simultaneous-move game.26

3. In a similar vein, Morris et al. (1998) find that the temporal order of moves affects cooperation even when the decision of the first-mover is not revealed. Consistent with magical thinking being the belief that one may directly influence the (yet-unchosen) action of one’s opponent, they find greater cooperation when players move first compared to second. Other-regarding preferences (as well as the evidentiary-reasoning interpretation of the beliefs in our model—see the Introduction) provide no rationale for this discrepancy, as play should be invariant to this strategically irrelevant difference in the games.

4. In a number of studies, experimental subjects played Prisoners’ Dilemma games and were also asked to predict the behavior of their opponents. Subjects who defected were more likely to predict that their opponents would defect.27 This feature is implied by the interpretation of our model, but absent from models with other- regarding preferences. While there may seem to be a sense in which it is consistent with reciprocity—players are more likely to cooperate when they expect cooperation from others—it is clearly inconsistent with standard notions of equilibrium, even if players care about reciprocity. Either the cooperators are too optimistic or the defectors are too pessimistic about their opponents’ behavior.

26 In the sequential-move game almost no second-movers cooperated after observing defection by their opponent. Perhaps more surprisingly, only about 15% of second-movers cooperated after observing cooperation. At the same time, and in line with other PD experiments (Dawes and Thaler, 1988), 37% of subjects cooperated in the standard, simultaneous-move PD. In the study only a small subset of the games each subject played were Prisoners’ Dilemma games. There is some evidence that repeated play of the one-shot, sequential Prisoners’ Dilemma can reverse their observation (Clark and Sefton, 2001), possibly because ethical considerations, like reciprocity, become more salient through frequent, uninterrupted repetition.

27 See Dawes et al. (1977); Orbell and Dawes (1991); Engel and Zhurakhovska (2012); Rubinstein and Salant (2014). Rubinstein and Salant (2014) suggest that players’ ex-post reported beliefs will accurately reflect the beliefs their choices were based upon in Prisoners’ Dilemma games (which feature a dominant strategy), but that this may not be the case in games such as Hawk-Dove (where either action is a best response to some belief).
Finally, it is clear that magical thinking introduces a perceived benefit from cooperation. One could, of course, consider a reduced-form model in which each player may have a direct utility gain from choosing $c$ over $d$. This gain might be interpreted as a form of “warm glow” (as introduced by Andreoni (1989) in the context of public-good games). As an alternative explanation for our data, such a model would have the following flaws.

First, to align with the expected-payoff calculations in our model on $PD^0$, this utility gain would have to be independent of $x$ and $y$, but increase proportionally when simultaneously scaling $r$ and $p$. That is, even though the “warm glow” a player would obtain by cooperating would have to vary across games, it would not depend on how strong were the motives for defection that he overcame—including them being arbitrarily small. The reduced-form model would give the analyst no intuition for this seemingly curious form of dependence. In contrast, our model provides a psychological mechanism, that of magical thinking, which generates it. Second, this model of warm glow would be at odds with the evidence in (1)-(4) above. Third, in the next section we extend our model beyond the Prisoners’ Dilemma to games in which it is unclear how to interpret as warm glow the utility gain a player would need to receive from selecting one action over the other in order to match the predictions of our model. For example, for any Battle of the Sexes game there would need to be no warm glow attached to either action choice, but there are two games, arbitrarily nearby, such that a player would need to receive a warm glow from selecting his preferred meeting event (instead of his opponent’s) in one game but the reverse in the other game.

5 Beyond Prisoners’ Dilemma Games
We now extend our game-theoretic analysis beyond PD games. Section 5.1 takes the behavioral model characterized by Theorem 1 and investigates its predictions for all symmetric $2\times2$ games.\textsuperscript{28} Section 5.2 extends the model to accommodate arbitrary finite action sets.

5.1 Symmetric $2\times2$ Games
Let $S^0 := \{(r,p,x,y)\mid r \geq p\}$, with labels as in Figure 1, denote the set of all symmetric $2\times2$ games. As discussed in the Introduction, such games give strategic meaning to the notion that magical thinkers believe they influence others to select “the same action as they do,” without having to rely on arbitrary labels of actions. Therefore without loss, $c$ (respectively, $d$) still corresponds to the action leading to the weakly superior (inferior) symmetric outcome, but outside the Prisoners’ Dilemma we no longer refer to it as “cooperate” (“defect”).

We find that our model provides a unified explanation of the experimental evidence in several of the most often studied games: Hawk-Dove/Chicken, the Stag Hunt, and the Battle

\textsuperscript{28}In §S.3, we explore the extension of the axiomatic component to this domain.
of the Sexes (in addition to the Prisoners’ Dilemma). We also compare the predictions of our model to Nash equilibrium in the standard model (hereafter, simply Nash equilibrium).29

We consider first the generic case in which \( r \neq p \), followed by the nongeneric complement.

5.1.1 If Symmetric Outcomes are not Payoff Equivalent

Let \( S_G^0 \) be the (generic) set of games \( \{(r, p, x, y)|r > p\} \), of which \( PD^0 \) is a subset. Lemma 1 remains valid, so we normalize \( r = 1, p = 0 \), and again denote this normalized subset as \( S_G \) (represented as the plane in Figure 4). It remains true that for any \( g \in S_G \) all equilibria are cutoff equilibria (Definition 2.2), and that a player of type \( \alpha \) satisfies the indifference equation (2) if and only if

\[
g \in \tilde{M}_\alpha := \left\{(x, y) \mid y = \frac{\alpha}{1 - \alpha} F(\alpha) - x \left(1 - \frac{F(\alpha)}{F(\alpha)}\right)\right\}.
\]

Let \( B : \mathbb{R}_- \to \mathbb{R} \) be the lower envelop of the \( \tilde{M}_\alpha \) on the domain \( x \leq 0 \). Notice that i) \( B(0) \geq 0 \), ii) \( B \) is decreasing and concave, and iii) \( \lim_{x \to -\infty} = \infty \). Figure 4 depicts \( B \) (and four sample \( \tilde{M} \)-lines) for the case of \( F(\alpha) = \sqrt{\alpha} \).

**Proposition 5** For any \( g \in S_G \), an equilibrium exists, all equilibria are cutoff, and

- if \( x > 0 \), then the equilibrium cutoff \( \alpha^* \) is unique, interior (i.e., \( \alpha^* \in (0, 1) \)), and characterized by (2).

- if \( x \leq 0 \), then \( \alpha^* = 0 \) is an equilibrium cutoff. It is unique if and only if \( y < B(x) \).

29We maintain our focus on symmetric equilibria (of both our model and the standard one). In a truly symmetric, anonymous, one-shot setting, asymmetric equilibria seem implausible as neither player would have any way of knowing if he were “player 1” or “player 2.”
The labeled quadrants of Figure 4 serve as a useful taxonomy for our discussion of the games in $S_G$. Quadrant I corresponds to $PD$, which we have focused on up to now. We proceed clockwise. For brevity, we focus the discussion on the interiors of each quadrant.

**Quadrant IV**: The defining feature of Prisoners’ Dilemma games is that there are strict gains to a player for selecting $d$ whether his opponent is playing $d$ or $c$ (i.e., $x, y > 0$). The games of quadrant IV retain the latter, meaning there are still gains from unilaterally deviating away from the better symmetric outcome $(c,c)$. A particularly well-know example of such games are Hawk-Dove (also known as Chicken) games, where $y \in (-1,0)$. Action $c$ corresponds to *dove* and $d$ to *hawk*.

Proposition 5 establishes that the equilibrium characterization results for $PD$ (Section 2.1) extend unchanged to these games, and it is straightforward to show that Proposition 3 extends verbatim as well. In addition, we find the following result. For any $g \in S_G^0$ with $x > 0$, let $\pi_g$ be the probability with which a player selects $d$ in the unique symmetric Nash equilibrium of $g$. The corresponding probability in our behavioral model is $F(\alpha_g^*)$.

**Proposition 6** For any $g \in S_G^0$ with $x > 0$, $\pi_g > F(\alpha_g^*)$. In addition, if $x$ and $y$ are held fixed and $(r - p) \to 0$ (or, more generally, if $\frac{r - p}{x + |y|} \to 0$), then $(\pi_g - F(\alpha_g^*)) \to 0$.

The result states that players are drawn to the action producing the superior symmetric outcome more often than is predicted by the symmetric Nash equilibrium. This is consistent with experimental findings in the Hawk-Dove game (for example, Rubinstein and Salant (2014)). However, as the difference between the symmetric outcomes disappears so too does the difference in the two models’ predictions. Intuitively, as the difference between the symmetric outcomes disappears, the magical-thinking component has a vanishing impact on any player’s ranking between $c$ and $d$ (even though players with different $\alpha$-types still differ in their expectations over opponent play). Section 5.1.2 covers the limit case where $r = p$.

**Quadrant III**: In these games $c$ is *both* the action leading to the better symmetric outcome and a dominant strategy (even without magical thinking). It seems natural that all players should choose $c$ then—as they do in the unique equilibrium of our behavioral model by Proposition 5.

**Quadrant II**: Quadrant II consists of coordination games, such as the Stag Hunt, in which both symmetric outcomes constitute Nash equilibria, but $(c,c)$ Pareto dominates all other outcomes. The choice of $d$ in such games seems empirically implausible if the loss $x \leq 0$ of playing $d$ rather than $c$ against an opponent playing $c$ is large, and the gain $y > 0$ of playing $d$ rather than $c$ against an opponent playing $d$ is small. Players should find it natural to coordinate on $c$ in such a game. At the same time, if the gain $y$ of playing $d$ against $d$ is
large compared to the loss $x \leq 0$ of playing $d$ against $c$, then it becomes risky to rely on the opponent to play $c$, and $d$ also becomes a plausible choice. These intuitions are supported by experimental evidence (for example, Straub (1995)).

From Proposition 5, our behavioral model is consistent with all players selecting $c$, and it uniquely predicts this behavior for a subset of those games where coordinating on $c$ is not “too risky” in the sense just described. This set is precisely characterized as the strict-lower contour set of $B$. Hence, our behavioral model generates a unique equilibrium prediction in more games than does the standard model. More generally, the set of games for which our model makes a unique prediction is larger (in the sense of set inclusion) the more magical thinking there is in the population (in the sense of a first-order stochastically dominant shift of $F$). For games in the upper contour set of $B$, where the tradeoff between the overall payoffs (higher under $(c, c)$) and riskiness is more pronounced, our model does not make a unique prediction and can accommodate a significant proportion of players selecting $d$.\footnote{For $(x, y), y \geq B(x), \alpha \in (0, 1)$ is an equilibrium cutoff if and only if $(x, y) \in \tilde{M}_\alpha$. Hence, the number of equilibria in $(x, y)$ in which not all types select $c$ is the number of $\tilde{M}_\alpha$-lines that pass through $(x, y)$.}

The intuition we gave for the implausibility of selecting $d$ when $|x|$ is large compared to $y$, is reminiscent of the motivation for the risk dominance criterion (Harsanyi and Selten, 1988). It is easy to verify that, in the standard model, $(c, c)$ is risk dominant when $|x| > y$ and $(d, d)$ is risk dominant when $|x| < y$. Our boundary, $B$, is more nuanced than the fixed linear one implied by risk dominance, as it depends on the (perceived) distribution of $\alpha$-types. Our model, therefore, provides flexibility, though within constraints, for explaining behavioral data in this quadrant of games by varying $F$, and at the same time connects behavior in this quadrant to behavior in other games. For example, collections with higher rates of cooperation in Prisoners’ Dilemma games also have a larger set of quadrant-II coordination games in which the payoff-dominant Nash equilibrium is uniquely selected in our model.

### 5.1.2 If Symmetric Outcomes are Payoff Equivalent

Consider now the (nongeneric) set of games $S_N^0 := \{(r, p, x, y) | r = p\}$. In such games our model of magical thinking is not behaviorally distinct from the standard model.

**Proposition 7** For any $g \in S_N^0$, an equilibrium exists. Further,

- If $(\sigma, P)$ is an equilibrium (of our model), then there exists a symmetric Nash equilibrium characterized by $\pi_g = P$.

- If $\pi_g$ characterizes a symmetric Nash equilibrium, then there exists $\sigma$ such that $(\sigma, \pi_g)$ is an equilibrium (of our model).\footnote{More specifically: 1) If $\pi_g$ characterizes a weak Nash equilibrium, then any $\sigma$ such that $\int_0^1 \sigma(\alpha)dF(\alpha) = \pi_g$ constitutes an equilibrium; 2) If $\pi_g$ characterizes a strict Nash equilibrium, then in any equilibrium,}
In line with the limit property established in Proposition 6, when there is no payoff difference between the symmetric outcomes, magical thinking does not influence behavior in one direction or the other. Hence, the cutoff property is no longer a requirement for equilibrium, as there is no reason that players with higher \( \alpha \)-types are more drawn to \( c \).

Though not always labeled as a symmetric game, Battle of the Sexes games are a subset of \( S_N^0 \) in which \( x > 0 > y, x \neq -y \). In our theory, action labels are only for the convenience of the analyst; it is the symmetry of the game that determines what “taking the same action” means. In a Battle of the Sexes game then, \( c \) and \( d \) do not correspond to “go to the ballet/boxing match,” but to “go to my own/my opponent’s preferred event,” (with the labeling depending on the ranking of \( x \) and \( -y \)).

For a magical thinker \( i \), therefore, both \( c \) and \( d \) are self-defeating: by being “selfish” and choosing his preferred event, \( i \) believes it more likely that his opponent \( j \) will likewise choose \( j \)’s preferred event, but also analogously if \( i \) tries to be “accommodating” by choosing \( j \)’s preferred event. The magical-thinking component then has no effect on preferences over actions, and equilibrium play is just as in the standard model.

Any Battle of the Sexes game has a unique symmetric Nash equilibrium, and hence our behavioral model predicts the same distribution of observed behavior. Notably, this common prediction is substantiated by the experimental studies of Battle of the Sexes games.\(^{32}\) The ability to explain experimental findings across the well-known games surveyed in this paper serves as another key distinction between our model and models of other-regarding preferences discussed in Section 4, each of which predict patterns of play in the Battle of the Sexes that differ from the prediction of the standard model.\(^{33}\)

5.2 Accommodating Arbitrary Finite Action Sets

Allowing arbitrary finite action sets requires the following additional notation. Let \( A := \{0, 1, \ldots, K\} \) and \( v(k, k') \) be the (finite) game-payoff a player receives from selecting action \( k \) when his opponent selects \( k' \). Define \( s(k) := v(k, k) \), and, without loss, order the actions such that \( s(\cdot) \) is nondecreasing in \( k \). To avoid technicalities, we consider the generic case in \( \sigma(d|\alpha) = \pi_\alpha \) for all \( \alpha \in [0, 1) \), but \( \sigma(d|1) \) can be arbitrary since \( \alpha = 1 \) players are indifferent between \( c \) and \( d \) when \( r = p \).

\(^{32}\)Camerer (2003, Chap 7.2) summarizes the evidence and concludes, “Even if the subjects are not deliberately randomizing, the data are consistent with the idea that, as a population, they are mixing in the [symmetric Nash] equilibrium proportions.” Note that one could have imagined an alternative conception of magical thinking in the Battle of the Sexes: by choosing to go to the ballet, for example, a player believes it is more likely that his opponent will choose to go to the ballet as well. In addition to relying on strategically irrelevant action labels, this alternative model would predict that players select their preferred event more frequently than is found in the experimental evidence.

\(^{33}\)This result is not difficult to demonstrate. We omit the analysis for the sake of brevity.
which \( s(k) < s(k+1) \) for all \( k < K \).\(^{34}\) Let \( \Gamma \) denote the set of such games.

In the extended behavioral model, each player \( i \in I \) is still privately endowed with a type \( \alpha_i \in [0, 1] \) and there is a common prior that types are drawn i.i.d. from a distribution with CDF \( F \in \mathcal{F} \). For each game, player \( i \) evaluates the expected payoff of action \( \alpha_i = k \in A \) as:

\[
V_i(k) = \alpha_i s(k) + (1 - \alpha_i) \sum_{k' \in A} P_i(k') v(k, k')
\]  

(3)

where \( P_i(k') \) is the probability \( i \) assigns to \( a_j = k' \), conditional on \( \alpha_i \) and \( a_j \) being determined independently. His strategy, \( \sigma_i \), is again characterized by the probability with which he selects each \( k \in A \) if his type is \( \alpha_i \) (denoted \( \sigma_i(k|\alpha_i) \)). The equilibrium notion is the immediate extension of Definition 2.1.\(^{35}\)

When evaluating a potential action \( k \), it is clear from (3) that the importance given to \( s(k) \), the payoff generated by the symmetric profile \( (k, k) \), is increasing in \( \alpha_i \). This leads to the following increasing-in-type property of equilibrium.

**Definition 5.1** A strategy \( \sigma \) is **increasing** if there exists \( \alpha^*_0 = 0 \leq \alpha^*_1 \leq \ldots \leq \alpha^*_K+1 = 1 \) such that \( \sigma(k|\alpha) = 1 \) for all \( \alpha \in (\alpha^*_k, \alpha^*_{k+1}) \).

**Proposition 8** For any \( F \in \mathcal{F} \) and \( g \in \Gamma \), an equilibrium \((\sigma, P)\) exists, and in all equilibria \( \sigma \) is increasing.

### 5.2.1 Pure Dilemma Games

Given that the bulk of our analysis has focused on PD games, for brevity, we focus our remaining analysis on their natural extension, which we refer to as **pure dilemma** games.

**Definition 5.2** A game \( g \in \Gamma \) is a **pure dilemma** if \( v(k, k') > v(k+1, k') \) for all \( k, k' \).

In terms of game-payoffs then, there is always strict individual incentive to play a lower (indexed) action, but higher actions can be viewed as “more cooperative.” In the standard model there is a unique Nash Equilibrium: all players select \( a_i = 0 \). By comparison, in any equilibrium of our model, a positive measure of types select actions other than 0. Hence, if the ranking is not strict, our characterization (Proposition 8) holds only up to payoff equivalence: an equilibrium exists and for any equilibrium there exists a payoff-equivalent increasing equilibrium. Also, if \( s(1) = s(K) \), then Proposition 7 extends and the predictions of our model are not behaviorally distinct from symmetric Nash equilibrium in the standard model.

\(^{34}\)If the ranking is not strict, our characterization (Proposition 8) holds only up to payoff equivalence: an equilibrium exists and for any equilibrium there exists a payoff-equivalent increasing equilibrium. Also, if \( s(1) = s(K) \), then Proposition 7 extends and the predictions of our model are not behaviorally distinct from symmetric Nash equilibrium in the standard model.

\(^{35}\)With \( P = (P(k))_{k \in A} \) and requirement (3) generalizing to \( P_i(k) = P(k) = \int_0^1 \sigma(k|\alpha) dF(\alpha) \) for all \( i \in I \) and \( k \in A \). This specification represents the literal extension of magical thinking as “players believe they influence others to select ‘the same action’ as they do,” as discussed in the Introduction. With more than two actions, one could envision more general notions of magical thinking that still capture its essence: players believe they influence the opponent to select an action more similar to their own action than the opponent otherwise would have. For simplicity, we consider only the literal extension.
just as in PD games, magical thinking generates a positive degree of cooperation, while the standard model predicts none.

As in Section 3, we can investigate whether there are connections between notions of more magical thinking and more cooperative behavior. Again, it is immediate from the increasing property of equilibrium (Proposition 8) that in any pure dilemma, a player’s cooperation level is increasing in type. What about increases in population-wide magical thinking, as measured by a first-order shift in $F$? The following proposition identifies a sufficient condition on the underlying game under which this change leads to uniformly greater cooperation.

**Definition 5.3** For pure dilemma game $g \in \Gamma$, an increasing strategy $\sigma$ is more cooperative than increasing strategy $\hat{\sigma}$ if $\alpha_k^\sigma \leq \alpha_k^{\hat{\sigma}}$ for all $k$.\(^{36}\)

**Proposition 9** Let $g \in \Gamma$ be a supermodular pure dilemma game.\(^{36}\)

1. For any $F \in \mathcal{F}$, there exists a most and a least cooperative equilibrium, denoted $(\sigma^M_F, P^M_F)$ and $(\sigma^L_F, P^L_F)$ respectively.

2. If $F \in \mathcal{F}$ first-order-stochastically dominates $\tilde{F} \in \mathcal{F}$, then $\sigma^M_F$ is more cooperative than $\sigma^M_{\tilde{F}}$ and $\sigma^L_F$ is more cooperative than $\sigma^L_{\tilde{F}}$.

First consider PD games. A PD game is supermodular if and only if $y \geq x$. Notice that such a PD game satisfies the very common assumption that $(c, c)$ maximizes the sum of game-payoffs: $2r > (r+x)+(p-y)$ or equivalently, $y > x-(r-p)$. This feature generalizes: if $g \in \Gamma$ is supermodular, then $2s(k) > v(l, l') + v(l', l)$ for all $l \leq l' \leq k$ with $l < k$.

For a supermodular PD game, from Propositions 2 and 3, we know that if $F, \tilde{F}$ satisfy Condition $S$, then each has a unique equilibrium, $(\sigma_F, P_F)$ and $(\sigma_{\tilde{F}}, P_{\tilde{F}})$, and $F$ f.o.s.d. $\tilde{F}$ implies $\sigma_F$ is more cooperative than $\sigma_{\tilde{F}}$. Proposition 9 generalizes this comparative static along two dimensions: it does not assume Condition $S$—so there may be multiple equilibria even if $g \in PD^0$—and it applies to pure dilemma games beyond the PD. The intuition remains similar. Fix a game and imagine first that the equilibrium was unchanged following a shift from $\tilde{F}$ to $F$. Then a given $\alpha$-type would face a more cooperative (perceived) distribution of play. But, just as Section 3 demonstrated for PD games, depending on payoff parameters, this change could make the $\alpha$-type more or less cooperative (i.e., seeking to take advantage of the population’s greater degree of cooperation in the latter case). Supermodularity of $g$ implies the latter case never obtains.\(^{37}\)

\(^{36}\)That is, for all $k' \geq k$ and $l' \geq l$, $v(k', l') - v(k, l') \geq v(k', l) - v(k, l)$.

\(^{37}\)Notice that the notion of more cooperative in Definition 5.3 is a strong one: each $\alpha$-type plays a (weakly) higher action. Alternatively, for $F, \tilde{F} \in \mathcal{F}$ and corresponding equilibria $(\sigma, P), (\tilde{\sigma}, \tilde{P})$, one could view the first population’s behavior as more cooperative if its $F$-measured distribution of play, $P$, f.o.s.d. the second population’s $\tilde{F}$-measured distribution of play, $\tilde{P}$, even if fixed $\alpha$-types play less cooperative actions under $\sigma$ than under $\tilde{\sigma}$. Under Condition $S$, Proposition 3 shows that a first-order shift in $F$ implies more cooperation in this weaker sense for all PD games. It is not difficult to construct further examples outside PD games.
We conclude by identifying a class of games for which all of the analysis from preceding sections (both game-theoretic and axiomatic) applies.

**Definition 5.4** A pure dilemma game \( g \in \Gamma \) has **increasing returns to joint cooperation** if, for all \( k, k' \):

\[
\frac{s(k + 1) - s(k)}{v(k, k') - v(k + 1, k')} \geq \frac{s(k) - s(k - 1)}{v(k - 1, k') - v(k, k')}. 
\]

Notice that this condition is neither weaker nor stronger than supermodularity. It requires that the benefit to increased joint cooperation increase at a rate greater than the increase in the individual benefit from lowering one’s own action, independent of the action selected by the opponent. Consider, for example, a public-good game where both players can contribute any amount between 0 and \( K \) dollars to the public good, and the function \( \Phi \) (with \( \frac{1}{2} < \Phi' < 1 \)) measures the individual benefit derived from the amount of public good provided given the total contribution. That is, \( v(k, k') = K - k + \Phi(k + k') \). If \( \Phi \) is (weakly) convex, then the returns to joint cooperation are (weakly) increasing.

**Proposition 10** If a pure dilemma game \( g \in \Gamma \) has increasing returns to joint cooperation, then in any equilibrium, \( \alpha_1^* = \alpha_K^* \in (0, 1) \) and is equal to an equilibrium cutoff of the PD game generated by deleting actions \( \{1, \ldots, K - 1\} \).

Under increasing returns to cooperation, (almost) all types play either 0 or \( K \) and intermediate actions play no (meaningful) role. Therefore, our equilibrium characterization and tight condition for uniqueness, our comparative statics, and even our axiomatization all apply to this class of games with the obvious additional axiom that intermediate actions are never chosen.

An example of an experimental study of linear two-player public-good games is Capraro et al. (2014), who investigate how the distribution of play responds to changes in the social benefit from cooperation, captured by the scalar \( \phi \), where \( \Phi(k + k') = \phi(k + k') \). They find that, independent of the value of \( \phi \), about 20% of subjects contribute half their endowment \( (a_i = \frac{K}{2}) \), and argue that these subjects are likely following a simple heuristic. The predictions of our model are well aligned with behavior in the remaining 80% of subjects. The vast majority of these subjects (75% of the total population) choose an extreme action \( a_i \in \{0, K\} \) (in line with Proposition 10), and the proportion of them choosing \( K \) increases with \( \phi \), (as predicted by Axiom 3 (Monotonicity), in the aggregate).\(^{38}\)

\(^{38}\)In the experiment, \( v(k, k') = K - k + wk' \), with \( 2 \leq w \leq 10 \). So each game is a pure dilemma and strategically equivalent to our description of a linear public-good game with \( \phi = \frac{w}{w+1} \).
6 Discussion

Methodology: Our approach connects behavioral axioms on the observed play of a collection of players to a representation that suggests a procedural interpretation of individual behavior and an equilibrium concept. This is analogous to the standard axiomatic analysis of individual choice (see footnote 10). Throughout the paper we have stressed this analogy, as well as the differences that arise when leveraging our richer domain of group behavior.

At the outset, we discussed several benefits of this methodology. Here we compare it to related, alternative approaches. In (what we refer to as) the standard one for connecting behavioral axioms to strategic, multi-agent environments, first, axioms characterize a specific utility representation of individual preferences regarding (lotteries over) physical allocations; then second, physical games are described in terms of those utilities; and finally, strategic analysis is performed according to an exogenously given solution concept (usually an equilibrium notion). The prototypical example of the standard approach assumes that players care only about their own physical payoffs and have risk preferences as axiomatized by von Neumann and Morgenstern (1944). As another example, Rohde (2010) provides axioms for the inequity-averse utility function employed in the game-theoretic analysis of Fehr and Schmidt (1999). (See also Dillenberger and Sadowski (2012); Saito (2013)).

Relative to the benefits of our axiomatic approach that were listed (1)-(3) at the bottom of page 1, the standard approach has the following differences. In contrast to (1), it relies on the assumption that behavior observed in the individual context is tightly connected to behavior in the strategic context. Clearly, it cannot achieve (3), as it is not possible to derive that behavior corresponds to any particular equilibrium notion, or that prior beliefs are common, by axiomatizing the objective function of each player separately.

Notably, once preferences over physical allocations are accounted for, if a given physical game remains a PD when represented in terms of these utilities, then the standard approach cannot explain cooperative behavior in this game—as each player should like higher utility payoffs for himself and be indifferent toward the utility payoffs of others. That is, explanations of cooperative behavior via altruism or inequity-aversion merely establish that games that look like PDs in terms of physical game-payoffs may not actually be PDs in terms of utility payoffs. In contrast, our theory is robust to this alternative interpretation of game-payoffs as utilities. Insofar as cooperation in such games is plausible, the standard approach’s inability to explain it could be due to a discrepancy between preferences in the individual and strategic contexts (related to (1)), or to inappropriate assumptions about beliefs or the equilibrium notion (related to (3)).

Segal and Sobel (2007, 2008) go beyond the standard approach by fixing a single game and
using as an additional primitive individual preferences over own (mixed) strategies, which may depend on what the (mixed) strategy profile is “supposed to be,” referred to as the “context.” They axiomatize a representation that can, for example, accommodate reciprocal preferences (Rabin, 1993), and then employ the natural extension of Nash equilibrium as the solution concept. Clearly, their model differs from (3) in the same manner as does the standard approach. In addition, the elicitation of the primitive requires that for a single game the analyst uncover the player’s preferences over his own strategies given each possible mixed-strategy profile. In contrast to (2), this is not data that is commonly collected, and faces the potential difficulty that the analyst must meaningfully communicate to each subject (i.e., have them truly believe) that the opponents are actually using that profile. At the very least this is more involved than simply asking subjects how they would like to play.

Our approach also differs from well-known axiomatic treatments in bargaining and cooperative game theory, most prominently in Nash bargaining, where axioms directly characterize the outcome rather than a model of play in the strategic setting coupled with a solution concept.\textsuperscript{39} Given the difficulty of accurately capturing the nuances of, say, bilateral negotiations, that such analysis does not rely on explicit modeling of the strategic situation is often seen as a strength. However, in the simplified setting of simultaneous-move, one-shot games, characterizing play seems a more natural objective. In addition, our representation suggests an intuitive explanation of the individual decision-making process, which enables us to provide comparative statics in terms of the model’s parameters.

For any fixed physical game, our behavioral model resembles a Bayesian game, in that each player is endowed with a type that affects how he evaluates the expected payoff of a potential strategy. The difference, of course, is that in standard Bayesian games, a player’s type maps outcomes into payoffs, whereas in our model, type affects the player’s expectations about what outcomes will obtain depending on his action choice. Following Savage (1954), the derivation of subjective beliefs is a central concern in the context of individual choice. Our model provides an example where beliefs (here, both about the opponent’s type and action choice) are derived from behavior in a strategic setting.

\textsuperscript{39}See Thomson (2001) for a thorough review of this approach. In addition, while axiomatic approaches in cooperative game theory characterize solutions without a strategic model, axiomatic approaches in non-cooperative game theory typically take as given the structure of the strategic model—by assuming that all players and the analyst view the game in the same way and that players are “rational” (i.e., they maximize expected utility with respect to some [nonmagical] belief about opponents’ play)—with the aim of characterizing particular solution concepts (e.g., rationalizability, Nash equilibrium, correlated equilibrium, etc.). Again, see Thomson (2001, §12.3), and Blonski et al. (2011) for an application to equilibrium selection in repeated games. Outside the axiomatic literature, Bergemann et al. (2014) consider behavior in games to identify interdependent preferences over outcomes. Because the aim is identification of preferences, they take as given both the structure of the model and the solution concept.
Repetition: There are various experimental studies that report on the evolution of cooperation when the same one-shot Prisoners’ Dilemma is played repeatedly, with opponents randomly and anonymously rematched after every round. Many of these studies find an initial decline in the incidence of cooperation before it stabilizes at a non-zero level. For a sample of studies Table 1 reports each of their featured one-shot games \((x, y) \in \text{PD}\) (modulo positive affine transformations), the approximate number of periods after which stabilization was reached, and the approximate average levels of cooperation thereafter.\(^{40}\) Note that the stable levels of cooperation summarized in the table give further support to Monotonicity (Axiom 3) in the aggregate.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>Approx. periods until stabilization</th>
<th>Approx. stable % of cooperation</th>
<th>Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.00, 3.00)</td>
<td>study ended after 10</td>
<td>(\lesssim 7)%</td>
<td>Dal Bó et al. (2010)</td>
</tr>
<tr>
<td>(2.33, 2.33)</td>
<td>20 out of 200</td>
<td>10 %</td>
<td>Berbery-Meyer and Roth (2006)</td>
</tr>
<tr>
<td>(1.67, 1.33)</td>
<td>0 out of 200</td>
<td>19 %</td>
<td>Andreoni and Miller (1993)</td>
</tr>
<tr>
<td>(0.44, 0.78)</td>
<td>10 out of 20</td>
<td>22 %</td>
<td>Cooper et al. (1996)</td>
</tr>
<tr>
<td>(0.33, 0.11)</td>
<td>20 out of 75</td>
<td>37 %</td>
<td>Aoyagi and Fréchette (2009)</td>
</tr>
</tbody>
</table>

Table 1: Studies of One-Shot Prisoners’ Dilemma with Random, Anonymous Rematching

As with most theories of behavior in one-shot settings, our theory does not formally provide any explanation for the dynamics before steady state is reached. The typical explanation for a pattern of initially varying behavior followed by stability is that subjects are initially learning about the game (e.g., how it works, how others play, etc.)—see Camerer and Fehr (2003) for a discussion. The interesting feature in this particular instance is that initial play is systematically more cooperative than steady state. While a formal model along these lines is beyond the scope of this paper, one possible explanation for this pattern is that subjects (act as if they) revise their estimates of their own \(\alpha\)-types based on play. Because they are not in fact magical, the updating will be systematically biased downward, leading subjects to cooperate less.\(^{41}\)

Magical-Thinking-Like Notions in other Strategic Models: In models of oligopolistic competition, the notion of conjectural variation (Bowley, 1924; Pigou, 1924) bears some resemblance to magical thinking. However, in this literature a firm’s belief about how its

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\(^{40}\)Not all studies provided these numbers explicitly. In these cases, they are estimates based on the information the studies do provide.

\(^{41}\)This could perhaps be because the \(\alpha_i\) in our behavioral model (of the single-iteration game) represents only the expected influence \(i\) believes he possesses, but his beliefs allow that his influence may vary across subjects pools or other environmental features. That play stabilizes at non-zero levels of cooperation suggests that the lower bound on \(\alpha_i\) is believed to be positive by some \(i\).
rival will respond to its action is typically interpreted as capturing a sequential response. In Roemer’s (2010; 2013) *Kantian equilibrium* each player prefers the equilibrium to any strategy profile that features identical deviations by all players. Related features are found in Feddersen and Sandroni (2006), who introduce *rule-utilitarian* players into a model of voting (see also Coate and Conlin (2004); Ali and Lin (2013)). As suggested by their names, the modeling of both Kantian equilibrium and rule-utilitarian players are motivated by ethical concepts, in contrast to our psychological interpretation of magical thinking. While these different motivations may have similar behavioral consequences in some settings, our motivation more naturally allows for heterogeneity among players that is absent from these models.\(^{42}\) In addition, the interpretation of magical thinking is more in line with the evidence discussed in Section 4.\(^{43}\) Of course, as we have stressed throughout, this paper is also—and most importantly—distinguished by providing a tight axiomatic characterization of our behavioral model.

We conclude by noting that magical thinking is likely not an appropriate description of behavior in all games for which the standard game-theoretic predictions are unsatisfying, be they inaccurate and/or weak due to multiplicity (impeding applied/policy research, argues Pakes (2008)). An ideal axiomatization would alleviate both problems by avoiding false predictions and ruling out multiplicity where it is descriptively inappropriate. Our model alleviates the first concern in Prisoners’ Dilemma games, and improves on the second in coordination games by eliminating equilibrium multiplicity in games where coordination on the better symmetric outcome is intuitive.\(^{44}\) While our representation features an equilibrium concept, this need not be the case in other contexts. As in theories of individual choice, the goal should be to connect testable and plausible behavioral axioms to an intuitive, tractable, and identified representation which may, or may not, have the strategic flavor of equilibrium.

\(^{42}\)Specifically, because in these models all Kantians or rule-utilitarians evaluate strategies in the same way, any heterogeneity in nonstandard behavior is driven completely by asymmetry in the physical aspects of the game (e.g., variations in the cost of voting). In contrast, even in symmetric games, our model captures heterogeneity in behavior (e.g., in the sets of PD games that different players choose to cooperate in).

\(^{43}\)Additional models in which players’ beliefs about opponent play may be biased include Orbell and Dawes (1991); Bernheim and Thomadsen (2005); Masel (2007); Capraro and Halpern (2014); al Nowaihi and Dhami (2015).

\(^{44}\)In addition, our behavioral model introduces the possibility of equilibrium multiplicity even in the PD, depending on \(F\). It is then the axioms that rule out models with multiplicity, again showing that the second concern can also be addressed axiomatically.
References


A Proofs

Proof of Lemma 1. Fix any \((r, p, x, y) \in PD^0\), and suppose that \((\sigma, P)\) is an equilibrium according to Definition 2.1. Then, for any player \(i\),

\[ V_i(c) - V_i(d) = \alpha_i[r - p + (1 - P)x + Py] - [(1 - P)x + Py], \]

and player \(i\) strictly prefers \(c\), strictly prefers \(d\), or is indifferent if (4) is positive, negative, or zero respectively. Hence, it is sufficient to show that the sign of (4) is unchanged for all \(\alpha_i\) when the payoffs are transformed to \(\kappa(r + \xi, p + \xi, x, y)\), where \(\kappa > 0\). Then,

\[ V_i(c) - V_i(d) = \alpha_i[\kappa r + \kappa \xi - \kappa p - \kappa \xi + (1 - P)\kappa x + P\kappa y] - [(1 - P)\kappa x + P\kappa y] \]

\[ = \kappa(\alpha_i[r - p + (1 - P)x + Py] - [(1 - P)x + Py]) \]

Because \(\kappa > 0\), the signs of (4) and (5) are identical. ■

Proof of Proposition 1. Claim (i): Fix any \((x, y) \in PD\), and suppose that \((\sigma, P)\) is an equilibrium according to Definition 2.1. Then,

\[ V_i(c) - V_i(d) = \alpha_i[1 + (1 - P)x + Py] - [(1 - P)x + Py] \]

Player \(i\) strictly prefers \(c\), strictly prefers \(d\), or is indifferent if (6) is positive, negative, or zero respectively. For any \(P \in [0, 1]\), i) if \(\alpha_i = 1\), then (6) is positive, and ii) (6) is linear in \(\alpha_i\). It follows that the equilibrium must be a cutoff equilibrium and that \(\alpha^* < 1\). Suppose now that \(\alpha^* = 0\). Then, by Definition 2.1, \(P = 0\). But then \(V_i(c|\alpha_i = 0) - V_i(d|\alpha_i = 0) = -x < 0\), which contradicts \(\alpha^* = 0\), establishing the result.

Claims (ii) and (iii): That solutions to (2) and equilibrium cutoffs are identical follows immediately from the properties of (6) discussed in the proof of Claim (i). It is therefore sufficient to establish existence of a solution to (2). If \(x = y\), (2) has a unique solution: \(\alpha^* = \frac{x}{1 + x} = \frac{1}{1 + y}\). If \(x \neq y\), any solution to (2) is (implicitly) characterized by

\[ F(\alpha^*) = T(\alpha^*|x, y) := \frac{\alpha^* - (1 - \alpha^*)x}{(1 - \alpha^*)(y - x)}. \]

If \(x > y\), then \(\lim_{\alpha \to 0} T(\alpha|x, y) = \frac{x}{x - y} > 1\), and \(\lim_{\alpha \to 1} T(\alpha|x, y) = -\infty\). Further, \(T\) is continuous and strictly decreasing in \(\alpha\). Hence, it must intersect \(F\), a continuous CDF on \([0, 1]\), exactly once. If \(x < y\), then \(\lim_{\alpha \to 0} T(\alpha|x, y) = \frac{x}{x - y} < 0\), and \(\lim_{\alpha \to 1} T(\alpha|x, y) = \infty\). Further, \(T\) is continuous and strictly increasing in \(\alpha\). Hence, it must intersect \(F\), a continuous CDF on \([0, 1]\), at least once. ■

Proof of Proposition 2. From Proposition 1, the number of equilibrium cutoffs is the number of solutions to (2), and existence is established. For \(x \geq y\), the arguments given in the proof of Proposition 1 demonstrate uniqueness of the solution for any \(F \in \mathcal{F}\). Now, fix arbitrary \(x < y\) and suppose \(F'(\alpha) \leq \frac{F(\alpha)}{\alpha - \alpha^*}\) for all \(\alpha \in (0, 1)\). Consider a solution \(\alpha^* \in (0, 1)\).

\[ F'(\alpha^*) \leq \frac{F(\alpha^*)}{\alpha^* - (\alpha^*)^2} = \frac{T(\alpha^*|x, y)}{\alpha^* - (\alpha^*)^2} = \frac{\alpha^*(1 + x) - x}{\alpha^*(1 - \alpha^*)^2(y - x)} \]

Further,

\[ T'(\alpha^*|x, y) = \frac{1}{(1 - \alpha^*)^2(y - x)} \]
It is a matter of simple algebra to see that the rightmost term in (8) is strictly less than (9) for any \( x, y, \alpha^* \) such that \( 0 < x < y \) and \( \alpha^* \in (0, 1) \). Hence, at any solution to (2), \( T \) intersects \( F \) from below. Because both functions are continuous they can intersect at most once.

To see that uniqueness fails if the condition is not satisfied, suppose there exists \( \alpha_0 \in (0, 1) \) such that \( F'(\alpha_0) > \frac{F(\alpha_0)}{\alpha_0 - \alpha_0^2} \). For any \( (x, y) \in PD \) such that \( y > x \), \( T \) is continuous, \( \lim_{\alpha \to 0} T(\alpha|x, y) = \frac{x}{x-y} < 0 \), and \( \lim_{\alpha \to 1} T(\alpha|x, y) = \infty \). Hence, there must exist at least one solution in which \( T \) intersects \( F \) from below. Therefore, if for the same game there exists a solution in which \( T \) intersects \( F \) from above, then there are multiple solutions. Let \( Y(x|\alpha, F(\alpha)) \) be the function such that \( \alpha \) solves (2) given \( F(\alpha) \), \( x \), and \( y = Y(x|\alpha, F(\alpha)) \); that is

\[
Y(x|\alpha, F(\alpha)) = \frac{\alpha - (1 - \alpha)(1 - F(\alpha))x}{(1 - \alpha)F(\alpha)}
\]

Notice that given any \((\alpha, F(\alpha)) \in (0, 1)^2\), for all \( x < \frac{\alpha}{(1-\alpha)[1-F(\alpha)]} \), \( Y(x|\alpha, F(\alpha)) > 0 \), meaning for such \( x \), \((x, Y(x|\alpha, F(\alpha))) \in PD \). Finally, it is straightforward that

\[
\lim_{x \to 0} \left( T'(\alpha|x, Y(x|\alpha_0, F(\alpha_0))) \right)_{|\alpha=\alpha_0} = \frac{F(\alpha_0)}{\alpha_0 - \alpha_0^2}
\]

By supposition \( F'(\alpha_0) > \frac{F(\alpha_0)}{\alpha_0 - \alpha_0^2} \). Therefore, because \( T' \) is continuous in both \( x \) and \( y \), there exists \( x > 0 \) small enough such that \( T \) intersects \( F \) from above at \( \alpha_0 \) for the game \((x, Y(x|\alpha_0, F(\alpha_0)))\).

**Proof of Theorem 1.**

Representation \( \implies \) Axioms: Consider a collection \( I \) with primitive \((D_i^0, C_i^0)_{i \in I}\) that satisfies the representation. Because each game has a unique equilibrium cutoff, Lemma 1 immediately implies Axiom 1 is satisfied. To verify that the primitive satisfies the remaining axioms it is sufficient to focus only on \( PD \) and \((D_i, C_i)_{i \in I}\).

Propositions 1-2 immediately imply the following. First, if \( \alpha_i = 0 \), then \( D_i = PD \), and if \( \alpha_i = 1 \), then \( C_i = PD \). Second, if \( \alpha_i \in (0, 1) \), then \( M_i = \{(x, y) \in PD|\alpha_{x,y}^* = \alpha_i\} \). Third, if \( \alpha_i = \alpha_j \), then \((D_i, C_i) = (D_j, C_j)\).

Now, fix arbitrary \( \alpha_i \in (0, 1) \), and solve (2) to get that

\[
M_i = \{(x, y) \in PD|\alpha_{x,y}^* = \alpha_i\} = \left\{(x, y) \in PD|y = \frac{\alpha_i}{(1-\alpha_i)F(\alpha_i)} - x \left(1 - \frac{1 - F(\alpha_i)}{F(\alpha_i)}\right)\right\}
\]

That is, \( M_i \) forms a line in \( PD \). Define \( int_i = \frac{\alpha_i}{(1-\alpha_i)F(\alpha_i)} \), and \( slp_i = \frac{1 - F(\alpha_i)}{F(\alpha_i)} \). It follows that if \( 0 < \alpha_j < \alpha_i < 1 \), then \( int_i \geq int_j \), and \( slp_i < slp_j \). The latter is obvious since \( \alpha_i > \alpha_j \implies F(\alpha_i) > F(\alpha_j) \) because \( F \in \mathcal{F} \). To see the former,

\[
\frac{d}{d\alpha} \left( \frac{\alpha}{(1-\alpha)F(\alpha)} \right) = \frac{F(\alpha) - (1 - \alpha^2)F'(\alpha)}{((1-\alpha)F(\alpha))^2} \geq 0 \forall \alpha \in (0, 1) \iff \text{Condition } S
\]

For arbitrary player \( \alpha_i \in (0, 1) \), let \( MU_i \) and \( ML_i \) be the strict upper- and strict lower-contour sets of \( M_i \) (within \( PD \)) respectively. Now, consider \((x, y) \in MU_i \). From Proposition 2, there exists unique \( \alpha_{x,y}^* \) and it is distinct from \( \alpha_i \) by \((x, y) \notin M_i \). From the argument above, whenever \( \alpha_j \leq \alpha_i \) then \( M_{L_j} \subseteq ML_i \). Therefore, \( \alpha_{x,y}^* > \alpha_i \). By the cutoff form of the equilibrium, \((x, y) \in D_i \). Therefore, \( D_i = MU_i \). An analogous argument establishes \( C_i = ML_i \).

Having completed the description of the data, \((D_i, C_i)_{i \in I}\), that the representation generates, we are ready to verify the axioms. That extreme players, \( \alpha_i = 0, 1 \), satisfy Axioms 2-4 is
clear, so consider any player \( i \) such that \( \alpha_i \in (0, 1) \). Axiom 2 is satisfied since the sets \( C_i = \{(x, y) \in PD \mid y < int_t - x \cdot slp_i \} \) and \( D_i = \{(x, y) \in PD \mid y > int_t - x \cdot slp_i \} \) are open in \( PD \). For Axiom 3, if \( (x, y) \in \overline{D_i} \), then for any \( (x', y') \geq (x, y) \) such that \( (x', y') \neq (x, y) \) it follows that \( (x', y') \in MU_i = D_i \). To verify Axiom 4, suppose both \( (x, y) \) and \( (x', y') \) are elements of \( D_i \). Then,

\[
\begin{align*}
y > int_t - x \cdot slp_i & \implies \gamma y > \gamma (int_t - x \cdot slp_i) \\
y' > int_t - x' \cdot slp_i & \implies (1 - \gamma)y' > (1 - \gamma)(int_t - x' \cdot slp_i) \\
& \implies \gamma y + (1 - \gamma)y' > \gamma (int_t - x \cdot slp_i) + (1 - \gamma)(int_t - x' \cdot slp_i) \\
& \implies \gamma y + (1 - \gamma)y' > int_t - (\gamma x + (1 - \gamma)x') slp_i
\end{align*}
\]

Hence, \((\gamma x + (1 - \gamma)x', \gamma y + (1 - \gamma)y') \in D_i \). A symmetric argument holds if \( \{(x, y), (x', y')\} \subset C_i \).

Finally, Axiom 5. Suppose the hypotheses of the axiom are satisfied for two distinct players \( i \) and \( j \). Then it must be that \( 0 < \alpha_i < \alpha_j \). If \( \alpha_j = 1 \), then \( C_j = PD \) and the axiom is trivial. If \( \alpha_j < 1 \), then \( 0 < \alpha_i < \alpha_j \) implies \( slp_j < slp_i \) (above). Further, by hypotheses (ii) and (iii), \( slp_1 < \frac{\delta}{2} \). Finally, \( slp_j < \frac{\delta}{2} \) and \( (x', y') \in C_j \) imply \((x' + \varepsilon, y' - \delta) \subset C_j \), completing the proof.

Axioms \( \implies \) Representation: The majority of the proof concerns behavior in the set of games \( PD \) (that is \( (D_i, C_i)_{i \in I} \)). In a series of lemmas we establish that Axioms 2-5 imply the representation on this smaller domain. Lemmas A.1-A.2 demonstrate that if \( (D_i, C_i) \) satisfies Axioms 2-4, then there is a unique value for \( \alpha_i \) and a unique scalar \( F_i \) such that any behavioral model \( [F_i, (\alpha_i, \alpha_{-i})] \) with \( F \in F \) and \( F(\alpha_i) = F_i \) can explain the behavior of player \( i \). Lemmas A.3-A.5 then show that there exists \( F \in F_S \) that simultaneously satisfies the required values for all \( i \in I \). Therefore, by Proposition 2, for all \( (x, y) \in PD \), under this \( F \) there is a unique equilibrium cutoff. This ensures that in each game there is an equilibrium consistent with the behavior of all players; hence, the behavioral model using this assignment of \( F \) and the mandated \( \alpha_i \)'s can explain \( (D_i, C_i)_{i \in I} \) (Lemma A.6). It is then an immediate corollary that the addition of Axiom 1 implies the representation on the full domain, \( PD^0 \) (Lemma A.7). This completes the proof.

**Fact A.1** Fix any player \( i \). If \( (D_i, C_i) \) satisfies Axioms 2 and \( C_i \neq \emptyset, D_i \neq \emptyset \), then for any \( (x, y) \in C_i, (x', y') \in D_i \), and continuous path \( p : [0, 1] \rightarrow PD \) such that \( p(0) = (x, y) \) and \( p(1) = (x', y') \), there exists \( t \in (0, 1) \) such that \( p(t) \in M_i \).

**Proof.** Let \( \bar{t} = \sup \{ t \mid p(t') \in C_i \forall t' \in [0, t] \} \). Because \((x, y)\) is an arbitrary element of \( C_i \), it is sufficient to show that \( p(\bar{t}) \in M_i \). Suppose that \( p(\bar{t}) \in C_i \). Then, by definition of \( \bar{t} \), for any \( \varepsilon > 0 \) there exists \( t \in (\bar{t}, \bar{t} + \varepsilon) \) such that \( p(t) \notin C_i \). Because \( p \) is continuous, this contradicts \( C_i \) being open (and, hence, Axiom 2). Now, suppose that \( p(\bar{t}) \in D_i \). By definition of \( \bar{t} \), for all \( \varepsilon > 0 \) there exists \( t \in (\bar{t} - \varepsilon, \bar{t}) \) such that \( p(t) \in C_i \), and therefore \( p(t) \notin D_i \). Because \( p \) is continuous, this contradicts \( D_i \) being open (and, hence, Axiom 2). Hence, \( p(\bar{t}) \in M_i \).

**Lemma A.1** Fix any player \( i \) such that \( (D_i, C_i) \) satisfies Axioms 2-4. If \( D_i \neq \emptyset \) and \( C_i \neq \emptyset \), then there is a unique pair \((int_t, slp_i) \in (0, \infty)^2 \) such that \( D_i = \{(x, y) \in PD \mid y > int_t - slp_i \cdot x \} \) and \( C_i = \{(x, y) \in PD \mid y < int_t - slp_i \cdot x \} \). If \( D_i = \emptyset \), then \( C_i = PD \); and if \( C_i = \emptyset \), then \( D_i = PD \).

**Proof.** Consider the three possible cases.

Case 1: \( D_i \neq \emptyset \) and \( C_i \neq \emptyset \): Axiom 2 implies that not only is \( M_i \) nonempty, but non-singleton (see Fact A.1). Therefore, let \( \{(x_1, y_1) \neq (x_2, y_2) \} \subset M_i \), with \( x_1 \leq x_2 \). By Axiom 3, \( x_1 < x_2 \) and \( y_1 > y_2 \). Again employing Axioms 2 and 3, see that \( M_i \cap \{(x, y) \in PD \mid x \in [x_1, x_2] \} \) must consist of
a strictly decreasing function \( \bar{y} \), where \( \bar{y}(x_1) = y_1 \) and \( \bar{y}(x_2) = y_2 \). For any \( x \in [x_1, x_2] \), if \( y > \bar{y}(x) \) then \( (x, y) \in D_i \), and if \( y \in (0, \bar{y}(x)) \) then \( (x, y) \in C_i \). Hence, Axiom 4 implies that \( \bar{y} \) is linear. Let \( slp_i := \frac{y_1-y_2}{x_2-x_1} \in (0, \infty) \) and \( int_i := (y_1 + slp_i \cdot x_1) \in (0, \infty) \).

Since the above applies to any pair of games in \( M_i \), all games in \( M_i \) must fall on the same line: \( M_i \subseteq \{(x, y) \in PD | y = int_i - slp_i \cdot x \} \). But, if the inclusion were strict, Axiom 2 would be violated (again, see Fact A.1). Hence, \( M_i = \{(x, y) \in PD | y = int_i - slp_i \cdot x \} \). The claimed structures of \( D_i \) and \( C_i \) follow from Axiom 3.

Case 2: \( D_i = \emptyset \): It must be that \( M_i = \emptyset \). Suppose to the contrary that some \( (x, y) \in M_i \). Then, by Axiom 3, for \( x' > x \), \( (x', y) \in D_i \): a contradiction. Hence, \( C_i = PD \).

Case 3: \( C_i = \emptyset \): It must be that \( M_i = \emptyset \). Suppose to the contrary that some \( (x, y) \in M_i \). Consider then a game \( (x', y) \) where \( x' \in (0, x) \). By \( C_i = \emptyset \), \( (x', y) \in D_i \). Axiom 3 implies that \( (x, y) \in D_i \): a contradiction. Hence, \( D_i = PD \). □

**Lemma A.2** Fix any player \( i \). If \( (D_i, C_i) \) satisfies Axioms 2-4, then there exists a unique pair \( (\alpha_i, F_i) \in [0,1]^2 \) such that \( (D_i, C_i) \) can be explained by any behavioral model \( \{F, (\alpha_i, \alpha_{-i})\} \) such that \( F \in \mathcal{F} \) and \( F(\alpha_i) = F_i \). Further, \( (\alpha_i, F_i) \) is given by the following:

\[
(\alpha_i, F_i) = \begin{cases} 
(0, 0) & \text{if } D_i, C_i \neq \emptyset \\
\left( \frac{\text{int}_i}{1 + \text{int}_i + slp_i}, \frac{1}{1 + slp_i} \right) & \text{if } D_i = \emptyset \\
(1, 1) & \text{if } C_i = \emptyset
\end{cases}
\]  

(10)

**Proof.** First \( \text{int}_i, slp_i > 0 \) (Lemma A.1) implies that \( (\alpha_i, F_i) \) from (10) is always in \([0,1]^2\). Consider, again, the three possible cases.

Case 1: \( D_i \neq \emptyset \) and \( C_i \neq \emptyset \): Recall from Proposition 1, that in the behavioral model, for any \( (x, y) \in PD \), each equilibrium is of cutoff form, with \( \alpha_{x,y}^* \) being any solution to (2). So, it is sufficient to show that for arbitrary \( (x, y) \in PD \) and \( F \in \mathcal{F} \) such that \( F(\alpha_i) = F_i \): i) \( (x, y) \in M_i \) if and only if \( \alpha_i \) solves (2); ii) \( (x, y) \in C_i \) implies that there exists \( \alpha \in (0, \alpha_i) \) such that \( \alpha \) solves (2); iii) \( (x, y) \in D_i \) implies that there exists \( \alpha \in (\alpha_i, 1) \) such that \( \alpha \) solves (2). We take them in turn.

i) By Lemma A.1, \( (x, y) \in M_i \iff y = \text{int}_i - slp_i \cdot x > 0 \). Solving (2) for \( y \) gives \( y = \frac{\text{int}_i}{F(\alpha_i) - \text{int}_i F(\alpha_i)} - \frac{1 - F(\alpha_i)}{F(\alpha_i)} \cdot x \). The pair of equations \( \text{int}_i = \frac{\alpha_i}{F(\alpha_i) - \text{int}_i F(\alpha_i)} \) and \( slp_i = \frac{1 - F(\alpha_i)}{F(\alpha_i)} \) has a unique solution: \( \alpha_i = \frac{\text{int}_i}{1 + \text{int}_i + slp_i} \) and \( F(\alpha_i) = \frac{1}{1 + slp_i} \). This establishes the claim.

ii) Suppose that \( (x, y) \in C_i \). By Lemma A.1, this implies that \( y < \text{int}_i - slp_i \cdot x \). Let \( d(\alpha) := V(c(\alpha = \alpha^*) - V(d(\alpha = \alpha^*) = \alpha [1 + (1 - F(\alpha))x + F(\alpha)y] - [(1 - F(\alpha))x + F(\alpha)y]. Using the assignments of \( (\alpha_i, F(\alpha_i) = F_i) \) from (10), it follows that \( d(\alpha_i) > 0 \). Notice that \( d(0) = x(F(0) - 1) - yF(0) = -x < 0 \). \( F \in \mathcal{F} \) implies that \( d \) is continuous \([0, \alpha_i]\). Hence, there exists \( \alpha \in (0, \alpha_i) \) that achieves \( d(\alpha) = 0 \) and is therefore an equilibrium cutoff in the game \( (x, y) \in C_i \) (by Proposition 1).

iii) Suppose that \( (x, y) \in D_i \). By Lemma A.1, this implies that \( y > \text{int}_i - slp_i \cdot x \). Using the assignments of \( (\alpha_i, F(\alpha_i) = F_i) \) from (10), it follows that \( d(\alpha_i) < 0 \). Notice that \( d(1) = 1 \). \( F \in \mathcal{F} \) implies that \( d \) is continuous on \([\alpha_i, 1]\). Hence, there exists \( \alpha \in (\alpha_i, 1) \) that achieves \( d(\alpha) = 0 \) and is therefore an equilibrium cutoff in the game \( (x, y) \in D_i \) (by Proposition 1).
The following is relevant for the next two cases. In the behavioral model, for any \( F \in \mathcal{F} \), Proposition 1 establishes that a player with type \( \alpha_i = 1 \) strictly prefers \( c \) in all equilibria of all games, and that a player with type \( \alpha_i = 0 \) strictly prefers \( d \) in all equilibria of all games. Further, for any \( F \in \mathcal{F} \) and any \( \alpha \in (0,1) \), the game \( (x,y) = \left( \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} \right) \) is in PD and has a unique equilibrium cutoff \( \alpha^*_x, y = \alpha \).

Case 2: \( D_i = \emptyset \): From above, in the behavioral model, for any \( F \in \mathcal{F} \), a player \( i \) strictly prefers \( c \) in every \( (x,y) \in PD \) if and only if his type is \( \alpha_i = 1 \). Further, for all \( F \in \mathcal{F} \), \( F(1) = 1 \).

Case 3: \( C_i = \emptyset \): From above, in the behavioral model, for any \( F \in \mathcal{F} \), a player \( i \) strictly prefers \( d \) in every \( (x,y) \in PD \) if and only if his type is \( \alpha_i = 0 \). Further, for all \( F \in \mathcal{F} \), \( F(0) = 0 \).

**Lemma A.3** Fix any two players \( i \) and \( j \) such that \( (D_i, C_i) \) and \( (D_j, C_j) \) satisfy Axioms 2-5 and \( D_i, C_i, D_j, C_j \) are all nonempty. Using \((\text{int}_i, \text{slp}_i), (\text{int}_j, \text{slp}_j)\) from Lemma A.1, if \( \text{int}_i < \text{int}_j \), then \( \text{slp}_i > \text{slp}_j \).

**Proof.** By Lemma A.1, \( M_i = \{(x,y) \in PD \mid y = \text{int}_i - \text{slp}_i \cdot x \} \) and \( M_j = \{(x,y) \in PD \mid y = \text{int}_j - \text{slp}_j \cdot x \} \). Fix any \( (x,y) \in M_i \), and for \( \varepsilon \in (0, \frac{1}{\text{slp}_i}) \), let \( \delta = \varepsilon \cdot \text{slp}_i \). It follows that \( (x + \varepsilon, y - \delta) \in M_i \). Next, \( \text{int}_i < \text{int}_j \) implies that for sufficiently small choices of \( x \) and \( \varepsilon \) there exists \((x', y')\) such that \( (x', y') \in M_j \) and \( \{(x,y), (x + \varepsilon, y - \delta)\} < \{(x', y'), (x' + \varepsilon, y' - \delta)\} \). By Axiom 5, \((x' + \varepsilon, y' - \delta) \in C_j \). By (10) in Lemma A.2, \((\text{int}_j, \text{slp}_j)\) is strict. Thus, \( \text{slp}_j < \frac{\delta}{\varepsilon} = \text{slp}_i \). ■

**Definition A.1** Fix any player \( i \) such that \( (D_i, C_i) \) satisfies Axioms 2-4 and \( D_i \neq \emptyset, C_i \neq \emptyset \). Assign \((\alpha_i, F_i)\) as done by (10) in Lemma A.2. Define the function \( H_i : [0,1] \rightarrow \mathbb{R} \cup \infty \) as \( H_i(a) = F_i \frac{a(1-\alpha_i)}{(1-a)} \) for \( a \in [0,1] \) and \( H_i(1) = \infty \).

**Fact A.2** For all \( i \) such that \( H_i \) is defined: i) \( H_i(0) = 0 \), ii) \( H_i \) is strictly increasing, differentiable, and strictly convex on \([0,1]\), iii) \( H_i(\alpha_i) = F_i \), iv) \( \lim_{a \to 1} H_i(a) = \infty \), and v) \( H_i'(\alpha_i) = \frac{F_i}{\alpha_i - \alpha_i} \).

**Proof.** Direct calculations. ■

**Lemma A.4** Fix any two players \( i \) and \( j \) such \((D_i, C_i)\) and \((D_j, C_j)\) satisfy Axioms 2-5. Assign \((\alpha_i, F_i), (\alpha_j, F_j)\) as done by (10) in Lemma A.2. The following are valid:

i) If \( \alpha_j < \alpha_i < 1 \), then \( F_j \in [H_i(\alpha_j), F_i] \).

ii) If \( 0 < \alpha_i < \alpha_j \), then \( F_j \in (F_i, H_i(\alpha_j)] \).

iii) If \( \alpha_i = \alpha_j \), then \( F_j = F_i \).

**Proof.** First, if \( \alpha_i \in \{0,1\} \) then (i) and (ii) have no implications, and (iii) is immediate from Lemma A.2. Now fix \( \alpha_i \in (0,1) \). If \( \alpha_j \in \{0,1\} \), then the claims follow from Fact A.2. If \( \alpha_j \in (0,1) \), then from Lemma A.3,

\[ (\text{int}_j, \text{slp}_j) \in \{(\text{int}, \text{slp})|\text{int} \leq \text{int}_i \text{ and } \text{slop} > \text{slop}_i\} \cup \{(\text{int}, \text{slp})|\text{int} \geq \text{int}_i \text{ and } \text{slop} < \text{slop}_i\} \]

\[ \cup \{(\text{int}, \text{slop})|\text{int} = \text{int}_i \text{ and } \text{slop} = \text{slop}_i\} \]
Inverting the bijection from (10) in Lemma A.2, 

\[
(\alpha_j, F_j) \in \left\{ (\alpha, \phi) \bigg| \frac{\alpha}{1 - \alpha} \phi \leq \frac{\alpha_i}{1 - \alpha_i} F_i \text{ and } \frac{1 - \phi}{\phi} > \frac{1 - F_i}{F_i} \right\} \\
\cup \left\{ (\alpha, \phi) \bigg| \frac{\alpha}{1 - \alpha} \phi \geq \frac{\alpha_i}{1 - \alpha_i} F_i \text{ and } \frac{1 - \phi}{\phi} < \frac{1 - F_i}{F_i} \right\} \\
\cup \left\{ (\alpha, \phi) \bigg| \frac{\alpha}{1 - \alpha} \phi = \frac{\alpha_i}{1 - \alpha_i} F_i \text{ and } \frac{1 - \phi}{\phi} = \frac{1 - F_i}{F_i} \right\}
\]

Rearranging and using Definition A.1 gives,

\[
(\alpha_j, F_j) \in \left\{ (\alpha, \phi) \big| H_i(\alpha) \leq \phi \text{ and } \phi < F_i \right\} \cup \left\{ (\alpha, \phi) \big| H_i(\alpha) \geq \phi \text{ and } \phi > F_i \right\} \\
\cup \left\{ (\alpha, \phi) \big| H_i(\alpha) = \phi \text{ and } \phi = F_i \right\}
\]

Which, by (ii) and (iii) of Fact A.2, is equivalent to,

\[
(\alpha_j, F_j) \in \left\{ (\alpha, \phi) \big| \alpha \in [0, \alpha_i) \text{ and } \phi \in [H_i(\alpha), F_i) \right\} \cup \left\{ (\alpha, \phi) \big| \alpha \in (\alpha_i, 1] \text{ and } \phi \in (F_i, H_i(\alpha)] \right\} \\
\cup \left\{ (\alpha, \phi) \big| \alpha = \alpha_i \text{ and } \phi = F_i \right\}
\]

This establishes the result. ■

**Corollary A.1** Fix any two players \(i\) and \(j\) such (\(D_i, C_i\) and (\(D_j, C_j\)) satisfy Axioms 2-5. Assigning \((\alpha_i, F_i), (\alpha_j, F_j)\) as done by (10), if \(0 < \alpha_i < \alpha_j < 1\), then either \(H_i = H_j\) or \(H_i'(a) > H_j'(a)\) for all \(a \in [0, 1)\).

**Proof.**

\[
H_i'(a) - H_j'(a) = \frac{F_i \cdot \alpha_j (1 - \alpha_i) - F_j \cdot \alpha_i (1 - \alpha_j)}{\alpha_i \cdot \alpha_j (1 - a)^2}
\]

By Lemma A.4, either (11) is zero for all \(a \in [0, 1)\), in which case \(H_i = H_j\) since \(H_i(0) = H_j(0)\) from Fact A.2, or (11) is positive for all \(a \in [0, 1)\). ■

**Lemma A.5** Fix a primitive \((D_i, C_i)_{i \in I}\) that satisfies Axioms 2-5. Assign \((\alpha_i, F_i)_{i \in I}\) as done by (10) in Lemma A.2. There exists \(F \in F_S\) with \(F(\alpha_i) = F_i\) for all \(i \in I\).

**Proof.** Order and (re-)index the distinct pairs featuring \(\alpha_i \in (0, 1)\) such that: \((0, 0) \ll (\alpha_1, F_1) \ll (\alpha_2, F_2) \ll \ldots \ll (\alpha_m, F_m) \ll (1, 1)\), where \(m \leq n\) and the ordering is strict by Lemma A.4. For each \(k \in \{1, 2, \ldots, m\}\), set \(F(\alpha_k) = F_k\) and \(F'(\alpha_k) = \frac{f_k}{\alpha_k - \alpha_k^2}\). Set \(F(0) = 0, F(1) = 1, F'(1) = 0\). Next, we fill in \(F\) in the intervals between the pairs to produce a strictly increasing, differentiable CDF that satisfies Conditions S. In doing so, we say that a differentiable function \(f_1\) smoothly pastes the ordered pair of differentiable, increasing functions \((f_2, f_3)\) on an interval \((\bar{z}, \bar{z})\) if i) \(f_1(\bar{z}) = f_2(\bar{z})\), ii) \(f'_1(\bar{z}) = f'_2(\bar{z})\), iii) \(f_1(\bar{z}) = f_3(\bar{z})\), and iv) \(f'_1(\bar{z}) = f'_3(\bar{z})\).

1. On \((0, \alpha_1)\) set \(F = H_1\), which satisfies all of the necessary properties (see Fact A.2).

2. Identify all \(k \in \{1, 2, \ldots, m - 1\}\), such that \(H_k = H_{k+1}\). For all such \(k\), set \(F = H_k\) on \((\alpha_k, \alpha_{k+1})\), which satisfies all of the necessary properties (see Fact A.2).
3. Fix arbitrary $k < m$ such that $H_k \neq H_{k+1}$, and let $L$ be the linear function tangent to $H_k$ at $\alpha_k$. There are two cases: a) $L(\alpha_{k+1}) < F_{k+1}$, or b) $L(\alpha_{k+1}) \geq F_{k+1}$.

a) In this case, $L$ intersects $H_{k+1}$ at some $\alpha^0 \in (\alpha_k, \alpha_{k+1})$, where $L' < H'_{k+1}(\alpha^0)$. Now for any $\varepsilon > 0$ small enough, there exists an elliptical arc $E$ that smoothly pastes ($L, H_{k+1}$) on $(\alpha^0 - \varepsilon, \alpha^0 + \varepsilon)$. By construction, for sufficiently small $\varepsilon$, setting $F = L$ on $(\alpha_k, \alpha^0 - \varepsilon)$, $F = E$ on $(\alpha^0 - \varepsilon, \alpha^0 + \varepsilon)$, and $F = H_{k+1}$ on $(\alpha^0 + \varepsilon, \alpha_{k+1})$, satisfies differentiability and strict monotonicity. To see that it satisfies Condition $S$, let $\tilde{H}_{\alpha,F}$ be the $H_{\cdot}$-function of a hypothetical player with $(\alpha_i, F_i) = (\alpha, F)$. By (v) of Fact A.2, it is sufficient to demonstrate that $F'(\alpha) \leq \tilde{H}'_{\alpha,F(\alpha)}(\alpha)$ for all $\alpha \in (\alpha_k, \alpha_{k+1})$. Notice that Corollary A.1 implies that this holds with equality on $[\alpha^0 + \varepsilon, \alpha_{k+1})$. For $\alpha \in (\alpha_k, \alpha^0 - \varepsilon)$, $F$ is weakly concave and crosses the strictly convex function $H_{\alpha,F(\alpha)}$ from above at $\alpha$. Hence, the inequality must be satisfied. Finally, if $\varepsilon$ is small enough, then since $E > H_{k+1}$ on $(\alpha^0 - \varepsilon, \alpha^0 + \varepsilon)$, in order to smoothly paste with $H_{k+1}$, it must be that $E' < H'_{k+1}$ on this interval. From Corollary A.1, $\tilde{H}'_{\alpha,F(\alpha)} > H'_{k+1}$, which establishes the inequality.

b) In this case, let $\hat{L}$ be the line that passes through $(\alpha_k, F_k)$ and $(\alpha_{k+1}, F_{k+1})$, so $\hat{L}' \leq L'$ by hypothesis. Next, let $\hat{L}_\delta$ be the line that passes through the midpoint between $(\alpha_k, F_k)$ and $(\alpha_{k+1}, F_{k+1})$ with slope $\hat{L}' - \delta$. For any $\delta > 0$ small enough, there exists $\varepsilon > 0$ small enough such that $(H_k, \hat{L}_\delta)$ can be smoothly pasted by elliptical arc $E_1$ on $(\alpha_k, \alpha_k + \varepsilon)$, and $(\hat{L}_\delta, H_{k+1})$ can be smoothly pasted by elliptical arc $E_2$ on $(\alpha_k - \varepsilon, \alpha_{k+1})$. By construction, for sufficiently small $\delta$ and $\varepsilon$, setting $F = E_1$ on $(\alpha_k, \alpha_k + \varepsilon)$, $F = \hat{L}_\delta$ on $[\alpha_k + \varepsilon, \alpha_{k+1} - \varepsilon]$, and $F = E_2$ on $(\alpha_{k+1} - \varepsilon, \alpha_{k+1})$, satisfies differentiability and strict monotonicity. The arguments that it satisfies Condition $S$ are analogous to those made in (3.a) above, since $F$ is weakly concave on $(\alpha_k, \alpha_{k+1} - \varepsilon)$, and $E_2' < H'_{k+1}$ on $(\alpha_{k+1} - \varepsilon, \alpha_{k+1})$ when $\varepsilon$ is sufficiently small.

4. For $\alpha \in (\alpha_m, 1)$, given the properties of $H_m$ from Fact A.2, there exists an $\alpha^0 \in [\alpha_m, 1)$ and an elliptical arc $E$ that both smoothly pastes $(H_m, 1)$ on $(\alpha^0, 1)$ and is concave. Set $F = H_m$ on $[\alpha_m, \alpha^0]$ and $F = E$ on $(\alpha_0, 1)$ to satisfy differentiability, strict monotonicity, and Condition $S$ (by the same arguments from (3)). ■

Lemma A.6 If $(D_i, C_i)_{i \in I}$ satisfies Axioms 2-5, then it can be explained by a behavioral model $[F, (\alpha_i)_{i \in I}]$, where $F \in \mathcal{F}$ satisfies Condition $S$. Furthermore, $(\alpha_i, F(\alpha_i))_{i \in I}$ is unique.


Lemma A.7 If $(D^0_i, C^0_i)_{i \in I}$ satisfies Axioms 1-5, then it can be explained by a behavioral model $[F, (\alpha_i)_{i \in I}]$, where $F \in \mathcal{F}$ satisfies Condition $S$. Furthermore, $(\alpha_i, F(\alpha_i))_{i \in I}$ is unique.


Proof of Proposition 3. The proof is ordered: $(a) \iff (c), (b) \iff (c), (a) \iff (e), (d) \iff (e)$.

$(a) \iff (c)$: Suppose $F$ f.o.s.d. $\bar{F}$. Recall that if $x = y$, then $x^* y = \bar{x^* y}$ (Section 2.1.1), implying $F(x^* y) \leq \bar{F}(\bar{x^* y})$ by f.o.s.d. If $x \neq y$, then the cutoffs are implicitly characterized by $F(x^* y) = T(x^* y|x, y)$ and $\bar{F}(\bar{x^* y}) = T(\bar{x^* y}|x, y)$. Further, if $x > y$, then $T(0|x, y) > 0 = F(0) = \bar{F}(0)$,

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and \( T(x,y) \) is strictly decreasing. Hence, by f.o.s.d., both \( \alpha^{*}_{x,y} \leq \alpha^{*}_{x,y} \) and \( \bar{F}(\alpha^{*}_{x,y}) \geq F(\alpha^{*}_{x,y}) \).

If \( x < y \), then \( T(0,x,y) < 0 = F(0) = \bar{F}(0) \), and \( T \) is strictly increasing (but intersecting \( F \) and \( \bar{F} \)) exactly once since both satisfy Condition \( S \). Hence, by f.o.s.d., both \( \alpha^{*}_{x,y} \geq \alpha^{*}_{x,y} \) and \( \bar{F}(\alpha^{*}_{x,y}) \geq F(\alpha^{*}_{x,y}) \). So (a) implies (c).

Now suppose that (a) does not hold; there exists \( \alpha^{0} \in (0,1) \) such that \( F(\alpha^{0}) > \bar{F}(\alpha^{0}) \). In the game \( x = y = 1 \), we have that \( \alpha^{*}_{x,y} = \alpha^{*}_{x,y} = \alpha^{0} \), which then violates (c).

(b) \( \iff \) (c): Notice that \( k_{x,y} \) and \( \bar{k}_{x,y} \) are binomial random variables with \( n \) “trials” (i.e., each players’ action) and probabilities of “success” (i.e., cooperation) of \( (1-F(\alpha^{*}_{x,y})) \) and \( (1-\bar{F}(\alpha^{*}_{x,y})) \) respectively.

Because \( n \) is common between the two random variables, a simple “coupling” argument (Lindvall, 2002, Ch. 1) establishes that \( k_{x,y} \) f.o.s.d. \( \bar{k}_{x,y} \) if and only if \( 1 - F(\alpha^{*}_{x,y}) \geq 1 - \bar{F}(\alpha^{*}_{x,y}) \).

(a) \( \iff \) (e): That (a) implies (e) is shown in the proof of (a) \( \iff \) (c) above. Now suppose that (a) does not hold; there exists \( \alpha^{0} \in (0,1) \) such that \( F(\alpha^{0}) > \bar{F}(\alpha^{0}) \). In the game \( x = y = 1 \), we have that \( \alpha^{*}_{x,y} = \alpha^{*}_{x,y} = \alpha^{0} \). Because \( F, \bar{F}, T \) are all continuous in \( y \) and \( T \) is decreasing in \( \alpha \) when \( x > y \), there exists an \( \varepsilon > 0 \) small enough such that in the game \( (x,y) = \left( \frac{\alpha^{0}}{1-\alpha^{0}}, \frac{\alpha^{0}}{1-\alpha^{0}} - \varepsilon \right) \in PD \), \( \alpha^{*}_{x,y} < \alpha^{*}_{x,y} \), which violates (e).

(d) \( \iff \) (e): That (e) does not hold in that there exists \( x_{0} \leq y_{0} \) such that \( \alpha^{*}_{x_{0},y_{0}} > \alpha^{*}_{x_{0},y_{0}} \).

Let \( \alpha^{0} \in (\alpha^{*}_{x_{0},y_{0}}, \alpha^{*}_{x_{0},y_{0}}) \). Then,

\[
\{(x_{0}, y_{0})\} \subseteq C_{\alpha^{0},F} \cap D_{\alpha^{0},F} \cap \{(x,y) | x \leq y \} \neq \emptyset,
\]

violating (d). A symmetric argument applies if there exists \( x_{0} \geq y_{0} \) such that \( \alpha^{*}_{x_{0},y_{0}} < \alpha^{*}_{x_{0},y_{0}} \).

**Proof of Proposition 4.** When \( x = y \), the unique solution to (2) is \( \alpha^{*} = \frac{x}{1+\alpha} \). Hence, Proposition 1 implies that, for any \((x,x) \in PD\) and any player \( i \) (of any collection): \((x,x) \in D_{i}\) if and only if \( \alpha_{i} < \frac{1}{1+\alpha} \). Part (a) of the proposition follows.

For part (b), first suppose that \( I \) is more influenced by \( x \) relative to \( y \) than is \( \bar{I} \). Then the behavior of the two collections agree on the subset of games \( \{(x,y) \in PD | x = y \} \). By part (a) of the proposition, \( (\alpha_{i})_{i \in I} = (\bar{\alpha}_{i})_{i \in \bar{I}} \). Immediately, if \( \alpha_{i} = 0,1 \), then \( F(\alpha_{i}) = \bar{F}(\bar{\alpha}_{i}) \). In addition, \( I \) defecting more in \( \{(x,y) \in PD | x \leq y \} \) than does \( \bar{I} \) implies that, for each \( i \) with \( \alpha_{i} \in (0,1) \), it must be that the \( M_{i} \)-line is weakly steeper under \( F \) than under \( \bar{F} \), i.e., \( slp_{i} \geq \bar{slp}_{i} \) (Lemma A.1). Next, by Lemma A.2, \( F(\alpha_{i}) = \frac{1}{1+slp_{i}} \leq \frac{1}{1+slp_{i}} = \bar{F}(\bar{\alpha}_{i}) \).

Second, suppose that for all \( i \leq n, \alpha_{i} = \bar{\alpha}_{i} \) and \( F(\alpha_{i}) \leq \bar{F}(\bar{\alpha}_{i}) \). Immediately, if \( \alpha_{i} = 0,1 \), then \( i \)'s behavior is the same in \( I \) and \( \bar{I} \). For each \( i \) with \( \alpha_{i} \in (0,1) \), \( M_{i} \cap \bar{M}_{i} = \{\{(\alpha_{i}^{0}, \alpha_{i}^{1}, \alpha_{i}^{0})\}\} \).

Also, \( slp_{i} = \frac{1-F(\alpha_{i})}{F(\alpha_{i})} \geq \frac{1-\bar{F}(\bar{\alpha}_{i})}{\bar{F}(\bar{\alpha}_{i})} = \bar{slp}_{i} \). Hence, for any \((x,y) \in PD\), if \( x \geq y \) then \((x,y) \in \bar{D}_{i} \implies (x,y) \in D_{i} \implies (x,y) \in \bar{D}_{i} \), establishing that \( I \) is more influenced by \( x \) relative to \( y \) than is \( \bar{I} \).

**Proof of Proposition 5.** First, the proof of Proposition 1, Claim (i) remains valid and implies that all equilibria are cutoff with \( \alpha^{*} < 1 \) and that \( \alpha^{*} = 0 \) is an equilibrium cutoff if and only if \( x \leq 0 \). Second, if \( x > 0 \), the existence, uniqueness, and characterization of the equilibrium cutoff follow identically from the proofs of Propositions 1 and 2. Third, any \( \alpha \in (0,1) \) is an equilibrium
cutoff if and only if \( g \in \tilde{M}_\alpha \) (the argument given in the proof of Theorem 1, “Representation \( \implies \) Axioms,” immediately extends to establish this). Fix now \( x \leq 0 \). By definition of \( B, i \) if \( y < B(x) \), then there does not exist an interior equilibrium cutoff in game \((x,y)\), and ii) if \( y = B(x) \), then there does exist an interior equilibrium cutoff in game \((x,y)\). The final case is \( y > B(x) \). For \( \alpha \in (0,1) \), define \( y^*(\alpha|x) := \frac{\alpha}{(1-\alpha)F(\alpha)} - x \left(1 - \frac{F(\alpha)}{F(\alpha)} \right) \), or equivalently, \((x,y^*(\alpha|x)) \in \tilde{M}_\alpha \). For any \( y > B(x) \), there exists \( \alpha_1 < \alpha_2 \) such that \( y^*(\alpha_1|x) = B(x) < y < y^*(\alpha_2|x) \), where the final inequality follows from \( \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)F(\alpha)} = \infty \) and \( \lim_{\alpha \to 1} \frac{1-F(\alpha)}{F(\alpha)} = 0 \). Since \( y^*(\cdot|x) \) is continuous, the Intermediate Value Theorem implies that there exists \( \alpha_3 \in (\alpha_1, \alpha_2) \) such that \( y^*(\alpha_3|x) = y \), meaning \( \alpha_3 \) in an interior equilibrium cutoff in the game \((x,y)\).

Proof of Proposition 6. First consider \( g \in PD^0 \), so \( \pi_g = 1 \). By Propositions 1-2 (and Lemma 1), \( \alpha^*_g \in (0,1) \), implying \( F(\alpha^*_g) < 1 = \pi_g \). Second, consider \( g \) such that \( x > 0 \geq y \). It is straightforward to obtain \( \pi_g = \frac{x}{x-y} \). The analog to (7) in which \( r,p \) have not been normalized is \( F(\alpha^*_g) = \frac{\alpha^*_g(r-p) - (1-\alpha^*_g)x}{(1-\alpha^*_g)(y-x)} \). Therefore, \( \pi_g - F(\alpha^*_g) = \frac{\alpha^*_g(r-p)}{(1-\alpha^*_g)(x-y)} > 0 \).

The inequality is due to \( \alpha^*_g \in (0,1) \) (by Proposition 5), \( r > p \), and \( x > 0 \geq y \).

For the general limit result, observe that Lemma 1 implies that it is sufficient to establish the following: if \( r = 1 \) and \( p = 0 \), then for any \( \varepsilon > 0 \), there exists \( K \in \mathbb{R}^+ \) such that, if \( x+|y| > K \), then \( \pi_{x,y} - F(\alpha^*_{x,y}) < \varepsilon \). Fix \( \varepsilon > 0 \), and define the following terms: \( \alpha^\varepsilon = F^{-1}(1-\varepsilon) \), \( K_1 = \frac{\alpha^\varepsilon}{(1-\alpha^\varepsilon)} \), and, letting \( (\text{int}_t, \text{slp}_t) \) be the \((\text{int}_t, \text{slp}_t)\) generated by the equilibrium behavior of a player \( i \) with \( \alpha_i = \alpha^\varepsilon, K_2 = \max\{\text{int}_t, \text{slp}_t\} \).

Setting \( K = \max\{K_1, K_2\} \), establishes the claim. To see this, suppose that \( y > 0 \) and \( x + |y| > K \). Then \( y > K - x \geq K_2 > x \geq \text{int}_{\alpha^\varepsilon} - x \cdot \text{slp}_{\alpha^\varepsilon} \). Hence, \((x,y) \in D_{\alpha^{\varepsilon}} \), and \( \alpha^*_{x,y} > \alpha^{\varepsilon} \). Therefore, \( F(\alpha^{*}_{x,y}) > F(\alpha^{\varepsilon}) = 1 - \varepsilon \), and \( \pi_{x,y} - F(\alpha^{*}_{x,y}) = 1 - F(\alpha^{*}_{x,y}) < \varepsilon \). Suppose instead that \( y \leq 0 \) and \( x + |y| > K \). First, if \( \alpha^*_{x,y} > \alpha^{\varepsilon} \), then \( 1 \geq \pi_{x,y} > F(\alpha^{*}_{x,y}) > F(\alpha^{\varepsilon}) = 1 - \varepsilon \), and the result holds. Second, if \( \alpha^*_{x,y} \leq \alpha^{\varepsilon} \), then by (12), we have

\[
\pi_{x,y} - F(\alpha_{x,y}) = \frac{\alpha^*_{x,y}}{(1-\alpha^*_{x,y})(y-x)} \leq \frac{\alpha^{\varepsilon}}{(1-\alpha^{\varepsilon})(x-y)} < \frac{\alpha^{\varepsilon}}{(1-\alpha^{\varepsilon})K_1} = \varepsilon,
\]

establishing the claim.

Proof of Proposition 7. Let \( r = p \). If \( \alpha_i = 1 \), \( V_i(c) = V_i(d) \) and player \( i \) is indifferent between \( c \) and \( d \). However, such players are measure zero, and their behavior has no effect on the claims in the proposition. For the remainder, focus then on players with \( \alpha_i \in [0,1) \), and therefore \( \text{sign}(V_i(c) - V_i(d)) = \text{sign}((1-\alpha_i)(P_1 \cdot (-y) + (1 - P_1) \cdot x)) = \text{sign}(P_1 \cdot (-y) + (1 - P_1) \cdot x) \), which is independent of \( \alpha_i \). Suppose now that \( (\sigma, P) \) is an equilibrium. If \( \sigma(d|\alpha) \) is constant in \( \alpha \) on \([0,1)\), then by Condition 3 of Definition 2.1, \( \sigma(d|\alpha) = P \) for \( \alpha \in [0,1) \). Therefore, assigning probability \( P \) to \( d \) is a best response to \( P \), regardless of \( \alpha_i \), implying \( \pi_g = P \) characterizes a symmetric Nash equilibrium. If \( \sigma(d|\alpha) \) is not constant in \( \alpha \) on \([0,1)\), then, since preferences are independent of \( \alpha_i \), for any \( \alpha_i \in [0,1), any mixture over \( d \) and \( c \) is a best response to \( P \). Again, then, \( \pi_g = P \) characterizes a symmetric Nash equilibrium. Now suppose that \( \pi_g \) characterizes a symmetric Nash equilibrium. Let \( F(\tilde{\alpha}) = \pi_g \), and \( \tilde{\sigma}(d|\alpha) = 1 \) if \( \alpha < \tilde{\alpha} \), and 0 otherwise. It is trivial to verify that \((\tilde{\sigma}, \pi_g)\) is an equilibrium according to Definition 2.1. Finally, existence of an equilibrium follows from the existence of a symmetric Nash equilibrium (Osborne and Rubinstein, 1994, §20.4).
Proof of Proposition 8. Fix $g \in \Gamma$ and a (candidate) equilibrium $(\sigma, P)$. Let $\alpha < \alpha'$ and $ar{k}(\alpha) := \max \{ k : \sigma(k;\alpha) > 0 \}$. Hence, for all $k < \bar{k}(\alpha)$,

$$V_i(\bar{k}(\alpha)|\alpha_i = \alpha) \geq V_i(k|\alpha_i = \alpha)$$

$$\alpha s(\bar{k}(\alpha)) + (1 - \alpha) \sum_{k' \in A} P(k')v(\bar{k}(\alpha), k') \geq \alpha s(k) + (1 - \alpha) \sum_{k' \in A} P(k')v(k, k')$$

$$\frac{\alpha}{1 - \alpha} \left( s(\bar{k}(\alpha)) - s(k) \right) \geq \sum_{k' \in A} P(k')(v(k, k') - v(\bar{k}(\alpha), k'))$$

$$\frac{\alpha'}{1 - \alpha'} \left( s(\bar{k}(\alpha)) - s(k) \right) > \sum_{k' \in A} P(k')(v(k, k') - v(\bar{k}(\alpha), k'))$$

$$\alpha' s(\bar{k}(\alpha)) + (1 - \alpha') \sum_{k' \in A} P(k')v(\bar{k}(\alpha), k') > \alpha' s(k) + (1 - \alpha') \sum_{k' \in A} P(k')v(k, k')$$

$$V_i(\bar{k}(\alpha)|\alpha_i = \alpha') > V_i(k|\alpha_i = \alpha').$$

Because $A$ is finite, but the set of $\alpha$-types is continuous, $\sigma$ must therefore be increasing as described in Definition 5.1. Equilibrium existence is then an immediate application of the argument in Athey (2001) (Theorem 1, as the above establishes that its SCC condition holds in our model). The only difference is that, since we are looking for a symmetric fixed point, we apply Kakutani’s fixed point theorem to the single-player best-response correspondence (in that paper’s notation: $\Gamma_1 : \Sigma_i \times \Sigma_i$), rather than to the two-player best-response correspondence (i.e., $(\Gamma_1, \Gamma_2) : \Sigma_1 \times \Sigma_2 \times \Sigma_1 \times \Sigma_2$).

Proof of Proposition 9. Fix a supermodular pure dilemma game $g \in \Gamma$ and let $W(k, l, \alpha) := \alpha s(k) + (1 - \alpha) v(k, l)$ (i.e., $V_i(k)$ given $\alpha_i = \alpha$ and $P(l) = 1$). Then $W$ has increasing differences in $(k, \alpha)$, and has increasing differences in $(k, l)$. To see the first, let $k' \geq k$ and $\alpha' \geq \alpha$:

$$W(k', l, \alpha') - W(k, l, \alpha') = W(k', l, \alpha) - W(k, l, \alpha)$$

$$= (\alpha' - \alpha) \left( s(k') + v(k, l) - v(k', l) \right) \geq 0.$$

To see the second, let $k' \geq k$ and $l' \geq l$:

$$W(k', l', \alpha) - W(k, l', \alpha) = W(k', l, \alpha) - W(k, l, \alpha)$$

$$= (1 - \alpha) \left( v(k', l') - v(k, l') \right) \geq 0.$$

Both statements in the proposition are then implications of Van Zandt and Vives (2007) (VZV). (1) follows from VZV Theorem 14 (they note that for symmetric games/models such as ours, the greatest and least equilibria are symmetric, VZV p. 346). (2) follows from VZV Proposition 16.

Proof of Proposition 10. Fix $g \in \Gamma$ that has increasing returns to joint cooperation and a (candidate) equilibrium $(\sigma, P)$. From Proposition 8, $\sigma$ is increasing. Suppose now that $\alpha_i^* < \alpha_k^*$. Then, by Definition 5.1, there exists $k \notin \{0, K\}$ and $\alpha > \alpha'$ such that $\sigma(k|\alpha) = \sigma(k|\alpha') = 1$. So, $V_i(k|\alpha_i = \alpha') \geq V_i(k-1|\alpha_i = \alpha')$, which implies $V_i(k|\alpha_i = \alpha) > V_i(k-1|\alpha_i = \alpha)$ (see proof of Proposition 8). It follows that:
\[ \alpha s(k) + (1 - \alpha) \sum_{k' \in A} P(k') v(k, k') > \alpha s(k - 1) + (1 - \alpha) \sum_{k' \in A} P(k') v(k - 1, k') \]

\[ 1 > \frac{(1 - \alpha)}{\alpha} \sum_{k' \in A} P(k') \frac{v(k - 1, k') - v(k, k')}{s(k) - s(k - 1)}. \]

Increasing returns to joint cooperation implies,

\[ \sum_{k' \in A} P(k') \frac{v(k - 1, k') - v(k, k')}{s(k) - s(k - 1)} \geq \sum_{k' \in A} P(k') \frac{v(k, k') - v(k + 1, k')}{s(k + 1) - s(k)}. \]

Hence,

\[ 1 > \frac{(1 - \alpha)}{\alpha} \sum_{k' \in A} P(k') \frac{v(k, k') - v(k + 1, k')}{s(k + 1) - s(k)}. \]

\[ \alpha s(k + 1) + (1 - \alpha) \sum_{k' \in A} P(k') v(k + 1, k') > \alpha s(k) + (1 - \alpha) \sum_{k' \in A} P(k') v(k, k') \]

\[ V_i(k + 1|\alpha_i = \alpha) > V_i(k|\alpha_i = \alpha), \]

which contradicts that \( \sigma(k|\alpha) = 1 \) in equilibrium. Therefore, \( \alpha_i* = \alpha_K^* \). So, at most a measure-zero set of \( \alpha \)-types assigns positive probability to any action other than 0 or \( K \). This has no effect on the best responses of other types, so equilibrium analysis is identical to that done in the PD game \( (r, p, x, y) = (s(K), s(0), v(0, K) - s(K), s(0) - v(K, 0)) \in PD^0. \]