Estimation Error and Portfolio Optimization
Motivation

- The Markowitz Mean-Variance Efficiency is the standard optimization framework for modern asset management.
- Given the expected returns, standard deviations and correlations of assets (along with constraints), the optimization procedure solves for the set of portfolio weights that has the lowest risk for a given level of portfolio expected returns.
- Standard algorithms (linear programming/quadratic programing) are available to compute the efficient frontier with or without short-selling/borrowing constraints.
Motivation

• A number of objections to MV Efficiency have been raised:
  • **Investor Utility**: Utility might involve preferences for more than means and variances and might be a complex function.
  • **Multi-period framework**: The one-period nature of static optimization does not take dynamic factors into account.
  • **Liabilities**: Little attention is given to the liability side.
  • **Instability and Ambiguity**: Small changes in input assumptions often imply large changes in the optimized portfolio. The MVE procedure overuses statistically estimated information and magnifies estimation errors.

• The Michaud (1998) procedure described here addresses the last problem.
The Procedure

The procedure described below has U.S. Patent #6,003,018 by Michaud et al., December 19, 1999.

- Estimate the expected returns and the variance-covariance matrix (θ) [say, the MLE (average) or using Bayesian techniques]. Suppose there are \( m \) assets.

- Solve for the minimum-variance portfolio. Call the expected return of this portfolio \( L \). Solve for the maximum return portfolio. Call the expected return of this portfolio \( H \).

- Choose the number of discrete increments, in returns, for characterizing the frontier. That is, for the purpose of the Monte Carlo analysis, one can think of looking at a series of points on the frontier - rather than every point on the frontier.
The Procedure

• Suppose for $L=.05$ and $H=.20$ and we choose the number of increments, $K=16$. This means that we evaluate the frontier at expected return=$\{.05, .06, ..., .19, .20\}$, that is 16 different points.

• We will represent the ‘frontier’ as $a_k$, where ‘a’ represents weights. So for $m$ assets, $a_k$ is $KxK$ (rows represent the number of points on the frontier and columns are the assets). The pair $(a_k, \theta)$ then represents the efficient frontier. In our example, we would have 16 rows (discrete increments) and $m$ columns (number of assets).
The Procedure

- Now begin the Monte Carlo analysis. Generate $\theta_i$ from the likelihood function $L(\theta)$. Hence $\theta_i$ and $\theta$ are statistically equivalent.

- For example, suppose we have $m=5$ assets and $T=200$ observations. Give a random number generator, $\theta$ (means, and variance-covariance matrix), assume a multivariate normal (not necessary but simplest), and draw five returns 200 times. With the generated data, calculate the simulated means and variance-covariance matrix, $\theta_i$. Note that $\theta_i$ and $\theta$ are “statistically equivalent”.

- Using $\theta_i$, calculate the minimum variance portfolio (expected return $L_i$) and the maximum expected return portfolio (expected return $H_i$). Use these to determine the size of the expected return increments.
The Procedure

• Following our example of K=16, suppose using $\theta_i$, $L_i=.03$ and $H_i=.25$. Hence, the discrete points would be expected returns of {.030, .044, .058, ..., .236, .250}.

• Calculate the efficient portfolio weights at each of these K points. (Solve for the minimum variance weights given expected returns of .030, ...).

• With this information, we now have $a_{k,i}$. This is the same dimension, Kxm.

• Repeat the simulations, so that we have 1,000 $a_{k,i}$s.
The Procedure

- Average the 1000 $a_{k,i}$s. For each increment, this gives us average portfolio weights. Call this $a_k^*$

- We can redraw the efficient frontier, by using the original means and variances, i.e. $\theta$ combined with the new weights, $a_k^*$
The Procedure

• Note, the new efficient frontier is *inside* the original frontier. Why?

• If we look at any particular Monte Carlo draw, say $\theta_i$, we could draw a frontier (which could be to the right or left of the original frontier). However, we are keeping track of the weights at the discrete increments. We average the weights (not the frontiers) - and then apply these average weights to the original $\theta$.

• We know that the efficient weights for $\theta$, are $a_k$. If we apply, $a_k^*$, then we must be to the right of the original frontier. In other words, if $a_k$ is the best, then $a_k^*$ cannot be the best. However, importantly, $a_k^*$ is taking estimation error into account.
Variations

• Instead of using K increments for the indexation based on the high and low returns, assume a quadratic utility function is used.

• A function of the form $\Phi = \sigma^2 - \lambda \mu$ is minimized. Each value of $\lambda$ ranging from zero to infinity defines a portfolio on the mean-variance and simulated frontiers. Decide on the values of the $\lambda$s and then use for the indexation.
Does the resampled frontier outperform?

• Based on simulations, yes. Here is the evaluation.

• Given the first draw of $\theta_i$, do another Monte Carlo exercise to determine the resampled frontier defined by $a_{k,i}^*$ (notice difference in notation).

• Do this extra Monte Carlo on each $\theta_i$ - so we will have 1,000 different $a_{k,i}^*$

• Compare the average of $a_{k,i}^*$ to the average of the $a_{k,i}$, (that is, compare the average of the simulated portfolios based on the “true” value of $\theta_i$ to the average of the portfolios that do not incorporated any allowance for estimation error). Simply draw frontiers based on $\theta$. 