A liquidity haircut for hedge funds

Investors in hedge funds have learned to be cautious when making decisions due to problems of survivorship bias, autocorrelation and hidden optionality. Here, Hari Krishnan and Izzy Nelken show how to quantify such caution. By analysing the incentive structure of hedge fund managers using an option pricing approach, they derive a liquidity haircut to compensate for lockup periods, and an illiquidity premium that effectively increases volatility.

Since the 1990s, mean variance optimisation has been widely used to construct portfolios of traditional assets, such as stocks and bonds. Over the past 10 years, alternative investments such as hedge funds have risen in prominence. There are more than 6,000 registered hedge funds worldwide. Many have performed well in bull and bear markets and there has been a significant flow of assets into hedge funds.

It is natural to ask the question: how much of an investor’s portfolio should be allocated to a specific hedge fund or a portfolio of hedge funds? The usual approach has been to incorporate hedge funds in a mean variance framework. However, many hedge funds have outperformed stocks under normal conditions while retaining a small probability of a large negative return.

Second, we assume that an investor holds a portfolio consisting of a traditional fund and a hedge fund. While the hedge fund manager wants to maximise the value of his incentive clause, the investor wants to maximise the return/risk ratio of his portfolio. Thus, the hedge fund manager and investor are playing a cat and mouse game. If the investor had unrestricted liquidity, he could rebalance between the hedge fund and traditional assets in response to a change in leverage or the probability of a blow-up. Since he cannot rebalance, he needs to discount the return or increase the volatility of the hedge fund before he makes a decision to invest. We adapt Longstaff’s method (1999) to calculate the illiquidity premium.

We conclude the article with a numerical example, where we show that the volatility of a hedge fund should be increased by a factor of about 10% over its historical value, to account for illiquidity.

Valuing the hedge fund manager’s contract

Goetzmann, Ingersoll & Ross (2001) have modelled the evolution of the high water mark and calculated the expected value of the contract to a manager. The high water mark is the highest asset level reached, net of withdrawals, since initial investment. Suppose that a fund’s asset level is given by \( S(t) \) at time \( t \) and that \( S(t) \) evolves according to a discrete log-normal diffusion with drift \( \mu - c - W \) and volatility \( \sigma \). The asset level drift \( \mu \) (assumed to be greater than zero) and management fee \( c \) are constant.

Funds are added or redeemed at the rate \( W = W(t, S) \). If an investor redeems, then \( W > 0 \), otherwise \( W \leq 0 \). Thus:

\[
S(t + \Delta t) = S(t) + (\mu - W - c)S(t)\Delta t + \sigma S(t)\sqrt{\Delta t} \xi
\]

where \( \xi \sim N(0, 1) \) is a normally distributed random variable.

Suppose that, over time, the manager collects a percentage \( p \) of profits above a high water mark \( H(t) \). In particular, suppose a manager collects a performance fee of \( p \max(S(t) - H(t - \Delta t), 0) \) at time \( t \geq \Delta t \). Then, the value of the manager’s performance fee is equal to the value of \( p \) call options on \( S(t) \) struck at \( H(t) \). The value of the call can be expressed as the solution to a Black Scholes-type partial differential equation over some time horizon \( T \).

For our purposes, it is not necessary to specify the precise value of the high water mark contract. However, we can use the Goetzmann, Ingersoll & Ross (2001) model to determine a manager’s optimal use of leverage. Our leverage model is somewhat stylised. In our opinion, however, it captures a manager’s typical response to varying asset levels. Suppose that we denote a fund’s leverage by \( l \) and assume that \( l \) is capped by \( l_{\text{max}} \). (Usually, a fund’s offering documents set a strict limit on leverage.) If the manager’s strategy is scalable, then \( \mu = \mu(l) \) and \( \sigma = \sigma(l) \) increase linearly in \( l \). This means that slippage does not increase as more capital is put to work. Since

\[ \text{If a hedge fund has a one year lockup, funds can typically only be taken out at the end of the calendar year following the year of investment. Thus, an investor who allocates money in January 2002 can only take the money out in December 2003 and the effective lockup period is two years.} \]
the high water mark gives the manager a long call position, the manager's expected profit should be increasing in $\mu$ and $\sigma$. Naively, a manager would want to maximise his leverage in order to make his expected pay-out as large as possible.

This is a reasonable assumption whenever $S(t)$ is close to $H(t)$, since in this region the high water mark contract increases sharply as a function of volatility. However, a manager will typically lower his volatility at the extremes when $S(t)$ and $H(t)$ are far apart. When $S(t) \ll H(t)$, the fund may be close to liquidation. Although a manager would like to increase the expected value of his performance fee, he doesn't want to risk his management fee. Increasing leverage increases the probability of default; if this were to occur, the manager would no longer receive the percentage $c$.\footnote{Our anonymous referee has correctly pointed out that some managers will increase leverage near default to model the probability of default. If, at any time $t$, $S(t) \ll LH(t)$, where $0 < L < 1$ is a constant, then many investors will withdraw and the fund will be forced to liquidate its positions. For example, if $L = 0.85$, then the fund will be forced to liquidate as soon as assets decline 15% from the high water mark. After liquidation, we further assume that the investor is only able to recover some percentage of $LH(t)$, say 50%.

For example, suppose that the high water mark is $100$ million, the annual management fee is 1.5% and the performance fee is 20%. The manager's profit and loss profile and use of leverage are illustrated in figure 1. If the fund reaches the default threshold, we assume that the manager not only loses this year's performance fee, but also a performance fee for the next three years (we assume a discount rate of zero). Here, we have assumed that it will take some time for the manager to increase his assets to $100$ million again.

**Longstaff's method**

Now that we have a picture of the hedge fund manager's strategy, we can study the appropriate rebalancing strategy for an investor. We want to know how much the investor should be compensated (in volatility terms) for his inability to move funds out of the hedge fund. Our approach is to use a variation of Longstaff's method (1999).

Longstaff has developed a technique for calculating the illiquidity premium of an asset, such as a stock. The idea is to construct a portfolio consisting of a reference asset (for example, a money market account) and the restricted asset. First, it is assumed that there are no liquidity constraints. This is the benchmark case. The investor tries to maximise his expected utility by rebalancing his portfolio over time, as necessary. Next, the investor isn't allowed to transfer money between the liquid asset and the restricted one. Here, he chooses static weights to maximise the expected utility of his portfolio.

It is clear that the expected utility is at least as large in the case where the investor does not have any liquidity constraints. In general, it should be larger. We then ask, how much extra return should the restricted asset have to make the expected utility functions the same? According to Longstaff, this is the appropriate illiquidity premium. For example, suppose that the investor's utility function is a simple information ratio, such as portfolio return/portfolio volatility. Further, an investor is able to achieve an information ratio of 1.1 if both assets are liquid and a ratio of one if one of the assets is restricted. Then, he should haircut the illiquid asset's return by 10% or increase its volatility by 10% before equating it with a more liquid asset.

We need to modify Longstaff's method slightly when dealing with hedge funds, since the expected return and volatility of a hedge fund depend on the level of assets relative to the high water mark. If the hedge fund's return and volatility were constant over time, an investor would increase his allocation to the hedge fund after a drawdown. This is not consistent with the behaviour of most investors. Investors usually decrease their allocation to a hedge fund that has suffered a negative return, for the following reasons.

- The hedge fund investment is now considered riskier than before.
- The investor is worried that other investors will withdraw. If the withdrawal amounts are large enough, the hedge fund manager may have to unwind positions at unfavourable prices. This would compound the negative return.
- Since most hedge funds have high water marks, the investor is worried about organisational risk. If the key employees in the fund do not believe they will get reasonable bonuses for the next few years (bonuses are usually taken from the performance fee), they may leave.

We have incorporated investor behaviour into our model by using variable leverage and the knock-out feature to model the evolution of a fund's assets over time.

Before we can apply a variant of Longstaff's method to a concrete example, we need to specify the investor's profit and loss function, as follows:

- If $S(t) >> H(t)$, the value of the investor's allocation is $(1 - p)(S(t) - H(t)) + LH(t)$ and $l$ (the strategy's leverage) is moderate. The manager is coasting, since there is only a small amount of vega in the management and performance contract. Thus, $\mu$ and $\sigma$ are also moderate.
- If $S(t)$ is close to $H(t)$, then $l$ is large, since the value of the incentive clause to the manager increases sharply in $l$. Here, vega is large and positive.
- If $S(t)$ is close to a threshold default level, then $l$ is small, since the manager wants to continue to collect a percentage $c$ of assets. As volatility increases, the probability of default increases sharply, and so the manager keeps $I$ at a relatively low level. Vega is large and negative.
- If $S(t)$ drops below the default level, the fund liquidates. The value of the investor's allocation is then some fraction of the default level. In the example in the next section, we assume that the investors receive 50% of the threshold amount.

In our model, drift and volatility of assets are level-dependent and the leverage $l$ is a step function in the argument $S(t) - H(t)$.

\[ \text{Survival zone, leverage is low} \quad \text{High vega zone, leverage is high} \quad \text{Manager collects 1.5% of assets} \quad \text{Fund has defaulted} \quad \text{Manager loses future coupon payments on} \quad \text{Payout to manager} \quad \text{Coasting zone, leverage is moderate} \quad \text{Asset value ($\text{million}$)} \]

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Simulating the illiquidity premium
Here, we present a numerical example where an investor’s illiquidity premium can be directly simulated, using Longstaff’s method.

Suppose there is an investor who wants to invest in a hedge fund and a traditional mutual fund. The investor’s utility function is specified by portfolio return/portfolio volatility and he wishes to maximise his expected utility over two years. Initially, the mutual fund and hedge fund have identical risk/reward characteristics.

- The historical annual return and volatility of the mutual fund and hedge fund are 10% and 10%, respectively. We assume that the mutual fund returns evolve according to a random walk.
- The mutual fund and the hedge fund are uncorrelated.

However, for simplicity, we assume that the investor cannot move money in or out of the hedge fund over time.

In turn, the hedge fund manager varies his leverage according to the following schedule.
- The current level of assets is 100 (expressed in millions of dollars) and is equal to the current high water mark.
- The manager can vary his leverage once a month.
- If assets drop below 85, the fund liquidates and investors receive 50%.
- If assets rise above 110, the manager uses moderate leverage. His expected annual return is 5%. However, the effective asset volatility is larger than 5%, since there is a positive probability of default at the next time step.
- If assets are between 88.75 and 110, the manager uses maximal leverage. His expected annual return is 10% with 10% volatility.
- If assets are above 110, the manager increases the value of the high water mark contract. His expected annual return is 10% with 20% volatility.

We now create two sets of simulations, a benchmark set where the investor is allowed to rebalance once a month and another set where no rebalancing is allowed. In each case, we approximate the investor’s expected utility. We summarise the results below.

Simulation 1 (monthly rebalancing)
The investor is aware of the manager’s variable leverage and wants to maximise his expected utility over the next two years. At each time step, the manager’s leverage is known. For example, at the start, the manager’s assets are 100 so his expected return is 20% with 20% volatility. Thus, the investor initially optimises his expected utility by allocating 67% to the traditional mutual fund and 33% to the hedge fund.

Simulation 2 (no rebalancing)
In this case, we are not allowed to rebalance the portfolio for the entire two years. Beforehand, we do not know the optimal static allocation to the traditional fund and the hedge fund. By performing a separate simulation for different weights (in increments of 1%), we have found the op-

If, over time, the level of assets goes down, a more complicated calculation needs to be made. For example, if assets drop below 88.75, the expected return is 5%. However, the effective asset volatility is larger than 5%, since there is a positive probability of default at the next time step. We approximate the effective volatility by taking a probability weighted average, as follows.

Suppose that we are at time $t$ and asset level $S(t)$. We know that:

$$S(t + \Delta t) = S(t) + (\mu - c)S(t)\Delta t + \sigma S(t)\sqrt{\Delta t}\xi$$

(assuming no withdrawals) and wish to calculate the probability that $S(t + \Delta t) \leq 85$, where $\Delta t = 0.083$ years. Solving for $\xi$, we find that this probability is the same as the probability $p$ that:

$$\xi \leq \frac{85 - S(t) - (\mu - c)S(t)\Delta t}{\sigma S(t)\sqrt{\Delta t}}$$

which can be calculated explicitly since $\xi \sim N(0, 1)$. As the asset level moves close to default, $p$ approaches 50%, assuming that the drift is zero. If:

$$\xi \leq \frac{85 - S(t) - (\mu - c)S(t)\Delta t}{\sigma S(t)\sqrt{\Delta t}}$$

the fund defaults and we set:

$$\xi = \frac{42.5 - S(t) - (\mu - c)S(t)\Delta t}{\sigma S(t)\sqrt{\Delta t}}$$

The volatility of $S(t)$ can then be approximated by:

$$\sqrt{\text{var}((1-p)\hat{\xi}_1 + p\hat{\xi}_2)}$$

where $\xi_1 \sim N(0, \sigma_1^2)$ and:

$$\xi_2 = N\left(\frac{42.5 - S(t) - (\mu - c)S(t)\Delta t}{\sigma S(t)\sqrt{\Delta t}}, 0\right)$$

We have simulated the investor’s utility 40,000 times by calculating the realised annual return and volatility for each simulation. A histogram of information ratios appears in figure 2. In the histogram, we have partitioned the x-axis in 0.025 increments. The expected utility over these simulations is 1.5.

Simulation 2 (no rebalancing)
In this case, we are not allowed to rebalance the portfolio for the entire two years. Beforehand, we do not know the optimal static allocation to the traditional fund and the hedge fund. By performing a separate simulation for different weights (in increments of 1%), we have found the op-
timal allocation to the traditional fund should be 55%, with 45% to the hedge fund. We can then plot the information ratio histogram in the same way as simulation 1. The results appear in figure 3.

Here, the distribution is clearly bimodal, since there is a much larger probability that the hedge fund will blow up before an investor can transfer funds to the traditional fund. The expected utility over 40,000 simulations is 1.3, or about 10% smaller than the benchmark case. We can either express the 10% difference as a return premium or in volatility terms. We choose the latter. Thus, the historical 10% volatility for the hedge fund should really be considered to be 11% (again, assuming moderate leverage).

In certain instances, we can use the liquidity haircut to decide how much we should allocate to a given hedge fund.

As we have pointed out, it can be dangerous to characterise a hedge fund purely by its historical mean, variance and correlation with other funds. However, many funds of funds already use mean variance optimisers to make asset allocation decisions for traditional assets and are reluctant to develop entirely new models for alternatives. The result in this article provides a ‘correction’ term that can be applied before the optimisation step.

Fund of funds managers often make incremental changes to their portfolios using a ranking system. New funds are ranked according to a few summary statistics, such as historical mean, variance and maximum drawdown. When a change is made, a new fund is chosen from the list. Our liquidity adjustment puts funds that trade in liquid markets (and usually have shorter lockups) on a more equal footing with funds that ‘sell liquidity’ for premium and should result in a different ranking from the one that relies purely on historical returns.

Conclusion
We have developed a technique for calculating the illiquidity haircut for a hedge fund. Since the lockup period and redemption schedules vary from fund to fund, the volatility adjustment in our numerical example should only be viewed as a guideline. Typically, funds that trade in more liquid markets (for example, interest rate futures) offer more liquidity to an investor and should not be haircut as severely.

We have also made assumptions about the way in which a hedge fund manager varies his leverage according to the level of assets in the fund. From experience, we have found that these assumptions are qualitatively correct. In a companion paper, we plan to develop a way to determine a hedge fund manager’s optimal use of leverage more precisely.

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