Inside Liquidity

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July 27, 2015

Abstract

We study a model in which a special role for safe, short-term assets in facilitating financial intermediation encourages the private sector to issue these assets to supplement the public supply of safe, short-term assets, even though their role in providing liquidity can be put in jeopardy during financial panics that lead agents to question the low-risk nature of these assets. During “normal” times these assets are considered sufficiently safe so that they contribute to the supply of liquid assets, but in a financial panic their role in providing liquidity is greatly diminished, if not eliminated all together. In this paper we propose an endogenous mechanism by which assets are categorized as safe enough to offer liquidity services, or to lose such a status that had been previously acquired. Such a model seems to match key features of the data, both in normal times and during a financial panic. During a financial panic, e.g., the model matches the observed large rise in yield spread between commercial paper and T-bills, the fall in the supply of commercial paper, and a positive association between the real yield on T-bills and the overall supply of commercial paper plus T-bills. The model also reveals that the very mechanism that endogenously introduces a special role for assets in providing liquidity also opens up the economy to multiple equilibria, and that the features just mentioned are also consistent with an economy that jumps from a good equilibrium to a bad one, with no other change in fundamentals.

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1 Introduction

A defining characteristic of liquidity crises seems to be a “flight to quality” in which agents substitute away from private sources of relatively safe, short-term assets to public sources. For example, during the 2007-2008 Financial Crisis in the United States, private sources of safe assets as reflected in the supply of commercial paper fell from 11.4 percent of GDP just prior to BNP Paribas ceasing activity with three hedge funds specializing in U.S. mortgage debt in August 2007 to 7.8 percent just after the Lehman bankruptcy in September 2008 (and continued to fall to below 5 percent by the end of 2009), whereas public sources of safe assets as reflected in the supply of Treasury bills rose from 6 percent of GDP to 10.0 percent during this time period (with a peak of 14.1 percent in March 2009). This substitution was also reflected in the Commercial Paper - T-bill yield spread, which went from 0.44 percent at the end of July 2007 to over 2.71 percent at the end of September 2008. In this paper we also document that there is an important positive relation between the total supply of commercial paper plus T-bills and the real yield on T-bills, both in normal times and as well during a financial panic, that will help sort out key drivers of the data. Specifically, as it relates to the recent financial crisis, the total supply of commercial paper plus T-bills fell by 16 percent in real terms from July 2007 to June 2008 and then rose by 39 percent from June 2008 to November 2008, while the ex post real yield on T-bills fell 5.55 percentage points as the total supply fell and rose 1.96 percentage points as the total supply rose. These results are documented in Figs. 2, 4, and 7, which are discussed later in this paper.

Some of these results seem at odds with recent empirical findings regarding the relation in general between private and public sources of relatively safe, short-term assets, while others seem supportive, which is what one would expect if there is something “special” about a financial panic. Krishnamurthy and Vissing-Jorgensen (2012) document that in general a rise in the supply Treasury bills leads to a fall in the AAA Corporate Bond - T-bill spread, which is extended by Krishnamurthy and Vissing-Jorgensen (2013) to the Commercial Paper - T-bill spread. This is at odds with observations during the 2007-2008 Financial Crisis, during which the supply of T-bills rose and the yield spread rose too. Carlson, et. al, (2014) document that in general a rise in the supply of T-bills leads to a fall in the supply of commercial paper, which seems consistent with observations during a financial panic.

Understanding these results is important not only for understanding events driving a liquidity crisis with perhaps an eventual goal of designing an effective regulatory environment and policy response to mitigating these crises, but these results seem fundamental to
understanding what drives the valuation of assets, especially those assets whose valuation seems to be driven by a special role they fulfill in an economy. The possible departure of real T-bill yields from their fundamental determinants due to a change in demand associated with a flight to quality requires some explanation, and given the low rate of default on commercial paper, the large rise in the Commercial Paper - T-bill yield spread during a financial panic is puzzling. Moreover, while the rise in the supply of T-bills seems to reflect a deliberate policy choice of accommodating a flight to quality during times of heightened financial uncertainty, questions remain regarding the extent of the substitution of public for private liquidity, especially the ability of public policy to achieve a sustained rise in the overall supply of liquidity.

This paper develops and estimates a fully-articulated model for understanding the observed relation between private and public sources of relatively safe, short-term assets, especially during a financial panic. We show that this model is consistent with the observed relationship between supplies and yields of commercial paper and T-bills, both in general from 1950 to 2014 and during the 2007-2008 Financial Crisis. The model is grounded in the monetarist approach to understanding the special role some assets play in facilitating transactions. The novel feature introduced here is a dependence of an asset’s ability to serve a special role in facilitating transactions on the perceived riskiness of the asset. In a financial panic private sources of relatively safe assets lose their special role, thereby initiating a flight to quality to public sources that retain this role. Moreover, a policy-induced rise in public safe assets during a financial panic has the effect of crowding out private sources, thus accelerating the decline of private sources. Regarding yields, the loss of the special role of private sources of relatively safe assets explains the rising yield spread during a financial panic, and the relation between the overall (private plus public) supply of safe assets and the marginal transactions value of these assets explains the positive association between the overall supply of safe assets and real yields on public sources of safe assets during a financial panic.

Along lines related to Holmström and Tirole (2011), we refer to the private supply of relatively safe, short-term assets as *inside liquidity*, by which we mean assets issued by the private sector that are held in part to provide a non-pecuniary service. Of course traditional notions of inside money satisfy this definition, but we have in mind a much broader definition that goes well beyond the non-pecuniary services offered by a traditional bank. In parallel, we think of short-term public debt as *outside liquidity* that offers a non-pecuniary return similar to the non-pecuniary return of outside money, but they may not be perfect substitutes so...
their relative supply may affect their relative rate of return. Building on this perspective, we have in mind that inside money is to outside money what inside liquidity is to outside liquidity. A key issue is that due to its private nature inside liquidity is backed up by risky claims and hence is itself risky (indeed, without deposit insurance inside money would be risky too), so any model of inside liquidity will have to contend with its demand and supply being sensitive to perceptions of risk in the economy. Indeed, the connection between risk and the price and supply of inside liquidity is a key focus of this paper in examining the behavior of liquidity during a financial panic.

Motivated by our theoretical model, we re-examine some of the empirical relationships between commercial paper and T-bills that have been documented in the literature. We find too that there is a negative relation between commercial paper and T-bills as a fraction of GDP, but this prediction of our model requires a sufficiently large independent source of variation in the supply of T-bills. Our model suggests that a more robust relation conditions on the real interest rate, which we show adds to the empirical support of this substitution result. The model predicts that the real yield on government sources of safe, short-term assets should depend positively on the total supply of safe, short-term assets, which we find. However, the model predicts that the yield spread should not depend on the supplies of private and government sources of liquidity, but rather should only depend on the perceived riskiness of private sources. As predicted, we find that a measure of volatility is a statistically significant predictor of yield spreads. We find too that conditional on this measure of volatility, the relative supply of T-bills to Commercial Paper becomes statistically insignificant. However, while the significance of the supply of T-bills as a fraction of GDP is also greatly reduced, it remains to be a statistically significant determinant of yield spreads. We examine other empirical issues too. We find that simulations of an estimated version of the model perform well in terms of matching key features of the data.

The next section reviews the relevant literature. Following that we develop the model. We then describe key features of the data, both from a long-term perspective as well as the experience of the 2007-2008 Financial Crisis. Lastly we use the data to estimate the model’s parameters and compare simulations of the model to key features of the data.

## 2 Literature Review

This paper builds on the literature that views money’s value as stemming from its role in facilitating transactions. Especially with inside money, transactions are not always completed
instantaneously. It might take a few days for checks to clear, and purchasing goods on credit involves a promise to make a fixed nominal payment by a specific date in the near future. One could imagine other contractual arrangements too that commit one to making a fixed nominal payment at some point in the near future. One benefit of outside or inside money is the surety of being able to make this payment if one holds a sufficient balance of money for this purpose. As stated by Lucas and Stokey (2011): “cash and other securities that are close to cash in the sense that they can be exchanged for cash easily and at a predicable price ... have a special role because they are used, indeed required, for carrying out transactions.” Indeed, Bansal and Coleman (1997) developed a model in which the unique role of short-term government debt in offering a guaranteed nominal payoff in the near future allowed it to inherit a non-pecuniary return from money’s role in facilitating transactions. Here we build on this literature exploring money’s transactional role by also considering other assets that have a stable nominal payoff, but in a way that recognizes and deals with these assets not being as stable as either money or short-term government debt. In this way we endogenize the monetary value of an asset by relating its monetary value to the stability of its nominal payoff.

A closely related literature views the special role of assets not from a transactional perspective, but rather from a collateral perspective. In parallel to assuming that some assets have a special role in facilitating transactions, this literature assumes that some assets have a special role in serving as collateral. An early paper along these lines is Kiyotaki and Moore (1997). Building on the models of Lagos and Wright (2005) and Williamson and Wright (2010), in a recent paper Venkateswaran and Wright (2013) examined a variety of asset-pricing implications of this approach. In many ways the asset-pricing implications of the transactional and collateral approaches seem similar, and indeed a useful research agenda may be to endogenize the collateral value of an asset along the lines of endogenizing the monetary value of an asset that is pursued in this paper.

In terms of the issues addressed, this paper is related to the recent literature directed towards understanding the unique set of events that characterize a financial panic, specifically the 2007-2008 Financial Crisis. Papers describing the collapse of using safe private assets such as commercial paper as a source of liquidity include Covitz, Liang, and Suarez (2009) and Kacperczyk and Schnabl (2010). Papers that describe the substitution of public for private assets include Carlson, et. al., (2014). Gorton and Metrick (2012) examined the liquidity crisis from the perspective of a classic bank run with a flight to quality but applied to the Shadow (or Securitized) Banking Sector. Sunderam (forthcoming) developed and estimated
a reduced-form model to argue that the rise in shadow banking during the pre-crisis period 2001-2007 was due to a rise in demand for liquid assets.

The theoretical literature addressing issues related to the role of liquid assets in an economy is large and growing, surely because the role of liquid assets goes to the very heart of financial intermediation. Gorton and Pennacchi (1990) modeled the endogenous formation of financial intermediaries as a mechanism to create safe, liquid assets that protect uninformed traders. Kiyotaki and Moore (2005) examined the value of liquid assets for facilitating trade when the timing of investment opportunities is uncertain. Acharya, Gale, and Yorulmazer (2010) develop a model in which a small change in information leads to a liquidity crisis as financial intermediaries are unable to borrow to roll over their debt, even when this borrowing is secured by fairly safe assets.\footnote{See also Martin, Skeie, and von Thadden (2012), who also examine bank runs in a model with short-term collateralized borrowing.} Dang, Gorton, and Holmström (2012) also stress the role of information in the provision of liquidity and provide an argument as to why liquidity is best served by safe, information-insensitive assets such as highly-rate debt. In some respects, this paper is closely related to Stein (2012) and Greenwood, Hansen and Stein (forthcoming), who develop models in which banks’ ability to create safe assets and associated transaction services provides them a cheap source of funding, which in turn leads them to create excessive liquidity that exposes the economy to a maturity-mismatched-induced financial crisis. Greenwood, Hansen and Stein (forthcoming) in particular focus on the ability of public sources of liquidity to crowd out private sources. In some sense, one aspect of our paper can be thought of as extending these results by focusing more specifically on the role of non-bank financial intermediaries in providing liquidity and recognizing that their ability to borrow cheaply by creating transaction services with safe assets may come into question during a financial panic.

3 The Model

The model consists of firms, financial intermediaries, a government and households in a discrete-time, infinite-horizon setting subject to uncertainty. To focus the paper on the determinants of inside liquidity, we will abstract from inside and outside money and instead focus on the relation between inside liquidity and short-term government debt (outside liquidity). For this reason too we will think of financial intermediaries as non-bank financial intermediaries.
3.1 Firms

Identical firms exist for two periods. Claims on their output trade in the first period and production is realized in the second period. Production in the second period is an exogenous amount, which is fully perishable at the end of the period. Since firms are identical, we will simply think of there being one firm producing aggregate output. Aggregate output \( y \) follows a two-state Markov process, with levels of output given by either \( y_0 \) or \( y_1 \), where \( 0 < y_0 < y_1 \). Given a realization of \( y_i, i \in \{0, 1\} \), the probability that output will equal \( y_1 \) next period is \( \pi_i \), so the transition matrix is given by

\[
\begin{pmatrix}
1 - \pi_0 & \pi_0 \\
1 - \pi_1 & \pi_1
\end{pmatrix}.
\]

We assume \( 0 < \pi_0 < \pi_1 < 1 \). Denote the current state as \( y \) and the expectation operator over next period’s realizations of \( y' \) conditional on \( y \) as \( E_y \). It will be convenient to summarize the probabilities as \( \pi(y) \), where \( \pi_i = \pi(y_i) \).

Ownership in the firm will be represented by an equity claim on the firm’s output, denoted by \( z \). That is, owning \( z \) claims on a firm that produces \( y \) units of output entitles the owner to \( zy \) units of output. All claims are initially owned by households and they are traded one period prior to the production of output but after the realization of \( \pi \). Denote the price of a unit of such a claim by \( p_z \).

3.2 Financial Intermediaries

Households create, manage, and discontinue financial intermediaries. Each financial intermediary is a fixed size, so changes in aggregate financial intermediation will be entirely along an extensive margin. To create a financial intermediary a household incurs a cost \( \kappa \geq 0 \) in the current period, but this financial intermediary will not be available to provide financial services until the next period. This time-to-build technology in providing financial services will be important, as it will introduce an option value to retaining a financial intermediary during a financial panic that will help explain observations of real yields on relatively safe private assets during such a panic. Households thus begin a period with \( n \) financial intermediaries that had existed in the prior period, plus \( m \) new financial intermediaries that had been created in the prior period. They can choose to discontinue a financial intermediary at no cost. If households choose to continue an existing financial intermediary (or proceed with plans to open a new financial intermediary), they must invest (or keep) \( h > 0 \) as a capital investment in the financial intermediary, and they incur a cost \( \theta > 0 \) in the current period.
In the current period, each financial intermediary can issue one-period debt in the amount of \( \hat{d}' > 0 \) at per unit price \( q_d \). By a fixed-size financial intermediary is meant that both amounts \( h \) and \( q_d \hat{d}' \) are fixed, and their sum is normalized to one:

\[
h + q_d \hat{d}' = 1.
\]

Each financial intermediary must invest its entire portfolio in purchasing equities in firms, denoted by \( \hat{z}'_f \), so

\[
p_z \hat{z}'_f = 1.
\]

In the next period the financial intermediary will receive a cash flow of \( \hat{z}'_f y' \) that it distributes to its debt holders and owners. Debt holders receive either the face value of their debt or the entire cash flow, whichever is less. Let \( x' \) denote the payout on their debt, so that every unit of face value gets paid \( x' \). From the perspective of the prior period, this payout ratio is a random variable, which is given by the relation \( x' \hat{d}' = \min \left\{ \hat{d}', \hat{z}'_f y' \right\} \), and can be rewritten as

\[
x' = \min \left\{ 1, \frac{y'}{p_z} \frac{q_d}{1 - h} \right\}.
\]

The next-period payoff to owners of one financial intermediary can similarly be written as:

\[
\omega' = \max \left\{ 0, \frac{y'}{p_z} - \frac{1 - h}{q_d} \right\}.
\]

### 3.3 Government

The government fully honors existing debt obligations, issues new one-period debt, and levies a lump-sum tax or pays a lump-sum distribution. Denote the face value of outstanding debt due in the current period as \( b \). The government issues new one-period debt \( b' \) at a per-unit price of \( q_b \). The difference \( T = b - q_b b' \) is financed by a lump-sum tax (positive or negative) on households. We assume the government implements a stationary policy to achieve an amount of debt \( b' = B(y) \) based on realizations of \( y \) and associated \( \pi \). We assume \( B(y) > 0 \) for every \( y \).

### 3.4 Households

Risk-neutral households value an infinite sequence of consumption \( \{c, c', c'', \ldots\} \) via the expected utility function

\[
c + \beta E[c'] + \beta^2 E[c''] + \ldots \tag{1}
\]
where the subjective discount factor $0 < \beta < 1$.

Purchasing goods costs resources. Just as with money (in a monetary economy), we assume that any asset that offers a guaranteed payment in the next period can be used to facilitate transactions. In some sense, we think of a household as being integrated with a bank that offers financial services to facilitate transactions. Since one-period government bonds have the feature of a guaranteed payment, we assume that holding a larger fraction of wealth in the form of government debt reduces the cost incurred in purchasing consumption. However, we also consider that any asset that promises a fixed payment in the next period can reduce transaction costs. In this sense too we assume that holding a larger fraction of wealth in the form of private debt reduces the cost incurred in purchasing consumption. The key difference, though, is that while private debt offers the promise of a fixed payment, ex post private debt may not pay off at par. We thus assume that, ex post, households incur an additional transaction cost that is proportional to the amount of debt that is not paid as promised. In the current period, thus, transaction costs incurred by a household in purchasing $c$ units of consumption are given by

$$
\nu c + \frac{\psi}{\eta - 1} y^n (b' + d')^{1-\eta} + \xi (1-x)d,
$$

where $\nu > 0$, $\psi > 0$, $\eta > 1$, and $\xi \geq 0$. This is the key mechanism by which the riskiness of an asset limits its usefulness in providing transaction or liquidity services. While this is an ex-post feature, certainly ex-ante assets that have a relatively high likelihood of default will have a lower expected value in terms of transaction services. In principle the transaction cost stemming from not paying debt at par could apply to government debt as well, but recall that government debt is assumed to always pay in full. The assumption that $y$ and not $c$ affects the marginal value of transaction services from liquid assets is certainly restrictive, but the resulting constancy of marginal transaction cost with respect to $c$ will have a significant benefit in terms of being able to sharply characterize the equilibrium.

Households own all the financial intermediaries and initially own all the firms. During the period they sell claims on firms, keeping equity $z'_h$. At the beginning of each period households also make or receive lump-sum tax payments to the government.

The flow budget constraint for households in the current period is given by

$$
c + \nu c + \frac{\psi}{\eta - 1} y^n (b' + d')^{1-\eta} + \xi (1-x)d + q_z b' + q_d d' + p_z (z_h' - 1) + n' (h + \theta) + m' \kappa = b + xd + yz_h + n\omega + T.
$$

(2)

In choosing the number of financial intermediaries $n'$ to have in operation this period and
the next, households must satisfy the following restriction:

\[ n' \leq n + m. \] (3)

Households must also satisfy two non-negativity conditions:

\[ n' \geq 0 \] (4)

and

\[ m' \geq 0. \] (5)

### 3.5 Equilibrium Conditions

Households choose state-contingent sequences for \( c, b', d', n', m', \) and \( \hat{z}'_h \) to maximize preferences given by (1), subject to their flow budget constraints given by (2) and inequality constraints (3)-(5). The first-order conditions, after imposing market-clearing conditions, are as follows:

\[
y = c + \nu c + \frac{\psi}{\eta - 1} y^n (b' + d')^{1-\eta} + n' \theta + m' \kappa,
\]

\[
q_b = \beta + \psi y^n (b' + d')^{-\eta},
\]

\[
q_d = \beta E_y [x'] + \psi y^n (b' + d')^{-\eta} - \beta \xi (1 - E_y [x']),
\]

\[
p_z = \beta E_y [y'],
\]

\[
h + \theta \begin{cases} 
\leq \beta E_y [\omega'] + \beta E_y [\lambda] & \text{if } n' = n + m \\
= \beta E_y [\omega'] + \beta E_y [\lambda] & \text{if } 0 < n' < n + m \\
\geq \beta E_y [\omega'] + \beta E_y [\lambda] & \text{if } n' = 0
\end{cases}
\]

\[
k \begin{cases} 
= \beta E_y [\lambda] & \text{if } m' > 0 \\
\geq \beta E_y [\lambda] & \text{if } m' = 0
\end{cases}
\]

\[
\lambda = \max \{ \beta E_y [\omega'] + \beta E_y [\lambda] - (h + \theta), 0 \},
\]

\[
d' = (1 - h) n',
\]

\[
q_d > 0,
\]

as well as inequalities (3)-(5), and where

\[
E_y [y'] = y_0 (1 - \pi(y)) + y_1 \pi(y),
\]

\[
E_y [x'] = E_y \left[ \min \left\{ 1, \frac{y'}{p_z} \frac{q_d}{1 - h} \right\} \right],
\]

\[
E_y [\omega'] = E_y \left[ \max \left\{ 0, \frac{y'}{p_z} - \frac{1 - h}{q_d} \right\} \right].
\]
For given parameters \((b, h, \theta, \psi, \beta, \xi, \eta)\), this system determines a recursive, stationary equilibrium comprised of endogenous variables \((c, q_b, q_d, p_z, d', n', m')\) conditional on the state vector \((y, n + m)\). Note that consumption could become negative in some states, but this is not a problem with linear utility. Denote the stationary policy functions for \(n'\) and \(m'\) as \(n' = N(y, n + m)\) and \(m' = M(y, n + m)\), respectively.

### 3.6 Solving for the Equilibrium

To solve for the equilibrium, in the following sense let’s first solve for \(q_d\) given \(y\) and \(b' + d'\). Upon substituting eq. (16) into eq. (8), using eq. (9), we see that \(q_d\) must satisfy

\[
q_d = \psi y^n(b' + d')^{-\eta} - \beta \xi + \beta(1 + \xi) \left( \min \left\{ 1, \frac{y_0}{\beta E_y[y']} \frac{q_d}{1 - h} \right\} (1 - \pi) + \min \left\{ 1, \frac{y_1}{\beta E_y[y']} \frac{q_d}{1 - h} \right\} \pi \right).
\]

Fig. 1 graphs the left and right sides of eq. (18) as a function of \(q_d\) (the left side is just the 45 degree line). A solution is when these lines intersect for \(q_d > 0\). Note that the right side is piecewise linear, with a slope that starts at \((1 + \xi)/(1 - h) > 1\) and becomes flat for sufficiently high \(q_d\). In general, there could exist either zero, one, two, or an infinite number of solutions. There exists at least one solution if either kink is (weakly) above the 45 degree line. The first kink (lowest value of \(q_d\)) will be above the 45 degree line if

\[
\beta(h + \xi) \frac{y_0(1 - \pi) + y_1\pi}{y_1} \geq \beta \xi - \psi y^n(b' + d')^{-\eta}
\]

and the second kink will be above the 45 degree line if

\[
\beta(1 + \xi) - \beta(1 - h) \frac{y_0(1 - \pi) + y_1\pi}{y_0} \geq \beta \xi - \psi y^n(b' + d')^{-\eta}.
\]

For given \(y\) and \(b' + d'\), the solution \(q_d\) is unique if

\[
\beta \xi - \psi y^n(b' + d')^{-\eta} < 0.
\]

If inequality (21) is violated and either inequality (19) or (20) is strict, then there exists two solutions. At the low \(q_d\) solution, default occurs in more states than at the high \(q_d\) solution. In some sense, if households expect high default rates then interest rates are high, which leads to high default rates, and if households expect low default rates then interest rates are low, which leads to low default rates. It seems plausible that self-fulfilling equilibria of
this type, which seem closely related to bank runs, are a feature of financial panics. Note, in particular, the role in $\xi$ in this regard. If $\xi$ is sufficiently low (e.g., $\xi = 0$), then there always exists a solution, and this solution is unique. It is only for sufficiently high $\xi$ that multiplicity of equilibria arise in which one equilibrium has a self-fulfilling higher default rate for financial intermediaries than the other equilibrium. In a later section we will revisit the issue of multiplicity of equilibria, but for now, if there exists more than one solution, choose the right most solution (highest $q_d$), which is associated with lower default rates. Once we estimate $\xi$ from the data and can examine properties of the estimated model to see if multiplicity of equilibria may be an issue, we will examine the possibility that in a financial panic the economy transitions from a “good” equilibrium to a “bad” one.

![Graph](image_url)

Figure 1: Solution to $q_d$ given $b' + d'$

Define $\bar{L}(y)$ as follows. If

$$\min \left\{ \xi - (h + \xi) \frac{y_0(1 - \pi)}{y_1} + y_1 \pi, (1 - h) \frac{y_0(1 - \pi)}{y_0} + y_1 \pi \right\} \leq 0$$

then set $\bar{L}(y) = \infty$, otherwise set $\bar{L}(y)$ such that

$$\psi y^n(\bar{L}(y))^{-\eta} = \beta \min \left\{ \xi - (h + \xi) \frac{y_0(1 - \pi)}{y_1} + y_1 \pi, (1 - h) \frac{y_0(1 - \pi)}{y_0} + y_1 \pi \right\}.$$  

Clearly $\bar{L}(y) > 0$. The above results prove that there always exists a solution $q_d > 0$ if $b' + d' \leq \bar{L}(y)$ and that there does not exist a solution $q_d > 0$ if $b' + d' > \bar{L}(y)$. Note that
the existence of \( q_d \) is not an issue of large amounts of public liquidity completely crowding out private liquidity, but rather is an issue that private liquidity cannot be valued. In some sense, there does not exist an equilibrium \( q_d \) if at any interest rate the default rate suggests the interest rate must be higher. To ensure that there is some range within which private liquidity can be valued, whether it is issued or not, we will assume the following restriction holds on the issuance of public liquidity: \( B(y) < \tilde{L}(y) \) for every \( y \). Define \( \tilde{d}(y) = \tilde{L}(y) - B(y) \). Private liquidity can thus be valued for any \( 0 \leq d' \leq \tilde{d}(y) \).

Denote the solution for \( q_d \) as \( q_d = Q_d(y, d') \) for \( 0 \leq d' \leq \tilde{d}(y) \), which is given explicitly as

\[
Q_d(y, d') = \begin{cases} 
\psi y^n(B(y) + d')^{-\eta} + \beta 
& \text{if } \frac{y_0}{\beta E_y[y']} \frac{\psi y^n(B(y) + d')^{-\eta} + \beta}{1 - h} \geq 1, \\
\psi y^n(B(y) + d')^{-\eta} + \beta \pi - \beta \xi (1 - \pi) 
& \text{if } \frac{y_0}{\beta E_y[y']} \frac{\psi y^n(B(y) + d')^{-\eta} + \beta}{1 - h} < 1.
\end{cases}
\]

\( Q_d \) is a continuous, strictly positive, and strictly-decreasing function of \( d' \). Some of these properties are somewhat more straightforward to establish by examining Fig. 1. For example, if the second of the two possibilities listed above for \( Q_d(y, d') \) is selected, then it must be the case that

\[
\frac{1 + \xi y_0(1 - \pi)}{1 - h} \frac{\beta E_y[y']}{y_1} < 1,
\]

and hence that \( Q_d \) is a strictly-decreasing function of \( d' \). To see this, suppose inequality (22) is violated. Then, since the slope of the middle segment in Fig. 1 is steeper than the 45 degree line, if there exists a solution at all, one such solution would have to be the first possibility above, which would be chosen.

Indeed, to always admit the possibility that an equilibrium is such that partial default is expected to occur in the low \( y \) state, we will impose inequality (22) for the remainder of this paper. Without this inequality, the “good” equilibrium will always be such that default never occurs, and hence there is no distinction between public and private liquidity from the perspective of households. Note, however, that even with this restriction, it is still possible that the equilibrium is such that there is never default. The outcome will depend on the value of other parameters. Note that with inequality (22) holding, if \( \tilde{L}(y) < \infty \), then

\[
Q_d(y, \tilde{d}(y)) = (1 - h) \frac{\beta E_y[y']}{y_1}.
\]

Define

\[
\bar{n}(y) \leq \frac{\tilde{L}(y) - B(y)}{1 - h}.
\]
For any \( n' \leq \bar{n}(y) \), define \( \Omega \) as

\[
\Omega(y, n') = E_y \left[ \max \left\{ 0, \frac{y' \beta E_y[y']}{\beta E_y[y'] - Q_d(y, (1-h)n')} \right\} \right]
\]

and note that \( \beta E_y[\omega'] = \Omega(y, n') \). Note as well that \( \Omega(y, n') \) is a decreasing function of \( n' \).

In addition, given eq. (23), if \( \bar{L}(y) < \infty \), then

\[
\Omega(y, \bar{n}(y)) = 0.
\] (24)

If will help to extend \( \Omega \) to the entire real line as it relates to \( n' \), so define

\[
\Omega^+(y, n') = \begin{cases} 
\Omega(y, n') & \text{if } n' \leq \bar{n}(y), \\
0 & \text{if } n' > \bar{n}(y).
\end{cases}
\]

From eq. (11) it follows that

\[
\beta E_y[\lambda(y', n' + m')] = \min\{\beta E_y[\lambda(y', n')], \kappa\}.
\]

From eq. (10) it thus follows that

\[
\Omega(y, n') + \beta E_y[\lambda(y', n' + m')] = \max \left\{ \Omega(y, n + m) + \min \left\{ \beta E_y[\lambda(y', n + m)], \kappa \right\}, \right. \left. \min \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\lambda(y', 0)], \kappa \right\}, h + \theta \right\} \right\}
\]

and hence that eq. (12), with \( \Omega(y, n + m) \) replaced by \( \Omega^+(y, n + m) \), can be written as

\[
\lambda(y, n + m) = \max \left\{ \max \left\{ \Omega^+(y, n + m) + \min \left\{ \beta E_y[\lambda(y', n + m)], \kappa \right\}, \right. \right. \min \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\lambda(y', 0)], \kappa \right\}, \right. \right. h + \theta \right\} - \left( h + \theta, 0 \right). \] (25)

Eq. (25) has a special, recursive structure that simplifies its study. Specifically, the solution for \( \lambda(y, 0) \) does not depend on \( \lambda(y, n + m) \) for any value of \( n + m > 0 \), and given \( \lambda(y, 0) \), the solution for \( \lambda(y, n + m) \) does not depend on \( \lambda \) for any other value of \( n + m \). The appendix proves that there exists a unique solution \( \lambda \) to eq. (25). The appendix also proves that the solution \( \lambda(y, n) \) is a continuous, decreasing function of \( n \). Define \( n^* = \max_y \bar{n}(y) \).

The appendix proves that with inequality (22) satisfied, if \( \bar{L}(y_0) < \infty \) and \( \bar{L}(y_1) < \infty \), then \( \lambda(y, n') = 0 \) for every \( y \) and every \( n' \geq n^* \).

To derive the equilibrium policy functions, per above let \( \lambda \) be the unique solution to eq. (25). Derive \( n' = N(y, n + m) \) as follows. If

\[
h + \theta \leq \Omega^+(y, n + m) + \min\{\beta E_y[\lambda(y', n + m)], \kappa\},
\]

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then \(n' = n + m\). If

\[
\Omega(y, 0) + \min\{\beta E_y[\hat{\lambda}(y', 0)], \kappa\} > h + \theta > \Omega^+(y, n + m) + \min\{\beta E_y[\lambda(y', n + m)], \kappa\},
\]

then \(0 < n' < n + m\) is determined such that

\[
h + \theta = \Omega^+(y, n') + \min\{\beta E_y[\lambda(y', n')], \kappa\}.
\]

Lastly, if

\[
\Omega(y, 0) + \min\{\beta E_y[\hat{\lambda}(y', 0)], \kappa\} \leq h + \theta,
\]

then \(n' = 0\). In a somewhat similar fashion, define \(m' = M(y, n + m)\) as follows. If

\[
\kappa \geq \beta E_y[\lambda(y', N(y, n + m))],
\]

then \(m' = 0\), else \(m' > 0\) such that

\[
\kappa = \beta E_y[\lambda(y', N(y, n + m) + m')].
\]

The remaining policy functions for \((q_b, q_d, p_z, d')\) are derived in a straightforward manner.

As it regards \(q_d\), note that \(N(y, n + m) + M(y, n + m) < n^*\), and hence the economy will never attempt to enter a region for \(n + m\) in which private debt cannot be valued for any possible value of \(y\). However, this does not rule out the possibility that the economy could enter a region in which private debt can be valued for some values of \(y\) and not others. In some sense, the value of private liquidity is sufficiently high in some states in which \(q_d\) can be valued so that the economy chooses to provide liquidity in anticipation of the occurrence of these states, despite the lack of an ability to value private liquidity in other possible states. One way to rule this out is to assume \(y\) is not too different in both states, in which case \(\bar{L}(y) = \infty\) for all \(y\). One advantage of using the extended function \(\Omega^+\) is that the evolution of the state vector is still well-defined (or well-constructed) under this possibility, so it will be straightforward to examine this issue in simulations of the model for particular parameter values.

### 3.7 Two Special Cases

Two special cases admit a sharper characterization of their solution, which we will use to develop some insight into the properties of this model. One special case is \(\kappa = 0\), in which case the supply of inside liquidity fully responds to shocks in the period in which they occur,
as financial intermediaries are not concerned about future capacity limitations. The other special case is \( \kappa = \infty \), in which case the supply of inside liquidity does not respond at all to shocks, but rather shocks affect only prices. The intermediate case in which both the supply and price of liquidity respond to shocks will be studied in the simulations of the estimated model.

For both special cases we will restrict ourselves to equilibria in which financial intermediaries do not default on their debt in the high \( y \) state \((y = y_1)\) and partially default on their debt in the low \( y \) state \((y = y_0)\). In this case, \( E_y[x'] \), \( E_y[\omega'] \) and \( q_d \) are given by

\[
E_y[x'] = \left( \frac{y_0}{\beta E_y[y']} \frac{q_d}{1-h} \right) (1-\pi) + \pi,
\]

\[
E_y[\omega'] = \left( \frac{y_1}{\beta E_y[y']} - \frac{1-h}{q_d} \right) \pi,
\]

\[
q_d = \frac{\psi y^n (b' + d')^{-\eta} + \beta \pi - \beta \xi (1-\pi)}{1 - \frac{1+\xi y_0 (1-\pi)}{1-h E_y[y']}}.
\]

From these relations it follows that

\[
\Omega(y, n') = \left( \frac{y_1}{\beta E_y[y']} - \frac{1-h - (1+\xi) y_0 (1-\pi)}{\psi y^n (B(y) + (1-h)n')^{-\eta} + \beta \pi - \beta \xi (1-\pi)} \right) \pi.
\]

In order for this to be an equilibrium with default in the low \( y \) state, it must be that, for \( y = y_0 \),

\[
\frac{y_0}{\beta E_y[y']} \frac{q_d}{1-h} < 1.
\]

For comparison purposes, define \( q_d^* \) as

\[
q_d^* = \beta E_y[x'],
\]

which is the discounted value of only the pecuniary return on private debt. We will measure the liquidity value of private debt as a departure from this value.

### 3.7.1 Special Case 1: \( \kappa = 0 \)

If \( \kappa = 0 \) then \( \lambda \) is given by

\[
\lambda(y, n+m) = \max\{\Omega^+(y, n+m) - (h+\theta), 0\}.
\]

From eq. (11) we see that \( E_y[\lambda'] = 0 \), which sets a lower bound on \( m' \). Since \( m' \) is otherwise arbitrary (as \( \kappa = 0 \)), choose \( m' = n^* \), as that leads to \( \Omega^+(y', n' + m') = 0 \) for any \( y' \) and
\( n' \geq 0 \) and consequently \( E_y[X'] = 0 \). Define \( N(y) \) as follows. If

\[
h + \theta \geq \Omega(y, 0)
\]

then set \( N(y) = 0 \), otherwise set \( 0 < N(y) < \bar{n}(y) \) such that

\[
h + \theta = \Omega(y, N(y)),
\]

for then \( n' = N(y) \) and \( d' = (1 - h)N(y) \) if \( m \geq n^* \) (which will hold after any initial period). Also, after any initial period,

\[
q_d = \frac{\beta \pi (1 - h)}{E_y[y'] - (h + \theta)} \tag{27}
\]

and

\[
q_d^* = \frac{\beta \pi (1 - (h + \theta))}{E_y[y'] - (h + \theta)},
\]

and hence

\[
q_d = \frac{1 - h}{1 - (h + \theta)} q_d^*.
\]

Here we can see that all the variation in \( q_d \) is thus due to variation in its expected pecuniary payout: interest rates (continuously compounded) on private liquidity move one-for-one with interest rates associated with \( q_d^* \). This is essentially because the interest rate paid by financial intermediaries on private liquidity is fully determined by free-entry and the consequent zero profit condition for financial intermediaries, which is unaffected by liquidity considerations. This is the chief reason for introducing a time-to-build technology for financial intermediaries, as financial intermediaries may then be willing to incur losses due to higher borrowing costs in a financial panic, as they perceive an option value of maintaining capacity that pays off during a subsequent recovery.

We can use eq. (27) to examine how \( q_d \) depends on \( y \). If, and only if,

\[
\frac{y_1}{y_0(1 - \pi_0) + y_1 \pi_0} - \frac{h + \theta}{\pi_0} > \frac{y_1}{y_0(1 - \pi_1) + y_1 \pi_1} - \frac{h + \theta}{\pi_1}, \tag{28}
\]

will a transition from \( y_0 \) to \( y_1 \) lead to a rise in \( q_d \). From eq. (17) we see that this inequality essentially reflects a tradeoff between the effect of a transition from \( y_0 \) to \( y_1 \) on equity prices (which tends to raise \( q_d \)) and the probability of a financial intermediary receiving payment in a high \( y \) state (which tends to lower \( q_d \)). If the transition from \( y_0 \) to \( y_1 \) leads to a large rise in equity prices because \( y_0 \) is low relative to \( y_1 \), then inequality (28) will tend to hold, and a rise in \( y \) will lead to a higher \( q_d \) (lower interest rate on private liquidity).
It can be shown that $b' + d'$ must satisfy the following equation:

$$\psi y^\eta (b' + d')^{-\eta} = \beta \xi + \left(1 - \frac{(1 + \xi)(1 - h - \theta)}{1 - h}\right) q_d.$$ 

If, and only if,

$$\xi > \frac{\theta}{1 - h - \theta},$$

(29)

will a rise in $q_d$ be associated with a rise in liquidity as a fraction of output as measured by $(b' + d')/y$. If inequality (29) holds, then a low $y$ state is associated with a fall in liquidity as a fraction of output, but other outcomes are possible too.

Interest rates on government debt can be written as

$$q_b = \beta (1 + \xi) + \left(1 - \frac{(1 + \xi)(1 - h - \theta)}{1 - h}\right) q_d.$$ 

Just as with $(b' + d')/y$, if and only if inequality (29) holds will a rise in $q_d$ be associated with a rise in $q_b$. Thus, if inequalities (28) and (29) both hold, then a low $y$ state is associated with a fall in interest rates on public liquidity.

To summarize, once adjusted for a change in the expected payout, private interest rates do not change during the transition from a high output state to a low output state. However, with a sufficiently large rise in equity prices and a sufficiently large liquidity parameter $\xi$, liquidity falls during this transition. Note the important role of the fall in liquidity in keeping private interest rates from responding to a shock to output. Essentially, in response to an adverse output shock the number of financial intermediaries and the associated supply of private liquidity contract to such an extent that private interest rates do not respond, thereby maintaining the zero-profit condition for financial intermediaries. In some sense, the fall in liquidity is too sharp to be consistent with a rise in interest rates on private liquidity during a fall in output. The next section studies an example in which this is not the case.

### 3.7.2 Special Case 2: $\kappa = \infty$

Consider the special case $\kappa = \infty$. In this case $m' = 0$ and $n$ can only fall over time. For illustration purposes, we will choose a constant $b$ and an initial $n > 0$ such that households choose to maintain that level of $n$ under all shocks, and hence the economy will exhibit a constant supply of private liquidity $d = (1 - h)n$.

For a constant $n$, $\lambda$ solves

$$\lambda(y, n) = \max\{\Omega(y, n) + \beta E_y[\lambda(y', n)] - (h + \theta), 0\}.$$ 

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A sufficient condition for the economy to maintain a constant level of $n$ is

$$\Omega(y, n) > h + \theta$$

for all $y$, which will hold for a sufficiently small $b$ and $n$.

Bond prices $q_b$ and $q_d$ are then given by

$$q_b = \beta + \psi y^n (b + d)^{-n},$$

$$q_d = \psi y^n (b + d)^{-n} - \beta \xi + \beta (1 + \xi) E_y[x'],$$

and note also that

$$q_d = \psi y^n (b + d)^{-n} - \beta \xi + (1 + \xi) q_d^*.$$

Although $b + d$ is held constant in this example, the marginal value of liquidity still changes due to a change in $y$, and per above this change in the value of liquidity affects the price of private liquidity. Note too that if $\xi$ is sufficiently large such that $\psi y^n (b + d)^{-n} - \beta \xi < 0$, then a change in the yield associated with $q_d^*$ due to a change in the expected payout on private liquidity will lead to a larger than one-for-one change in the yield on private liquidity. In this sense, the change in yield on private liquidity would seem excessive relative to a change in default rates.

Since private liquidity may partially default in the low $y$ state, in an obvious sense a rise in the probability of a realization of the high $y$ state (due to a current realization of a high $y$ state) will tend to raise the expected payout of private liquidity, but as reflected in eq. (26), the dependence of the realized payout on private liquidity in the low $y$ state on this probability admits the possibility that the fall in this payout could be sufficiently large so that the expected payout actually falls. Indeed, this expected payout can be explicitly written as

$$E_y[x'] = \frac{y_0^{(1-\pi)}}{E_y[y'(1-h)]} \psi y^n (b + d)^{-n} - \beta \xi + \pi \left(1 - \frac{y_0^{(1-\pi)}}{E_y[y'(1-h)]} \beta (1 + \xi) \right).$$

For sufficiently low $y_0$ or for $\pi_0$ sufficiently close to $\pi_1$, $E_y[x']$ will rise with $y$.

To summarize, in the previous example with $\kappa = 0$ a change in $y$ is associated with a sufficiently large change in private liquidity to prevent any change in the yield on private liquidity that cannot be accounted for by a change in default rates. In contrast, with $\kappa = \infty$ the supply of private liquidity is constant by construction, which leads to an equilibrium that magnifies the response of the yield on private liquidity to a change in $y$, thereby leading to a change in the yield on private liquidity that is in excess of what’s due to a change in default
rates. Moreover, this magnification depends on ξ, which captures the special role of private liquidity. In the simulations we will consider an intermediate case in which both the supply and price of private liquidity depend on y in an important way. To focus the simulations on the relevant range of parameters, we will first examine the data and estimate the model’s parameters.

4 The Data

We use quarterly data from 1950 to 2014. We obtain the supply of short-term U.S. Treasury bills and short-term private debt (open-market paper, but mainly commercial paper) from the Federal Reserve’s Flow of Funds accounts, which has continuous data on commercial paper only beginning in 1950. Nominal 3-month Treasury bill yields and AA-rated commercial paper yields are obtained from the Federal Reserve’s Selected Interest Rate Release via St. Louis FRED. For the inflation adjusting we use the year-over-year changes in the U.S. Consumer Price Index. Equity market returns are obtained from CRSP. Data on equity valuation, total operating cost, and total assets for non-bank financial firms are obtained from COMPUSTAT, which are only available from 1974 to the present.

![Figure 2: T-bill and Commercial Paper Quantities](image)

The supplies of T-bill and Commercial Paper as a fraction of GDP are presented in Fig. 2. This figure exhibits a clear negative relation between the supply of T-bills and Commercial Paper. Table 1 reports results of regressing log(Commercial Paper/GDP) on
Table 1: Linear Regression: Commercial Paper on T-bill Supply and Real Yield

log(T-bill/GDP) with a Newey-West correction to the standard errors. The coefficient is negative (-1.736) and significant ($t = -2.994$). This seems consistent with our model with sufficient independent variation in the supply of short-term government debt, as a rise in the supply of short-term government debt will tend to crowd out relatively safe short-term private debt. As suggested by eq. (7), Table 1 also reports results of regressing log(Commercial Paper/GDP) on log(T-bill/GDP) and the real yield on T-bills. The coefficient on log(T-bill/GDP) remains negative (-1.520) and significant ($t = -2.621$) and as well the coefficient on the real T-bill yield is positive (0.129) and significant (2.611), which seems consistent with eq. (7).
Real yields on T-bills and Commercial Paper are presented in Fig. 3, along with data on the overall supply of liquidity as measured by (Commercial Paper + T-bill)/GDP. Clearly the real T-bill and Commercial Paper rates move together, and they would be expected to in this model since they both depend in the same way on the overall supply of liquidity. Table 2 reports results of regressing the real T-bill yield on the supply of liquidity. The coefficient is positive (2.537) and significant ($t = 2.553$), as predicted by the model. A relatively high supply of overall liquidity is associated with a relatively low marginal non-pecuniary return on liquidity and thereby is associated with a high yield on riskless sources of liquidity.

<table>
<thead>
<tr>
<th>Dependent Variable: Real T-bill Yield</th>
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<tbody>
<tr>
<td>1950-2014</td>
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<tr>
<td></td>
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<tr>
<td>Constant</td>
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<tr>
<td>Std. Err.</td>
</tr>
<tr>
<td>t-Stat</td>
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<tr>
<td>Log(Total Liquidity/GDP)</td>
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<tr>
<td>Std. Err.</td>
</tr>
<tr>
<td>t-Stat</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
</tr>
</tbody>
</table>

Total Liquidity = T-bill + Commercial Paper. Standard Errors reflect a Newey-West adjustment with 4 quarterly lags

Table 2: Linear Regression: Real T-bill Yield on Total Liquidity

The model predicts that the Commercial Paper − T-bill spread should not depend on the relative supply of T-bills to Commercial Paper, but rather should depend only on the riskiness of Commercial Paper. Given the low frequency of default on Commercial Paper, it is difficult to obtain a reliable time series inferring the expectation of default from observed default rates. Instead, we relate the probability of default to overall market uncertainty as measured by Market Volatility derived from an EGARCH Conditional Variance of total equity market return using CRSP data. Data on the Commercial Paper − T-bill spread is presented in Fig. 4, along with our measure of Market Volatility. Results from regressing the Commercial Paper − T-bill spread on Market Volatility are presented in Table 3. The regression coefficient is positive (4.072) and significant ($t = 2.660$). If we add to this regression the relative supply ratio Commercial Paper/T-bill, we see that this variable enters insignificantly ($t = -1.430$) while the Market Volatility measure retains its significance, just as predicted by the model. However, when we include just the T-bill/GDP ratio, which is a predictor used in Krishnamurthy and Vissing-Jorgenson (2012, 2013), we find that the significance of this variable is substantially reduced by the inclusion of Market Volatility, but it remains significant. This suggests that either Market Volatility is missing some information relevant to expected default rates on Commercial Paper, or there is some aspect of T-bills as it relates to liquidity that is not represented in the model.
5 Model Estimation and Simulation

Eq. (7) is one non-linear equation in the price of riskless liquidity, $q_b$, and the overall supply of liquidity as a fraction of GDP, $(b' + d')/y$. Table 4 reports results from fitting a nonlinear regression to this equation. The standard errors are created by a bootstrap method with 1000 random draws. We experienced some difficulty in producing reliable estimates of $\beta$, so we simply chose a value of $\beta = .96$. Both $\eta$ and $\psi$ are estimated to be positive and significant. As required by the model (but not imposed in the estimation), $\eta$ is estimated to be larger than one.

From eqs. (7) and (8), the difference in price between Commercial Paper and T-bills can be written as

$$q_d - q_b = \beta(1 + \xi)(E_y[x'] - 1). \tag{30}$$

As mentioned, it is difficult to infer $E_y[x']$ from observed data. Instead, we posit the following relationship between the expected payout rate, $E_y[x']$, and Market Volatility, $\sigma$, estimated from an EGARCH(1,1) fitted to CRSP total return data:

$$E_y[x'] = \frac{1}{1 + \alpha \sigma}. \tag{31}$$

Substituting eq. (31) into eq. (30), we derive the relation

$$q_d - q_b = \beta(1 + \xi) \left( \frac{1}{1 + \alpha \sigma} - 1 \right). \tag{32}$$
Dependent Variable: Commercial Paper – T-bill Yield Spread
1950-2014

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Standard Error</th>
<th>t-Stat</th>
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<td>Adjusted R-squared</td>
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<td>Log(CP Debt/T-bill Debt)</td>
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<td>Equity Market Volatility</td>
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<td>1.558</td>
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<td>Adjusted R-squared</td>
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<tr>
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<td>Log(T-bill Debt/GDP)</td>
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<td>Equity Market Volatility</td>
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<td>Adjusted R-squared</td>
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Standard Errors reflect a Newey-West adjustment with 4 quarterly lags

Table 3: Linear Regression: Yield Spread on Equity Market Volatility

Even given $\beta$, it will difficult to use eq. (32) to estimate both $\xi$ and $\alpha$, as both serve similar roles. Ideally, we would like to have data on $x'$ so that eq. (31) could help to pin down $\alpha$, and then use eq. (32) to estimate $\xi$. Unfortunately, given the very low likelihood of observing $x'$ below one and the relatively short sample, we are unable to use robust statistical methods to estimate $\alpha$. Instead, note from Fig. 4 that $\sigma$ ranges from about 10 to 30 with an average around 15. At $\sigma = 15$, using eq. (32) a value of $\alpha = .005$ leads to $E_y[x'] = .9926$ ($\sigma = 10$ leads to .9950 and $\sigma = 30$ leads to .9852). Based on these “reasonable” values, Table 5 presents estimates of $\xi$ for three different values of $\sigma$: .003, .005, and .007 (and $\beta = .96$). As before, the standard errors are created by a bootstrap method with 1000 random draws. The parameter $\xi$ is always estimated to be very large and significant ($\xi = 6.8914$ with $t = 15.5951$ for $\alpha = .005$, which is the value we use in the simulations).

The remaining parameters are calibrated and presented in Table 6. The parameter
$y_0/y_1$ is estimated based on the Great Depression and the 2007-2008 Financial Crisis. From peak to trough, output fell by about 30 percent during the Great Depression and 3 percent during the 2007-2008 Financial Crisis. We chose a value of .9 for this ratio. We chose $\pi_0 = .9$ and $\pi_1 = .99$, so a financial panic occurs about once a century and lasts a little over 1.1 years. Based on Fig. 2 the average ratio of T-bills to GDP is around 8 percent, so we initially set $b$ constant to .08. In a subsequent simulation we will consider varying $b'/y$ between the two states to reflect the tendency for government policy to issue more T-bills during a financial crisis. Fig. 5 displays data on leverage of financial firms, which shows leverage ranges from .04 to .15. We chose .06, which is roughly the average from 1979 up to the 2007-2008 Financial Crisis, but excludes the time prior to 1979 during which time there seems to be a downward trend. We chose $\theta = .02$ as at that value (given the remaining parameters), the amount of private short-term debt in the simulations roughly equals the amount of government short-term debt, which holds true in the data. Fig. 6 displays data on operating costs of financial intermediaries as a fraction of assets, which ranges from .04 to .12, so our parameter is a bit lower than the data in this regard. We do not have reliable data on the fixed cost of starting a financial intermediary, so while initially we set $\kappa = \theta$, 

### Table 4: Non-Linear Estimation of $\psi$ and $\eta$

<table>
<thead>
<tr>
<th>1950-2014</th>
<th>Estimate</th>
<th>Standard Error (Bootstrap)</th>
<th>t-stat</th>
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<td>$\beta$</td>
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<tr>
<td>$\psi$</td>
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<td>0.0014</td>
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<tr>
<td>$\eta$</td>
<td>1.1025</td>
<td>0.1845</td>
<td>5.9758</td>
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### Table 5: Non-Linear Estimation of $\xi$

<table>
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<tr>
<th>1950-2014</th>
<th>Estimate</th>
<th>Standard Error (Bootstrap)</th>
<th>t-stat</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>0.0030</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi$</td>
<td>12.1476</td>
<td>0.7169</td>
<td>16.9455</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0050</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi$</td>
<td>6.8914</td>
<td>0.4419</td>
<td>15.5951</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0070</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi$</td>
<td>4.6388</td>
<td>0.3176</td>
<td>14.6056</td>
</tr>
</tbody>
</table>
we also vary this parameter in the simulations to get a sense of how the results depend on various choices.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{y_0}{y_1} )</td>
<td>GDP(bad state) GDP(good state)</td>
<td>.9</td>
<td>= .7 for Great Depression, .97 in 2007-2008</td>
</tr>
<tr>
<td>((\pi_0, \pi_1))</td>
<td>Prob good state</td>
<td>(.9, .99)</td>
<td>business cycle freq.</td>
</tr>
<tr>
<td>( b )</td>
<td>T-bill GDP</td>
<td>.08</td>
<td>aver. 1970-2014</td>
</tr>
<tr>
<td>( h )</td>
<td>Equity Assets</td>
<td>.06</td>
<td>aver. 1979-2007</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Operating Cost Assets</td>
<td>.02</td>
<td>match ( d = b ) (data range: .04 to .12)</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Start-Up Cost Assets</td>
<td>range .01 to .04</td>
<td>no data</td>
</tr>
</tbody>
</table>

Table 6: Calibration of Remaining Parameters

![Equity / Total Assets](image)

Figure 5: Leverage

The results of simulating this model are presented in Table 7, which presents the average value of key variables in the model economy in a high \( y \) vs. a low \( y \) state. This table reports interest rates and not bond prices, so \( r_b = 1/q_b - 1 \) and \( r_d = 1/q_d - 1 \). The model is simulated for three values of \( \kappa: .01, .02, \) and \( .04 \). The top three panels consider a government policy in which \( b' \) is held constant at \(.08\). Consider first the top left panel in which \( \kappa = .01\). Here we see that a fall in output leads to a fall in \( d' \), the private supply of liquidity,
and therefore in a fall in total liquidity as a fraction of output, \((b' + d')/y\). Consequently, government interest rates as measured by \(r_h\) fall during a fall in output. Here, and in all the panels, the fall in the expected payout on private liquidity, \(E_y[x']\) as magnified by \(\xi\) leads to a rise in interest rates on private liquidity during a fall in output.

In the top middle panel \(\kappa = .02\) and as a consequence of the higher cost of re-starting a financial intermediary during a subsequent rise in output, the fall in private liquidity is less during a fall in output. Otherwise, the model behaves qualitatively similar to that when \(\kappa = .01\). In the top right panel \(\kappa = .04\) and this time the supply of liquidity is constant across the two states. Total liquidity is thus also constant, however total liquidity as a fraction of output rises during a fall in output. Because of this, interest rates on government liquidity actually rise during a fall in output.

The bottom three panels consider a government policy in which \(b'\) rises dramatically during a fall in output, from .08 to .2. Short-term government debt thus rises by a factor of 2.5 in the simulations, and based on Fig. 2 government debt from Aug. 2007 to March 2009 rose from 6 percent of GDP to 14.1 percent, which is a factor of 2.35. In all three panels this rise in government liquidity crowds out the supply of private liquidity, but the rise in government liquidity is sufficiently large that total liquidity rises during a fall in output. Consequently, for any value of \(\kappa\), government interest rates rise during a fall in output.
Table 7: Simulations

<table>
<thead>
<tr>
<th>$\kappa = .01$</th>
<th>$\kappa = .02$</th>
<th>$\kappa = .04$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{low}$</td>
<td>$y_{high}$</td>
<td>$y_{low}$</td>
</tr>
<tr>
<td>$y$</td>
<td>0.90</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.90</td>
<td>0.99</td>
</tr>
<tr>
<td>$b'$</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>$d'$</td>
<td>0.05</td>
<td>0.11</td>
</tr>
<tr>
<td>$\frac{b'+d'}{y}$</td>
<td>0.15</td>
<td>0.19</td>
</tr>
<tr>
<td>$r_b$</td>
<td>0.0142</td>
<td>0.0208</td>
</tr>
<tr>
<td>$r_d$</td>
<td>0.0366</td>
<td>0.0227</td>
</tr>
<tr>
<td>$E_y[x']$</td>
<td>0.9972</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

6 Liquidity and the 2008 Lehman Bankruptcy

Fig. 7 shows the monthly behavior of the ex-post real yield on 3-month T-bills along with the real supply of Commercial Paper + T-bills just before, during, and just after the financial panic caused by the September 2008 Lehman bankruptcy. This figure shows that the overall supply of liquidity rose following September 2008 and the real yield on T-bills rose as well. From Fig. 2 we see that the overall supply of liquidity rose because the rise in the supply of T-bills was larger than the drop in Commercial Paper. The model matches all of these observations, which is reflected in the simulations as reported in Table 7. In the model, a fall in output and the associated fall in the probability of a high output realization in the next period tends to lower the expected payout on private liquidity, thereby leading to a rise in the yield spread between public and private liquidity as well as a contraction in private liquidity. The rise in the yield spread is well in excess of the change in expected payouts and is due to a loss of liquidity value of private liquidity. The large rise in public liquidity
more than compensated for the fall in private liquidity, thereby leading to a rise in overall liquidity, thus lowering the marginal value of liquidity and thereby raising the real yield on public liquidity. Indeed, the tendency for private liquidity to fall as public liquidity rises meant that public liquidity had to rise substantially in order for total liquidity to rise. In this regard, note that the model suggests that if the government had not issued such a large amount of T-bills, then the overall supply of liquidity would have fallen and the real yield on 3-month T-bills would have fallen too.

Figure 7: Total Liquidity and Real Interest Rates during 2008 Financial Panic

7 Financial Panic as a ‘Bad’ Equilibrium

Section 3.6 discussed the possibility that there may exist two equilibria, one in which yields on private liquidity are low and the expected rate of default of financial intermediaries is low, and one in which yields on private liquidity are high and the expected rate of default of financial intermediaries is high, and the difference in default rates are self fulfilling. Along with other parameters, this possibility required a sufficiently high value of $\xi$, and in particular this multiplicity of equilibria could not occur for $\xi = 0$. The estimated value of $\xi$ is 6.89, which suggests multiple equilibria may indeed be a feature of the estimated model. Indeed, from Fig. 1 we see that a necessary condition for multiple equilibria to exist is if the horizontal intercept in that figure, given by $\psi y^2(b' + d')^{-\eta} - \beta \xi$, is negative. Based on the estimated parameters and using values of $y/(b' + d') = 1/.18$ (from Table 7 for the high $y$ state with a constant $b'$ and $\kappa = .02$), this value is -6.6.
Figure 8 shows Fig. 1, but at the estimated parameter values, and with \( y/(b' + d') = 1/1.18 \), constant \( b' = .08 \) and \( \kappa = .02 \). This figure shows the right side of eq. (18) for both low and high values of \( y \). Here we can see that both states exhibit multiple equilibria. For the high \( y \) state, the yield on private equilibria could equal .02 (as shown in Table 7), in which case financial intermediaries would only default if the low \( y \) state was realized in the next period, or the yield could equal approximately .11, in which case financial intermediaries would partially default in both states of the world next period. Starting from the low \( y \) state, the yield on private liquidity could either equal about .05 or again .11. The bad equilibrium would of course have associated with it a contraction in private liquidity.

The bad equilibrium can be computed just as Table 7 was constructed, but with the private bond price equation replaced by

\[
Q_d(y, d') = \frac{\beta \xi - \psi y^\eta (B(y) + d')^{-\eta}}{1 - \frac{1 + \xi}{1 - h} - 1}.
\]

In simulations of this version of the model, it turns out that for all the parameter configurations considered in Table 7, \( d' = 0 \) under all states of the world, as at the higher interest rate on private liquidity (now about .12), financial intermediaries do not find it profitable to enter. An interesting case to consider is the bad equilibrium, but when \( b' = .20 \) under all states of the world (this is essentially the top panel of Table 7 but with \( b' = .20 \) and at the
bad equilibrium). The results are reported in Table 8 (only results for $\kappa = .02$ are reported as the other panels are similar). The intriguing result is that a shift from a high output, good equilibrium with $b' = .08$ to a bad equilibrium with $b' = .20$ (and no other change in current or expected output) produces very similar results as the shift from a high level of output to a low level of output (and associated revision in expectations of future output): private liquidity/output falls (from 0.10 to 0), overall liquidity/output rises (from 0.18 to 0.20), public interest rates rise (from .0201 to .0222), and the yield spread rises (from .0018 to .0987).

\begin{table}[h]
\centering
\begin{tabular}{llll}
\hline
 & $y_{low}$ & $y_{high}$ \\
$y$ & 0.90 & 1 \\
$\pi$ & 0.90 & 0.99 \\
b' & 0.20 & 0.20 \\
d' & 0.00 & 0.00 \\
$\frac{b'+d'}{y}$ & 0.22 & 0.20 \\
r_b & 0.0243 & 0.0222 \\
r_d & 0.1206 & 0.1209 \\
$E_y[x']$ & 0.9889 & 0.9886 \\
\hline
\end{tabular}
\caption{Simulations for the Bad Equilibrium}
\end{table}

8 Conclusion

Foundational to any model of a financial panic would seem to be a flight to quality in which some assets that served a useful role in supporting a valuable financial-intermediation activity during normal times no longer serves that role in a financial panic, thereby leading to a collapse of this activity with collateral effects felt throughout the economy. In this paper, the strong demand for safe, short-term assets encourages the private sector to issue these assets to supplement the public supply of safe, short-term assets, even though their role in providing liquidity can be put in jeopardy during financial panics that lead agents to question the low-risk nature of these assets. During “normal” times these assets are considered sufficiently safe so that they contribute to the supply of liquid assets, but in a financial panic their role in providing liquidity is great diminished, if not eliminated all
together. The endogenous mechanism by which assets can be categorized as safe enough to offer liquidity services, or to lose such a status that had been previously acquired, is the novel aspect of this paper. The distinguishing event-dependent characteristic is their ex-ante perception of default as magnified by the costs of default borne by special, “liquid” assets. Rather than assuming this cost is infinite, which would lead to only default-free public sources of liquidity (provided these assets are perceived as riskfree), this paper assumes this cost is finite, which opens the door to the private-sector supply of these assets.

Such a model seems to match key features of the data, especially during a financial panic. The large rise in yield spread between commercial paper and T-bills, and the fall in the supply of commercial paper, both suggest that commercial paper loses some aspect of providing liquidity that limits their demand during a financial panic. The fall in commercial paper is further diminished by a rise in the supply of T-bills, as public sources of liquidity tend to crowd out private sources. Moreover, the rise in the overall supply of liquidity in a financial panic due to the dramatic rise in T-bills leads to a rise in the real yield on T-bills, which seems supported in the data. That the model gets these observations right suggests it may be useful for thinking about events underlying a flight to quality in particular and for thinking about the special role some assets serve in an economy in general.

References


This appendix proves the existence of a unique solution $\lambda$ to eq. (25) and establishes properties of this solution. As mentioned, eq. (25) has a special, recursive structure that simplifies its study. First, $\lambda(y, 0)$ need only solve the following equation

$$
\lambda(y, 0) = \max \left\{ \max \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\lambda(y', 0)], \kappa \right\}, \kappa \right\}, \kappa \right\} - (h + \theta), 0 \right\}. \quad (33)
$$

Denote by $\lambda(0)$ the pair $(\lambda(y_0, 0), \lambda(y_1, 0))$. Define the function $A_0 : R_{\geq 0}^2 \to R_{\geq 0}^2$ as follows.

For any $\hat{\lambda}(0) \in R_{\geq 0}^2$, define $\lambda(0) = A_0(\hat{\lambda}(0))$ as

$$
\lambda(y, 0) = \max \left\{ \max \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\hat{\lambda}(y', 0)], \kappa \right\}, \kappa \right\} - (h + \theta), 0 \right\}.
$$
\[
\min \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\hat{\lambda}(y', 0)], \kappa \right\}, h + \theta \right\} - (h + \theta), 0 \right\}. \quad (34)
\]

Define
\[
\tilde{\lambda} = \max_y \left\{ \max \left\{ \Omega(y, 0) + \kappa - (h + \theta), 0 \right\} \right\}.
\]

**Proposition 1:** There exists one, and only one, \( \lambda(0) \in [0, \tilde{\lambda}]^2 \) such that \( \lambda(0) = A_0(\lambda(0)) \).

**Proof:** \([0, \tilde{\lambda}]^2\) is a complete metric space and it straightforward to established that \( A_0([0, \tilde{\lambda}]^2) \subset [0, \tilde{\lambda}]^2 \). Clearly \( A_0 \) is monotone and satisfies the discounting property of Blackwell’s Theorem 5, hence \( A_0 \) is a contraction mapping on \([0, \tilde{\lambda}]^2\). \( \text{Q.E.D.} \)

For \( n + m > 0 \), define \( \lambda(n + m) \) as the pair \( (\lambda(y_0, n + m), \lambda(y_1, n + m)) \). Define the function \( A : \mathbb{R}_{\geq 0}^2 \to \mathbb{R}_{\geq 0}^2 \) in a similar manner as \( A_0 \), except with \( \lambda(0) \) already being known: for any \( \hat{\lambda}(n + m) \in \mathbb{R}_{\geq 0}^2 \) and the solution \( \lambda(y, 0) \) to eq. (33), define \( \lambda(n + m) = A(\hat{\lambda}(n + m)) \) as

\[
\lambda(y, n + m) = \max \left\{ \max \left\{ \Omega^+(y, n + m) + \min \left\{ \beta E_y[\hat{\lambda}(y', n + m)], \kappa \right\}, \min \left\{ \Omega(y, 0) + \min \left\{ \beta E_y[\lambda(y', 0)], \kappa \right\}, h + \theta \right\} \right\} - (h + \theta), 0 \right\}. \quad (35)
\]

**Proposition 2:** There exists one, and only one, \( \lambda(n + m) \in [0, \tilde{\lambda}]^2 \) such that \( \lambda(n + m) = A(\lambda(n + m)) \).

**Proof:** Similar to Proposition 1. \( \text{Q.E.D.} \)

We can now summarize various essential properties of \( \lambda \) in the following proposition.

**Proposition 3:** The unique solution \( \lambda(y, n + m) \) to eq. (25) is a decreasing function of \( n + m \) for any value of \( y \). Moreover, for any \( n + m \) such that \( \Omega^+(y, n + m) = 0 \) for all values of \( y \), \( \lambda(y, n + m) = 0 \) for all values of \( y \).

**Proof:** Due to the monotonicity of \( A \), \( \lambda \) inherits the monotonicity of \( \Omega \) with respect to \( n + m \). It is straightforward to establish that if \( n + m \) is such that \( \Omega^+(y, n + m) = 0 \) for all values of \( y \), then \( 0 = A(0) \). \( \text{Q.E.D.} \)