Dynamic Pricing of Relocating Resources in Large Networks
(Online Appendix)

Santiago R. Balseiro\textsuperscript{1}, David B. Brown\textsuperscript{2}, and Chen Chen\textsuperscript{2}

\textsuperscript{1}Graduate School of Business, Columbia University
\textsuperscript{2}Fuqua School of Business, Duke University
srb2155@columbia.edu, dbrown@duke.edu, cc459@duke.edu

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EC1 Proofs

EC1.1 Proof of Proposition 1

Let $N(T)$ be the number of requests during time $[0, T]$. Using the notation of Section 2.2, the long-run average number of relocations per resource per unit time $\Phi^\pi$ is

$$\Phi^\pi = \frac{1}{m} \cdot \lim_{T \to \infty} \mathbb{E}\left\{ \frac{1}{T} \cdot \sum_{t=1}^{N(T)} \sum_{i,j \in [n]} y_{ij,t} \cdot 1[\xi_t \leq d_{ij,t}^\pi] \right\}$$

(i)

$$\Phi^\pi = \frac{1}{m} \cdot \mathbb{E}\left\{ \lim_{T \to \infty} \frac{1}{T} \cdot \sum_{t=1}^{N(T)} \sum_{i,j \in [n]} y_{ij,t} \cdot 1[\xi_t \leq d_{ij,t}^\pi] \right\}$$

(ii)

$$\Phi^\pi = \frac{\sum_{i,j \in [n]} \eta_{ij}}{m} \cdot \mathbb{E}\left\{ \lim_{N \to \infty} \frac{1}{N} \cdot \sum_{t=1}^{N} \sum_{i,j \in [n]} y_{ij,t} \cdot 1[\xi_t \leq d_{ij,t}^\pi] \right\}$$

(iii)

$$\Phi^\pi = \frac{\sum_{i,j \in [n]} \eta_{ij}}{m} \cdot \lim_{N \to \infty} \mathbb{E}\left\{ \sum_{t=1}^{N} \sum_{i,j \in [n]} y_{ij,t} \cdot d_{ij,t}^\pi \right\}$$

where (i) follows from the generalized dominated convergence theorem (Theorem 19 in [Royden and Fitzpatrick 2010]) because $\sum_{i,j \in [n]} y_{ij,t} \cdot 1[\xi_t \leq d_{ij,t}^\pi] \in [0, 1]$ and $\lim_{T \to \infty} \mathbb{E}\left[ \frac{N(T)}{T} \right] = \mathbb{E}\left[ \lim_{T \to \infty} \frac{N(T)}{T} \right]$, (ii) from $\lim_{T \to \infty} \frac{N(T)}{T} = \sum_{i,j \in [n]} \eta_{ij}$ and $\lim_{T \to \infty} N(T) = \infty$, and (iii) from the bounded convergence theorem. In the last equation, $\Theta^\pi$ represents the expected throughput per period. Using $\Theta^\pi \leq 1$ and $\sum_{i,j \in [n]} \eta_{ij} \leq \bar{\eta} n$, we have

$$\Phi^\pi \leq \frac{\eta}{m}.$$

For the other direction, let $V^\pi$ denote the long-run average revenue of policy $\pi$. We have

$$V^\pi = \lim_{N \to \infty} \frac{1}{N} \cdot \mathbb{E}\left\{ \sum_{t=1}^{N} \sum_{i,j \in [n]} y_{ij,t} \cdot r_{ij}(d_{ij,t}^\pi) \right\} \leq \bar{\omega} \cdot \lim_{N \to \infty} \frac{1}{N} \cdot \mathbb{E}\left\{ \sum_{t=1}^{N} \sum_{i,j \in [n]} y_{ij,t} \cdot d_{ij,t}^\pi \right\} = \bar{\omega} \cdot \Theta^\pi,$$

where the inequality is due to the mean value theorem and the facts that $r_{ij}(0) = 0$ and $\bar{\omega} > 0$ is a uniform bound on the derivatives of the one-period revenue functions by Assumption 1. Using the above inequality on $V^\pi$ and $\sum_{i,j \in [n]} \eta_{ij} \geq \eta n$, we obtain

$$\Phi^\pi \geq \frac{\eta V^\pi}{\bar{\omega}} \cdot \frac{n}{m},$$

which completes the proof. The same result also holds with relocation times because incorporating relocation times only affects the controls $d_{ij,t}^\pi$ (as resources are blocked while relocating), and we can express resource utilization in terms of the controls as before.

EC1.2 Proof of Proposition 2

Since the network topology of the hub-and-spoke structure is strongly connected and the one-period revenue functions $r_{ij}(d)$ and $r_{0i}(d)$ are uniformly bounded by Assumption 1, Assumption 5.6.1 of
Bertsekas (2012) holds. According to Proposition 5.6.2 of Bertsekas (2012), the average revenue $V^{\text{OPT}}$ of an optimal policy does not depend on the initial state of the system, and moreover, there exists a solution to the Bellman equation (2) if randomized controls are allowed. Since the one-period revenue functions are concave, randomization does not improve performance and there must be a solution to (2) with controls being deterministic. Thus (2) has a solution. Finally, according to Proposition 5.6.1 of Bertsekas (2012), if $d^*(x,s)$ attains the maximum in (2) for each state $(x,s)$, the stationary policy $d^*(x,s)$ is optimal.

**EC1.3 Proof of Proposition 3**

It is easy to see that (3) decomposes over spokes with each spoke problem being

$$
\max_{\pi \in \Pi} \lim_{T \to \infty} \frac{1}{T} \cdot \mathbb{E}\left\{ \sum_{t=1}^{T} \left( y_{i0,t} \cdot r_{i0}(d_{i0,t}^\pi) + y_{0i,t} \cdot r_{0i}(d_{0i,t}^\pi) - \lambda \cdot x_{i,t}^\pi \right) \right\}
\quad \text{s.t. } x_{i,t+1}^\pi = x_{i,t}^\pi - y_{i0,t} \cdot \mathbb{I}[\xi_t \leq d_{i0,t}^\pi] + y_{0i,t} \cdot \mathbb{I}[\xi_t \leq d_{0i,t}^\pi], \quad \forall \; t \geq 1,
\quad 0 \leq x_{i,t}^\pi \leq m, \quad \forall \; t \geq 1.
$$

(EC-1)

We can interpret (EC-1) as an average revenue problem for spoke $i$ where in each time period, one request arrives following the same request rates as in the original problem, and every resource in spoke $i$ incurs a holding cost $\lambda$. It is easy to see that Assumption 5.6.1 of Bertsekas (2012) holds for the spoke problem. By the same argument as in the proof of Proposition 2, the optimal average revenue $h_i^\lambda$ does not depend on the initial state of spoke $i$, and moreover, $h_i^\lambda$ together with some differential value functions $v_i^\lambda(x,i,0)$, $v_i^\lambda(x,0,i)$, and $v_i^\lambda(x,\emptyset)$ satisfies the Bellman equation (5). Since (3) decomposes into (EC-1), the optimal average revenue $\bar{V}^\lambda$ of the Lagrangian relaxation does not depend on the initial state as well, and

$$
\bar{V}^\lambda = m\lambda + \sum_{i=1}^{n} h_i^\lambda.
$$

Finally, since every feasible policy to the original problem is feasible to the Lagrangian relaxation and attains an objective value that is no smaller, we have $\bar{V}^\lambda \geq V^{\text{OPT}}$.

**EC1.4 Proof of Proposition 4**

We first show that (6) is equivalent to the dual problem of the LP formulation of (5) in Section EC1.4.1.
EC1.4.1 Equivalence Between (5) and (6)

First, note that the optimal average revenue $h^λ_i$ in (5) can be solved by the following linear program (EC-2) following Section 5.5 of Bertsekas (2012).

$$\min_{h^λ_i, v^λ_i(x, i, 0), v^λ_i(x, 0, i), v^λ_i(x, \emptyset)} h^λ_i$$

s.t.

$$h^λ_i + v^λ_i(x, i, 0) \geq r_{i0}(d) + d \cdot v^λ_i(x - 1) + (1 - d) \cdot v^λ_i(x) - \lambda \cdot x, \quad \forall x \in [0 : m], d \in [0, 1 \land x],$$

(EC-2)

$$h^λ_i + v^λ_i(x, 0, i) \geq r_{0i}(d) + d \cdot v^λ_i(x + 1) + (1 - d) \cdot v^λ_i(x) - \lambda \cdot x, \quad \forall x \in [0 : m], d \in [0, 1 \land (m - x)],$$

$$h^λ_i + v^λ_i(x, \emptyset) \geq v^λ_i(x) - \lambda \cdot x, \quad \forall x \in [0 : m].$$

From Proposition 5.1.6 in Bertsekas (2012), any solution to (5) is an optimal solution to (EC-2). Problem (EC-2) is a semi-infinite linear program (see Section 4 in Anderson and Nash 1987) with a finite number of decision variables and infinitely many constraints.

According to Section 4.4 in Anderson and Nash (1987), the dual problem of (EC-2) can be written as

$$\max_{F_{i,0}^{(i,0)}(d) \geq 0, F_{i,m}^{(i,0)}(d) \geq 0, p_i(x, \emptyset), p_i(x) \geq 0} \sum_{x=0}^{m} \left\{ \int_0^1 r_{i0}(d) \cdot dF_{i,x}^{(i,0)}(d) + \int_0^1 r_{0i}(d) \cdot dF_{i,x}^{(0,i)}(d) \right\} - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x)$$

s.t.

$$\sum_{x=0}^{m} p_i(x) = 1,$$

$$p_i(x) \cdot q_{i0} = \int_0^1 dF_{i,x}^{(i,0)}(d), \quad \forall x \in [0 : m],$$

$$p_i(x) \cdot q_{0i} = \int_0^1 dF_{i,x}^{(0,i)}(d), \quad \forall x \in [0 : m],$$

(EC-3)

$$p_i(x) \cdot (1 - q_i) = p_i(x, \emptyset), \quad \forall x \in [0 : m],$$

$$p_i(x) = 1 \left[ x \leq m - 1 \right] \cdot \int_0^1 d \cdot dF_{i,x+1}^{(i,0)}(d) + 1 \left[ x \geq 1 \right] \cdot \int_0^1 d \cdot dF_{i,x-1}^{(i,0)}(d)$$

$$+ \int_0^1 (1 - d) \cdot dF_{i,x}^{(i,0)}(d) + \int_0^1 (1 - d) \cdot dF_{i,x}^{(0,i)}(d) + p_i(x, \emptyset), \quad \forall x \in [0 : m],$$

$$F_{i,0}^{(i,0)}(d) = p_i(0) \cdot q_{i0}, \quad \forall d \in (0, 1],$$

$$F_{i,m}^{(i,0)}(d) = p_i(m) \cdot q_{0i}, \quad \forall d \in (0, 1],$$

$$F_{i,x}^{(i,0)}(d), F_{i,x}^{(0,i)}(d) \in M[0, 1], \quad \forall x \in [0 : m],$$

where $M[0, 1]$ is the set of Lebesgue-Stieltjes measures on interval $[0, 1]$ with every $g(d) \in M[0, 1]$ an increasing and right-continuous function with $g(0-) = 0$. We can interpret the variables $F_{i,x}^{(i,0)}(d)$ and $F_{i,x}^{(0,i)}(d)$ as the joint probability that $x$ resources are in spoke $i$, a request $(i, 0)$ or $(0, i)$ arrives, and the service provider selects a demand level no larger than $d$. $p_i(x, \emptyset)$ is the probability that $x$ resources are in spoke $i$ and the request is one of any other types, and $p_i(x)$ is the probability with
EC1.4.2 Equivalence Between (6) and (7)

We now show (EC-3) and (6) are equivalent. To see this, note that every feasible solution to (EC-3) represents a randomized control to the spoke problem. Specifically, at every state \( x \) with \( p_i(x) > 0 \), the provider selects a demand level according to the cumulative distribution \( F_{i,x}^{(0,0)}(d)/(q_{0i} \cdot p_i(x)) \) if a request \((0,i)\) arrives, and cumulative distribution \( F_{i,x}^{(0,i)}(d)/(q_{0i} \cdot p_i(x)) \) if a request \((i,0)\) arrives. Since the one-period revenue functions \( r_{i0}(d) \) and \( r_{0i}(d) \) are concave, selecting the mean values \( d_i(x,i,0) = \int_0^1 d \cdot dF_{i,x}^{(i,0)}(d)/(q_{0i} \cdot p_i(x)) \) for a request \((i,0)\) and \( d_i(x,0,i) = \int_0^1 d \cdot dF_{i,x}^{(0,i)}(d)/(q_{0i} \cdot p_i(x)) \) for a request \((0,i)\) can only be better. This implies that we can simply focus on deterministic controls in (EC-3), which corresponds to (6). Thus, (EC-3) and (6) are equivalent.

Finally, if we let \( h_i^\lambda > \bar{r} \) and all the differential values be zero, we get a feasible solution to (EC-2) with all constraints in (EC-2) satisfied with strict inequality; thus strong duality holds according to Theorem 1 of Section 8.6 in Luenberger [1997].

Finally we point out that, from the complementary slackness property elaborated in the same theorem in Luenberger [1997], for all \( x \in [0 : m] \) with \( p_i(x) > 0 \), we have:

\[
\begin{align*}
    h_i^\lambda + v_i^\lambda(x,i,0) &= r_{i0}\left(d_i(x,i,0)\right) + d_i(x,i,0) \cdot v_i^\lambda(x) - 1 + \left(1 - d_i(x,i,0)\right) \cdot v_i^\lambda(x) - \lambda \cdot x, \quad \text{(EC-4)} \\
    h_i^\lambda + v_i^\lambda(x,0,i) &= r_{0i}\left(d_i(x,0,i)\right) + d_i(x,0,i) \cdot v_i^\lambda(x) + 1 + \left(1 - d_i(x,0,i)\right) \cdot v_i^\lambda(x) - \lambda \cdot x, \quad \text{(EC-5)}
\end{align*}
\]

and

\[
    h_i^\lambda + v_i^\lambda(x,\emptyset) = v_i^\lambda(x) - \lambda \cdot x, \quad \text{(EC-6)}
\]

where \( h_i^\lambda, v_i^\lambda(x,i,0), v_i^\lambda(x,0,i) \) and \( v_i^\lambda(x,\emptyset) \) is an optimal solution to (EC-2) and \( p_i(x) \), \( d_i(x,i,0) \) and \( d_i(x,0,i) \) is an optimal solution to (6).

EC1.4.2 Equivalence Between (6) and (7)

From Lemma EC2.1 the support of the optimal probability distribution in (6) is \( I_i = [0 : H_i] \) for some integer \( H_i \in \mathbb{N}_+ \). Introduce new variables \( \beta_i(x) \geq 0 \) for \( x \in [0 : m - 1] \) such that \( p_i(x+1) = \beta_i(x) \cdot p_i(x) \). We can write (6) as

\[
\begin{align*}
    h_i^\lambda &= \max_{\substack{d_i(x,i,0) \in [0,1], \\
    d_i(x,0,i) \in [0,1], \\
    p_i(x) \cdot \beta_i(x) \geq 0}} \sum_{x=0}^{m} p_i(x) \left[ q_{0i} \cdot r_{i0}\left(d_i(x,i,0)\right) + q_{0i} \cdot r_{0i}\left(d_i(x,0,i)\right) \right] - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x) \\
\text{s.t.} \quad &\sum_{x=0}^{m} p_i(x) = 1, \\
    p_i(x) \cdot \beta_i(x) &= p_i(x+1), \quad \forall x \in [0 : m - 1], \\
    q_{0i} \cdot d_i(x,0,i) &= \beta_i(x) \cdot q_{0i} \cdot d_i(x+1,i,0), \quad \forall x \in [0 : m - 1], \\
    d_i(0,i,0) &= 0, \\
    d_i(m,0,i) &= 0.
\end{align*}
\]
The first part of the objective equals
\[
\sum_{x=0}^{m} p_i(x) \left[ q_{i0} \cdot r_{i0} \left( d_i(x, i, 0) \right) + q_{oi} \cdot r_{oi} \left( d_i(x, 0, i) \right) \right]
\]
\[
= \sum_{x=0}^{m-1} \left\{ p_i(x) \cdot q_{oi} \cdot r_{oi} \left( d_i(x, 0, i) \right) + p_i(x+1) \cdot q_{i0} \cdot r_{i0} \left( d_i(x+1, i, 0) \right) \right\} \tag{EC-8}
\]
\[
= \sum_{x=0}^{m-1} p_i(x) \left[ q_{oi} \cdot r_{oi} \left( d_i(x, 0, i) \right) + \beta_i(x) \cdot q_{i0} \cdot r_{i0} \left( d_i(x+1, i, 0) \right) \right],
\]

where (i) is due to the constraints \(d_i(0, i, 0) = 0\) and \(d_i(m, 0, i) = 0\) and the fact that \(r_{ij}(0) = 0\).

According to (EC-8) and the constraints of (EC-7), given \(\beta_i(x)\), it is easy to solve \(d_i(x, 0, i)\) and \(d_i(x+1, i, 0)\) from the concave problem \(\gamma_i(\beta)\) in (8), which is
\[
\gamma_i(\beta) = \max_{d_{i0}, d_{oi} \in [0, 1]} q_{oi} \cdot r_{oi} (d_{oi}) + \beta \cdot q_{i0} \cdot r_{i0} (d_{i0})
\]
\[
s.t. \quad q_{oi} \cdot d_{oi} = \beta \cdot q_{i0} \cdot d_{i0}.
\]

Thus, (EC-7) is equivalent to (EC-9)
\[
h_i^\lambda = \max_{p_i(x), \beta_i(x) \geq 0} \sum_{x=0}^{m-1} p_i(x) \cdot \gamma_i \left( \beta_i(x) \right) - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x)
\]
\[
s.t. \quad \sum_{x=0}^{m} p_i(x) = 1,
\]
\[
\quad p_i(x) \cdot \beta_i(x) = p_i(x+1), \quad \forall x \in [0 : m - 1].
\]

Eliminating \(\beta_i(x)\) from (EC-9) yields (7), which is
\[
h_i^\lambda = \max_{p_i(x) \geq 0} \sum_{x=0}^{m-1} p_i(x) \cdot \gamma_i \left( \frac{p_i(x+1)}{p_i(x)} \right) - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x)
\]
\[
s.t. \quad \sum_{x=0}^{m} p_i(x) = 1,
\]

where we set \(x \cdot \gamma_i \left( \frac{y}{x} \right) = 0\) if \(x = 0\). Since the support of an optimal probability distribution in (7) is a sequence of consecutive integers starting from zero, an optimal solution to (7) can be converted into a feasible solution to (EC-9) and vice versa, thus the equivalence between (EC-9) and (7). Lemma EC1.3 shows that the function \(\gamma_i(\beta)\) is concave in \(\beta\); this implies that (7) is a convex optimization problem, as we show in Lemma EC1.1.

**Lemma EC1.1.** (7) is a convex optimization problem.

*Proof.* It suffices to show the objective of (7) is concave in \(p_i(x)\). Since \(\gamma_i(\beta)\) is concave in \(\beta\) by Lemma EC1.3 and \(x \cdot \gamma_i \left( \frac{y}{x} \right)\) is the perspective of \(\gamma_i(\beta)\), \(x \cdot \gamma_i \left( \frac{y}{x} \right)\) is jointly concave in \((x, y)\) from Section 3.2.6 of Boyd and Vandenberghe (2004). This implies that the objective of (7) is concave in \(p_i(x)\). \(\square\)

We can solve (7) efficiently with multiple methods. In Section EC1.4.3 we provide an specialized
algorithm for solving (7) through its first-order optimality conditions. We provide some useful properties for \( \gamma_i(\beta) \) as a preparation.

**Lemma EC1.2.** \( \gamma_i(\beta) \) is increasing in \( \beta \in \mathbb{R}_+ \).

**Proof.** We show that for any \( 0 \leq \beta_1 < \beta_2 \), we have \( \gamma_i(\beta_1) \leq \gamma_i(\beta_2) \). Let \( d^1_{0i} \) and \( d^1_{0i} \) be an optimal solution to \( \gamma_i(\beta_1) \). We have \( q_{0i} \cdot d^1_{0i} = \beta_1 \cdot q_{0i} \cdot d^1_{0i} \). It is easy to see that \( d^1_{0i} \) and \( \frac{\beta_1}{\beta_2} d^1_{0i} \) is feasible to \( \gamma_i(\beta_2) \). Thus,

\[
\gamma_i(\beta_2) \geq q_{0i} \cdot r_{0i}(d^1_{0i}) + \beta_2 \cdot q_{0i} \cdot r_{0i}\left(\frac{\beta_1}{\beta_2} d^1_{0i}\right)
\geq q_{0i} \cdot r_{0i}(d^1_{0i}) + \beta_2 \cdot q_{0i} \cdot \beta_1 \beta_2 \cdot r_{0i}(d^1_{0i})
= \gamma_i(\beta_1),
\]

where (i) is due to \( r_{0i}\left(\frac{\beta_1}{\beta_2} d^1_{0i}\right) \geq \beta_1 \beta_2 \cdot r_{0i}(d^1_{0i}) \) because \( r_{0i}(d) \) is concave and \( r_{0i}(0) = 0 \).

**Lemma EC1.3.** \( \gamma_i(\beta) \) is strictly concave in \( \beta \in \mathbb{R}_+ \).

**Proof.** For any \( 0 \leq \beta_1 < \beta_2 \) and \( \alpha_1, \alpha_2 \in (0, 1) \) with \( \alpha_1 + \alpha_2 = 1 \), we let \( \beta = \alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2 \) and show that \( \gamma_i(\beta) > \alpha_1 \cdot \gamma_i(\beta_1) + \alpha_2 \cdot \gamma_i(\beta_2) \).

Let \( d^1_{0i} \) and \( d^1_{0i} \) be an optimal solution to \( \gamma_i(\beta_1) \), and \( d^2_{0i} \) and \( d^2_{0i} \) an optimal solution to \( \gamma_i(\beta_2) \). Since \( q_{0i} \cdot d^1_{0i} = \beta_1 \cdot q_{0i} \cdot d^1_{0i} \) and \( q_{0i} \cdot d^2_{0i} = \beta_2 \cdot q_{0i} \cdot d^2_{0i} \), it is easy to see that \( d^1_{0i} = \alpha_1 \cdot d^1_{0i} + \alpha_2 \cdot d^2_{0i} \) and \( d^2_{0i} = (\alpha_1 \cdot \beta_1 \cdot d^1_{0i} + \alpha_2 \cdot \beta_2 \cdot d^2_{0i})/(\alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2) \) is feasible to \( \gamma_i(\beta) \). Thus,

\[
\gamma_i(\beta) \geq q_{0i} \cdot r_{0i}\left(\alpha_1 \cdot d^1_{0i} + \alpha_2 \cdot d^2_{0i}\right) + \beta \cdot q_{0i} \cdot r_{0i}\left(\frac{\alpha_1 \cdot \beta_1 \cdot d^1_{0i} + \alpha_2 \cdot \beta_2 \cdot d^2_{0i}}{\alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2}\right)
> q_{0i} \cdot \left(\alpha_1 \cdot r_{0i}(d^1_{0i}) + \alpha_2 \cdot r_{0i}(d^2_{0i})\right) + q_{0i} \cdot \left(\alpha_1 \cdot \beta_1 \cdot r_{0i}(d^1_{0i}) + \alpha_2 \cdot \beta_2 \cdot r_{0i}(d^2_{0i})\right)
= \alpha_1 \cdot \gamma_i(\beta_1) + \alpha_2 \cdot \gamma_i(\beta_2)
\]

where the second inequality is due to the strict concavity of the revenue functions and Jensen’s inequality.

For ease of exposition, in the following, we assume that \( \gamma_i(\beta) \) is differentiable in \( \beta \); otherwise, we can simply replace the derivatives of \( \gamma_i(\beta) \) with its sub-gradients in the analysis.

**Lemma EC1.4.** \( \gamma_i(\beta) \) and its derivatives are bounded from above: \( 0 = \gamma_i(0) \leq \gamma_i(\beta) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}), \)

and \( 0 \leq \gamma_i'(0) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}) \).

**Proof.** It is easy to see from (8) that \( \gamma_i(0) = 0 \). Moreover, the objective of (8) satisfies that

\[
q_{0i} \cdot r_{0i}(d_{0i}) + \beta \cdot q_{0i} \cdot r_{0i}(d_{0i}) \\
\leq q_{0i} \cdot (\bar{r} + \beta \cdot q_{0i} \cdot d_{0i} \cdot \bar{\omega}) \\
\leq q_{0i} \cdot (\bar{r} + \bar{\omega}),
\]

where the first inequality is due to the mean value theorem and the facts that \( r_{0i}(0) = 0 \) and that \( \bar{\omega} \) is the uniform bound on the derivatives of \( r_{ij}(d) \) by Assumption 1. Thus \( \gamma_i(\beta) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}) \).
Finally, note that
\[
\gamma_i(\beta) \leq q_{0i} \cdot d_{0i} \cdot \bar{\omega} + \beta \cdot q_{i0} \cdot \bar{r}
\]
\[
\leq \beta \cdot q_{i0} \cdot (\bar{r} + \bar{\omega}),
\]
where (i) is from \( r_{0i}(d_{0i}) \leq d_{0i} \cdot \bar{\omega} \) and (ii) from \( q_{0i} \cdot d_{0i} = \beta \cdot q_{i0} \cdot d_{i0} \leq \beta \cdot q_{i0} \). We have
\[
\gamma_i'(0+) = \lim_{\beta \to 0} \frac{\gamma_i(\beta) - \gamma_i(0)}{\beta} = \lim_{\beta \to 0} \frac{\gamma_i(\beta)}{\beta} \leq q_{i0} \cdot (\bar{r} + \bar{\omega}).
\]

The remaining of Lemma EC1.4 is directly from Lemmas EC1.2 and EC1.3.

**Lemma EC1.5.** Let \( z(\beta) = \beta \cdot \gamma_1'(\beta) - \gamma_2(\beta) \) be a function of \( \beta \in \mathbb{R}_+ \). \( z(\beta) \) is strictly decreasing in \( \beta \) and \( z(0) = 0 \).

**Proof.** \( z(0) = 0 \) because \( \gamma_i(0) = 0 \) and \( \gamma_i'(0) \) is bounded from Lemma EC1.4. To see that \( z(\beta) \) is strictly decreasing in \( \beta \), for any \( 0 \leq \beta_1 < \beta_2 \) we have
\[
z(\beta_1) - z(\beta_2) = \gamma_i(\beta_2) - \gamma_i(\beta_1) + \beta_1 \cdot \gamma_i'(\beta_1) - \beta_2 \cdot \gamma_i'(\beta_2)
\]
\[
= \left\{ \gamma_i(\beta_2) - \gamma_i(\beta_1) - \gamma_i'(\beta_2) (\beta_2 - \beta_1) \right\} + \beta_1 \cdot \left( \gamma_i'(\beta_1) - \gamma_i'(\beta_2) \right)
\]
\[
\geq \gamma_i(\beta_2) - \gamma_i(\beta_1) - \gamma_i'(\beta_2) (\beta_2 - \beta_1) > 0,
\]
where the first inequality is because \( \gamma_i'(\beta_1) - \gamma_i'(\beta_2) \geq 0 \) by the concavity of \( \gamma_i(\beta) \), and the second inequality is due to the first-order condition of the strictly concave function \( \gamma_i(\beta) \).

**EC1.4.3 A Specialized Algorithm to Solve (7)**

Since (7) is a convex program and all constraints are linear, strong duality holds. Let \( f(p) = \sum_{x=0}^{m-1} p_x \cdot \gamma_i \left( \frac{p_{x+1}}{p_x} \right) \), \( \lambda \cdot \sum_{x=0}^{m} x \cdot p_x \) with \( p = (p_x)_{x \in [0:m]} \in \mathbb{R}_{+}^{m+1} \) denote the objective of (7). Relax the equality constraint \( \sum_{x=0}^{m} p_i(x) = 1 \) with a dual variable \( r \in \mathbb{R} \) and let \( L(p, r) = f(p) + r \cdot (1 - \sum_{x=0}^{m} p_x) \) denote the corresponding Lagrangian function and \( r^* \) denote the Lagrange multiplier. Proposition EC1.6 provides the optimality condition for (7).

**Proposition EC1.6.** The following hold for (7).

1. The derivative of \( f \) is \( \partial f / \partial p_x = -\lambda \cdot x + \gamma_i' \left( \frac{p_{x+1}}{p_x} \right) \cdot 1 \left[ x \geq 1 \right] - z \left( \frac{p_{x+1}}{p_x} \right) \cdot 1 \left[ x \leq m - 1 \right] \);
2. \( p = (p_x)_{x \in [0:m]} \in \mathbb{R}^{m+1}_+ \) is optimal to (7) and \( r \in \mathbb{R} \) is a Lagrange multiplier if and only if (a) \( p \) is feasible to (7), and (b) \( \partial f / \partial p_x = r \) for all \( p_x > 0 \) and \( \partial f / \partial p_x \leq r \) for all \( p_x = 0 \);
3. The Lagrange multiplier \( r^* = h^*_i \) equals the optimal value of (7).

**Proof.** Part 1 can be verified directly. Part 2 is from Proposition 6.2.5 in [Bertsekas et al.] (2003).
For part 3, suppose \( p = (p_x)_{x \in [0:m]} \) is an optimal solution to (7) with support \([0:H]\). We have

\[
r^* = \sum_{x \in [0:H]} p_x \cdot r^*(i) \quad \sum_{x \in [0:H]} p_x \cdot \left( \frac{\partial f}{\partial p_x} \right)
\]

\[
= -\lambda \sum_{x \in [0:H]} p_x \cdot x + \sum_{x \in [1:H]} p_x \cdot \gamma_i \left( \frac{p_x}{p_{x-1}} \right) - H \lambda (m-1) \left( p_{x+1} - \gamma_i \right) \left( \frac{p_{x+1}}{p_x} \right) + H \lambda (m-1) \left( p_{x+1} - \gamma_i \right) \left( \frac{p_{x+1}}{p_x} \right)
\]

\[
= -\lambda \sum_{x \in [0:H]} p_x \cdot x + \sum_{x = 0}^{H \lambda (m-1)} p_x \cdot \gamma_i \left( \frac{p_x}{p_{x-1}} \right)
\]

\[
= f(p) = h_i^\lambda,
\]

where (i) is from part 2, (ii) from part 1 and the definition of \( z(\beta) = \beta \cdot \gamma'_i(\beta) - \gamma_i(\beta) \), (iii) from \( p_{H+1} = 0 \) and thus \( \sum_{x = 0}^{H \lambda (m-1)} p_{x+1} - \gamma_i \left( \frac{p_{x+1}}{p_x} \right) = \sum_{x = 1}^{H \lambda (m-1)} p_x \cdot \gamma_i \left( \frac{p_x}{p_{x-1}} \right) \), and (iv) from the fact that \( p_x = 0 \) for all \( x > H \).

Lemma EC1.7 provides a bisection method to solve the optimality condition in Proposition EC1.6 part 2 efficiently.

**Lemma EC1.7.** For any \( r \geq 0 \), let

\[
m^* = \begin{cases} 
0 & \text{if } \gamma'_{\ell}(0) \leq \lambda + r, \\
m & \text{if } \gamma'_{\ell}(0) > \lambda m + r, \\
\lceil \nu - 1 \rceil & \text{otherwise},
\end{cases}
\]

with \( \nu = \frac{\gamma'_{\ell}(0) - r}{\lambda} \) and \( \lceil x \rceil \) denoting the minimum integer that is no smaller than \( x \).

Let \( \beta_x = 0 \) for all \( x \geq m^* \). If \( m^* \geq 1 \), set \( \beta_{m^*-1} \) to be the value that satisfies

\[
\gamma'_{\ell}(\beta_{m^*-1}) = \lambda m^* + r,
\]

(10)

and set \( \beta_x \) for \( x \leq m^* - 2 \) recursively in the backward manner with

\[
\gamma'_{\ell}(\beta_x) = z(\beta_{x+1}) + r + \lambda(x + 1),
\]

(11)

where \( z(\beta) \) is defined in Lemma EC1.5. We have

1. \( \beta_x \) is decreasing in \( x \): \( \beta_0 \geq \cdots \geq \beta_{m^*-1} > 0 = \beta_{m^*} = \cdots = \beta_{m-1} \); and
2. if \( r + z(\beta_0) = 0 \), \( r = r^* \) and the probabilities \( p_i(x) \) that satisfy \( p_i(x+1) = \beta_x \cdot p_i(x) \) for all \( x \in [0:m-1] \) are optimal to (7); and
3. \( r > r^* \) if \( r + z(\beta_0) > 0 \) and \( r < r^* \) if \( r + z(\beta_0) < 0 \).

From Lemma EC1.7 parts 2 and 3, we can solve the Lagrange multiplier \( r^* = h_i^\lambda \) using a bisection method. Moreover, letting \( \beta_x^* \) be the values from (EC-10) and (EC-11) with \( r = r^* \), the

\[\text{EC1}^1\] Equivalently, in case 3, \( m^* \) is the unique integer satisfying \( \lambda m^* + r < \gamma'_{\ell}(0) \leq \lambda(m^* + 1) + r \).

\[\text{EC2}^2\] If the right-hand side value of (EC-11) is non-positive, return \( r < r^* \).
probabilities \( p_i^r(x) \) that satisfy \( p_i^r(x + 1) = \beta_x^r \cdot p_i^r(x) \) for all \( x \in [0 : m - 1] \) are optimal to (7). Since

\[
p_i^r(x) = \left( \prod_{y=0}^{x-1} \beta_y^r \right) \cdot p_i^r(0), \quad \forall x \in [m],
\]

(EC-12)

and these probabilities sum up to one, we have

\[
p_i^r(0) = \left( 1 + \sum_{x=1}^{m} \prod_{y=0}^{x-1} \beta_y^r \right)^{-1}.
\]

(EC-13)

From (EC-12) and (EC-13), we can compute \( p_i^r(x) \) for all \( x \leq m \). Finally, since the ratios \( \beta_x^r \) are decreasing in \( x \in [0 : m - 1] \) from Lemma [EC1.7] part 1, the probability distribution \( p_i^r(x) \) is discrete log-concave as defined in Definition [EC2.1]; this is the unique solution to (6) according to Proposition [EC1.6] part 1, the derivatives of the objective \( f \) are decreasing in \( x \) and these probabilities sum up to one, we have

\[
p_i^r(x) \leq p_i^r(x + r) \quad \text{for all } 0 \leq r \leq m - 1.
\]

Part 1: we prove by induction. As a base case, \( \beta_{m^* - 1} \geq \beta_{m^* - 1} \) satisfies (EC-11) because \( \beta_{m^*} = 0 \), \( z(0) = 0 \) by Lemma [EC1.5] and (EC-10). Now for any \( x \leq m^* - 2 \), we have

\[
\gamma'_i(\beta_x) \overset{(a)}{=} z(\beta_{x+1}) + r + \lambda(x + 1)
\]

\[
= z(\beta_{x+1}) + r + \lambda(x + 2) - \lambda
\]

\[
\overset{(b)}{=} z(\beta_{x+1}) - z(\beta_{x+2}) + \gamma'_i(\beta_{x+1}) - \lambda
\]

\[
\overset{(c)}{\leq} \gamma'_i(\beta_{x+1}),
\]

where (a) and (b) are because (EC-11) holds at \( x \) and \( x + 1 \), and (c) is from the facts that \( \lambda \geq 0 \), \( z(\beta) \) is decreasing in \( \beta \) by Lemma [EC1.5] and \( \beta_{x+1} \geq \beta_{x+2} \) by assumption. Since \( \gamma_i(\beta) \) is concave by Lemma [EC1.3] we have \( \beta_x \geq \beta_{x+1} \).

Part 2: this is essentially verifying the optimality condition as in Proposition [EC1.6] part 2. Note that from Proposition [EC1.6] part 1, the derivatives of the objective \( f \) only depend on the ratios \( \beta_x = \frac{p_{x+1}}{p_x} \). If \( r + z(\beta_0) = 0 \), it is easy to check that the probabilities \( p_i(x) \) with \( p_i(x+1) = \beta_x \cdot p_i(x) \) for all \( x \leq m - 1 \) and the value \( r \) satisfy the optimality condition in Proposition [EC1.6] part 2, with \( \frac{\partial f}{\partial p_x} = r \) for all \( x \leq m^* \) and \( \frac{\partial f}{\partial p_x} \leq r \) for all \( x \geq m^* + 1 \). Thus, \( p_i(x) \) are optimal to (7) and \( r = r^* \) equals the Lagrange multiplier.

Part 3: let \( \beta_x(r) \) for all \( 0 \leq x \leq m - 1 \) and \( m^*(r) \) denote the values of \( \beta_x \) and \( m^* \) with a specific \( r \). Since \( z(\beta) \) is decreasing in \( \beta \) from Lemma [EC1.5], it suffices to show \( \beta_x(r) \) is decreasing in \( r \) for all \( 0 \leq x \leq m - 1 \), which we will prove by induction. Clearly, \( m^*(r) \) decreases in \( r \). Thus, for any \( r_1 > r_2 \), \( \beta_x(r_1) \leq \beta_x(r_2) \) for all \( x \geq m^*(r_1) \). Now, for any \( x \leq m^*(r_1) - 1 \), suppose that \( \beta_{x+1}(r_1) \leq \beta_{x+1}(r_2) \). We have

\[
\gamma'_i(\beta_x(r_1)) \overset{(i)}{=} z(\beta_{x+1}(r_1)) + r_1 + \lambda(x + 1)
\]

\[
\overset{(ii)}{\geq} z(\beta_{x+1}(r_2)) + r_2 + \lambda(x + 1)
\]

\[
\overset{(iii)}{= \gamma'_i(\beta_x(r_2)),
\]

EC-9
EC1.5 Proof of Proposition 5

Let \( h^λ_1, v^λ_1(x, i, 0), v^λ_1(x, 0, i) \) and \( v^λ_1(x, \emptyset) \) be an optimal solution to (EC-2). By the complementary slackness properties in (EC-4)-(EC-6) and the fact that \( I_i \) is closed under the Lagrangian policy by Proposition EC2.2, for all resource levels \( x \in I_i = [0 : H_i] \) we have

\[
h^λ_1 + v^λ_1(x, i, 0) = \max_{d \in [0,1]} \left\{ r_{10}(d) + d \cdot \left( v^λ_1(x - 1) - v^λ_1(x) \right) \right\} + v^λ_1(x) - \lambda \cdot x,
\]

\[
h^λ_1 + v^λ_1(x, 0, i) = \max_{d \in [0,1]} \left\{ r_{0i}(d) + d \cdot \left( v^λ_1(x + 1) - v^λ_1(x) \right) \right\} + v^λ_1(x) - \lambda \cdot x,
\]

and the controls \( d_i(x, i, 0) \) and \( d_i(x, 0, i) \) of the Lagrangian policy attain the maximum in (EC-14). We can interpret the Bellman equation (EC-14) as an average revenue problem with states restricted to be in set \( I_i \). Lemma EC1.8 shows the differential value functions in (EC-14) are concave in \( x \).

**Lemma EC1.8.** The differential value functions \( v^λ_1(x, i, 0), v^λ_1(x, 0, i) \) and \( v^λ_1(x, \emptyset) \) in (EC-14) are concave in \( x \) for \( x \in I_i \).

We defer proof of Lemma EC1.8 to the end of this section. Let \( \Delta v^λ_i(x) = v^λ_1(x) - v^λ_1(x - 1) \) be the difference of the average differential values of two adjacent states. Lemma EC1.8 implies that \( \Delta v^λ_i(x) \) is decreasing in \( x \) for \( x \leq H_i \). Since the one-period revenue functions \( r_{10}(d) \) and \( r_{0i}(d) \) are strictly concave, the demand levels \( d_i(x, i, 0) \) and \( d_i(x, 0, i) \) that attain the maximum in (EC-14) are unique. Moreover, since \( d_i(x, i, 0) = \arg\max_{d \in [0,1]} \{ r_{10}(d) - d \cdot \Delta v^λ_i(x) \} \) and the objective has increasing differences in \( d \) and \( -\Delta v^λ_i(x) \), by the theory of monotone comparative statics (e.g., Topkis 1978, Milgrom and Shannon 1994, Topkis 2011), the unique optimal solution is increasing in \( x \) because \( -\Delta v^λ_i(x) \) is increasing in \( x \) for \( x \leq H_i \) and \( d_i(0, i, 0) = 0 \). Similar analysis implies the demand level \( d_i(x, 0, i) = \arg\max_{d \in [0,1]} \{ r_{0i}(d) + d \cdot \Delta v^λ_i(x + 1) \} \) is decreasing in \( x \) for \( x \in I_i \).

**Proof of Lemma EC1.8** Since the Lagrangian policy is optimal to (EC-14) and is a unichain policy by Lemma EC2.1, Proposition 5.2.4 of Bertsekas (2012) implies that the differential value functions in (EC-14) are unique up to a constant.

We now show the differential value functions are concave using a value iteration argument. Let \( v^λ_i = \{ v^λ_i(x, s) : x \in I_i, s \in \{(1, 0), (0, 1), \emptyset\} \} \) be a set of value functions for states in the problem (EC-14), and let \( T v^λ_i \) be the one-step iteration with \( v^λ_i \) being the terminal values, i.e.,

\[
Tv^λ_i(x, i, 0) = \max_{d \in [0,1]} \left\{ r_{10}(d) + d \cdot \left( v^λ_i(x - 1) - v^λ_i(x) \right) \right\} + v^λ_i(x) - \lambda \cdot x,
\]

\[
Tv^λ_i(x, 0, i) = \max_{d \in [0,1]} \left\{ r_{0i}(d) + d \cdot \left( v^λ_i(x + 1) - v^λ_i(x) \right) \right\} + v^λ_i(x) - \lambda \cdot x,
\]

\[
Tv^λ_i(x, \emptyset) = v^λ_i(x) - \lambda \cdot x,
\]

for all \( x \in I_i \), with \( v^λ_i(x) = q_0 \cdot v^λ_i(x, i, 0) + q_{0i} \cdot v^λ_i(x, 0, i) + (1 - q_i) \cdot v^λ_i(x, \emptyset) \) being the average terminal values over request types. Lemma EC1.9 shows that the map \( T \) preserves concavity.
**Lemma EC1.9.** If the value function \( v_i^\lambda = \{ v_i^\lambda(x, s) : x \in I_i, s \in \{(1, 0), (0, 1), \varnothing\} \} \) is concave in \( x \), the one-step iteration \( T v_i^\lambda \) is concave in \( x \) as well.

**Proof.** The proof is standard (e.g., Proposition 5.2 of Talluri and Van Ryzin 2006) but we include it for completeness. Let \( \Delta v_i^\lambda(x) = v_i^\lambda(x) - v_i^\lambda(x - 1) \) for \( x \leq H_i \) be the difference of average values of two adjacent states. By assumption \( \Delta v_i^\lambda(x) \) is decreasing in \( x \). Let

\[
\Phi := \left\{ T v_i^\lambda(x + 2, i, 0) - T v_i^\lambda(x + 1, i, 0) \right\} - \left\{ T v_i^\lambda(x + 1, i, 0) - T v_i^\lambda(x, i, 0) \right\}.
\]

We need to show that \( \Phi \leq 0 \). Let \( d_i(x, i, 0) \) and \( d_i(x, 0, i) \) be the demand levels that attain the maximum in (EC-15). From (EC-15), for any \( x \geq 0 \) we have

\[
\Phi = \Delta v_i^\lambda(x + 2) - \Delta v_i^\lambda(x + 1) + \left\{ r_{i0} \left( d_i(x + 2, i, 0) \right) - d_i(x + 2, i, 0) \cdot \Delta v_i^\lambda(x + 2) \right\} - \left\{ r_{i0} \left( d_i(x + 1, i, 0) \right) - d_i(x + 1, i, 0) \cdot \Delta v_i^\lambda(x + 1) \right\} - \left\{ r_{i0} \left( d_i(x + 1, i, 0) \right) - d_i(x + 1, i, 0) \cdot \Delta v_i^\lambda(x + 1) \right\} + \left\{ r_{i0} \left( d_i(x, i, 0) \right) - d_i(x, i, 0) \cdot \Delta v_i^\lambda(x) \right\}.
\]

Since \( d_i(x + 1, i, 0) \) attains the maximum in (EC-15),

\[
r_{i0} \left( d_i(x + 1, i, 0) \right) - d_i(x + 1, i, 0) \cdot \Delta v_i^\lambda(x + 1) \geq r_{i0} \left( d_i(x, i, 0) \right) - d_i(x, i, 0) \cdot \Delta v_i^\lambda(x + 1)
\]

and

\[
r_{i0} \left( d_i(x + 1, i, 0) \right) - d_i(x + 1, i, 0) \cdot \Delta v_i^\lambda(x + 1) \geq r_{i0} \left( d_i(x + 2, i, 0) \right) - d_i(x + 2, i, 0) \cdot \Delta v_i^\lambda(x + 1)\]

Thus,

\[
\Phi \leq \Delta v_i^\lambda(x + 2) - \Delta v_i^\lambda(x + 1) + \left\{ r_{i0} \left( d_i(x + 2, i, 0) \right) - d_i(x + 2, i, 0) \cdot \Delta v_i^\lambda(x + 2) \right\} - \left\{ r_{i0} \left( d_i(x + 2, i, 0) \right) - d_i(x + 2, i, 0) \cdot \Delta v_i^\lambda(x + 1) \right\} - \left\{ r_{i0} \left( d_i(x, i, 0) \right) - d_i(x, i, 0) \cdot \Delta v_i^\lambda(x + 1) \right\} + \left\{ r_{i0} \left( d_i(x, i, 0) \right) - d_i(x, i, 0) \cdot \Delta v_i^\lambda(x) \right\} = \left( 1 - d_i(x + 2, i, 0) \right) \left( \Delta v_i^\lambda(x + 2) - \Delta v_i^\lambda(x + 1) \right) + d_i(x, i, 0) \left( \Delta v_i^\lambda(x + 1) - \Delta v_i^\lambda(x) \right) \leq 0,
\]

where the last inequality follows because \( \Delta v_i^\lambda(x) \) is decreasing in \( x \) and demand levels are between...
zero and one. This implies $Tv^\lambda_i(x, i, 0)$ is concave in $x$. The same analysis on $Tv^\lambda_i(x, 0, i)$ implies $Tv^\lambda_i(x, 0, i)$ is concave in $x$ as well. Finally, $Tv^\lambda_i(x, \emptyset)$ is concave in $x$ because $Tv^\lambda_i(x, \emptyset) = v^\lambda(x) - \lambda \cdot x$ and $v^\lambda(x)$ is concave in $x$.

Lemma EC1.9 and the convergence of value iteration in the proof of Proposition 5.6.2 in Bertsekas (2012) imply that the differential value functions in (EC-14) are concave in $x$.

**EC1.6 Proof of Proposition 6**

According to Danskin’s Theorem (Proposition 4.5.1 in Bertsekas et al. 2003), the fact that the optimal probability distribution to (6) is unique (see Proposition EC2.8), and Proposition 4.2.4 in Bertsekas et al. (2003), the sub-differential of $\bar{V}\lambda$ at any $\lambda \geq 0$ is a singleton

$$\partial \bar{V}\lambda = \left\{ m - \sum_{i=1}^{n} \sum_{x=0}^{m} x \cdot p_i(x): p_i(x) \text{ is optimal to (6) with } \lambda \right\}. \quad (EC-16)$$

Thus by standard optimality conditions for convex optimization (Proposition 4.7.2 in Bertsekas et al. 2003), the dual variable $\lambda^*$ is an optimal solution to (9) if and only if

$$\sum_{i=1}^{n} \sum_{x=0}^{m} x \cdot p_i(x) \leq m,$$

$$\lambda^* \geq 0,$$

$$\lambda^* \cdot \left( m - \sum_{i=1}^{n} \sum_{x=0}^{m} x \cdot p_i(x) \right) = 0,$$

$$p_i(x) \text{ is an optimal solution to (6) with } \lambda^*,$$

which we can equivalently write as \[10].

**EC1.7 Proof of Lemma 1**

First, we show that $\lambda^*(\delta) \leq \bar{r}/(m - \delta)$ if $\delta < m$. Since $\lambda^*(\delta)$ is an optimal solution to (11), $V^\mu(\delta) = (m - \delta) \cdot \lambda^*(\delta) + \sum_{i=1}^{n} h_i^\lambda(\delta)$. It then suffices to show $V^\mu(\delta) \leq \bar{r}$ and $h_i^\lambda \geq 0$ for all spokes $i \in [n]$ and dual variables $\lambda \geq 0$.

First, the optimality condition of (11) implies that (11) is equivalent to the problem of maximizing the average revenue subject to the constraint that the hub has at least $\delta$ resources in expectation. Since $\bar{r}$ is the uniform bound on the one-period revenue functions, $V^\mu(\delta) \leq \bar{r}$. Second, $h_i^\lambda$ is equal to the optimal value of (6) by Proposition 4. Let $p_i(0) = 1$, $p_i(x) = 0$ for all $x \geq 1$, $d_i(x, 0, i) = 0$ for all $x$, and $d_i(x, i, 0) = 1$ for all $x \geq 1$. This provides a feasible solution to (6) with an objective value of zero, thus $h_i^\lambda \geq 0$.

Combining the fact that $\lambda^*(\delta) \leq \bar{r}/(m - \delta)$ with (12) leads to the result.

**EC1.8 Proof of Lemma 2**

We prove the result by first showing that the value function of the Lagrangian policy in the relaxed system approximately solves the Bellman equation of the original system along the path induced by the Lagrangian policy in the original system. We then use a verification theorem to bound the total loss between these two systems. Finally, we extend to infinite horizon settings using a value iteration argument.
Step 1 (approximate Bellman equation). Let \( v_{i,t}(x,i,0) \), \( v_{i,t}(x,0,i) \) and \( v_{i,t}(x,\emptyset) \) denote the value functions of the Lagrangian policy in each spoke \( i \) problem, with \( x \) resources and \( t \) time periods ahead, and the request type being \((i,0),(0,i)\), or one of any other types, respectively. Let

\[
v_{i,t}(x,q_i) = q_{0i} \cdot v_{i,t}(x,i,0) + q_{0i} \cdot v_{i,t}(x,0,i) + (1 - q_i) \cdot v_{i,t}(x,\emptyset)
\]

be the average value functions over request types. The Bellman equation for each spoke problem with the Lagrangian policy is

\[
\begin{align*}
v_{i,t}(x,i,0) &= r_{i0}(d_i(x,i,0)) + d_i(x,i,0) \cdot \left( v_{i,t-1}(x-1) - v_{i,t-1}(x) \right) + v_{i,t-1}(x), \\
v_{i,t}(x,0,i) &= r_{0i}(d_i(x,0,i)) + d_i(x,0,i) \cdot \left( v_{i,t-1}(x+1) - v_{i,t-1}(x) \right) + v_{i,t-1}(x), \quad \text{(EC-17)} \\
v_{i,t}(x,\emptyset) &= v_{i,t-1}(x),
\end{align*}
\]

for all \( x \in [0 : m] \), where \( d_i(x,0,0) \) and \( d_i(x,0,i) \) are the demand values of the Lagrangian policy. Let \( \Delta v_{i,t}(x) = v_{i,t}(x) - v_{i,t}(x-1) \) for \( x \in [m] \) be the difference of the continuation values of two adjacent states. Lemma \[EC1.10\] shows that \( \Delta v_{i,t}(x) \leq \omega \) are uniformly bounded from above by the derivative bound \( \bar{\omega} \) as defined in Assumption 1.

**Lemma EC1.10.** The difference of the continuation values \( \Delta v_{i,t}(x) \) satisfies \( \Delta v_{i,t}(x) \leq \bar{\omega} \) for all spokes \( i \), time periods \( t \), and resource levels \( x \in [m] \), where \( \bar{\omega} \) is the uniform bound on the derivatives of the one-period revenue functions as defined in Assumption 1.

We prove Lemma \[EC1.10\] at the end of this section. Let \( V^R_t(x,s) \) and \( V^R_t(x,s) \) denote the continuation values of the Lagrangian policy in the relaxed and the original systems, with \( x = (x_i)_{i \in [n]} \) being the state of resources and \( s \) being the request type. Let \( V^R_t(x) = \mathbb{E}_s[V^R_t(x,s)] \) and \( V^R_t(x) = \mathbb{E}_s[V^R_t(x,s)] \) denote the average values over request types. The boundary conditions of the two systems are \( V^R_0(x,s) = V_0(x,s) = 0 \). Because we relax the capacity constraint of the hub in the relaxed system, \( V^R_t(x) \) decouples over spokes with

\[
V^R_t(x) = \sum_{i=1}^{n} v_{i,t}(x_i), \quad \text{(EC-18)}
\]

where \( v_{i,t}(x) \) are the average value functions of spoke \( i \) as in \[EC-17\].

The Lagrangian policy takes different actions in the two systems at the same state \((x,s)\) only when \( x_0 = 0, s = (0,i) \), and \( x_i \leq m - 1 \) for some spoke \( i \in [n] \); let

\[
\mathcal{A} = \left\{ (x,s) : x_0 = 0, s = (0,i), x_i \leq m - 1, i \in [n] \right\}
\]

be the set of states in which the policy can take different actions. Let \( R(x,s) \) be the expected one-period revenue in the original system at state \((x,s)\) in \( \mathcal{X} \times \{(i,0),(0,i) : i \in [n]\} \), i.e.,

\[
R(x,i,0) = r_{i0}(d_i(x,i,0)), \\
R(x,0,i) = r_{0i}(d_i(x,0,i)) \cdot \left( 1 - \mathbb{1}[(x,0,i) \in \mathcal{A}] \right). \quad \text{(EC-19)}
\]

Moreover, let

\[
\bar{R}(x,s) = V^R_t(x,s) - R(x,s) - \mathbb{E}\left[V^R_{t-1}(\tilde{x},\tilde{s}) \mid x,s\right], \quad \text{(EC-20)}
\]

where the expectation is taken with respect the random variable \((\tilde{x},\tilde{s})\), which is the next state in the original system under the Lagrangian policy when the current state is \((x,s)\). This value can be interpreted as the ex-ante compensation that needs to be given to the provider in the relaxed
system at state \((x, s)\) in order for her to be willing to switch from the current action to the action in the original system.

Fix a state \((x, s)\) of the original system. We can write the value function of the relaxed system as follows:

\[
V_t^R(x, s) = \mathbb{1}[(x, s) \not\in \mathcal{A}] \cdot V_t^R(x, s) + \mathbb{1}[(x, s) \in \mathcal{A}] \cdot V_t^R(x, s)
\]

\[
= \mathbb{1}[(x, s) \not\in \mathcal{A}] \cdot \left( R(x, s) + \mathbb{E}\left[V_{t-1}^R(\tilde{x}, \tilde{s})\right](x, s) \right)
\]

\[
+ \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \left( R(x, s) + \mathbb{E}\left[V_{t-1}^R(\tilde{x}, \tilde{s})\right](x, s) + \tilde{R}_t(x, s) \right)
\]

\[
= R(x, s) + \mathbb{E}\left[V_{t-1}^R(\tilde{x}, \tilde{s})\right](x, s) + \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \tilde{R}_t(x, s),
\]

where the second equation follows from the Bellman equation for the relaxed system together with the fact that the evolution in the relaxed and original system coincide for all states not in \(\mathcal{A}\) and using \([\text{EC-20}]\). We proceed by bounding the terms \(\varepsilon_t\). Let \(z(s)\) be the index of the spoke involved in type \(s\) and let \(n(x, s)\) be the resource level of spoke \(z(s)\) when the state of resources is \(x\). We have

\[
\varepsilon_t = \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \tilde{R}_t(x, s)
\]

\[
\overset{(i)}{=} \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \left( V_t^R(x, s) - \mathbb{E}\left[V_{t-1}^R(\tilde{x}, \tilde{s})\right](x, s) \right)
\]

\[
\overset{(ii)}{=} \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \left( V_t^R(x, s) - V_{t-1}^R(x) \right)
\]

\[
\overset{(iii)}{=} \mathbb{1}[(x, s) \in \mathcal{A}] \cdot \left[ r_s \left( d_{z(s)}(n(x, s), s) \right) + d_z(s) \left( n(x, s), s \right) \cdot \Delta v_{z(s), t-1} \left( n(x, s) + 1 \right) \right]
\]

\[
\overset{(iv)}{\leq} \mathbb{1}[(x, s) \in \mathcal{A}] \cdot (\bar{r} + \bar{\omega})
\]

where (i) follows from \([\text{EC-20}]\) and the fact that \(R(x, s) = 0\) when \((x, s) \in \mathcal{A}\) by \([\text{EC-19}]\). (ii) from \(\tilde{x} = x\) when \((x, s) \in \mathcal{A}\), (iii) from the Bellman equation of \(V_t^R(x, s)\), the fact that the value function decomposes over spokes by \([\text{EC-18}]\) and the transition only involves \(s = (0, i)\), (iv) from the definition of \(\bar{r}\) in Assumption 1 and Lemma \([\text{EC1.10}]\). Putting everything together, we obtain that the value function for the relaxed system satisfies the following approximate Bellman equation in the original system

\[
V_t^R(x, s) \leq R(x, s) + \mathbb{E}\left[V_{t-1}^R(\tilde{x}, \tilde{s})\right](x, s) + \mathbb{1}[(x, s) \in \mathcal{A}] \cdot (\bar{r} + \bar{\omega}).
\]

**Step 2 (verification).** Let \(\{x_t, s_t\}_{t \leq t}\) be the path of states of the original system with the Lagrangian policy. By taking expectations over the states \(\{x_t, s_t\}_{t \leq t}\) and using the boundary condi-
tion on the value function we have

\[ V_{t}^{R}(x_{t}, s_{t}) = \mathbb{E}\left[ \sum_{\tau=1}^{t} V_{\tau}^{R}(x_{\tau}, s_{\tau}) - V_{\tau-1}^{R}(x_{\tau-1}, s_{\tau-1}) \right] \]

\[ = (i) \mathbb{E}\left[ \sum_{\tau=1}^{t} V_{\tau}^{R}(x_{\tau}, s_{\tau}) \right] - \mathbb{E}\left[ V_{\tau-1}^{R}(x_{\tau}, s_{\tau}) \right] \]

\[ \leq (ii) \mathbb{E}\left[ \sum_{\tau=1}^{t} R(x_{\tau}, s_{\tau}) \right] + \left( \bar{r} + \bar{\omega} \right) \sum_{\tau=1}^{t} \mathbb{P}[x_{\tau}, s_{\tau}] \in A] \] \hspace{1cm} (EC-22)

\[ = (iii) V_{t}(x_{t}, s_{t}) + \left( \bar{r} + \bar{\omega} \right) \sum_{\tau=1}^{t} \mathbb{P}[x_{\tau}, s_{\tau}] \in A] \]

\[ \leq (iv) V_{t}(x_{t}, s_{t}) + \left( \bar{r} + \bar{\omega} \right) \sum_{\tau=1}^{t} \mathbb{P}[x_{0, \tau} = 0], \]

where the (i) follows by the tower rule for conditional expectations and using the fact that the dynamics are Markovian, (ii) follows from (EC-21) over \( \tau \in [t] \) together with linearity of expectations, (iii) because \( R(x_{\tau}, s_{\tau}) \) is the expected one-period revenue in the original system at state \( (x_{\tau}, s_{\tau}) \), and (iv) because \( A \subseteq \{(x, s) : x_{0} = 0\} \).

**Step 3 (value iteration).** Taking an average over \( t \) time periods and letting \( t \) go to infinity gives

\[ V^{R}(\delta) = (a) \lim_{t \to \infty} \frac{1}{t} V_{t}^{R}(x_{t}, s_{t}) \]

\[ \leq (b) \lim_{t \to \infty} \frac{1}{t} \left( V_{t}(x_{t}, s_{t}) + \left( \bar{r} + \bar{\omega} \right) \sum_{\tau=1}^{t} \mathbb{P}[x_{0, \tau} = 0] \right) \]

\[ = (c) V^{\pi}(\delta) + \left( \bar{r} + \bar{\omega} \right) \mathbb{P}[X_{0}(\delta) = 0], \]

where (a) is due to the fact that the long-run time average of the total revenue converges to the average revenue of the policy by a value iteration argument (see Proposition 5.3.1 in Bertsekas 2012), (b) is from (EC-22), and (c) from the same value iteration argument and the fact that the time-average limiting distribution converges to the stationary distribution because the Markov chain has a single recurrent class by Corollary EC2.11.

**Proof of Lemma EC1.10.** According to Proposition 5, the demand values of the Lagrangian policy are monotone in the resource levels of the spokes: for each spoke \( i \) and for all \( x \in [m] \), we have \( 0 \leq d_{i}(x, 0, i) \leq d_{i}(x-1, 0, i) \leq 1 \) and \( 0 \leq d_{i}(x-1, i, 0) \leq d_{i}(x, i, 0) \leq 1 \). By coupling the private value of the arriving request when the number of resources is \( x-1 \) and \( x \) respectively, we can write
the difference of the continuation values $\Delta v_{i,t}(x)$ in the following recursive way:

$$
\Delta v_{i,t}(x) = q_{i0} \left\{ (1 - d_i(x, 1, 0, i)) \cdot \Delta v_{i,t-1}(x) \\
+ \left( d_i(x, 1, 0, i) - d_i(x, 0, i) \right) \cdot \left( -G_{0i}(d_i(x - 1, 0, i)) \right) \\
+ d_i(x, 0, i) \cdot \left( -G_{0i}(d_i(x - 1, 0, i)) + G_{0i}(d_i(x, 0, i)) + \Delta v_{i,t-1}(x + 1) \right) \right\} \\
+ q_{i0} \left\{ (1 - d_i(x, i, 0)) \cdot \Delta v_{i,t-1}(x) \\
+ \left( d_i(x, i, 0) - d_i(x - 1, i, 0) \right) \cdot G_{i0}(d_i(x, i, 0)) \\
+ d_i(x - 1, i, 0) \cdot \left( -G_{0i}(d_i(x - 1, i, 0)) + G_{0i}(d_i(x, i, 0)) + \Delta v_{i,t-1}(x - 1) \right) \right\} \\
+ \left( 1 - q_{i} \right) \cdot \Delta v_{i,t-1}(x) \\
= q_{i0} \left( r_{0i}(d_i(x, 0, i)) - r_{0i}(d_i(x - 1, 0, i)) + d_i(x, 0, i) \cdot \Delta v_{i,t-1}(x + 1) \right) \\
+ q_{i0} \left( r_{i0}(d_i(x, i, 0)) - r_{i0}(d_i(x - 1, i, 0)) + d_i(x - 1, i, 0) \cdot \Delta v_{i,t-1}(x - 1) \right) \\
+ \left( 1 - q_{i0} \cdot d_i(x - 1, 0, i) - q_{i0} \cdot d_i(x, i, 0) \right) \Delta v_{i,t-1}(x),
$$

(EC-23)

with boundary conditions $\Delta v_{i,0}(x) = 0$ for all $x$.

We prove by induction. Clearly this is true for $t = 0$ by the boundary conditions that $\Delta v_{i,0}(x) = 0$. Now suppose $\Delta v_{i,t-1}(x) \leq \bar{\omega}$ for all spokes $i$ and resource levels $x$. We show that $\Delta v_{i,t}(x) \leq \bar{\omega}$. From (EC-23) we have

$$
\Delta v_{i,t}(x) \leq q_{i0} \left( r_{0i}(d_i(x, 0, i)) - r_{0i}(d_i(x - 1, 0, i)) \right) + q_{0i} \cdot d_i(x, 0, i) \cdot \bar{\omega} \\
+ q_{i0} \left( r_{i0}(d_i(x, i, 0)) - r_{i0}(d_i(x - 1, i, 0)) \right) + q_{i0} \cdot d_i(x - 1, i, 0) \cdot \bar{\omega} \\
+ \left( 1 - q_{i0} \cdot d_i(x - 1, 0, i) - q_{i0} \cdot d_i(x, i, 0) \right) \cdot \bar{\omega}.
$$

To show $\Delta v_{i,t}(x) \leq \bar{\omega}$, it suffices to show that

$$
q_{i0} \left( r_{0i}(d_i(x, 0, i)) - r_{0i}(d_i(x - 1, 0, i)) \right) + q_{i0} \left( r_{i0}(d_i(x, i, 0)) - r_{i0}(d_i(x - 1, i, 0)) \right) \\
\leq \left\{ q_{i0} \left( d_i(x - 1, 0, i) - d_i(x, 0, i) \right) + q_{i0} \left( d_i(x, i, 0) - d_i(x - 1, i, 0) \right) \right\} \cdot \bar{\omega}.
$$
This is true because the left-hand side satisfies that
\[
q_0 \left( r_{0i}(d_i(x,0,i)) - r_{0i}(d_i(x-1,0,i)) \right) + q_0 \left( r_{i0}(d_i(x,i,0)) - r_{i0}(d_i(x-1,i,0)) \right) \\
\leq q_0 \left| r_{0i}(d_i(x,0,i)) - r_{0i}(d_i(x-1,0,i)) \right| + q_0 \left| r_{i0}(d_i(x,i,0)) - r_{i0}(d_i(x-1,i,0)) \right| \\
\leq \left\{ q_0 \left( d_i(x-1,0,i) - d_i(x,0,i) \right) + q_0 \left( d_i(x,i,0) - d_i(x-1,i,0) \right) \right\} \cdot \bar{\omega},
\]
where the last inequality is due to the mean value theorem, the monotonicity property – i.e., \( d_i(x,0,i) \leq d_i(x-1,0,i) \) and \( d_i(x-1,i,0) \leq d_i(x,i,0) \) – and the fact that \( \bar{\omega} \) is the uniform bound on the derivatives of the one-period revenue functions as in Assumption 1.

EC1.9 Proof of Lemma 3

Let \( \hat{x}_{i,t} \) and \( x_{i,t} \) denote the number of resources in locations \( i \in [0:n] \) at time \( t \) in the relaxed and original systems, respectively. Lemma EC1.11 shows that if the two systems start at the same state and have the same sequence of requests and private values, \( \hat{x}_{i,t} \geq x_{i,t} \) for all spokes \( i \in [n] \) and time periods \( t \).

**Lemma EC1.11.** If the relaxed and original systems start at the same state and have the same sequence of requests and private values, for any time period \( t \), \( \hat{x}_{i,t} \geq x_{i,t} \) for all spokes \( i \in [n] \) and \( \hat{x}_{0,t} \leq x_{0,t} \).

**Proof.** We prove by induction. Since the two systems start at the same state, \( \hat{x}_{i,0} = x_{i,0} \) for all \( i \in [n] \). For each spoke \( i \), first suppose \( \hat{x}_{i,t-1} = x_{i,t-1} \). If request \( (i,0) \) arrives at time \( t \), the Lagrangian policy takes the same action in the two systems, hence \( \hat{x}_{i,t} = x_{i,t} \). If request \( (0,i) \) arrives, the Lagrangian policy takes different actions in the two systems only when the hub of the original system runs out of resources, in which case we have \( \hat{x}_{i,t} \geq \hat{x}_{i,t-1} = x_{i,t-1} = x_{i,t} \). Next suppose \( \hat{x}_{i,t-1} \geq x_{i,t-1} + 1 \) for spoke \( i \). If request \( (i,0) \) arrives at time \( t \), we have \( \hat{x}_{i,t} \geq \hat{x}_{i,t-1} - 1 \geq x_{i,t-1} \). If request \( (0,i) \) arrives, \( \hat{x}_{i,t} \geq \hat{x}_{i,t-1} \geq x_{i,t-1} + 1 \geq x_{i,t} \). Thus by induction, \( \hat{x}_{i,t} \geq x_{i,t} \) for all spokes \( i \in [n] \) and time periods \( t \). Finally, since \( \sum_{i=0}^{n} x_{i,t} = m \) and \( \sum_{i=0}^{n} \hat{x}_{i,t} = m \), we have \( \hat{x}_{0,t} \leq x_{0,t} \) for all time periods \( t \). \( \square \)

Lemma EC1.11 implies that for any integer \( k \),
\[
P \left[ \hat{X}_i(\delta) \leq k \right] = \lim_{t \to \infty} P \left[ \hat{x}_{i,t} \leq k \right] \leq \lim_{t \to \infty} P \left[ x_{i,t} \leq k \right] = P \left[ X_i(\delta) \leq k \right],
\]
and
\[
P \left[ X_0(\delta) \leq k \right] = \lim_{t \to \infty} P \left[ x_{0,t} \leq k \right] \leq \lim_{t \to \infty} P \left[ \hat{x}_{0,t} \leq k \right] = P \left[ \hat{X}_0(\delta) \leq k \right],
\]
where the equations are because in both systems, the limiting distribution of the Markov chain converges to the unique stationary distribution, independently of the initial state, due to Corollaries EC2.10 and EC2.11. The inequalities follow from Lemma EC1.11 when we start the two systems with the same state and couple the sequence of requests and their private values. Note that in the proof we only use the fact that the Lagrangian policy only depends on the state of resources through the resource level of the spoke involved in the request type.
EC1.10 Proof of Proposition 7

Proposition 7 comes from Lemma EC1.12, which provide a concentration inequality for a sequence of independent random variables with discrete log-concave distributions (defined in Definition EC2.1) and uniformly bounded means.

Lemma EC1.12. Let \( \{X_i\}_{i=1}^{n} \) be a sequence of independent discrete log-concave random variables each with mean value \( \mu_i = \mathbb{E}[X_i] \). If \( \mu_i \leq c \) for all \( i \leq n \) are uniformly bounded from above by some constant \( c > 0 \), then for any \( \lambda \geq 1 \) and letting \( X = \sum_{i=1}^{n} X_i \) and \( \mu = \mathbb{E}[X] = \sum_{i=1}^{n} \mu_i \), we have

\[
P[X \geq \lambda \mu] \leq \exp \left\{ - \frac{(\lambda - 1)\mu}{1 + c} - \frac{n + \mu}{1 + c} \ln \left( 1 - \frac{\lambda \mu - \mu}{\lambda \mu + n} \right) \right\}.
\] (EC-24)

We prove Lemma EC1.12 at the end of this section. We can apply Lemma EC1.12 to \( \tilde{X}_i(\delta) \) for \( i \in [n] \) because each \( \tilde{X}_i(\delta) \) is log-concave by Proposition EC2.9, and \( \tilde{X}_i(\delta) \) are independent because the joint distribution is equal to the product of their marginal distributions by Corollary EC2.10.

Let \( \mu = \mathbb{E}\left[\sum_{i=1}^{n} \tilde{X}_i(\delta)\right] \) be the expected number of resources in the spokes of the relaxed system. We have \( 0 < \mu \leq m - \delta \). Applying (EC-24) with \( \lambda = \frac{m}{\mu} \) and \( b = \frac{1}{1+c} \) gives

\[
P[\tilde{X}_0(\delta) \leq 0] = P\left[\sum_{i=1}^{n} \tilde{X}_i(\delta) \geq m\right]
\leq \exp \left\{ - b \cdot \left( \lambda \mu - \mu + (n + \mu) \cdot \ln \left( 1 - \frac{\lambda \mu - \mu}{\lambda \mu + n} \right) \right) \right\}
= \exp \left\{ - b \cdot \left( m - \mu + (n + \mu) \cdot \ln \left( \frac{n + \mu}{m + n} \right) \right) \right\}
= \exp \left\{ b \cdot \left( (n + \mu) \cdot \ln \left( \frac{m + n}{n + \mu} \right) - (m - \mu) \right) \right\}.
\] (EC-25)

Since \( \ln x \leq \frac{x-1}{\sqrt{x}} \) for \( x \geq 1 \), we have

\[
\blacklozenge \leq (n+\mu) \cdot \frac{m-\mu}{n+\mu} \cdot \sqrt{\frac{n+\mu}{m+n}} - (m-\mu) = (m-\mu) \cdot \left( \sqrt{\frac{1-m-\mu}{m+n}} - 1 \right) \leq - \frac{(m-\mu)^2}{2 \cdot (m+n)} \leq - \frac{\delta^2}{2 \cdot (m+n)},
\]

where the second-to-last inequality is due to \( \sqrt{1-x} - 1 \leq -\frac{x}{2} \) for \( x \leq 1 \). Thus from (EC-25) we have

\[
P[\tilde{X}_0(\delta) \leq 0] \leq \exp \left( - \frac{b}{2} \cdot \frac{\delta^2}{m+n} \right).
\]

Proof of Lemma EC1.12. We first provide an upper bound on the probability generating function of a log-concave random variable in Lemma EC1.14. The proof is based on the classic inequality bounding the factorial moments of a log-concave random variable, as we state in Lemma EC1.13.

Definition EC1.1 (Factorial Moment). Let \( p = \{p_i\}_{i=0}^{\infty} \) be a discrete distribution with all its support
on non-negative integers. The factorial moment of \( p \) of order \( r \geq 1 \) is
\[
\mu_{[r]} = \sum_{i=0}^{\infty} p_i \cdot \left\{ i \cdot (i-1) \cdots (i-r+1) \right\} = \sum_{i=r}^{\infty} p_i \cdot \left\{ i \cdot (i-1) \cdots (i-r+1) \right\} = \sum_{i=r}^{\infty} p_i \cdot \frac{i!}{(i-r)!}.
\]
We set \( \mu_{[0]} = 1 \) for convenience.

**Lemma EC1.13** (Theorem 2 in [Keilson 1972]). Let \( p = \{p_i\}_{i=0}^{\infty} \) be a discrete log-concave distribution and let \( \mu_{[r]} \) denote its order-\( r \) factorial moment. For any \( r \geq 1 \) we have
\[
\left\{ \frac{\mu_{[r+1]}}{(r+1)!} \right\}^{1/(r+1)} \leq \left\{ \frac{\mu_{[r]}}{r!} \right\}^{1/r} \leq \cdots \leq \frac{\mu_{[1]}}{1!} = \mu,
\]
where \( \mu \) denotes the mean value of \( p \). All inequalities in (EC-26) hold with equalities when \( p \) is a geometric distribution, i.e., when \( p_i = \theta(1-\theta)^i \) for some \( 0 < \theta \leq 1 \).

**Lemma EC1.14.** Let \( X \) be a discrete log-concave random variable (defined in Definition [EC2.1]) and let \( \mu = \mathbb{E}[X] \) denote its mean value. We have \( \mathbb{E}[z^X] \leq \frac{1}{1 - \mu(z-1)} \) for all \( 1 \leq z < 1 + \frac{1}{\mu} \).

**Proof.** First, we have
\[
\mathbb{E}[z^X] = \sum_{i=0}^{\infty} p_i \cdot z^i = \sum_{i=0}^{\infty} p_i \sum_{j=0}^{i} \binom{i}{j} (z-1)^j = \sum_{j=0}^{\infty} \frac{1}{j!} \cdot (z-1)^j \sum_{i=j}^{\infty} p_i \cdot \frac{i!}{(i-j)!} = \sum_{j=0}^{\infty} (z-1)^j \cdot \frac{\mu^j}{j!},
\]
where (i) is due to the binomial expansion that \( z^i = [(z-1)+1]^i = \sum_{j=0}^{i} \binom{i}{j} \cdot (z-1)^j \) for all \( i \geq 0 \) and (ii) follows from switching the order of summations by Tonelli’s theorem because all terms are non-negative. (EC-26) then implies that
\[
\mathbb{E}[z^X] \leq \sum_{j=0}^{\infty} \mu^j (z-1)^j = \frac{1}{1 - \mu(z-1)}.
\]

We now prove Lemma [EC1.12]. The proof follows Theorem 2.1 in [Janson 2018], which provides a concentration inequality for summations of independent geometric random variables using Chernoff inequality. Their results can be easily extended to random variables with log-concave distributions, as we present it here.

From Lemma [EC1.14] the moment generating function of each random variable \( X_i \) can be bounded from above by
\[
\mathbb{E}[e^{tX_i}] \leq \frac{1}{1 - \mu_i(e^t - 1)} = \frac{e^{-t}}{(1 + \mu_i)e^{-t} - \mu_i}, \quad 0 \leq t < \ln \left( 1 + \frac{1}{\mu_i} \right).
\]
Since for all \( 0 \leq t < \frac{1}{1+\mu_i} \leq \ln \left( 1 + \frac{1}{\mu_i} \right) \), the denominator satisfies
\[
(1 + \mu_i)e^{-t} - \mu_i \geq (1 + \mu_i) \cdot (1 - t) - \mu_i = 1 - (1 + \mu_i)t > 0,
\]
we have
\[
\mathbb{E}[e^{tX_i}] \leq \frac{e^{-t}}{1 - (1 + \mu_i)t}, \quad 0 \leq t < \frac{1}{1 + \mu_i}.
\]
As a result, for all $0 \leq t < \frac{1}{1+c} \leq \min_{i} \frac{1}{1+r_{i}}$, we have

$$\mathbb{E}[e^{tX}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] \leq e^{-nt} \prod_{i=1}^{n} \left(1 - (1 + \mu_i)t\right)^{-1}$$

because the random variables are independent. By the Chernoff inequality, for all $0 \leq t < \frac{1}{1+c}$,

$$\mathbb{P}[X \geq \lambda \mu] \leq e^{-t\lambda \mu} \mathbb{E}[e^{tX}] \leq \exp\left(-t\lambda \mu - tn - \sum_{i=1}^{n} \ln\left(1 - (1 + \mu_i)t\right)\right)$$

$$\leq \exp\left(-t\lambda \mu - tn - \sum_{i=1}^{n} \frac{1 + \mu_i}{1 + c} \ln\left(1 - (1 + c)t\right)\right)$$

$$\leq \exp\left(-t\lambda \mu - \frac{n + \mu}{1 + c} \ln\left(1 - (1 + c)t\right)\right),$$

where (a) is due to the fact that the function $-\ln(1-x)$ is convex on $(0,1)$ and is 0 at $x = 0$, thus by Jensen’s inequality,

$$-\ln(1-x) \leq -\frac{x}{y} \ln(1-y), \ \forall 0 \leq x \leq y < 1.$$  

By choosing $t = \frac{(\lambda-1)\mu}{(1+c)(\lambda\mu+n)}$ (which is optimal here), we obtain (EC-24).

**EC1.11 Proof of Lemma 4**

We prove a more general result stated in Lemma EC1.15 that under some regularity conditions on the function $\gamma_i(\beta)$ as defined in [8]. Assumption 2 holds. We then show Lemma 4 are sufficient for the assumptions on $\gamma_i(\beta)$ in Lemma EC1.15 to hold.

**Lemma EC1.15.** Suppose that function $\gamma_i(\beta)$ is differentiable, and on $\beta \in [0, 1]$, is strongly concave with parameter $\ell_i > 0$ and has Lipschitz continuous gradient with parameter $L_i > 0$. Further assume that $n\ell_i \geq \bar{\ell}$, $nL_i \leq \bar{L}$, and $q_{0i}, q_{0i} \leq \frac{d}{n}$ for all $i \in [n]$ and some positive constants $\bar{\ell}$, $\bar{L}$ and $\bar{q}$. Then $\lambda^*(\delta) \geq \frac{\lambda}{n}$ for all $\delta \geq 0$ and some constants $\lambda > 0$ and Assumption 2 holds.

We prove Lemma EC1.15 in Appendix EC1.11.1; an overview of the key steps of the proof is as follows. Letting $p_i(x)$ be the optimal probabilities to [6], we can lower bound the ratio $\beta(x) = p_i(x+1)/p_i(x)$ through the first-order optimality conditions of [7] and show that these ratios are close to one for a large number of states $x$ if the dual variable $\lambda$ is sufficiently small. This implies that the expected number of resources at every spoke grows unbounded as $\lambda$ goes to zero. Since the total number of resources at the spokes with $\lambda = \lambda^*(\delta)$ is no larger than $m - \delta$, $\lambda^*(\delta)$ cannot be too small, which in turn implies that the spoke resources are uniformly bounded.

We now show Lemma 4 provides sufficient conditions for the assumptions on $\gamma_i(\beta)$ in Lemma EC1.15 to hold based on the primitives of the problem, thus finished the proof.

Let $d_{\bar{a}}(\beta)$ denote an optimal solution to $\gamma_i(\beta)$. We can set $d_{\bar{a}}(0)$ arbitrarily because it does not affect the objective when $\beta = 0$. When $\beta > 0$, we can express $d_{\bar{a}_i}$ in terms of $d_{\bar{a}}$ and rewrite [8] as

$$\gamma_i(\beta) = \max_{d_{\bar{a}} \in [0,1]} q_{0i} \cdot r_{0i} \left(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d_{\bar{a}} \right) + \beta \cdot q_{0i} \cdot r_{\bar{a}}(d_{\bar{a}}).$$  

(EC-27)
Since the revenue functions \( r_{0i}(d) \) and \( r_{0i}(d) \) are strictly concave, \( d_{i0}(\beta) \) is unique for any \( \beta > 0 \). Let

\[
f(\beta, d) = q_{0i} \cdot r_{0i}(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d) + \beta \cdot q_{i0} \cdot r_{i0}(d)
\]

be the objective of (EC-27). \( f(\beta, d) \) is concave in \( d \) with partial derivative

\[
\frac{\partial f(\beta, d)}{\partial d} = \beta \cdot q_{i0} \cdot \left( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d) + r_{i0}'(d) \right)
\]

decreasing in \( d \). Since \( \frac{\partial f(\beta, d)}{\partial d} |_{d=0} = \beta \cdot q_{i0} \cdot (r_{i0}'(0) + r_{i0}'(0)) > 0 \), the optimal solution satisfies \( d_{i0}(\beta) > 0 \) when \( \beta > 0 \). \( d_{i0}(\beta) \) may equal to the right end-point \( \min \{1, \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \} \). In the following, we study \( \gamma_i(\beta) \) depending on whether the optimal solution is interior or one of the upper boundaries is binding.

**Case 1** We have \( d_{i0}(\beta) = 1 \) if \( 1 \leq \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \) and \( \frac{\partial f(\beta, d)}{\partial d} |_{d=1} \geq 0 \), which is equivalent to \( \beta \leq \frac{q_{0i}}{q_{0i}} \) and \( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}}) + r_{i0}'(1) \geq 0 \). Since \( r_{i0}(d) \) and \( r_{0i}(d) \) are strictly concave, there exists some \( \beta > 0 \) such that \( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}}) + r_{i0}'(1) = 0 \) if and only if \( r_{i0}'(0)+r_{i0}'(1) > 0 \). If it is the case, since \( r_{i0}'(1)+r_{i0}'(1) < 0 \), we must have \( \beta < \frac{q_{0i}}{q_{0i}} \). Moreover, \( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}}) + r_{i0}'(1) \geq 0 \) for all \( \beta \leq \beta \), and thus \( d_{i0}(\beta) = 1 \) when \( 0 < \beta \leq \beta \). If \( r_{i0}'(0)+r_{i0}'(1) \leq 0 \), we set \( \beta = 0 \). When \( \beta \in [0, \beta] \), since \( d_{i0}(\beta) = 1 \), we have

\[
\gamma_i(\beta) = f(\beta, 1) = q_{0i} \cdot r_{0i}(\beta \cdot \frac{q_{0i}}{q_{0i}}) + \beta \cdot q_{i0} \cdot r_{i0}(1),
\]

which is concave and differentiable in \( \beta \). The derivative

\[
\gamma_i'(\beta) = q_{0i} \cdot \left( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}}) + r_{i0}'(1) \right)
\]

is continuous on \([0, \beta]\) because \( r_{i0}(d) \) and \( r_{0i}(d) \) are twice differentiable. This implies that if \( \beta > 0 \),

\[
\gamma_i'(\beta) = q_{0i} \cdot \left( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}}) + r_{i0}'(1) \right).
\]

(EC-28)

The second-order derivative satisfies

\[
- \gamma_i''(\beta) = -\frac{q_{0i}}{q_{0i}} \cdot r_{i0}''(\beta) \in \left[ \frac{q^2 u}{q n}, \frac{q^2 U}{q n} \right], \quad \forall \beta \in [0, \beta].
\]

(EC-29)

**Case 2** \( d_{i0}(\beta) = \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \) if \( \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \leq 1 \) and \( \frac{\partial f(\beta, d)}{\partial d} |_{d=\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}} \geq 0 \), which is equivalent to \( \beta \geq \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \) and \( r_{i0}'(1)+r_{i0}'\left(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}\right) \geq 0 \). Since \( r_{i0}(d) \) and \( r_{0i}(d) \) are strictly concave, there exists some \( \beta > 0 \) such that \( r_{i0}'(1)+r_{i0}'\left(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}\right) = 0 \) if and only if \( r_{i0}'(1)+r_{i0}'(0) > 0 \). In this case, \( \beta \) has to be larger than \( \frac{q_{0i}}{q_{0i}} \) and \( r_{i0}'(1)+r_{i0}'\left(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}\right) \geq 0 \) for all \( \beta \geq \beta \). Thus, \( d_{i0}(\beta) = \frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta} \) when \( \beta \geq \beta \). If \( r_{i0}'(1)+r_{i0}'(0) \leq 0 \),
we set $\bar{\beta} = \infty$. When $\beta \in [\bar{\beta}, \infty)$, since $d_{i^0}(\beta) = \frac{q_{i^0}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}$, we have

$$\gamma_i(\beta) = f(\beta, d_{i^0}(\beta)) = q_{0i} \cdot r_{i^0}(1) + \beta \cdot q_{i0} \cdot r_{i0}'(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}).$$

Hence,

$$\gamma'_i(\beta) = q_{i0} \cdot r_{i0}'(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}) - \frac{1}{\bar{\beta}} \cdot q_{0i} \cdot r_{i0}'(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}),$$

and

$$\gamma''_i(\beta) = \frac{1}{\bar{\beta}^3} \cdot \frac{q_{i0}^2}{q_{0i}} \cdot r_{i0}''(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}) < 0.$$

The first-order derivative is continuous on $[\bar{\beta}, \infty)$ because $r_{i0}(d)$ and $r_{0i}(d)$ are twice differentiable. Thus if $\beta < \infty$,

$$\gamma'_i(\beta+) = q_{i0} \cdot r_{i0}'(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}) - \frac{1}{\beta} \cdot q_{0i} \cdot r_{i0}'(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}) = q_{i0} \cdot \left( r_{i0}(d_{i^0}(\beta)) - d_{i^0}(\beta) \cdot r_{i0}'(d_{i^0}(\beta)) \right). \quad (EC-30)$$

If $\beta < 1$, since $\bar{\beta} \geq \frac{q_{i0}}{q_{0i}}$, the second-order derivative satisfies

$$- \gamma''_i(\beta) = -\frac{1}{\bar{\beta}^3} \cdot \frac{q_{i0}^2}{q_{0i}} \cdot r_{i0}''(\frac{q_{0i}}{q_{0i}} \cdot \frac{1}{\beta}) \in \left[ \frac{q^2 \bar{\alpha}}{q \cdot n}, \frac{q^2 \bar{\alpha}}{q \cdot n} \right], \forall \beta \in [\bar{\beta}, 1]. \quad (EC-31)$$

**Case 3** When $\beta \in [\bar{\beta}, \bar{\beta})$, $d_{i^0}(\beta)$ satisfies

$$\frac{\partial f(\beta, d)}{\partial d} \bigg|_{d=d_{i^0}(\beta)} = \beta \cdot q_{0i} \cdot \left( r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d_{i^0}(\beta)) + r_{i0}'(d_{i^0}(\beta)) \right) = 0.$$

Thus,

$$r_{i0}'(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d_{i^0}(\beta)) + r_{i0}'(d_{i^0}(\beta)) = 0. \quad (EC-32)$$

From (EC-32) and the implicit function theorem, we have

$$\frac{d_{i^0}(\beta)}{\partial \beta} = -\frac{q_{i0} \cdot d_{i^0}(\beta) \cdot r_{i0}''(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d_{i^0}(\beta))}{q_{i0} \cdot \beta \cdot r_{0i}''(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d_{i^0}(\beta)) + q_{0i} \cdot r_{0i}''(d_{i^0}(\beta))} \leq 0, \quad (EC-33)$$

thus $d_{i^0}(\beta)$ is decreasing in $\beta$.

Let us first assume $\bar{\beta} < \infty$ and consider the function $f(\beta, d)$ over $[\bar{\beta}, \bar{\beta}) \times [0, 1]$; since $d$ in $f(\beta, d)$ is restricted to be in $[0, 1 \wedge \frac{q_{i0}}{q_{0i}} \cdot \frac{1}{\bar{\beta}}]$, we extend $r_{0i}(d)$ smoothly over $[1, \infty]$ making sure $f(\beta, d)$ is well-defined on the support $[\bar{\beta}, \bar{\beta}) \times [0, 1]$. Note that in the extension we ensure that $r_{0i}(d)$ is strongly concave and has a Lipschitz continuous gradient with the same parameters. Since the partial derivative of $f(\beta, d)$ with respect to $\beta$ is

$$\frac{\partial f(\beta, d)}{\partial \beta} = q_{i0} \cdot \left( d \cdot r_{0i}'(\beta \cdot \frac{q_{0i}}{q_{0i}} \cdot d) + r_{i0}(d) \right),$$

which is continuous in $(\beta, d)$, by the envelope theorem, especially Corollary 4 in Milgrom and Segal.
differentiable in any bounded interval \([0, \bar{\beta}]\), where (i) is due to the first-order condition \((EC-32)\). From \((EC-33)\) and \((EC-34)\), the second-order derivative is
\[
\gamma''(\beta) = \frac{q_0^2 \cdot (d_{l0}(\beta))^2 \cdot r_{l0}''(d_{l0}(\beta)) \cdot r_{l0}''(\beta \cdot \frac{q_0}{\bar{q}_0} \cdot d_{l0}(\beta))}{q_{l0} \cdot \beta \cdot r_{l0}''(\beta \cdot \frac{q_0}{\bar{q}_0} \cdot d_{l0}(\beta)) + q_{l0} \cdot r_{l0}''(d_{l0}(\beta))} \leq 0, \forall \beta \leq \bar{\beta},
\]
thus \(\gamma(\beta)\) is concave on \([\beta, \bar{\beta}]\). If \(\bar{\beta} > 0\), from Corollary 4 in Milgrom and Segal \((2002)\), we have
\[
\gamma_{l+}(\beta) = \frac{\partial f(\beta, d)}{\partial \beta} \bigg|_{d=d_{l0}(\beta) = 1} = q_{l0} \cdot \left( r_{l0}(\beta \cdot \frac{q_0}{\bar{q}_0}) + r_{l0}(1) \right) = \gamma'(\beta),
\]
where (a) is due to \((EC-28)\); hence \(\gamma'(\beta)\) is differentiable at \(\beta = \bar{\beta}\). Analogously, if \(\bar{\beta} < \infty\), again from Corollary 4 in Milgrom and Segal \((2002)\), we have
\[
\gamma_{l-}(\beta) = \frac{\partial f(\beta, d)}{\partial \beta} \bigg|_{d=d_{l0}(\beta)} = q_{l0} \cdot \left( r_{l0}(d_{l0}(\beta)) - d_{l0}(\beta) \cdot r_{l0}'(d_{l0}(\beta)) \right) = \gamma'(\beta),
\]
where (b) is from \((EC-34)\) and (c) is from \((EC-30)\); thus \(\gamma'(\beta)\) is differentiable at \(\beta = \bar{\beta}\) as well. Combining three segments together, we know \(\gamma(\beta)\) is differentiable everywhere. Note that we assume \(\bar{\beta} < \infty\) in case 3. If this is not the case, following the same argument, we can show \(\gamma(\beta)\) is differentiable in any bounded interval \([0, M]\) with \(M > 0\); thus again, \(\gamma_{l}(\beta)\) is differentiable on \(\mathbb{R}_+\).

Finally, from \((EC-32)\) and the mean value theorem we have
\[
r_{l0}'(0) + r_{l0}''(\varepsilon_1) \cdot \beta \cdot \frac{q_0}{\bar{q}_0} \cdot d_{l0}(\beta) + r_{l0}'(0) + r_{l0}''(\varepsilon_2) \cdot d_{l0}(\beta) = 0
\]
for some \(\varepsilon_1 \in [0, \beta \cdot \frac{q_0}{\bar{q}_0}, d_{l0}(\beta)]\) and \(\varepsilon_2 \in [0, d_{l0}(\beta)]\). This implies that
\[
d_{l0}(\beta) = \frac{-r_{l0}'(0) + r_{l0}''(0)}{r_{l0}''(\varepsilon_1) \cdot \beta \cdot \frac{q_0}{\bar{q}_0} + r_{l0}''(\varepsilon_2)} \geq \frac{r_{l0}'(0) + r_{l0}''(0)}{U} \cdot \frac{q}{\beta \cdot q + q} \geq \frac{\bar{U} \cdot \frac{q}{\beta \cdot q + q}}{U}.
\]

where in the last inequality we use the facts that \(r_{ij}(1) \geq 0, r_{ij}(0) = 0\), and \(r_{ij}(1) \leq r_{ij}(0) + r_{ij}'(0) \cdot (1 - 0) - \frac{(1-0)^2}{2} \cdot \bar{U} \) by strong concavity of \(r_{ij}''(d)\). Thus if \(\bar{\beta} < 1\), on the interval \([\beta, 1 \land \bar{\beta}]\), \((EC-35)\) and \((EC-36)\) implies that the second-order derivative satisfies
\[
-\gamma''(\beta) \in \left[ \frac{1}{2n} \frac{\bar{U}^3}{\bar{U}^2 \bar{q} (q+q)^2} \cdot \frac{q^2 U}{q} \cdot n, \forall \beta \in [\beta, 1 \land \bar{\beta}] \right]
\]
From \((EC-29)\), \((EC-31)\), \((EC-37)\), the monotonicity of \(\gamma_{l}(\beta)\), and the mean value theorem on \(\gamma_{l}(\beta)\), \(\gamma_{l}(\beta)\) is \(\ell_{ij}\)-strongly concave and has a \(L_{ij}\)-Lipschitz continuous gradient on \(\beta \in [0, 1]\) with some
Proof. Part 1 is because both \( g \) and \( \ell \) are small. Lemmas EC1.17 and EC1.18 serve as preliminary results. Since \( L > 0 \), the objective \( \bar{\rho}_i \) is strictly decreasing and the inverse \( \gamma \) is unique by Proposition EC2.8, the objective \( \bar{\rho}_i \) is strictly decreasing. Moreover, from Lemma EC1.3, \( \bar{\rho}_i \) is differentiable. Finally, by assumption, \( \beta \in [0, 1] \) and \( \gamma(\beta) \) is strongly concave with a constant \( \ell > 0 \), i.e.,

\[
\gamma(\beta') \leq \gamma(\beta) + \gamma'(\beta) \cdot (\beta' - \beta) - \frac{\ell}{2} \cdot (\beta' - \beta)^2, \forall \beta, \beta' \in [0, 1],
\]

and has a Lipschitz continuous gradient with a constant \( L > 0 \), i.e.,

\[
|\gamma'(\beta) - \gamma'(\beta')| \leq L \cdot |\beta - \beta'|, \forall \beta, \beta' \in [0, 1].
\]

EC1.11.1 Proof of Lemma EC1.15

We let \( \mathbb{E}[^X_i(q)] \) be the expected number of resources in the spoke problem with \( \lambda \geq 0 \). Lemma EC1.15 shows that \( \mathbb{E}[^X_i(q)] \) can be arbitrarily large by choosing a sufficiently small \( \lambda \).

Lemma EC1.16. Suppose the assumptions on \( \gamma_i(\beta) \) and \( q_{ij} \) in Lemma EC1.15 hold. Then for any \( \rho > 0 \), there exists a constant \( c(\rho) \) such that \( \mathbb{E}[^X_i(q)] \geq \rho \) for \( \lambda = \frac{c(\rho)}{m} \), all spokes \( i \in [n] \) and \( m \) large enough.

We prove Lemma EC1.16 in Appendix EC1.11.2. Since the perturbed problem (11) is a convex program in \( \lambda \) and the optimal probability distribution to each spoke problem with a given \( \lambda \) is unique by Proposition EC2.8, the objective \( \mathbb{V} - \delta \lambda \) is differentiable in \( \lambda \) with derivative \( \frac{\partial (\mathbb{V} - \delta \lambda)}{\partial \lambda} = (m - \delta) - \sum_{i \in [n]} \mathbb{E}[^X_i(q)] \). From Lemma EC1.16, there exists a constant \( \bar{\lambda} \) such that \( \mathbb{E}[^X_i(q)] \geq \frac{2m}{n} \) for \( \lambda = \frac{\bar{\lambda}}{n} \) and all \( i \in [n] \). Thus, the derivative at \( \lambda = \frac{\bar{\lambda}}{n} \) is negative and as a result \( \lambda^*(\delta) \geq \frac{2}{n} \). Finally, since \( 0 \leq \lambda_i^*(\delta) \leq (q_{ii} + q_{ii}) \cdot \bar{r} - \lambda^*(\delta) \cdot \mathbb{E}[^X_i(q)] \), we have \( \mathbb{E}[^X_i(q)] \leq \bar{r} \cdot \frac{2q}{n} \) for all spokes \( i \in [n] \).

EC1.11.2 Proof of Lemma EC1.16

For ease of notation, in the proof, we consider a spoke \( i \) and drop the subscript \( i \) for \( \gamma_i(\beta) \) by letting \( \gamma(\beta) \triangleq \gamma_i(\beta) \); we drop the subscript \( i \) for \( \ell \) and \( L \) as well. By assumption, \( \gamma(\beta) \) is differentiable. Moreover, from Lemma EC1.3, \( \gamma(\beta) \) is strictly concave; thus, the derivative \( \gamma'(\beta) \) is strictly decreasing and the inverse \( \gamma'^{-1}(e) \) exists and strictly decreases as well. Finally, by assumption, on \( \beta \in [0, 1] \) \( \gamma(\beta) \) is strongly concave with a constant \( \ell > 0 \), i.e.,

\[
\gamma(\beta') \leq \gamma(\beta) + \gamma'(\beta) \cdot (\beta' - \beta) - \frac{\ell}{2} \cdot (\beta' - \beta)^2, \forall \beta, \beta' \in [0, 1],
\]

and has a Lipschitz continuous gradient with a constant \( L > 0 \), i.e.,

\[
|\gamma'(\beta) - \gamma'(\beta')| \leq L \cdot |\beta - \beta'|, \forall \beta, \beta' \in [0, 1].
\]

EC1.17. Let \( g(\beta, y) = (\gamma')^{-1}(\beta \cdot \gamma(0) - \gamma(\beta)) \) with \( \lambda > 0 \) and \( \gamma(\beta) = \beta \cdot \gamma'(\beta) - \gamma(\beta) \) as in Lemma EC1.15. Let \( \bar{y} = \frac{\gamma(0) - \gamma(1)}{\bar{y}} \) and \( \beta^*_y = \inf \{ \beta \geq 0 : g(\beta, y) \leq \beta \} \).

1. \( g(\beta, y) \) is strictly increasing in \( \beta \) and \( y \);
2. \( \beta^*_y = g(\beta^*_y, 0) = 0 \), \( \bar{y} > 0 \) and \( \beta^*_y = g(\beta^*_y, \bar{y}) = 1 \);
3. for \( y \in [0, \bar{y}] \), \( 0 \leq \beta^*_y \leq 1 \), \( g(\beta^*_y, y) = \beta^*_y \), and \( \beta^*_y \) is increasing in \( y \);
4. for \( \beta \in [0, \beta^*_y] \), \( g(\beta, y) \geq \alpha_y \beta + (1 - \alpha_y) \beta^*_y \) with \( \alpha_y = 1 - \frac{\ell}{L} \) and \( \beta^*_y \).

Proof. Part 1 is because both \( (\gamma')^{-1}(e) \) and \( \gamma(\beta) \) are strictly decreasing and \( \lambda > 0 \). For part 2, since \( g(0, 0) = 0 \), we have \( \beta^*_0 = g(\beta^*_0, 0) = 0 \). Moreover, since \( \gamma(0) = 0 \) and \( \gamma(1) = \gamma(0) + \gamma'(0) \cdot (1 - 0) \) by
strict concavity of $\gamma(\beta)$, $\gamma'(0) > \gamma(1)$ and hence $\bar{y} > 0$. Finally, let $\tilde{g}(\beta) \triangleq g(\beta, \bar{y}) = (\gamma')^{-1}(\gamma(1) + z(\beta))$. We show $\tilde{g}(1) = 1$ and $\tilde{g}(\beta) > \beta$ for all $\beta \in \mathbb{R}_+$ and $\beta \neq 1$; thus $\beta_y^* = g(\beta_y^*, \bar{y}) = 1$. To see this, first, note that

$$
\tilde{g}(1) = (\gamma')^{-1}(\gamma(1) + z(1)) = (\gamma')^{-1}(\gamma'(1)) = 1.
$$

Second, since $\gamma(\beta)$ is strictly concave, for any $\beta \neq 1$ we have

$$
\gamma(1) < \gamma(\beta) + \gamma'(\beta) \cdot (1 - \beta) = \gamma'(\beta) - z(\beta).
$$

Thus, $\gamma'(\tilde{g}(\beta)) = \gamma(1) + z(\beta) < \gamma'(\beta)$. Since $\gamma'(\beta)$ is strictly decreasing, we have $\tilde{g}(\beta) > \beta$.

Part 3: we already proved this for $y = 0$ and $y = \bar{y}$ in part 2. We now show that for any $y \in (0, \bar{y})$, $0 \leq \beta_y^* \leq 1$ and $g(\beta_y^*, y) = \beta_y^*$. First, $g(0, y) > g(0, 0) = 0$ and $g(1, y) < g(1, \bar{y}) = 1$ from part 1. Second, note that $g(\beta, y)$ is jointly continuous in $(\beta, y)$ for $\beta \in [0, 1]$ and $y \in [0, \bar{y}]$. To see this, since $\gamma(\beta)$ has Lipschitz continuous gradient on $\beta \in [0, 1]$, $\gamma'(\beta)$ is continuous on $[0, 1]$ and $(\gamma')^{-1}(\beta)$ is continuous on $[\gamma'(1), \gamma'(0)]$. The continuity of $g(\beta, y)$ then follows from the definition of $z(\beta)$ and the fact that $g(\beta, y) \in [0, 1]$ when $\beta \in [0, 1]$ and $y \in [0, \bar{y}]$. Now let $h(\beta) = g(\beta, y) - \beta$. From above we know $h(0) > 0$, $h(1) < 0$, and $h(\beta)$ is continuous on $\beta \in [0, 1]$. The intermediate value theorem then implies the existence of a point $\beta_y^* \in [0, 1]$ that satisfies $g(\beta_y^*, \bar{y}) = \beta_y^*$. Finally, the monotonicity of $\beta_y^*$ follows from the fact that $g(\beta, y)$ is increasing in $y$.

Part 4: from $\beta_y^* = g(\beta_y^*, y)$, we have $\gamma'(\beta_y^*) = z(\beta_y^*) + \gamma'(0) - \lambda y$. Thus,

$$
g(\beta, y) = (\gamma')^{-1}(z(\beta) + \gamma'(0) - \lambda y) = (\gamma')^{-1}(z(\beta) - z(\beta_y^*) + \gamma'(\beta_y^*)) - \gamma'(\beta_y^*).$$

Since $\gamma'(\beta)$ is strictly decreasing in $\beta$, $g(\beta, y) \geq \alpha_y \beta + (1 - \alpha_y) \beta_y^*$ if and only if

$$z(\beta) - z(\beta_y^*) + \gamma'(\beta_y^*) \leq \gamma'(\alpha_y \beta + (1 - \alpha_y) \beta_y^*).$$

Letting $h(\beta) = \gamma'(\beta) - z(\beta)$, above is equivalent to

$$\gamma'(\beta) - \gamma'(\alpha_y \beta + (1 - \alpha_y) \beta_y^*) \leq h(\beta) - h(\beta_y^*).$$

(E-40)

Since $\gamma(\beta)$ has Lipschitz continuous gradient on $[0, 1]$ with $L > 0$, the left-hand side is no larger than $L(1 - \alpha_y)(\beta_y^* - \beta)$. On the other hand, the right-hand side satisfies

$$h(\beta) - h(\beta_y^*) = (1 - \beta)\gamma'(\beta) - (1 - \beta_y^*)\gamma'(\beta_y^*) + \gamma(\beta) - \gamma(\beta_y^*) \geq (1 - \beta)\gamma'(\beta) - (1 - \beta_y^*)\gamma'(\beta_y^*) + \gamma'(\beta - \beta_y^*) = (1 - \beta_y^*)(\gamma'(\beta) - \gamma'(\beta_y^*)) \geq \ell(1 - \beta_y^*)(\beta_y^* - \beta),$$

where the first inequality is from $\gamma(\beta) + \gamma'(\beta)(\beta_y^* - \beta) \geq \gamma(\beta_y^*)$ because $\gamma(\beta)$ is concave, and the second inequality is from the $\ell$-strong concavity of $\gamma(\beta)$ on $[0, 1]$. Finally, (E-40) holds because $\ell(1 - \beta_y^*)(\beta_y^* - \beta) = L(1 - \alpha_y)(\beta_y^* - \beta)$ by the choice of $\alpha_y$.

In Lemma ECI.18 we provide lower and upper bounds on $\beta_y^*$ for $y \in [0, \bar{y}]$.

**Lemma ECI.17.** $\beta_y^*$ defined in Lemma ECI.17 satisfies the following for $y \in [0, \bar{y}]$.

1. $1 - \sqrt{\frac{2\lambda}{T} (\bar{y} - y)} \leq \beta_y^* \leq 1 - \sqrt{\frac{2\lambda}{T} (\bar{y} - \bar{y})}$;
2. $1 - \sqrt{1 - \frac{2\lambda}{T} y} \leq \beta_y^* \leq 1 - \sqrt{1 - \frac{2\lambda}{T} \bar{y}}$, where the second inequality holds for $y \leq \frac{1}{2\lambda}$.

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Proof. We first prove inequality \((\text{EC-41})\) as a preparation.

\[
\frac{\lambda}{L} \cdot \frac{1}{1 - \beta_y^*} \leq \frac{d\beta_y^*}{dy} \leq \frac{\lambda}{\ell} \cdot \frac{1}{1 - \beta_y^*}, \quad \forall \ 0 \leq \beta_y^* \leq 1. \tag{EC-41}
\]

To see this, note that \(\gamma'(\beta_y^*) = z(\beta_y^*) + \gamma'(0) - \lambda y\) from \(\beta_y^* = g(\beta_y^*, y)\) in Lemma \(\text{EC1.17}\) part 2; thus letting \(h(\beta) = \gamma'(\beta) - z(\beta)\), we have \(h(\beta_y^*) = \gamma'(0) - \lambda y\). For any \(\beta, \beta + \Delta \beta \in [0, 1]\),

\[
h(\beta + \Delta \beta) - h(\beta) = \gamma'(\beta + \Delta \beta) \cdot (1 - \beta - \Delta \beta) + \gamma(\beta + \Delta \beta) - \gamma(\beta) - \gamma'(\beta) \cdot (1 - \beta) - \gamma(\beta) \]

\[
= \gamma'(\beta + \Delta \beta) \cdot (1 - \beta - \Delta \beta) - \gamma'(\beta) \cdot (1 - \beta) + \gamma'(\beta + \epsilon \Delta \beta) \cdot \Delta \beta
\]

\[
= \left(\gamma'(\beta + \Delta \beta) - \gamma'(\beta)\right) \cdot (1 - \beta) + \Delta \beta \cdot \left(\gamma'(\beta + \epsilon \Delta \beta) - \gamma'(\beta + \Delta \beta)\right),
\]

\(\tag{EC-42}\)

where the second equation is because \(\gamma(\beta + \epsilon \Delta \beta) = \gamma(\beta) + \gamma'(\beta + \epsilon \Delta \beta) \cdot \Delta \beta\) for some \(\epsilon \in [0, 1]\) by the mean value theorem. For any \(y, y + \Delta y \in [0, \bar{y}]\), let \(\Delta \beta_y^* = \beta_y^* + \Delta \beta - \beta_y^*\); \(\Delta \beta_y^* > 0\) if and only if \(\Delta y > 0\) because \(\beta_y^*\) is increasing in \(y\) by Lemma \(\text{EC1.17}\) part 3. Since \(h(\beta_y^*) = \gamma'(0) - \lambda y\) for all \(y \in [0, \bar{y}]\), we have

\[
\lambda \Delta y = \lambda(y + \Delta y) - \lambda y = h(\beta_y^*) - h(\beta_y^* + \Delta \beta_y^*) = h(\beta_y^*) - h(\beta_y^* + \Delta \beta_y^*). \tag{EC-43}
\]

Combining \(\text{(EC-42)}\) and \(\text{(EC-43)}\) with the fact that \(\gamma(\beta)\) is \(\ell\)-strongly concave and has \(L\)-Lipschitz continuous gradient on \([0, 1]\) gives

\[
\frac{\ell}{\lambda} \cdot \left(1 - \beta_y^*\right) + O(\Delta \beta_y^*) \leq \frac{\Delta y}{\beta_y^*} \leq \frac{L}{\lambda} \cdot \left(1 - \beta_y^*\right) + O(\Delta \beta_y^*).
\]

Letting \(\Delta \beta_y^*\) go to zero and rearranging gives \(\text{(EC-41)}\).

Part 1: since \(\beta_y^* \leq 1\), from \(\text{(EC-41)}\) we have \((1 - \beta_y^*) \cdot d\beta_y^* \leq \frac{\lambda}{\ell} \cdot dy\). Since \(\beta_y^*\) is increasing in \(y\), integrating both sides over \([y, \bar{y}]\) gives

\[
\int_{\beta_y^*}^{\bar{y}} (1 - \beta_y^*) \cdot d\beta_y^* = \frac{(1 - \beta_y^*)^2}{2} \leq \int_y^{\bar{y}} \frac{\lambda}{\ell} \cdot dy = \frac{\lambda}{\ell} \cdot (\bar{y} - y).
\]

Rearranging gives the lower bound. A similar analysis applied to the first inequality in \(\text{EC-41}\) yields the upper bound.

Part 2: proof is analogous to part 1 by integrating over \([0, y]\). \hfill \Box

Now we are ready to provide lower bounds on the ratios of the optimal probabilities to \([6]\) in Lemma \(\text{EC1.19}\).

**Lemma EC1.19.** Let \(\tilde{\beta}_1 = 0\) and consider \(\tilde{\beta}_{y+1} = g(\beta_y, y)\). Let \(\beta_x = \frac{p_i(x)}{p_i(x+1)}\) for \(x \in [0 : m^* - 1]\) be the ratio of the optimal probabilities to the spoke problem \([6]\), with \(m^*\) being the end point of the support of \(p_i(x)\) as defined in Lemma \(\text{EC1.7}\). Then,

1. \(\tilde{\beta}_y\) is increasing in \(y\);
2. \(\tilde{\beta}_{y+1} \leq \beta_y^*\) for all integers \(y \leq \bar{y}\);
3. \(\tilde{\beta}_y \leq \beta_{m^* - y}\) for all \(y \in [1 : m^*]\) when \(m \geq \frac{\gamma'(0)}{\lambda}\);

EC-26
4. \( m^* \geq \bar{y} - 1 \) when \( m \geq \frac{\gamma(0)}{\lambda} \).

**Proof.** We prove parts 1 to 3 by induction. For part 1, as a base case, let \( \bar{\beta}_1 = 0 \); it is easy to see that \( \bar{\beta}_1 = g(\bar{\beta}_0, 0) = 0 \) satisfies the iteration. By induction, \( \bar{\beta}_y = g(\bar{\beta}_{y-1}, y - 1) = \bar{\beta}_y \) because \( g(\beta, y) \) is increasing in both \( \beta \) and \( y \).

Part 2: as a base case, we have \( \bar{\beta}_1 = 0 \leq \beta^*_0 = 0 \). Now suppose \( \bar{\beta}_y \leq \beta^*_y \). Since \( \beta^*_y \) increases in \( y \) by Lemma EC1.17, part 3, \( \bar{\beta}_y \leq \beta^*_y \) and thus \( \bar{\beta}_y + 1 = g(\bar{\beta}_y, y) \leq g(\beta^*_y, y) = \beta^*_y \) by monotonicity of \( g(\beta, y) \).

Part 3: when \( m \geq \frac{\gamma(0)}{\lambda} \), \( m^* < m \) and is the unique integer satisfying:

\[
\lambda m^* + h^\lambda_i < \gamma'(0) \leq \lambda (m^* + 1) + h^\lambda_i.
\]  

(EC-44)

From the first-order optimality conditions as in Lemma EC1.7, we have:

\[
\gamma'(\beta_{m^* - 1}) = \lambda m^* + h^\lambda_i, \\
\gamma'(\beta_{x - 1}) = z(\beta_x) + \lambda x + h^\lambda_i, \quad \forall \, x \leq m^* - 1.
\]

As a base case, we have \( \beta_{m^* - 1} = (\gamma')^{-1}(\lambda m^* + h^\lambda_i) \geq (\gamma')^{-1}(\gamma'(0)) = 0 = \bar{\beta}_1 \), where the inequality follows from the facts that \( \gamma'(\beta) \) decreases in \( \beta \) and \( \lambda m^* + h^\lambda_i \leq \gamma'(0) \) by \( \text{EC-44} \). In the induction step, suppose \( \beta_{m^* - 1} \geq \beta_y \).

Then

\[
\beta_{m^* - 1} = (\gamma')^{-1}(z(\beta_{m^* - 1}) + h^\lambda_i + \lambda m^* - \gamma(0) - \lambda y) \geq (\gamma')^{-1}(z(\beta_y) + \gamma'(0) - \lambda y) = \bar{\beta}_{y+1},
\]

where the inequality follows from the facts that \( \gamma'(\beta) \) decreases in \( \beta \), \( \lambda m^* + h^\lambda_i \leq \gamma'(0) \) by \( \text{EC-44} \), \( z(\beta) \) decreases in \( \beta \) by Lemma EC1.5, and \( \beta_{m^* - 1} \geq \beta_y \) by assumption.

Part 4: first, note that the optimal value of the spoke problem \( h^\lambda_i \) satisfies \( h^\lambda_i \leq h^\lambda_i = 0 - \lambda \mathbb{E}[X^\lambda] \leq \gamma(1) \), where the second inequality follows from the facts that \( \mathbb{E}[X^\lambda] \geq 0 \) and \( \gamma(1) \) is the flow relaxation to \( h^\lambda_i = 0 \). From \( \text{EC-44} \) we have \( m^* \geq \frac{\gamma'(0) - h^\lambda_i}{\lambda} \geq \frac{\gamma'(0) - \gamma(1)}{\lambda} = 1 - \bar{y} - 1 \).

As a final preparation to the proof of Lemma EC1.16, we provide two more lemmas. Lemma EC1.20 bounds the gap \( e_y = \beta^*_y - \bar{\beta}_y \) from above.

**Lemma EC1.20.** For any integer \( y \leq \bar{y} \), the gap \( e_y = \beta^*_y - \bar{\beta}_y \geq 0 \) satisfies:

1. \( e_y \leq \alpha_{y-1} \cdot e_{y-1} + \beta^*_y - \beta^*_{y-1} \leq \alpha_{y-1} \cdot e_{y-1} + \frac{\lambda}{\ell} \cdot \frac{1}{1 - \beta^*_y}; \)

2. for any \( \tilde{\beta} \in [0, 1] \), let \( \bar{y} = \inf \{ y \geq 0 : \beta^*_y \geq \tilde{\beta} \} \) and \( \tilde{\alpha} = (1 - \frac{\ell}{\ell}) + \frac{\ell}{\ell} \tilde{\beta} \); then for any integer \( 1 \leq y \leq \bar{y} \), we have \( e_y \leq \tilde{\alpha} e_{y-1} + e_1 + \frac{\lambda}{\ell} \cdot \frac{\lambda}{1 - \beta^*_y} \cdot \frac{1}{1 - \tilde{\alpha}}. \)

**Proof.** Part 1: from Lemma EC1.17 part 4 we have

\[
e_y = \beta^*_y - g(\bar{\beta}_{y-1}, y - 1) \leq \beta^*_y - \alpha_{y-1} \cdot \bar{\beta}_{y-1} - (1 - \alpha_{y-1}) \bar{\beta}_{y-1} = \alpha_{y-1} \cdot e_{y-1} + \beta^*_y - \beta^*_{y-1}.
\]

Moreover, from (EC-41) we have

\[
\beta^*_y - \beta^*_{y-1} = \int_{y-1}^y \left( \frac{d\beta^*_y}{ds} \right) \, ds \leq \int_{y-1}^y \frac{\lambda}{\ell} \cdot \frac{1}{1 - \beta^*_y} \, ds \leq \frac{\lambda}{\ell} \cdot \frac{1}{1 - \beta^*_y}.
\]

(45)

**EC3** From (EC-44), \( m^* < \frac{\gamma'(0) - h^\lambda_i}{\lambda} \leq \frac{\gamma'(0)}{\lambda} \leq m \).
Part 2: since $\beta^*_y$ is increasing in $y$ by Lemma EC1.17 part 3, for any integer $y$ with $1 \leq y \leq \tilde{y}$, $\beta^*_y \leq \beta_y^* = \tilde{\beta}$ and hence $\alpha_y = (1 - \frac{\ell}{\bar{\alpha}}) + \frac{\ell}{\bar{\alpha}} \beta^*_y \leq \bar{\alpha}$ and $\epsilon_y \leq \alpha_{y-1} \cdot \epsilon_{y-1} + \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*_y} \leq \bar{\alpha} \cdot \epsilon_{y-1} + \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*_y}$. We now prove the inequality by induction. First, it is trivially true with $y = 1$. Now suppose $y \leq \tilde{y}$ and the inequality holds at $y - 1$. We have

$$e_y \leq \bar{\alpha} \cdot \epsilon_{y-1} + \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*_y} \leq \tilde{\alpha}^{y-1} \cdot e_1 + \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*_y} \cdot \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} + 1 \right) = \bar{\alpha}^{y-1} \cdot e_1 + \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*_y} \cdot \frac{1}{1 - \tilde{\alpha}}.$$  

Lemma EC1.21 shows that the expected value of a discrete random variable is increasing in the ratio of adjacent probabilities.

**Lemma EC1.21.** Consider two integer-valued random variables $X_i$ with $i \in \{1, 2\}$, each with support $a_i \leq x \leq b_i$ and probability mass function $g_i(x)$. If $a_1 \geq a_2$, $b_1 \geq b_2$, and $\frac{g_1(x+1)}{g_1(x)} \geq \frac{g_2(x+1)}{g_2(x)}$ for all $a_1 \leq x \leq b_2 - 1$, then $\mathbb{E}[X_1] \geq \mathbb{E}[X_2]$.

**Proof.** It is easy to check that $g_1(x')g_2(x) \geq g_2(x')g_1(x)$ for all $x' \geq x$. Thus by Section 1.C in Shaked and Shanthikumar (2007), $X_1$ dominates $X_2$ in the monotone likelihood ratio order, and this implies that $X_1$ first-order stochastically dominates $X_2$. 

**Proof of Lemma EC1.16:** We show in Section EC4.1 that for any $\beta \in (0, 1)$, the distribution of the resources in spoke $i$ with $\beta_x = \frac{p_i(x+1)}{p_i(x)}$ being $\beta_x = \beta$ for $0 \leq x \leq k - 1$ and $\beta_x = 0$ for $x \geq k$ has mean $B^k(\beta) = \frac{\beta^k + k \beta^{k+1}}{(1 - \beta) (1 - \beta^k)}$. Moreover, $\lim_{k \to \infty} B^k(\beta) = B^\infty(\beta) = \frac{\beta}{1 - \beta}$. For any $\rho > 0$, pick $\tilde{\beta}$ be such that $B^\infty(\tilde{\beta}) = \frac{\beta}{1 - \beta} = 2\rho$, i.e., $\tilde{\beta} = \frac{2\rho}{1 + 2\rho}$. Since $\lim_{k \to \infty} B^k(\tilde{\beta}) = B^\infty(\tilde{\beta}) = 2\rho$, there exists an integer $N_1 \in \mathbb{N}$ such that $B^k(\beta) \geq \rho$ for all $k \geq N_1$.

Select $\tilde{\beta} \in (\beta, 1)$ and let $\Delta = \tilde{\beta} - \bar{\beta} > 0$. Let $\bar{y} = \gamma(0)^{-1} \gamma(1)^{-1}$, $\tilde{y} = \inf \{ y \geq 0 : \beta^*_y \geq \tilde{\beta} \}$, and $\tilde{y}^0$ the minimum integer that is at least $\tilde{y}$. Let $m = \gamma(0)^{-1}$ (we will specify sufficient conditions for this later), hence $m^* \geq \bar{y} - 1$ from Lemma EC1.19 part 4. For any integer $y$ satisfying $0 \leq y \leq \bar{y} - \tilde{y}^0 - 1 \leq m^* - \tilde{y}^0$, the ratio $\beta_y = \frac{p_i(y+1)}{p_i(y)}$ (with $p_i(y)$ optimal to (6)) satisfies $\beta_y \geq \tilde{\beta} y - \tilde{y}^0$, where the first inequality is from Lemma EC1.19 part 3 and the second one is because $\tilde{\beta}_y$ is increasing in $y$ by Lemma EC1.19 part 1. Since (a): $\beta_{y^{0-1}} = \beta_{y^{0-1}}^* \epsilon_{\rho^{0-1}}$, (b): $\beta_{y^{0-1}}^* \geq \beta_{y^{0-1}} \geq \tilde{\beta} - \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*}$ analogous to (EC-45) and noting that $\beta_{y^{0-1}}^* = \tilde{\beta}$, (c): $\epsilon_{\rho^{0-1}}$ can be bounded from above by Lemma EC1.20 in terms of $e_1$ with $\bar{\alpha}$ defined therein satisfying $\bar{\alpha} \leq (1 - \frac{\ell}{n}) + \frac{\mathbf{I}}{\bar{\alpha}} < 1$, and (d): $e_1 = \beta_1^* - \tilde{\beta}_1 = \beta_1^* \leq 1 - \sqrt{\frac{2\lambda}{\ell}}$ by the upper bound in Lemma EC1.18 part 2, we have

$$\beta_y \geq \beta_{y^{0-1}} \geq \tilde{\beta} - \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*} - \left( 1 - \sqrt{1 - \frac{2\lambda}{\ell}} \right) - \frac{\lambda}{\bar{\alpha}} \cdot \frac{1}{1 - \beta^*} \cdot \frac{1}{1 - \tilde{\alpha}}, \forall 0 \leq y \leq \bar{y} - \tilde{y}^0 - 1.$$  

Since the right-hand side of above converges to $\tilde{\beta}$ when $\lambda$ diminishes and recall that $\ell \geq \frac{\tilde{y}^0}{n}$, there exists a constant $c_1 > 0$ such that when $\lambda \leq \frac{c_1}{n}$, $\beta_y \geq \tilde{\beta} - \Delta = \beta$ for all $0 \leq y \leq \bar{y} - \tilde{y}^0 - 1$. Moreover, since $\bar{y} - \tilde{y} \geq \frac{\ell}{\bar{\alpha}} \cdot (1 - \beta)^2$ from Lemma EC1.18 part 1, there exists a constant $c_2 > 0$ such that $\bar{y} - \tilde{y}^0 \geq N_1$ when $\lambda \leq \frac{c_2}{n}$.

Combining everything together, since the mean value $\mathbb{E}[X^\lambda]$ increases in $\beta_x = \frac{p_i(x+1)}{p_i(x)}$ by Lemma EC1.21, we have $\mathbb{E}[X^\lambda] \geq B^{N_1}(\beta) \geq \rho$ when $\lambda = \frac{c_1}{n}$ with $c = \min\{c_1, c_2\}$. To ensure that $m \geq \gamma(0)^{-1}$, note that since $\gamma(0)^{-1} \leq q_0 \cdot (\bar{r} + \tilde{\omega})$ by Lemma EC1.4 and $q_0 \leq \frac{c}{n}$, it suffices to set $m \geq \frac{\tilde{\omega} (\bar{r} + \tilde{\omega})}{c}$. 

EC-28
**EC1.12 Proof of Theorem 2**

The proof is analogous to the proof of Theorem 1. First, the same sensitivity analysis for $V^R(\delta)$ in Section 4.1 implies that Lemma 1 still holds, i.e.,

$$V^R(\delta) \leq V^R(0) = V^R \leq V^R(\delta) + \bar{r} \cdot \frac{\delta}{m - \delta}. \quad (EC-46)$$

Second, we can bound $V^R(\delta) - V^\pi(\delta)$ from above in a similar manner as in Lemma 2. Specifically, the same argument in Lemma EC1.10 implies that the difference of the continuation values $\Delta v_{i,t}(x)$ for each spoke is still bounded from above by the derivative bound $\bar{\omega}$ in Assumption 1. This implies that every time the Lagrangian policy differs in the two systems, the difference in continuation values is at most $\bar{r} + \bar{\omega}$ if the request is from a hub to a spoke; if the request is between hubs, the difference in continuation values is at most $\bar{r}$.

Since the Lagrangian policy takes different actions in the relaxed and original systems at the same state $(x, s)$ only when $x_j = 0$ and hub $j$ is the originating location for some $j \in [J]$, following the same proof of Lemma 2, we have

$$V^R(\delta) - V^\pi(\delta) \leq (\bar{r} + \bar{\omega}) \cdot \sum_{j \in [J]} q_j \cdot \mathbb{P}[X_j(\delta) = 0], \quad (EC-47)$$

where $q_j = \sum_{i \in [n]} q_{ij} + \sum_{j' \in [J]} q_{jj'}$ is the probability that hub $j$ is the originating location of the request. Combining (EC-46) and (EC-47) gives the desired result.

**EC2 Additional Results**

**Lemma EC2.1 (Lagrangian Policy in the Spoke Problem).** For each spoke problem and using the Lagrangian policy, we have

1. Set $I_i$ is the single positive recurrent class and the chain is aperiodic;
2. $p_i(x)$ is the unique stationary distribution;
3. Set $I_i$ takes the form of $I_i = [0 : H_i]$ for some non-negative integer $0 \leq H_i \leq m$;
4. The Lagrangian policy is optimal to each spoke problem.

**Proof.** We prove Lemma EC2.1 through a sequence of properties. We say a set of states is closed if the state remains in the set when started at a state in the set. Proposition EC2.2 shows the set $I_i$ is closed with the Lagrangian policy.

**Proposition EC2.2.** Set $I_i$ is closed with the Lagrangian policy.

**Proof.** Suppose not. Without loss of generality we assume $x \in I_i$ and $x + 1 \in I_i^c$ but $d_i(x, 0, i) > 0$. From the balance constraint in (6) we have $p_i(x + 1) \cdot q_{0i} \cdot d_i(x + 1, i, 0) = p_i(x) \cdot q_{0i} \cdot d_i(x, 0, i) > 0$. This implies $p_i(x + 1) > 0$ and a contradiction.

**Proposition EC2.3.** $p_i(x)$ is a stationary distribution with the Lagrangian policy.

**Proof.** This is a direct result from the balance constraint in (6):

$$p_i(x) \cdot q_{0i} \cdot d_i(x, 0, i) = p_i(x + 1) \cdot q_{0i} \cdot d_i(x + 1, i, 0), \forall x \in [0 : m - 1].$$
Proposition EC2.3 implies all states in $I_i$ are positive recurrent.

**Corollary EC2.4.** All states $x \in I_i$ are positive recurrent.

**Proof.** Since $p_i(x)$ is a stationary distribution from Proposition EC2.3 and $p_i(x) > 0$ for all states $x \in I_i$, states $x \in I_i$ are recurrent according to Theorem 6.5.4 in Durrett (2010). Moreover, each state $x \in I_i$ is positive recurrent because the set $I_i$ is finite. \qed

**Proposition EC2.5.** States $x \in I^c_i$ are transient.

**Proof.** This is due to Proposition EC2.2 and the construction of the Lagrangian policy for states outside the set $I_i$. \qed

**Proposition EC2.6.** Set $I_i$ is irreducible.

**Proof.** Suppose not. Since all states in set $I_i$ are positive recurrent (Corollary EC2.4), $I_i$ must contain at least two recurrent classes. Let $C \subseteq I_i$ be a recurrent class and a strict subset of $I_i$. Without loss of generality, we assume state $x \in C$ whereas state $x-1 \in I_i \setminus C$ lies in another recurrent class. Since states $x-1$ and $x$ do not reach each other, we must have $d_i(x-1,0,i) = d_i(x,i,0) = 0$. However, by the complementary slackness properties (EC-4)–(EC-6), we have $d_i(x-1,0,i) = \arg\max_{x \in [0,1]} \{ r_{0i}(d) + d \cdot (v_i^*(x) - v_i^*(x-1)) \}$ and $d_i(x,i,0) = \arg\max_{x \in [0,1]} \{ r_{i0}(d) + d \cdot (v_i^*(x-1) - v_i^*(x)) \}$ with $v_i^*(x)$ being the average differential value functions in (EC-2). Since the maximum points $d^*_{bi} = \arg\max_{d \in [0,1]} r_{0i}(d)$ and $d^*_{i0} = \arg\max_{d \in [0,1]} r_{i0}(d)$ are unique and strictly positive by Assumption 1, either $d_i(x-1,0,i) \geq d^*_{bi} > 0$ or $d_i(x,i,0) \geq d^*_{i0} > 0$; thus a contradiction. \qed

**Proposition EC2.7.** The Markov chain is aperiodic.

**Proof.** The chain stays at the current state in every time period when the request type is neither $(i,0)$ nor $(0,i)$, with probability $1 - q_i > 0$. \qed

From Corollary EC2.4 and Propositions EC2.5 and EC2.6, set $I_i$ is the single positive recurrent class and all states outside $I_i$ are transient. As a result, $p_i(x)$ is the unique stationary distribution. Since $I_i$ is irreducible by Proposition EC2.6 and every transition can only increase or decrease the current state by one, $I_i$ must take the form of $I_i = [L_i : H_i]$ which incorporates a sequence of consecutive integers. We now show $L_i = 0$. If $\lambda > 0$ but $L_i > 0$, shifting the probabilities and controls to the left by $L_i$ with $\tilde{p}_i(x) = p_i(x+L_i)$, $\tilde{d}_i(x,i,0) = d_i(x+L_i,i,0)$ and $\tilde{d}_i(x,0,i) = d_i(x+L_i,0,i)$ yields a feasible solution to (6) with a strictly better objective value, thus a contradiction. If $\lambda = 0$, Lemma EC1.7 implies that $I_i = [0 : m]$, i.e., the optimal distribution spans the whole range.

Finally, since the Markov chain has a single positive recurrent class and $p_i(x)$ is the unique stationary distribution, the average revenue of the Lagrangian policy does not depend on the initial state and can be expressed as the objective of (6). The strong duality in Proposition 4 implies that the average revenue of the Lagrangian policy is equal to the optimal average revenue $h_i^\lambda$, hence the Lagrangian policy is optimal to the spoke problem. \qed

**Proposition EC2.8.** The stationary distribution $p_i(x)$ that is optimal to (6) is unique.

**Proof.** Suppose not. Let distribution $p_i^a(x)$ together with controls $d^a_i(x,i,0)$ and $d^a_i(x,0,i)$ and distribution $p_i^b(x)$ together with controls $d^b_i(x,i,0)$ and $d^b_i(x,0,i)$ be two optimal solutions to (6). $p_i^a(x)$ and $p_i^b(x)$ are not identical and we denote their supports by $[0 : H^a_i]$ and $[0 : H^b_i]$, respectively.
Without loss of generality, let \(d_i^0(x, i, 0) = d_i^0(x, 0, i) = 0\) for all states \(x\) with \(p_i^0(x) = 0\), and \(d_i^0(x, i, 0) = d_i^0(x, 0, i) = 0\) for all states \(x\) with \(p_i^0(x) = 0\).

We first show that there exists a state \(x\) such that \(p_i^0(x), p_i^0(x) > 0\) and either \(d_i^0(x, i, 0) \neq d_i^0(x, 0, i)\) or \(d_i^0(x, 0, i) \neq d_i^0(x, 0, i)\). To see this, note that if \(H_i^a \neq H_i^b\), taking \(x = \min \{H_i^a, H_i^b\}\), we have \(p_i^0(x), p_i^0(x) > 0\) and \(d_i^0(x, 0, i) \neq d_i^0(x, 0, i)\). Otherwise if \(H_i^a = H_i^b\), since the distributions \(p_i^0(x)\) and \(p_i^0(x)\) are not identical and both of them sum up to one, there must exist a state \(x \in [0 : H_i^a - 1]\) such that the ratios \(\frac{p_i^0(x+1)}{p_i^0(x)}\) and \(\frac{p_i^0(x+1)}{p_i^0(x)}\) are not equal. Since \(p_i^0(x) \cdot q_0 \cdot d_i^0(x, 0, i) = p_i^0(x+1) \cdot q_0 \cdot d_i^0(x+1, i, 0)\) and \(p_i^0(x) \cdot q_0 \cdot d_i^0(x, 0, i) = p_i^0(x+1) \cdot q_0 \cdot d_i^0(x+1, i, 0)\), we have either \(d_i^0(x, 0, i) \neq d_i^0(x, 0, i)\) or \(d_i^0(x+1, i, 0) = d_i^0(x+1, i, 0)\).

For any \(\alpha_1, \alpha_2 > 0\) with \(\alpha_1 + \alpha_2 = 1\), let \(p_i(x) = \alpha_1 \cdot p_i^0(x) + \alpha_2 \cdot p_i^0(x)\) for all states \(x\), \(d_i(x, i, 0) = \frac{\alpha_1 \cdot p_i^0(x)}{p_i(x)} \cdot d_i^0(x, i, 0) + \frac{\alpha_2 \cdot p_i^0(x)}{p_i(x)} \cdot d_i^0(x, i, 0)\) and \(d_i(x, 0, i) = \frac{\alpha_1 \cdot p_i^0(x)}{p_i(x)} \cdot d_i^0(x, 0, i) + \frac{\alpha_2 \cdot p_i^0(x)}{p_i(x)} \cdot d_i^0(x, 0, i)\) for all states \(x\) with \(p_i(x) > 0\), and \(d_i(x, i, 0) = d_i(x, 0, i) = 0\) for all states \(x\) with \(p_i(x) = 0\). It is easy to see that \(p_i(x), d_i(x, i, 0)\) and \(d_i(x, 0, i)\) are feasible to \(\mathcal{L}\). Moreover, since the revenue functions \(r_{00}(d)\) and \(r_{00}(d)\) are strictly concave by Assumption 1, due to Jensen’s inequality, the objective value with \(p_i(x)\) and controls \(d_i(x, i, 0)\) and \(d_i(x, 0, i)\) is strictly larger than the objective values with the probability distributions \(p_i^0(x)\) and \(p_i^0(x)\). This violates the optimality of \(p_i^0(x)\) and \(p_i^0(x)\) and thus a contradiction.

Proposition \[EC2.9\] shows that the stationary distribution for each spoke is (discrete) log-concave.

**Definition EC2.1** (Discrete Log-concavity, c.f., [Keilson and Gerber 1971, Keilson 1972]). A discrete probability distribution \(p = \{p_i\}_{i=0}^\infty\) with all its support on non-negative integers is discrete log-concave (or simply log-concave) if (i) its support \(I_p = \{i \geq 0 : p_i > 0\}\) is a sequence of consecutive integers, i.e., for all \(0 \leq n_1 \leq n \leq n_2\), if \(n_1, n_2 \in I_p\), then \(n \in I_p\); and (ii) \(p_i^2 \geq p_{i-1} \cdot p_{i+1}\) for all \(i \geq 1\).

**Proposition EC2.9.** For each spoke \(i \in [n]\), the stationary distribution \(p_i(x)\) solved from \(\mathcal{L}\) is discrete log-concave.

**Proof.** From Lemma \[EC2.1\], the support of \(p_i(x)\) is \(I_i = [0 : H_i]\) that is a sequence of consecutive integers. Secondly, from the flow balance constraint in \(\mathcal{L}\)

\[
p_i(x) \cdot q_{00} \cdot d_i(x, 0, i) = p_i(x+1) \cdot q_{00} \cdot d_i(x+1, i, 0), \quad \forall x \in [0 : m - 1],
\]

we have

\[
(p_i(x))^2 \cdot d_i(x, i, 0) \cdot d_i(x, 0, i) = p_i(x-1) \cdot p_i(x+1) \cdot d_i(x+1, i, 0) \cdot d_i(x-1, 0, i)
\]

for all \(x \in [1 : m - 1]\). Since the demand level \(d_i(x, 0, i)\) is decreasing in \(x\) and \(d_i(x, i, 0)\) is increasing in \(x\) for \(x \in I_i\) by Proposition 5 and the support of \(p_i(x)\) is a sequence of consecutive integers, we have \((p_i(x))^2 \geq p_i(x-1) \cdot p_i(x+1)\) for all \(x \in [1 : m - 1]\). □

Corollary \[EC2.10\] shows that the Lagrangian policy in the Lagrangian relaxation has a unique stationary distribution, which factors across spokes.

**Corollary EC2.10** (Lagrangian Policy in the Relaxation). The Lagrangian policy is optimal to the Lagrangian relaxation. Moreover, let the system state be the resource levels \(x \in \mathcal{X}\). Using the Lagrangian policy, the following hold:
1. The set \( \prod_{i=1}^{n} I_i \) is a positive recurrent class and is aperiodic, and all states outside the set \( \prod_{i=1}^{n} I_i \) are transient; and

2. \( q(x) = \prod_{i=1}^{n} p_i(x_i) \) is the unique stationary distribution.

Proof. Since the Lagrangian relaxation decomposes over spokes and the Lagrangian policy is optimal to each spoke problem, the Lagrangian policy is optimal to the Lagrangian relaxation as well. From Lemma EC2.1, it is easy to see that the set \( \prod_{i=1}^{n} I_i \) is positive recurrent and all states outside \( \prod_{i=1}^{n} I_i \) are transient. To show the aperiodicity, let \( x \in X \) be a boundary state of the set \( \prod_{i=1}^{n} I_i \). Without loss of generality we assume \( x \in \prod_{i=1}^{n} I_i \), whereas \( x + e_i' \notin \prod_{i=1}^{n} I_i \) for some spoke \( i' \in [n] \). State \( x \) stays unchanged when a request \( (0, i') \) arrives, which occurs with probability \( q_{0i'} > 0 \). Thus the chain is aperiodic.

Since the Markov chain has a single positive recurrent class \( \prod_{i=1}^{n} I_i \), the stationary distribution is unique. \( q(x) \) is the stationary distribution if and only if \( \sum_{x \in X} q(x) = 1 \) and for all \( x \in X \),

\[
q(x) \cdot \sum_{i \in [n]} \left[ q_{0i} \cdot d_i(x_i, i, 0) + q_{0i} \cdot d_i(x_i, 0, i) \right] = \sum_{i \in [n]} \left\{ \mathbb{1}[x_i \geq 1] \cdot q(x - e_i) \cdot q_{0i} \cdot d_i(x_i - 1, 0, i) \right\} + \sum_{i \in [n]} \left\{ \mathbb{1}[x_i \leq m - 1] \cdot q(x + e_i) \cdot q_{0i} \cdot d_i(x_i + 1, i, 0) \right\}.
\]

(EC-48)

It is easy to see that \( q(x) = \prod_{i=1}^{n} p_i(x_i) \) satisfies (EC-48) because of the flow balance constraint in (6). Thus, \( q(x) = \prod_{i=1}^{n} p_i(x_i) \) is the unique stationary distribution in the Lagrangian relaxation.

Corollary EC2.11 shows that using the Lagrangian policy in the original problem also leads to a unichain policy; the proof is analogous to proof of Corollary EC2.10.

Corollary EC2.11 (Lagrangian Policy in the Original Problem). Let the system state be the resource levels \( x \in X \). Using the Lagrangian policy, the set \( \prod_{i=1}^{n} I_i \cap X \) is the single positive recurrent class and is aperiodic, and all states outside the set \( \prod_{i=1}^{n} I_i \cap X \) are transient.

Proposition EC2.12 formalizes the decomposition across spokes and hubs for general networks.

Proposition EC2.12. The Lagrangian relaxation bound \( \bar{V}^{\lambda, \mu, \nu} \) described in Section 6.2 decomposes over spokes with

\[
\bar{V}^{\lambda, \mu, \nu} = m \lambda + \sum_{i=1}^{n} h_i^{\lambda, \mu, \nu} + \sum_{j,j' \in [J]} g_{jj'}^{\mu},
\]

where \( g_{jj'}^{\mu} \triangleq \max_{d \in [0, 1]} \left\{ r_{jj'}(d) + d \cdot (\mu_{j'} - \mu_j) \right\} \) denotes the average revenue earned from a hub-to-hub request \( (j, j') \) as in (14), and \( h_i^{\lambda, \mu, \nu} \) denotes the average revenue of an optimal policy to each spoke \( i \) problem, which is equal to the optimal value of (EC-49)

\[
\max_{d_i(x, i, j) \in [0, 1]} \sum_{x=0}^{m} p_i(x) \cdot \left\{ \sum_{j=1}^{J} \left[ q_{ij} \cdot r_{ij}(d_i(x, i, j)) + q_{ji} \cdot r_{ji}(d_i(x, j, i)) \right] + \sum_{i' = 1}^{n} q_{ii'} \cdot r_{ii'}(d_i(x, i, i')) \right\} + \sum_{j' = 1}^{J} \sum_{x=0}^{m} p_i(x) \cdot \left( \nu_{ij} \cdot q_{ij} \cdot d_i(x, j, i) - \nu_{ij} \cdot q_{ij} \cdot d_i(x, j', i) \right) + \sum_{j=1}^{J} \mu_j \sum_{x=0}^{m} p_i(x) \cdot \left( q_{ij} \cdot d_i(x, j, i) - q_{jj} \cdot d_i(x, j, j) \right) - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x).
\]

EC-32
\[ \sum_{x=0}^{m} p_i(x) = 1, \quad \text{(EC-49)} \]

\[
p_i(x) \cdot \left( \sum_{j=1}^{J} q_{ji} \cdot d_i(x, j, i) + \sum_{i'=1}^{n} q_{ii'} \cdot d_i(x, i', i) \right) = p_i(x + 1) \cdot \left( \sum_{j=1}^{J} q_{ji} \cdot d_i(x + 1, i, j) + \sum_{i'=1}^{n} q_{ii'} \cdot d_i(x + 1, i, i') \right), \quad \forall \, x \in [0 : m - 1],
\]

\[
d_i(0, i, j) = 0, d_i(m, j, i) = 0, \quad \forall \, j \in [J],
\]

\[
d_i(0, i, i') = 0, d_i(m, i', i) = 0, \quad \forall \, i' \in [n].
\]

**Proof.** To simplify the notation, in the proof we suppress the superscript \( \lambda, \mu, \nu \) throughout that specifies the specific dual variables to use in the Lagrangian relaxation.

Since we assume the network topology is strongly connected, the Lagrangian relaxation bound \( \bar{V} \) does not depend on the initial state of the system by the same argument as in Proposition 2. Moreover, since we relax the capacity constraint of each hub, we can express \( \bar{V} \) as

\[
\bar{V} = m \lambda + h + \sum_{j,j' \in [J]} q_{jj'} \cdot g_{jj'},
\]

where \( g_{jj'} = \max_{d \in [0,1]} \{ r_{jj'}(d) + d \cdot (\mu_{j'} - \mu_j) \} \) denotes the average revenue from a hub-to-hub request \( (j, j') \), and \( h \) denotes the average revenue of an optimal control to requests that involve any of the spokes. Furthermore, \( h \) and some differential value functions \( v(x_s, i, j), v(x_s, j, i), v(x_s, i, i') \) and \( v(x_s, j, j') \) satisfy the following Bellman equation

\[
h + v(x_s, i, j) = \max_{d \in [0,1]} \left\{ r_{ij}(d) + d \cdot \left( v(x_s - e_i) - v(x_s) + \mu_j \right) \right\} + v(x_s) - \lambda \cdot \sum_{i \in [n]} x_i,
\]

\[
ah + v(x_s, j, i) = \max_{d \in [0,1]} \left\{ r_{ji}(d) + d \cdot \left( v(x_s + e_i) - v(x_s) - \mu_j \right) \right\} + v(x_s) - \lambda \cdot \sum_{i \in [n]} x_i,
\]

\[
ah + v(x_s, i, i') = \max_{d \in [0,1], \{a_1,a_2\} \in \{0,1\}^{(m-x_i)}} \left\{ r_{ii'}(d) + d \cdot \left( v(x_s - e_i + a_1 \cdot e_i' + a_2 \cdot e_i') - v(x_s) - \mu_{ii'} \right) \right\}
\]

\[
+ (1 - d) \cdot \left( v(x_s + a_2 \cdot e_i') - a_2 \cdot v_{ii'} \right) - \lambda \cdot \sum_{i \in [n]} x_i, \quad \forall \, i \in [n], i' \in [n] \setminus \{i\},
\]

\[
h + v(x_s, j, j') = v(x_s) - \lambda \cdot \sum_{i \in [n]} x_i, \quad \forall \, j, j' \in [J],
\]

\[
\text{(EC-50)}
\]

for all \( x_s \in [0 : m]^n \), where \( v(x_s) = \mathbb{E}_s[v(x_s, s)] \) denotes the average differential value function over request types, and the binary variables \( a_1 \) and \( a_2 \) in the third equation denote the decision of adding one resource in the destination when a spoke-to-spoke request \( (i, i') \) arrives, and the request is fulfilled or not, respectively.

It is easy to verify that the average revenue \( h \) and the differential value functions decompose
over spokes with

\[ h = \sum_{i \in [n]} h_i, \]

\[ v(x_s, i, j) = v_i(x_i, i, j) + \sum_{k \neq i} v_k(x_k, \emptyset), \]

\[ v(x_s, j, i) = v_i(x_i, j, i) + \sum_{k \neq i} v_k(x_k, \emptyset), \]

\[ v(x_s, i, i') = v_i(x_i, i, i') + v_{i'}(x_{i'}, i, i') + \sum_{k \neq i, i'} v_k(x_k, \emptyset), \]

\[ v(x_s, j, j') = \sum_{k \in [n]} v_k(x_k, \emptyset), \]

where \( h_i \) denotes the average revenue of an optimal policy to each spoke \( i \) problem, and the differential value functions \( v_i(x, i, j) \), \( v_i(x, j, i) \), \( v_i(x, i, i') \), \( v_i(x, i', i) \) and \( v_i(x, \emptyset) \) correspond to the state with \( x \) resources in spoke \( i \) and the request type being \( (i, j) \), \( (j, i) \), \( (i, i') \), \( (i', i) \), or one of any other types, respectively. Moreover, \( h_i \) and the differential value functions satisfy the following Bellman equation

\[
h_i + v_i(x, i, j) = \max_{d \in [0,1 \wedge x]} \left\{ r_{ij}(d) + d \cdot \left( v_i(x - 1) - v_i(x) + \mu_j \right) \right\} + v_i(x) - \lambda \cdot x, \forall j \in [J],
\]

\[
h_i + v_i(x, j, i) = \max_{d \in [0,1 \wedge (m-x)]} \left\{ r_{ji}(d) + d \cdot \left( v_i(x + 1) - v_i(x) - \mu_j \right) \right\} + v_i(x) - \lambda \cdot x, \forall j \in [J],
\]

\[
h_i + v_i(x, i, i') = \max_{d \in [0,1 \wedge (m-x)]} \left\{ r_{ii'}(d) + d \cdot \left( v_i(x - 1) - v_i(x) + \nu_{ii'} \right) \right\} + v_i(x) - \lambda \cdot x, \forall i' \in [n] \setminus \{i\},
\]

\[
h_i + v_i(x, i', i) = \max_{a \in \{0,1 \wedge (m-x)\}} \left\{ v_i(x + a) - a \cdot \nu_{ii'} \right\} - \lambda \cdot x, \forall i' \in [n] \setminus \{i\},
\]

\[
h_i + v_i(x, \emptyset) = v_i(x) - \lambda \cdot x,
\]

for all \( x \in [0 : m] \) and each spoke \( i \), with \( v_i(x) \) being the average differential value function over request types. Here we only verify the decomposition of \( v(x_s, i, i') \) that involves a spoke-to-spoke request \( (i, i') \). Suppose the average revenues and differential value functions decompose over spokes
by [EC-51] and [EC-52] holds. The right-hand side of the third equation in [EC-50] is equal to

$$\max_{d \in [0,1] \cap x_i, a_1, a_2 \in (0,1) \cap (m-x_i)} r_{ii'}(d) + d \cdot \left(v_i(x_i - 1) + v_{ii'}(x_{i'} + a_1) + \nu_{ii'} - a_1 \cdot \nu_{ii'}\right) + (1 - d) \cdot \left(v_i(x_i) + v_{ii'}(x_{i'} + a_2) - a_2 \cdot \nu_{ii'}\right) + \sum_{k \neq i, i'} v_k(x_k) - \lambda \cdot \sum_{i \in [n]} x_i$$

$$= \max_{d \in [0,1] \cap x_i} r_{ii'}(d) + d \cdot \left(v_i(x_i - 1) + \nu_{ii'} + \max_{a_1 \in (0,1) \cap (m-x_i)} v_{ii'}(x_{i'} + a_1) - a_1 \cdot \nu_{ii'}\right) + (1 - d) \cdot \left(v_i(x_i) + \max_{a_2 \in (0,1) \cap (m-x_i)} v_{ii'}(x_{i'} + a_2) - a_2 \cdot \nu_{ii'}\right) + \sum_{k \neq i, i'} v_k(x_k) - \lambda \cdot \sum_{i \in [n]} x_i$$

$$\overset{(i)}{=} \max_{a \in (0,1) \cap (m-x_i)} v_{ii'}(x_{i'} + a_1) - a \cdot \nu_{ii'}$$

$$+ \max_{d \in [0,1] \cap x_i} r_{ii'}(d) + d \cdot \left(v_i(x_i - 1) + \nu_{ii'} - \nu_i(x_i)\right) + \sum_{k \neq i, i'} v_k(x_k) - \lambda \cdot \sum_{i \in [n]} x_i$$

$$= \sum_{i \in [n]} h_i + v_i(x_i, i, i') + v_{ii'}(x_{i'}, i, i') + \sum_{k \neq i, i'} v_k(x_k, \emptyset)$$

$$= h + v(x_{s,i}, i, i').$$

Note that (i) implies that although the provider can make the decision of adding one resource in the destination after knowing the outcome of the fulfillment, it loses nothing if she instead makes the decision before the outcome, by comparing the cost $\nu_{ii'}$ and the marginal value of having one more resource in the destination. Thus we can make the decisions at the origin and destination independently.

Finally, following the same argument as in Proposition 4, $h_i$ is equal to the optimal value of [EC-49], which is the dual formulation of the spoke problem.

For an arbitrary network, we can divide the locations into hubs and spokes and consider the Lagrangian relaxation bound and policy in Section 6.2. Proposition 9 shows that the Lagrangian relaxation provides tighter bounds than the fluid relaxation bound $V^p$. We provide the proof here.

**Proof of Proposition 9.** The optimality condition of $\min_{\mu, \nu} V^{\lambda=0, \mu, \nu}$ implies that it is equivalent to [EC-53], the problem of maximizing the average revenue subject to the constraints that the in-flow and out-flow of each hub $j$ is balanced in expectation, and the out-flow of spoke $i$ through requests $(i, i')$ is equal to the in-flow of spoke $i'$ through requests $(i, i')$ for each spoke-to-spoke connection $(i, i')$. 

EC-35
\[
\begin{align*}
\max & \quad \sum_{i=1}^{n} \sum_{x=0}^{m} p_i(x) \cdot \sum_{j=1}^{J} \left\{ q_{ij} \cdot r_{i,j} \left( d_i(x, i, j) \right) + q_{ji} \cdot r_{j,i} \left( d_i(x, j, i) \right) \right\} + \sum_{i=1}^{n} \sum_{x=0}^{m} p_i(x) \cdot \sum_{i'=1}^{n} q_{ii'} \cdot r_{ii'} \left( d_i(x, i, i') \right) + \sum_{j=1}^{J} \sum_{j'=1}^{J} q_{jj'} \cdot r_{jj'} \left( d_{jj'} \right) \\
\text{s.t.} & \quad \sum_{x=0}^{m} p_i(x) = 1, \; \forall \; i \in [n], \\
& \quad \sum_{i=1}^{n} q_{ij} \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, j) + \sum_{j'=1}^{J} q_{jj'} \cdot d_{jj'} \\
& \quad = \sum_{i=1}^{n} q_{ij} \sum_{x=0}^{m} p_i(x) \cdot d_i(x, j, i) + \sum_{j'=1}^{J} q_{jj'} \cdot d_{jj'}, \; \forall \; j \in [J], \quad (EC-53) \\
& \quad p_i(x) \cdot \left( \sum_{j=1}^{J} q_{ij} \cdot d_i(x, j, i) + \sum_{i'=1}^{n} q_{ii'} \cdot d_i(x, i', i) \right) = \\
& \quad p_i(x+1) \cdot \left( \sum_{j=1}^{J} q_{ij} \cdot d_i(x+1, i, j) + \sum_{i'=1}^{n} q_{ii'} \cdot d_i(x+1, i, i') \right), \quad \forall \; x \in [0 : m - 1], i \in [n], \\
& \quad q_{ii'} \cdot \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, i') = q_{ii'} \cdot \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, i'), \; \forall \; i, i' \in [n], \\
& \quad d_i(0, i, j), d_i(0, i, i'), d_i(0, j, i), d_i(0, i', i), \; \forall \; i \in [n], j \in [J], i' \in [n].
\end{align*}
\]

For any optimal solution to \( (EC-53) \), let \( d_{ij} = \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, j) \), \( d_{ji} = \sum_{x=0}^{m} p_i(x) \cdot d_i(x, j, i) \) and \( d_{ii'} = \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, i') \) denote the average demand values. We show these demand values plus \( d_{jj'} \) are feasible to the fluid relaxation. First, by the second constraint in \( (EC-53) \) we have

\[
\sum_{i=1}^{n} q_{ij} d_{ij} + \sum_{j=1}^{J} q_{jj'} d_{jj'} = \sum_{i=1}^{n} q_{ji} d_{ji} + \sum_{j'=1}^{J} q_{jj'} d_{jj'}, \; \forall \; j \in [J], \quad (EC-54)
\]

which implies that the flow at each hub is balanced. Second, summing both sides of the third constraint over \( x \in [0 : m - 1] \) plus the fourth and last constraints gives

\[
\sum_{j=1}^{J} q_{ji} d_{ji} + \sum_{i'=1}^{n} q_{ii'} d_{ii'} = \sum_{j=1}^{J} q_{ij} d_{ij} + \sum_{i'=1}^{n} q_{ii'} d_{ii'}, \; \forall \; i \in [n], \quad (EC-55)
\]

which implies that the flow at each spoke is balanced as well. Thus from \( (EC-54) \) and \( (EC-55) \), the demand values \( d_{ij}, d_{ji}, d_{ii'}, \) and \( d_{jj'} \) are feasible to the fluid relaxation. Finally, by Jensen’s
inequality
\begin{align*}
V^F \geq & \sum_{i=1}^{n} \sum_{j=1}^{J} \left( q_{ij} \cdot r_{ij}(d_{ij}) + q_{ji} \cdot r_{ji}(d_{ji}) \right) + \sum_{i=1}^{n} \sum_{j'=1}^{J} q_{ij'} \cdot r_{ij'}(d_{ij'}) \\
& \sum_{i=1}^{n} \sum_{j'=1}^{J} \left( q_{ij} \cdot r_{ij'}(d_{ij}) + q_{ji} \cdot r_{ji'}(d_{ji'}) \right) \\
& \geq \min_{\mu, \nu} \tilde{V}^{\lambda=0, \mu, \nu}.
\end{align*}

In the case with general relocation times, Proposition [EC2.13] further relax the spoke problem to provide a tractable upper bound.

**Proposition EC2.13.** With general relocation times, the Lagrangian relaxation bound \( \tilde{V}^{\lambda, \mu, \nu} \) decomposes over spokes as

\[
\tilde{V}^{\lambda, \mu, \nu} = m\lambda + \sum_{i=1}^{n} h_i^{\lambda, \mu, \nu} + \sum_{j,j' \in [J]} q_{jj'} \cdot g_{jj'}^{\mu},
\]

where \( g_{jj'}^{\mu} \triangleq \max_{d \in [0,1]} \{ r_{jj'}(d) + d \cdot (\mu_j - \mu_j - \lambda \cdot \Lambda_{jj'}) \} \) denotes the average revenue earned from a hub-to-hub request \((j, j')\) and \( h_i^{\lambda, \mu, \nu} \) denotes the average revenue of an optimal policy to each spoke \(i\) problem. Moreover, \( h_i^{\lambda, \mu, \nu} \) is no larger than \( h_i \) which is the optimal value of \((EC-56)\).

\[
\max_{d_i(x,i,j) \in [0,1], d_i(x,j,i) \in [0,1], d_i(x'i,i) \in [0,1],} \sum_{x=0}^{m} \sum_{j=1}^{J} p_i(x) \cdot q_{ij} \cdot r_{ij}(d_i(x,i,j)) + \left( \mu_j - \lambda \Lambda_{ij} \right) \cdot d_i(x,i,j) \\
+ \sum_{x=0}^{m} \sum_{j=1}^{J} p_i(x) \cdot q_{ji} \cdot r_{ji}(d_i(x,j,i)) - \mu_j \cdot d_i(x,j,i) \\
+ \sum_{x=0}^{m} \sum_{j'=1}^{J} p_i(x) \cdot q_{i'j'} \cdot r_{i'j'}(d_i(x,i',i')) + \nu_{i'j'} \cdot d_i(x,i',i') \\
- \sum_{x=0}^{m} \sum_{j'=1}^{J} p_i(x) \cdot q_{i'j} \cdot \nu_{i'j} \cdot d_i(x',i,i') + \left( \sum_{x=0}^{m} p_i(x) \right) \cdot q_{ii} \cdot r_{ii}^{*} - \lambda \cdot \sum_{x=0}^{m} x \cdot p_i(x) \\
\text{s.t.} \quad \sum_{x=0}^{m} p_i(x) = 1, \quad (EC-56)
\]

\[
p_i(x) \cdot \left\{ \sum_{j=1}^{J} q_{ij} \cdot d_i(x,j,i) + \sum_{i' \neq i} q_{i'j} \cdot d_i(x,i',i) \right\} = p_i(x + 1) \cdot \left\{ \sum_{j=1}^{J} q_{ij} \cdot d_i(x+1,j,i) + \sum_{i' \neq i} q_{i'j} \cdot d_i(x+1,i,i') \right\}, \quad \forall \ x \in [0 : m - 1],
\]

\[
d_i(0, i, j) = 0, d_i(m, j, i) = 0, \ \forall \ j \in [J],
\]

\[
d_i(0, i, i') = 0, d_i(m, i', i) = 0, \ \forall \ i' \in [n] \setminus \{i\}.
\]

**Proof.** The decomposition is analogous to Proposition [EC2.12]. Note that compared to Proposition [EC2.12] the extra term \( \lambda \cdot \Lambda_{jj'} \) in \( g_{jj'}^{\mu} \) comes from the fact that the relocation \((j, j')\) takes \( \Lambda \cdot \tau_{jj'} \) periods on average (this is because requests follow a Possion process of rate \( \Lambda \) that is independent
of the relocation times) and each period incurs a penalty $\lambda$.

We now show the average revenue $h_i^{\lambda,\mu,\nu}$ of the spoke problem is no larger than $\hat{h}_i$. To see this, we first set the binding condition of the spoke problem to be the sum of resources in the spoke and transiting to it is no larger than $m$; then the resources that are moving out of the spoke are irrelevant. We then allow that the resources that are moving to the spoke can be instantaneously available at the spoke. Because a resource incurs a penalty $\lambda$ per period no matter it is in the spoke or moving to it, it is always better to keep the resources at the spoke as this increases the opportunity to serve the requests; this yields (EC-56). We conjecture that the relaxation works well when incoming relocation times are not long.

The Lagrangian relaxation with the optimal dual variable corresponds to maximizing the average revenue subject to the sum of resources that are in the spokes and transiting to the hubs no larger than $m$ in expectation.

**EC3 More Discussions on the Lagrangian Dual Problem**

Recall that the Lagrangian dual problem (9) is

$$V^R = \min_{\lambda \geq 0, \bar{V}^\lambda},$$

which is a convex optimization problem. According to (4) and (EC-2), $V^R$ is equal to the optimal value of (EC-57).

$$\begin{align*}
\min_{\lambda \geq 0, h_i^\lambda, v_i^\lambda(x,i,0), v_i^\lambda(x,0,i), v_i^\lambda(x,\varnothing)} & \quad m\lambda + \sum_{i=1}^{n} h_i^\lambda \\
\text{s.t.} & \quad h_i^\lambda + v_i^\lambda(x,i,0) \geq \max_{d \in [0,1]} \left\{ r_{i0}(d) + d \cdot (v_i^\lambda(x-1) - v_i^\lambda(x)) \right\} + v_i^\lambda(x) - \lambda \cdot x, \quad \forall x \leq m, i \in [n], \\
& \quad h_i^\lambda + v_i^\lambda(x,0,i) \geq \max_{d \in [0,1]} \left\{ r_{0i}(d) + d \cdot (v_i^\lambda(x+1) - v_i^\lambda(x)) \right\} + v_i^\lambda(x) - \lambda \cdot x, \quad \forall x \leq m, i \in [n], \\
& \quad h_i^\lambda + v_i^\lambda(x,\varnothing) \geq v_i^\lambda(x) - \lambda \cdot x, \quad \forall x \leq m, i \in [n].
\end{align*}$$

(EC-58)

Analogous to Proposition 4, $V^R$ is the optimal value of (EC-58) as well, which is maximizing the average revenue subject to the constraint that the expected number of resources in the hub is non-negative.

$$\begin{align*}
\max_{d_i(x,i,0) \in [0,1], d_i(x,0,i) \in [0,1], p_i(x) \geq 0} & \quad \sum_{i=0}^{n} \sum_{x=0}^{m} p_i(x) \left[ q_{i0} \cdot r_{i0}(d_i(x,i,0)) + q_{0i} \cdot r_{0i}(d_i(x,0,i)) \right] \\
\text{s.t.} & \quad \sum_{i=1}^{n} \sum_{x=0}^{m} x \cdot p_i(x) \leq m,
\end{align*}$$

(EC-58)
Theorem EC4.1. The average revenue $V^S(\delta)$ of the static pricing policy $\pi^S(\delta)$ satisfies

$$0 \leq V^{S,R} - V^{S}(\delta) \leq (\bar{r} + \bar{\omega}) \cdot \mathbb{P}[X_0(\delta) = 0] + \bar{r} \cdot \frac{\delta}{m - \delta},$$

where $\mathbb{P}[X_0(\delta) = 0]$ is the stationary probability that the hub runs out of resources in the original problem under the policy $\pi^S(\delta)$. Moreover, if there exist some constants $\bar{q} > 0$, $\beta \in \left(\frac{m}{m+n}, 1\right)$ and $\varepsilon > 0$ such that $q_{i0}, q_{0i} \leq \frac{b}{n}$ and $\left(1 - \beta\right)^2 \cdot \min_{i \leq n} \gamma_i(\beta) \geq \frac{b}{n}$, then

$$\mathbb{P}[X_0(\delta) \leq 0] \leq \exp \left( - \frac{b}{2} \cdot \frac{\delta^2}{m + n} \right).$$

Note that we restrict to static pricing in the spoke problem (EC-59). We can solve the best possible bound $V^{S,R} = \min_{\lambda \geq 0} V^{S,\lambda}$ and compute a static policy $\pi^S(\delta)$ from a perturbed problem $V^{S,R}(\delta) = \min_{\lambda \geq 0} V^{S,\lambda} - \delta \lambda$. Let $V^S(\delta)$ denote the performance of policy $\pi^S(\delta)$ in the original problem. Analogous to Section 4, Theorem EC4.1 shows that $\pi^S(\delta)$ converges to the optimal static policy in the large network regime with a proper choice of $\delta$.

**Theorem EC4.1.** The average revenue $V^S(\delta)$ of the static pricing policy $\pi^S(\delta)$ satisfies

$$0 \leq V^{S,R} - V^S(\delta) \leq (\bar{r} + \bar{\omega}) \cdot \mathbb{P}[X_0(\delta) = 0] + \bar{r} \cdot \frac{\delta}{m - \delta},$$

where $\mathbb{P}[X_0(\delta) = 0]$ is the stationary probability that the hub runs out of resources in the original problem under the policy $\pi^S(\delta)$. Moreover, if there exist some constants $\bar{q} > 0$, $\beta \in \left(\frac{m}{m+n}, 1\right)$ and $\varepsilon > 0$ such that $q_{i0}, q_{0i} \leq \frac{b}{n}$ and $\left(1 - \beta\right)^2 \cdot \min_{i \leq n} \gamma_i(\beta) \geq \frac{b}{n}$, then

$$\mathbb{P}[X_0(\delta) \leq 0] \leq \exp \left( - \frac{b}{2} \cdot \frac{\delta^2}{m + n} \right).$$

Since the functions $\gamma_i(\beta)$ are concave, if $\gamma_i(\beta)$ are continuously differentiable, this is equivalent to requiring $\min_{i \leq n} \gamma_i'(\frac{m}{m+n}) \geq \frac{2}{n} \cdot (\frac{m+n}{n})^2$ for some $\varepsilon > 0$. 

EC-39
As a result, we can rewrite $h_\leq 0$ where

$$V^R - V^\beta(\delta) \leq O \left( \sqrt{\frac{\ln n}{n}} \right)$$

if we set $\delta = \sqrt{\frac{1}{b} \cdot (m + n) \cdot \ln n}$.

**EC4.1 Proof of Theorem EC4.1**

We first rewrite the spoke problem (EC-59). An optimal solution to (EC-59) satisfies $d_{i0}, d_{i1} > 0$, and we let $\beta = \frac{q_{i0} d_{i0}}{q_{i0} d_{i0}}$. By the flow balance constraint in (EC-59), we have $p_i(x + 1) = \beta \cdot p_i(x)$, and hence

$$p_i(x) = \beta^x \cdot p_i(0), \quad \forall \ x \in [m].$$ (EC-60)

Since these probabilities sum up to one, we have

$$p_i(0) = \left(1 + \sum_{x=1}^{m} \beta^x \right)^{-1} = \begin{cases} \frac{1}{m+1} & \text{if } \beta = 1, \\ \frac{1}{1 - \beta} & \text{otherwise.} \end{cases}$$ (EC-61)

The first part of the objective of $h^i_\beta(\lambda)$ can be written as

$$\sum_{x=0}^{m-1} p_i(x) \cdot \left[ q_{i0} \cdot r_{i0}(d_{i0}) + \beta \cdot q_{i0} \cdot r_{i0}(d_{i0}) \right] = \left(1 - p_i(m)\right) \cdot \left[ q_{i0} \cdot r_{i0}(d_{i0}) + \beta \cdot q_{i0} \cdot r_{i0}(d_{i0}) \right].$$

As a result, we can rewrite $h^i_\beta(\lambda)$ as

$$h^i_\beta(\lambda) = \max_{\beta \geq 0} A^m(\beta) \cdot \gamma_i(\beta) - \lambda \cdot B^m(\beta),$$ (EC-62)

where

$$A^m(\beta) = 1 - p_i(m) = 1 - \beta^m \cdot p_i(0) = \begin{cases} \frac{m+1}{1 - \beta} & \text{if } \beta = 1, \\ \frac{1}{1 - \beta^m} & \text{otherwise,} \end{cases}$$

and

$$B^m(\beta) = \sum_{x=0}^{m} x \cdot p_i(x) = p_i(0) \cdot \sum_{x=1}^{m} x \beta^x = \begin{cases} \frac{m}{2} \cdot \beta^{m+2} & \text{if } \beta = 1, \\ \frac{m}{(1 - \beta)(1 - \beta^{m+1})} & \text{otherwise.} \end{cases}$$

We first provide some useful properties for $A^m(\beta)$ and $B^m(\beta)$.

**Lemma EC4.2 (Monotonicity).** $B^m(\beta)$ is strictly increasing in $\beta \geq 0$.

**Proof.** For any $\beta_1 < \beta_2$ and $i \in \{1, 2\}$, let $Z_i$ be a discrete random variable with support $[0 : m]$ and density function $g_i(\cdot)$ specified by (EC-60) and (EC-61) using a parameter $\beta_i$. Since for any $0 \leq x < y \leq m$, $\frac{g_i(y)}{g_i(x)} = \beta_1^{y-x} < \beta_2^{y-x} = \frac{g_i(y)}{g_i(x)}$, $Z_2$ dominates $Z_1$ in the monotone likelihood ratio order (see Section 1.C of Shaked and Shanthikumar [2007]). Hence, $Z_2$ first-order stochastically dominates $Z_1$ and as a result, $B^m(\beta_2) = \mathbb{E}[Z_2] > \mathbb{E}[Z_1] = B^m(\beta_1)$; the strict inequality is because $Z_1$ and $Z_2$ have distinct density functions (see Theorem 1.A.8 of Shaked and Shanthikumar [2007]). \( \square \)

**Lemma EC4.3 (Uniform convergence of $A^m(\beta)$ and $B^m(\beta)$).** Let $A^\infty(\beta) = \begin{cases} 1 & \text{if } 0 \leq \beta \leq 1, \\ \frac{1}{\beta} & \text{if } \beta > 1 \end{cases}$ and
\[ B^\infty(\beta) = \begin{cases} \frac{\beta}{1-\beta} & \text{if } 0 \leq \beta < 1 \\ \infty & \text{if } \beta \geq 1 \end{cases} . \]

Then,

1. \( 0 \leq A^\infty(\beta) - A^m(\beta) \leq \frac{1}{m+1} \) for all \( \beta \geq 0 \);

2. \( 0 \leq B^\infty(\beta) - B^m(\beta) \leq \frac{(m+1)\beta^{m+1}}{1-\beta} \) for all \( \beta \in [0,1) \), and \( \lim_{m \to \infty} B^m(\beta) = \infty \) for all \( \beta \geq 1 \).

**Proof.** To see the first part, note that for any \( \beta \), we have

\[ 0 \leq A^\infty(\beta) - A^m(\beta) = \frac{\beta - \beta^{m+1}}{1-\beta^{m+1}} = \frac{\beta m^m}{\sum_{i=0}^m \beta^i} \leq \frac{1}{m+1} . \]

If \( \beta = 1 \), \( A^\infty(\beta) - A^m(\beta) = 1 - \frac{m}{m+1} = \frac{1}{m+1} . \) If \( \beta > 1 \), we have

\[ 0 \leq A^\infty(\beta) - A^m(\beta) = \frac{\beta - 1}{\beta \cdot (\beta^{m+1} - 1)} = \frac{1}{\sum_{i=0}^m \beta^{i+1}} \leq \frac{1}{m+1} . \]

For the second part, if \( \beta < 1 \), we have

\[ 0 \leq B^\infty(\beta) - B^m(\beta) = \frac{(m+1) \cdot \beta^{m+1}}{1-\beta^{m+1}} \leq \frac{(m+1) \cdot \beta^{m+1}}{1-\beta} . \]

If \( \beta \geq 1 \), it is easy to see that \( \lim_{m \to \infty} B^m(\beta) = \infty \). \( \square \)

**Lemma EC4.4 (Derivatives of \( A^m(\beta) \) and \( B^m(\beta) \)).** The derivatives of \( A^m(\beta) \) and \( B^m(\beta) \) satisfy

1. \( 0 \geq \frac{d}{d\beta} A^m(\beta) \geq -m\beta^{m-1} \) for all \( \beta \in [0,1) \); and

2. \( 0 \leq \frac{d}{d\beta} B^m(\beta) \leq \frac{d}{d\beta} B^\infty(\beta) = \frac{1}{(1-\beta)^2} \) for all \( \beta \in [0,1) \).

**Proof.** For the first part, note that for any \( \beta \in [0,1) \) we have

\[ \frac{d}{d\beta} A^m(\beta) = \frac{\beta^{m-1}}{(1-\beta^{m+1})^2} \cdot \left((m+1)\beta - \beta^{m+1} - m\right) . \]

Since \( 1 - \beta^{m+1} = (1 - \beta) \cdot (1 + \beta + \cdots + \beta^m) \) and

\[
(m+1)\beta - \beta^{m+1} - m = -(1 - \beta) \cdot (m + \beta^{m+1} - \beta) = -(1 - \beta) \cdot (m - \beta^2 - \cdots - \beta^{m})
= -(1 - \beta)^2 \cdot (m + (m-1)\beta + (m-2)\beta^2 + \cdots + \beta^{m-1}),
\]

we have

\[ 0 \geq \frac{d}{d\beta} A^m(\beta) = -\beta^{m-1} \cdot \frac{(m + (m-1)\beta + (m-2)\beta^2 + \cdots + \beta^{m-1})}{(1 + \beta + \cdots + \beta^m)^2} \geq -m\beta^{m-1} , \]

where the last inequality is because

\[ m + (m-1)\beta + (m-2)\beta^2 + \cdots + \beta^{m-1} \leq m \cdot (1 + \beta + \cdots + \beta^m) \leq m \cdot (1 + \beta + \cdots + \beta^m)^2 . \]

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For the second part, since \( B^\infty(\beta) - B^m(\beta) = \frac{(m+1)^2 \beta^m}{1 - \beta^{m+1}} \), we have
\[
\frac{d}{d\beta} B^\infty(\beta) - \frac{d}{d\beta} B^m(\beta) = \frac{(m+1)^2 \beta^m}{(1 - \beta^{m+1})^2} \geq 0.
\]
Thus,
\[
0 \leq \frac{d}{d\beta} B^m(\beta) \leq \frac{1}{(1 - \beta)^2},
\]
where the first inequality is because \( B^m(\beta) \) is increasing in \( \beta \) by Lemma EC4.3.

**EC4.1.1 Proof of Part One**

Let \( \lambda(\delta) \) denote an optimal solution to the perturbed problem \( V^{s,R}(\delta) \), \( d_{i0} \) and \( d_{0i} \) an optimal solution to the spoke problem \( h^s_2(\lambda(\delta)) \), and \( \beta_i = \frac{q_{0i} d_{0i}}{q_{0i} d_{i0}} \). Let random variables \( X_i(\delta) \) and \( \tilde{X}_i(\delta) \) denote an optimal solution to the perturbed problem \( \pi^s(\delta) \) in the original and the relaxed systems, respectively. Then,
\[
V^s(\delta) = \sum_{i=1}^{n} \left\{ q_{0i} \cdot \mathbb{P}[X_i(\delta) \geq 1] \cdot r_{i0}(d_{i0}) + q_{0i} \cdot \mathbb{P}[X_0(\delta) \geq 1] \cdot r_{0i}(d_{0i}) \right\}
= \sum_{i=1}^{n} \left\{ q_{0i} \cdot \mathbb{P}[X_0(\delta) \geq 1] \cdot \frac{d_{0i}}{d_{i0}} \cdot r_{i0}(d_{i0}) + q_{0i} \cdot \mathbb{P}[X_0(\delta) \geq 1] \cdot r_{0i}(d_{0i}) \right\}
= \sum_{i=1}^{n} \mathbb{P}[X_0(\delta) \geq 1] \cdot \left\{ q_{0i} \cdot r_{0i}(d_{0i}) + \beta_i \cdot q_{0i} \cdot r_{i0}(d_{i0}) \right\}
= \mathbb{P}[X_0(\delta) \geq 1] \cdot \sum_{i=1}^{n} \gamma_i(\beta_i),
\]
where the second equality is from the flow balance equation \( q_{0i} \cdot \mathbb{P}[X_i(\delta) \geq 1] \cdot d_{0i} = q_{0i} \cdot \mathbb{P}[X_0(\delta) \geq 1] \cdot d_{i0} \), and the last equality is because \( d_{0i} \) and \( d_{0i} \) are optimal to \( \gamma_i(\beta_i) \).

On the other hand, analogous to Section 4.2, \( V^{s,R}(\delta) \) is equal to the performance of \( \pi^s(\delta) \) in the relaxed system, thus
\[
V^{s,R}(\delta) = \sum_{i=1}^{n} A^m(\beta_i) \gamma_i(\beta_i) \leq \sum_{i=1}^{n} \gamma_i(\beta_i).
\]
Combining (EC-63) and (EC-64) implies
\[
V^{s,R}(\delta) - V^s(\delta) \leq \left( 1 - \mathbb{P}[X_0(\delta) \geq 1] \right) \cdot \sum_{i=1}^{n} \gamma_i(\beta_i) \leq \mathbb{P}[X_0(\delta) = 0] \cdot (\bar{r} + \bar{\omega}) \cdot \sum_{i=1}^{n} q_{0i},
\]
where the last inequality is due to Lemma EC1.4. Finally, analogous to Lemma 1, we have
\[
V^{s,R}(\delta) \leq V^{s,R} \leq V^{s,R}(\delta) + \bar{r} \cdot \frac{\delta}{m - \delta}.
\]

We consider the same relaxed system as in Section 4.2.
From (EC-65) and (EC-66) we have

\[ V^{S,R} - V^S(\delta) = \left( V^{S,R} - V^{S,R}(\delta) \right) + \left( V^{S,R}(\delta) - V^S(\delta) \right) \leq (\bar{r} + \bar{\omega}) \cdot P[X_0(\delta) = 0] + \bar{r} \cdot \frac{\delta}{m - \delta}, \]

which is analogous to Theorem 1.

**EC4.1.2 Proof of Part Two**

From Corollary 1 we have

\[ \text{EC4.1.2 Proof of Part Two} \]

From (EC-65) and (EC-66) we have

\[ \beta \geq \frac{2q_{0i} \cdot (\bar{r} + \bar{\omega})}{\lambda + 2q_{0i} \cdot (\bar{r} + \bar{\omega})} \]

with \( b = \frac{1}{1 + c} \). Moreover, if we choose \( \delta = \sqrt{\frac{m+n}{b} \cdot \ln n} \), then \( V^S - V^S(\delta) \leq O\left( \sqrt{\frac{\ln n}{m}} \right) \) when \( m \) and \( n \) grow at the same rate. We now show that under the additional assumptions regarding \( q_{ij} \) and \( \gamma_i(\beta) \), we can find such a constant \( c \).

Let \( \beta_i^m(\lambda) \) denote an optimal solution to the spoke problem \( h_i^S(\lambda) \) with some \( \lambda \geq 0 \). Lemma **EC4.5** shows that \( \beta_i^m(\lambda) \) shrinks towards zero when \( \lambda \) becomes large.

**Lemma EC4.5.** For any \( \lambda > 0 \) and \( i \in [n] \), \( \beta_i^m(\lambda) \leq \frac{2q_{0i} \cdot (\bar{r} + \bar{\omega})}{\lambda + 2q_{0i} \cdot (\bar{r} + \bar{\omega})} < 1 \) when \( m \) is large enough.

**Proof.** Let \( h_i^S(\beta, \lambda) = A^m(\beta) \gamma_i(\beta) - \lambda B^m(\beta) \). It suffices to show that \( h_i^S(\beta, \lambda) \leq h_i^S(0, \lambda) = 0 \) for all \( \beta \geq \bar{\beta} \triangleq \frac{2q_{0i} \cdot (\bar{r} + \bar{\omega})}{\lambda + 2q_{0i} \cdot (\bar{r} + \bar{\omega})} \) and large enough \( m \). Since \( \lim_{m \to \infty} B^m(\bar{\beta}) = \frac{\bar{\beta}}{1 - \bar{\beta}} \) by Lemma **EC4.3**, there exists some \( \bar{m} \in \mathbb{N}_+ \) such that for all \( m \geq \bar{m} \), \( B^m(\bar{\beta}) \geq \frac{1}{2} \cdot \frac{\bar{\beta}}{1 - \bar{\beta}} \). Thus, for any \( \beta \geq \bar{\beta} \) and \( m \geq \bar{m} \),

\[ h_i^S(\beta, \lambda) = A^m(\beta) \gamma_i(\beta) - \lambda B^m(\beta) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}) - \lambda \gamma_i(\beta) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}) - \frac{\lambda}{2} \cdot \frac{\bar{\beta}}{1 - \bar{\beta}} = 0, \]

where the first inequality is due to the facts that \( A^m(\beta) \leq 1 \), \( \gamma_i(\beta) \leq q_{0i} \cdot (\bar{r} + \bar{\omega}) \) from Lemma **EC1.4**, and \( B^m(\bar{\beta}) \) is increasing in \( \beta \) by Lemma **EC4.2**.

Lemma **EC4.6** shows that \( \beta_i^m(\lambda) \) is large when \( \lambda \) is small.

**Lemma EC4.6.** For any \( \bar{\beta} \leq 1 \), \( \beta_i^m(\lambda) > \bar{\beta} \) if \( \lambda < (1 - \bar{\beta})^2 \gamma_i'(\bar{\beta}) \) and \( m \) is large enough.

**Proof.** Let \( \varepsilon = (1 - \bar{\beta})^2 \gamma_i'(\bar{\beta}) - \lambda > 0 \) and \( \delta = \frac{1}{2} \cdot \frac{\varepsilon}{(\bar{r} + \bar{\omega}) \cdot (q_{0i} + q_{0j})} \). Since by Lemmas **EC4.3** and **EC4.4**, \( A^m(\beta) \) and \( \frac{d}{dm} A^m(\beta) \) converge to 1 and 0 uniformly on \([0, \bar{\beta}]\) when \( m \) grows to infinity, there exists a constant \( m \) independent of spoke \( i \) such that for all \( m \geq m \) and \( \beta \leq \bar{\beta}, A^m(\beta) \geq 1 - \delta \) and...
\( \frac{d}{d\beta} A^m(\beta) \geq -\delta \). Thus, the derivative of the objective of the spoke problem satisfies

\[
\frac{d}{d\beta} \left( A^m(\beta) \gamma_i(\beta) - \lambda B^m(\beta) \right) = \left( \frac{d}{d\beta} A^m(\beta) \right) \gamma_i(\beta) + A^m(\beta) \gamma_i'(\beta) - \lambda \frac{d}{d\beta} B^m(\beta) \\
\geq -\delta \cdot \gamma_i(\beta) + (1 - \delta) \cdot \gamma_i'(\beta) - \lambda \frac{d}{d\beta} B^m(\beta) \\
\geq \gamma_i'(\beta) - \frac{\lambda}{(1 - \beta)^2} - \delta \cdot (\bar{r} + \bar{\omega}) \cdot (q_{i0} + q_{i1}) \\
\geq \frac{\varepsilon}{(1 - \beta)^2} - \frac{\varepsilon}{2} > 0,
\]

where (i) is because \( \frac{d}{d\beta} B^m(\beta) \leq \frac{1}{(1 - \beta)^2} \) by Lemma EC4.4 and \( \gamma_i(\beta) \leq q_{i0}(\bar{r} + \bar{\omega}) \) and \( \gamma_i'(\beta) \leq q_{i0}(\bar{r} + \bar{\omega}) \) by Lemma EC1.4, and (ii) is from the definitions of \( \varepsilon \) and \( \delta \). Therefore, the objective is strictly increasing in \([0, \beta]\) and as a result, \( \beta^m(\lambda) > \bar{\beta} \) when \( m \geq m \).

Let \( \lambda(\delta) \) denote an optimal solution to the perturbed problem \( V^{s,R}(\delta) \). First, we have \( \lambda(\delta) \geq \frac{\varepsilon}{n} \). If not, then from Lemma EC4.6 \( \beta_i^m(\lambda(\delta)) > \beta \) for all \( i \in [n] \) when \( m \) is large enough. Since \( B^m(\beta) \) is increasing in \( \beta \) by Lemma EC4.2 \( \sum_{i=1}^{n} B^m(\beta_i^m(\lambda(\delta))) \geq nB^m(\bar{\beta}) \). Moreover, since \( \lim_{m \to \infty} B^m(\beta) = \frac{\beta}{1 - \beta} \) for all \( \beta < 1 \) and \( \frac{\beta}{1 - \beta} > m \), \( \sum_{i=1}^{n} B^m(\beta_i^m(\lambda(\delta))) > m \) when \( m \) is large enough. But this contradicts with the fact that \( \sum_{i=1}^{n} B^m(\beta_i^m(\lambda(\delta))) \leq m - \delta \) by the optimality condition of \( \lambda(\delta) \). Hence, \( \lambda(\delta) \geq \frac{\varepsilon}{n} \), and from Lemma EC4.5, we have

\[
\beta_i^m(\lambda(\delta)) \leq \frac{2q_{i0} \cdot (\bar{r} + \bar{\omega})}{\lambda(\delta) + 2q_{i0} \cdot (\bar{r} + \bar{\omega})} \leq \frac{2q \cdot (\bar{r} + \bar{\omega})}{\varepsilon + 2q \cdot (\bar{r} + \bar{\omega})} < 1.
\]

Finally, letting \( c = \frac{\bar{\beta}}{1 - \beta} \), we have \( \mathbb{E}[X_i(\delta)] = B^m(\beta_i^m(\lambda(\delta))) \leq \frac{\beta_i^m(\lambda(\delta))}{1 - \beta_i^m(\lambda(\delta))} \leq \frac{\bar{\beta}}{1 - \beta} = c \).

**EC4.2 Proof of Proposition 8**

Since spokes are identical, we drop the index \( i \) and let \( \gamma(\beta) = \gamma_i(\beta) \) and \( h^s(\lambda) = h^s_i(\lambda) \) for ease of notation. First, it is easy to see that \( V^F = \hat{\gamma}(1) \) and thus \( V(\pi^F) = \frac{m}{m + n} \hat{\gamma}(1) \).

Let \( \rho = \frac{m}{n} \) and \( \lambda = n\lambda \). Then

\[
V^{s,R} = \min_{\lambda \geq 0} \left\{ m\lambda + n \cdot h^s(\lambda) \right\} \\
= \min_{\lambda \geq 0} \left\{ m\lambda + n \cdot \max_{\beta \geq 0} \left\{ A^m(\beta) \cdot \gamma(\beta) - \lambda \cdot B^m(\beta) \right\} \right\} \\
= \min_{\lambda \geq 0} \left\{ \rho \lambda + \max_{\beta \geq 0} \left\{ A^m(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^m(\beta) \right\} \right\}.
\]

Lemma EC4.7 shows that solving the problem with \( m = \infty \) provides an upper bound on \( V^{s,R} \) in the large network limit.
Lemma EC4.7 (Interchange of operations).

\[
\lim_{n \to \infty} V_{s,r} \leq \min_{\lambda \geq 0} \left\{ \rho \lambda + \max_{\beta \geq 0} \left\{ A^\infty(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^\infty(\beta) \right\} \right\}. \tag{EC-67}
\]

We prove Lemma EC4.7 at the end of this section. In the following, we optimize the right-hand side of (EC-67). Note that

\[
\min_{\lambda \geq 0} \left\{ \rho \lambda + \max_{\beta \geq 0} \left\{ A^\infty(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^\infty(\beta) \right\} \right\} = \min_{\lambda \geq 0} \max_{\beta \in [0,1]} \left\{ \rho \lambda + \hat{\gamma}(\beta) - \hat{\lambda} \cdot \frac{\beta}{1-\beta} \right\} \tag{EC-68}
\]

where the first equation is because we can restrict the domain to \( \beta \in [0,1] \) by Lemma EC4.9. Since \( \hat{\gamma}(\beta) \) is concave by Lemma EC1.3, the second equation is due to the fact that the right-hand side is a convex program and strong duality holds. Since \( \hat{\gamma}(\beta) \) together with \( \frac{\beta}{1-\beta} \) is increasing in \( \beta \) by Lemma EC1.2, the optimal solution \( \beta^* \) satisfies \( \frac{\beta^*}{1-\beta^*} = \rho \) and thus \( \beta^* = \frac{\rho}{1+\rho} = \frac{m}{m+n} \). Combining (EC-67) and (EC-68), we have \( \lim_{n \to \infty} V_{s,r} \leq \hat{\gamma}(\beta^*) = \hat{\gamma}\left(\frac{m}{m+n}\right) \), and this provides an upper bound on the performance of any static pricing policy in the large network regime.

On the other hand, for any \( n \), let \( \tilde{\rho} = \rho - O\left(\sqrt{\ln n/n}\right) \), \( \tilde{\beta} = \frac{\tilde{\rho}}{1+\tilde{\rho}} = \frac{m}{m+n} \), and \( \tilde{d}_0 \) and \( \tilde{d}_0^i \) an optimal solution to \( \hat{\gamma}(\beta^*) \). Based on the analysis in Appendix EC4.1 (especially equation (EC-63) and Section EC4.1.2), the performance of the static policy using \( d_0 \) and \( d_0^i \) converges to the upper bound \( \hat{\gamma}\left(\frac{m}{m+n}\right) \) in the large network regime. Thus, the bound is tight and \( V^s = \hat{\gamma}\left(\frac{m}{m+n}\right) \) in the limit.

**EC4.2.1 Proof of Lemma EC4.7**

We let \( h^m(\beta, \hat{\lambda}) \triangleq A^m(\beta) \hat{\gamma}(\beta) - \hat{\lambda} B^m(\beta) \), \( h^m(\hat{\lambda}) = \max_{\beta \geq 0} h^m(\beta, \hat{\lambda}) \) and \( \beta^m(\hat{\lambda}) \) denote an optimal solution to \( h^m(\hat{\lambda}) \). Let \( h^\infty(\beta, \hat{\lambda}) \triangleq A^\infty(\beta) \hat{\gamma}(\beta) - \hat{\lambda} B^\infty(\beta) \) and \( h^\infty(\hat{\lambda}) \) and \( \beta^\infty(\hat{\lambda}) \) denote the maximum value and point over \( \beta \) for a given \( \hat{\lambda} \). From Lemma EC1.4, we have \( \hat{\gamma}(\beta) \leq c_d \triangleq n \cdot q_0 \cdot (\bar{r} + \bar{\omega}) \) for all \( \beta \geq 0 \).

**Lemma EC4.8.** For any \( \hat{\lambda} > 0 \), \( \beta^m(\hat{\lambda}) \leq \frac{2c_d}{\hat{\lambda}+2c} < 1 \) when \( m \) is large enough.

**Proof.** This is exactly Lemma EC4.5 \( \square \)

**Lemma EC4.9.** \( \beta^\infty(0) = 1 \) and \( \beta^\infty(\hat{\lambda}) \leq \frac{c_d}{\hat{\lambda}+c} < 1 \) for any \( \hat{\lambda} > 0 \).

**Proof.** For any \( \hat{\lambda} > 0 \) and \( \beta \geq \tilde{\beta} = \frac{c_d}{\hat{\lambda}+c} \), analogous to the proof of Lemma EC4.5, we have

\[
h^\infty(\beta, \hat{\lambda}) = A^\infty(\beta) \hat{\gamma}(\beta) - \hat{\lambda} B^\infty(\beta) \leq c - \hat{\lambda} \cdot \frac{\beta}{1-\beta} = 0 = h^\infty(0, \hat{\lambda}).
\]

Thus, \( \beta^\infty(\hat{\lambda}) \leq \frac{c_d}{\hat{\lambda}+c} \). Suppose \( \hat{\lambda} = 0 \). Then \( h^\infty(\beta, \hat{\lambda}) = A^\infty(\beta) \hat{\gamma}(\beta) \). If \( \beta \leq 1 \), \( A^\infty(\beta) \hat{\gamma}(\beta) = \hat{\gamma}(\beta) \) is increasing by Lemma EC1.2. If \( \beta \geq 1 \), \( A^\infty(\beta) \hat{\gamma}(\beta) = \frac{\hat{\gamma}(\beta)}{\beta} \) with derivative \( \frac{d}{d\beta} \left( \frac{\hat{\gamma}(\beta)}{\beta} \right) = \frac{\beta \hat{\gamma}'(\beta) - \hat{\gamma}(\beta)}{\beta^2} \leq 0 \) because the numerator is non-positive by Lemma EC1.5. Thus, \( \beta^\infty(0) = 1 \). \( \square \)
Lemma EC4.10. For any \( \hat{\lambda} \geq 0 \), \( \lim_{m \to \infty} h^m(\hat{\lambda}) = h^\infty(\hat{\lambda}) \).

Proof. If \( \hat{\lambda} > 0 \), from Lemmas EC4.8 and EC4.9, we can restrict the domain to be \( \beta \in \left[ 0, \frac{2c}{2c+\lambda} \right] \). From Lemma EC4.3, \( A^m(\beta) \) and \( B^m(\beta) \) converge to \( A^\infty(\beta) \) and \( B^\infty(\beta) \) uniformly on \( \beta \in \left[ 0, \frac{2c}{2c+\lambda} \right] \) as \( m \) goes to infinity; this implies \( \lim_{m \to \infty} h^m(\hat{\lambda}) = h^\infty(\hat{\lambda}) \). If \( \hat{\lambda} = 0 \), the result follows analogously because \( A^m(\beta) \) converges to \( A^\infty(\beta) \) uniformly on \( \beta \geq 0 \) by Lemma EC4.3.

Now we are ready to prove Lemma EC4.7. By the definition of \( V^{S,R} \), for any \( \hat{\lambda} \geq 0 \) we have

\[
V^{S,R} \leq \rho \hat{\lambda} + \max_{\beta \geq 0} \left\{ A^m(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^m(\beta) \right\}.
\]

Taking limits on both sides and noting Lemma EC4.10, we have

\[
\lim_{n \to \infty} V^{S,R} \leq \rho \hat{\lambda} + \max_{\beta \geq 0} \left\{ A^\infty(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^\infty(\beta) \right\}.
\]

Finally, minimizing the right-hand side over \( \hat{\lambda} \geq 0 \) gives the desired result.

**EC4.3 More Details on Example 1**

Since \( q_i = q_0 = \frac{1}{2n} \) and all private values are uniformly distributed on \([0, 1]\), \( \gamma_i(\beta) = \frac{1}{2n} \cdot \frac{\beta}{1+\beta} \) and \( d_{0i}(\beta) = \frac{\beta}{1+\beta} \) and \( d_{i0}(\beta) = \frac{1}{1+\beta} \) are optimal to \( \gamma_i(\beta) \). Since spokes are identical, we drop the index \( i \) and let \( \gamma(\beta) = \gamma_i(\beta) \) for ease of notation. Let \( \rho = \frac{m}{n} = \frac{2}{3}, \hat{\gamma}(\beta) = n \cdot \gamma(\beta) = \frac{1}{2} \cdot \frac{\beta}{1+\beta} \) and \( \hat{\lambda} = n \lambda \).

Letting \( \beta^* = \frac{1}{1+\rho} = \frac{m}{m+n} = \frac{2}{5} \), from Proposition 8 and the proof therein, \( \hat{\gamma}(\beta^*) = \frac{1}{7} \) is a tight upper bound on the performance of the optimal static policy in the large network limit, and the asymptotically optimal static policy converges to \( d_{i0}(\beta^*) = \frac{5}{7} \) and \( d_{0i}(\beta^*) = \frac{2}{7} \).

We now provide a lower bound on the Lagrangian relaxation bound \( V^R \) and show that it is strictly larger than \( V^{S,R} \) in the large network regime. Since \( \lim_{n \to \infty} \{ V^R - V^{OPT} \} = 0 \) by Corollary 2, this implies that no static pricing policy is asymptotically optimal in the regime. We can also get a simple dynamic pricing policy that is strictly better than the optimal static policy as a byproduct.

Since the number of resources per location is relatively small, it is beneficial to keep the number of resources in each spoke to be small, thus retaining some resources in the hub. Motivated by this, we consider a family of cutoff policies with some parameter \( k \) that keeps at most \( k \) resources in a spoke and uses static controls to manage these resources. Analogously, we can provide a bound on the performance of any cutoff policy with a parameter \( k \) by relaxing the constraint that the hub has non-negative resources with a dual variable \( \lambda \geq 0 \); the best performance bound is given by

\[
V^{k,R} = \min_{\lambda \geq 0} \left\{ \rho \hat{\lambda} + \max_{\beta \geq 0} \left\{ A^k(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^k(\beta) \right\} \right\},
\]

which is equivalent to maximizing the average revenue subject to the constraints that the expected number of resources in the hub is non-negative and we restrict to cut-off policies. The max-min
inequality implies
\[ V^{k,R} \geq \max_{\beta \geq 0} \min_{\lambda \geq 0} \left\{ \rho \lambda + A^k(\beta) \cdot \hat{\gamma}(\beta) - \hat{\lambda} \cdot B^k(\beta) \right\} \]
\[ = \max_{\beta \geq 0} \left\{ A^k(\beta) \hat{\gamma}(\beta) \text{ s.t. } B^k(\beta) \leq \rho \right\} \]
\[ \geq A^k(\tilde{\beta}) \hat{\gamma}(\tilde{\beta}) \text{ where } \tilde{\beta} : B^k(\tilde{\beta}) = \rho. \]

Take \( k = 2 \). We can solve \( \tilde{\beta} = \frac{\sqrt{33} - 1}{8} \approx 0.593 \) from \( B^2(\tilde{\beta}) = \frac{2 \beta^2 + \beta}{\beta^2 + \beta + 1} = \rho = \frac{2}{3} \). Since \( V^R \geq V^{k,R} \) for any \( k \in \mathbb{N} \), we have
\[ V^R \geq V^{k=2,R} \geq A^{k=2}(\tilde{\beta}) \hat{\gamma}(\tilde{\beta}) \approx 0.152 > \frac{1}{7} \geq V^{s,R}. \]

Finally, analogous to the discussion at the end of Section EC4.2, we can construct cut-off policies with asymptotic performance equal to \( A^{k=2}(\tilde{\beta}) \hat{\gamma}(\tilde{\beta}) \) by using a perturbed \( \beta = \tilde{\beta} - O(\sqrt{\ln n/n}) \).

### EC5 Performance Analysis of Uniformly Related Hubs

In this section, we analyze the performance of policy \( \pi(\delta) \) for the special case with uniformly related hubs as described in Definition EC5.1.

**Definition EC5.1 (Uniformly Related Hubs).** A network with uniformly related hubs is a hub-and-spoke network with multiple hubs where:

1. for each spoke \( i \in [n] \), the revenue functions \( r_{ij}(d) \) and \( r_{ji}(d) \) are identical across hubs, i.e., \( r_{ij}(d) \) are identical across \( j \in [J] \) and so are \( r_{ji}(d) \);
2. for each spoke \( i \in [n] \), the request rates satisfy \( q_{ij} = c_i \cdot q_{ji} \geq 0 \) for all hubs and some constant \( c_i > 0 \);
3. for any two hubs \( j \) and \( j' \), the request rates and the maximum points of the revenue functions as defined in Assumption 1 satisfy \( d_{jj'}^* \cdot q_{jj'} = d_{jj'}^* \cdot q_{j'j} \).

Assumption EC5.1 is a form of symmetry in which, for each spoke, the revenue functions and ratio of requests are identical across hubs. We allow, however, revenue functions and request rates to be different across spokes. Note that a sufficient condition for part 3 is that, for any two hubs \( j \) and \( j' \), the request rates satisfy \( q_{jj'} = q_{j'j} \) and the revenue functions satisfy \( r_{jj'}(d) = r_{j'j}(d) \). Proposition EC5.1 shows that with uniformly related hubs, \( \mu_j = 0 \) for all \( j \in [J] \) constitutes an optimal solution to (16) (this is equivalent to the flow balance constraint in (16) being redundant) and the controls \( d_i(x,i,j) \) and \( d_i(x,j,i) \) in the Lagrangian policy derived from (16) are identical across hubs. We can show that when hubs are uniformly related, at optimality, the flow between each hub-spoke pair is balanced. Summing over each spoke we obtain that hubs are balanced. Moreover, using the Lagrangian policy in the original system, we obtain a unique stationary distribution with a special form.

**Proposition EC5.1.** For a network with uniformly related hubs, the following properties hold:

(a) The flow balance constraint of (16) is redundant;

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(b) For the Lagrangian policy \( \pi(\delta) \), all controls \( d_i(x, i, j) \) and \( d_i(x, j, i) \) are identical across hubs \( j \) and \( d_{jj} = d_{jj}' \), for all hub-to-hub requests;

(c) In the original system, the Markov chain with the policy \( \pi(\delta) \) has a single recurrent class \( \mathcal{C} \) and is aperiodic; thus with this policy we obtain a unique stationary distribution over states, which we denote by \( P(x_h, x_s) \), where \( x_h = (x_j)_{j \in [J]} \in \mathbb{N}^J \) and \( x_s = (x_i)_{i \in [n]} \in \mathbb{N}^n \) denote the number of resources in the hubs and the spokes, respectively;

(d) \( P(\cdot) \) is reversible in \( \mathcal{C} \); and

(e) Conditioned on a state \( x_s = (x_i)_{i \in [n]} \) of the spokes, \( P(\cdot) \) is uniform across the resources in the hubs, i.e., \( P(x_h, x_s) = P(x_h', x_s) \) for any \( x_h = (x_j)_{j \in [J]} \in \mathbb{N}^J \) and \( x_h' = (x_j')_{j \in [J]} \in \mathbb{N}^J \) with \( \sum_{j \in [J]} x_j = \sum_{j \in [J]} x_j' = m - \sum_{i \in [n]} x_i. \)

We prove Proposition [EC5.1] at the end of this section. Part (e) of this result implies that the number of resources in the spokes \( x_s \) only provides information on the number of resources in the hubs \( x_h \) through their summations \( \sum_{i \in [n]} x_i \), and vice versa; this follows directly from the reversibility property in part (d). We will use this fact to bound the probability that any hub is depleted in the following analysis. In particular, we consider a high multiplicity model and we show with a particular choice of the parameter \( \delta \), the depletion probability \( P[X_j(\delta) = 0] \) of each hub \( j \) of the original system diminishes to zero as the number of spokes \( n \) increases, the ratio \( \frac{m}{n} \) remains fixed, and the number of hubs \( J \) grows at rate \( o(n) \).

In the high multiplicity model, we assume the \( n \) spokes can be divided into \( S \) distinct spoke types and the number of spokes of each type \( s \) is fixed to be a proportion \( \alpha_s > 0 \) of \( n \). All spokes of a given type have the same revenue functions and have the same arrival rates into and out of each hub (these rates may vary across hubs).

From Corollary 1 and Proposition 7, we have that the probability that all hubs run out of resources in the original system goes to zero when \( n \) increases and the ratio \( \frac{m}{n} \) remains fixed. Because the resources in the hubs are uniformly distributed according to Proposition [EC5.1] part (e), the depletion probability of each hub \( j \) of the original system diminishes to zero as well, as we show in Proposition [EC5.2]

**Proposition EC5.2.** Let \( \alpha = \min_{s \in [S]} \alpha_s \) and let \( b = \frac{1}{1 + \frac{m}{(\alpha + 1)n}} \). In the high multiplicity model, the depletion probabilities of each hub \( j \) of the original system are equal and satisfy

\[
P[X_j(\delta) = 0] \leq \exp \left( - \frac{b}{8} \cdot \frac{\delta^2}{m + n - \frac{1}{2}\delta} \right) + \frac{J - 1}{2\delta + J - 1}.
\]

Putting Theorem 2 and Proposition [EC5.2] together, we obtain the following result.

**Corollary EC5.3.** Under the high multiplicity model, the Lagrangian policy \( \pi(\delta) \) with \( 0 \leq \delta < m \) satisfies

\[
V^\pi(\delta) \leq V^{OPT} \leq V^\pi(\delta) + \bar{r} \cdot \frac{\delta}{m - \delta} + (\bar{r} + \bar{\omega}) \cdot \left\{ \exp \left( - \frac{b}{8} \cdot \frac{\delta^2}{m + n - \frac{1}{2}\delta} \right) + \frac{J - 1}{2\delta + J - 1} \right\},
\]

where \( b \) is as in Proposition [EC5.2]. Moreover, if \( m \) and \( n \) grow at the same rate and \( J \) grows at a
that these revenue functions are identical across hubs by Definition EC5.1, Jensen’s inequality implies

\[ V^{\text{OPT}} - V^*(\delta) \leq O\left( \max \left\{ \sqrt{\frac{J}{n}}, \sqrt{\frac{\ln n}{n}} \right\} \right). \]

**Proof.** We first show in Lemma EC5.4 that the Lagrangian policies with dual variables \( \mu_j = 0 \) for all \( j \in [J] \) and any \( \lambda \geq 0 \) have the desired properties as stated in Proposition EC5.1 parts (a) and (b).

**Lemma EC5.4.** Let \( d_i(x, i, j) \), \( d_i(x, j, i) \), and \( d_{jj'} \) be the controls of a Lagrangian policy with dual variables \( \mu_j = 0 \) for all \( j \in [J] \) and some \( \lambda \geq 0 \). We have

1. \( d_i(x, i, j) \) and \( d_i(x, j, i) \) are identical across hubs; and
2. \( d_{jj'} = d_{jj'}^* \); and
3. the in-flow and out-flow of each hub \( j \) is balanced in expectation, i.e., for each \( j \in [J] \),

\[
\sum_{i=1}^{n} q_{ij} \sum_{x=0}^{m} p_i(x) \cdot d_i(x, i, j) + \sum_{j'=1}^{J} q_{jj'} \cdot d_{jj'} = \sum_{i=1}^{n} q_{ij} \sum_{x=0}^{m} p_i(x) \cdot d_i(x, j, i) + \sum_{j'=1}^{J} q_{jj'} \cdot d_{jj'}.
\]

**Proof.** First, \( d_{jj'} = \arg\max_{d \in [0, 1]} \left\{ r_{jj'}(d) + d \cdot (\mu_j' - \mu_j) \right\} = d_{jj'}^* \) when \( \mu_j = 0 \) for all \( j \in [J] \). Next, let \( p_i(x) \) be an optimal probability distribution to the spoke \( i \) problem. The controls \( d_i(x, j, i) \) and \( d_i(x + 1, i, j) \) are optimal to the concave problem

\[
\max_{d_{jj}, d_{ij} \in [0, 1]} \quad p_i(x) \cdot \sum_{j=1}^{J} q_{ji} \cdot r_{ji}(d_{ji}) + q_{ij} \cdot r_{ij}(d_{ij}) \quad \text{s.t.} \quad p_i(x) \cdot \sum_{j=1}^{J} q_{ji} \cdot d_{ji} = p_i(x + 1) \cdot \sum_{j=1}^{J} q_{ij} \cdot d_{ij}.
\]

Since \( r_{ij}(d) \) and \( r_{ji}(d) \) are strictly concave by Assumption 1, the solution is unique. Moreover, since these revenue functions are identical across hubs by Definition EC5.1, Jensen’s inequality implies that \( d_i(x, i, j) \) and \( d_i(x, j, i) \) are identical across hubs. Otherwise the average controls \( \frac{\sum_{j=1}^{J} q_{ji} \cdot d_{ij}}{\sum_{j=1}^{J} q_{ji}} \) and \( \frac{\sum_{j=1}^{J} q_{ij} \cdot d_{ij}}{\sum_{j=1}^{J} q_{ij}} \) are feasible and yield a strictly better objective. Finally, summing up the flow balance constraint in [15]:

\[
p_i(x) \cdot \sum_{j=1}^{J} q_{ji} \cdot d_i(x, j, i) = p_i(x + 1) \cdot \sum_{j=1}^{J} q_{ij} \cdot d_i(x + 1, i, j)
\]
on both sides over \( x \in [0 : m - 1] \) and noting that \( d_i(0, i, j) = 0 \) and \( d_i(m, j, i) = 0 \) for all \( j \in [J] \),

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we have
\[ \sum_{x=0}^{m} p_i(x) \cdot \sum_{j=1}^{J} q_{ji} \cdot d_i(x, j, i) = \sum_{x=0}^{m} p_i(x) \cdot \sum_{j=1}^{J} q_{ij} \cdot d_i(x, i, j), \]
which implies that the flow is balanced for each spoke. Since the controls are identical across hubs and the request rates satisfy \( q_{ij} = c_i \cdot q_{ji} \) for all \( j \in [J] \) and some constant \( c_i > 0 \) by Definition EC5.1, the flow between each hub-spoke pair is balanced, i.e., for any \( j \in [J] \) and \( i \in [n] \)
\[ \sum_{x=0}^{m} p_i(x) \cdot q_{ji} \cdot d_i(x, j, i) = \sum_{x=0}^{m} p_i(x) \cdot q_{ij} \cdot d_i(x, i, j). \]

Summing both sides over spokes plus the fact that \( q_{ij} > 0 \) and the request rates satisfy \( C \).

Part (d): The properties of the controls stated in parts (a) and (b) imply that the resulting Markov chain under the Lagrangian policy enjoys some helpful properties. Let \( x_h = (x_j)_{j=1}^{J} \in \mathbb{N}^{J} \) and \( x_s = (x_n)_{n=1}^{N} \in \mathbb{N}^{N} \) denote the number of resources in the hubs and the spokes, respectively, and let the system state be the resource levels \((x_h, x_s) \in \mathbb{N}^{J+N}\). Due to the assumption that the network topology is strongly connected and the fact that the controls for requests between a hub and a spoke are identical across the hubs, it is easy to see the Markov chain with the policy \( \pi(\delta) \) has a single recurrent class \( C = \{ (x_h, x_s) \in \mathbb{N}^{N+J} : \sum_{i \in [n]} x_i + \sum_{j \in [J]} x_j = m, \ x_i \in I_i \text{ for all } i \in [n] \} \) and is aperiodic. Hence the Lagrangian policy is a unichain policy, and with this policy the limiting distribution converges to a unique stationary distribution, which we denote by \( \mathbb{P}(x_h, x_s) \), independent of the initial state.

Part (c): The properties of the controls stated in parts (a) and (b) imply that the resulting Markov chain under the Lagrangian policy enjoys some helpful properties. Let \( x_h = (x_j)_{j=1}^{J} \in \mathbb{N}^{J} \) and \( x_s = (x_n)_{n=1}^{N} \in \mathbb{N}^{N} \) denote the number of resources in the hubs and the spokes, respectively, and let the system state be the resource levels \((x_h, x_s) \in \mathbb{N}^{J+N}\). Due to the assumption that the network topology is strongly connected and the fact that the controls for requests between a hub and a spoke are identical across the hubs, it is easy to see the Markov chain with the policy \( \pi(\delta) \) has a single recurrent class \( C = \{ (x_h, x_s) \in \mathbb{N}^{N+J} : \sum_{i \in [n]} x_i + \sum_{j \in [J]} x_j = m, \ x_i \in I_i \text{ for all } i \in [n] \} \) and is aperiodic. Hence the Lagrangian policy is a unichain policy, and with this policy the limiting distribution converges to a unique stationary distribution, which we denote by \( \mathbb{P}(x_h, x_s) \), independent of the initial state.

Part (d): We say that \( \mathbb{P}(\cdot) \) is reversible in \( C \) if for any two states \((x_h, x_s), (x'_h, x'_s) \in C\),
\[ \mathbb{P}(x_h, x_s) \cdot p(x_h, x_s, x'_h, x'_s) = \mathbb{P}(x'_h, x'_s) \cdot p(x'_h, x'_s, x_h, x_s), \]
where \( p(x_h, x_s, x'_h, x'_s) \) denotes the transition probability from a state \((x_h, x_s)\) to a state \((x'_h, x'_s)\) with the Lagrangian policy. This part is a direct consequence of Theorem 6.5.1 in [Durrett 2010], which provides a necessary and sufficient condition for an irreducible Markov chain to have a reversible measure.

Lemma EC5.5 (Theorem 6.5.1 in [Durrett 2010]). Consider an irreducible Markov chain with states denoted by \( x \) and transition probabilities denoted by \( p \). A necessary and sufficient condition for the existence of a reversible measure is that

1. \( p(x, x') > 0 \) implies \( p(x', x) > 0 \) for any two states \( x \) and \( x' \); and
2. for any loop of states \( x_0, x_1, \cdots, x_N = x_0 \) with \( \prod_{k=1}^{N} p(x_k, x_{k-1}) > 0 \), \( \prod_{k=1}^{N} \frac{p(x_{k-1}, x_k)}{p(x_k, x_{k-1})} = 1. \)

We show the transition probabilities \( p \) induced by the Lagrangian policy satisfy the conditions in Lemma EC5.5 within the recurrent class \( C \). The first part simply follows from the fact that
relaxed system is no larger than $m$ due to part (b).

We now prove the second part. Let $s_k$ denote the request that induces the transition from state $x_k$ to $x_{k+1}$, $\tilde{s}_k$ denote the reverse trip of $s_k$, and $x_{k,i}$ denote the number of resources in spoke $i$ when the state is $x_k$. Let $S_h = \{ s_k = (i, j') : j, j' \in [J] \}$ denote the set of requests between the hubs and $S_i = \{ s_k = (i, j) : j \in [J] \}$ denote the set of requests between a hub and spoke $i$.

By partitioning requests into these sets, it is equivalent to show that

$$\prod_{k : s_k \in S_h} q_{ij} \cdot d^*_{ij} = \prod_{i \in [n]} \prod_{k : s_k \in S_i} q_{ij} \cdot d_i(x_{k,i}, s_k) \cdot \prod_{i \in [n]} \prod_{k : s_k \in S_i} q_{ij} \cdot d_i(x_{k+1,i}, \tilde{s}_k).$$

It suffices to show that $\varepsilon_h = \varepsilon'_h$ and $\varepsilon_i = \varepsilon'_i$ for all $i \in [n]$. $\varepsilon_h = \varepsilon'_h$ is clear from the fact that $q_{ij} \cdot d^*_{ij} = q_{ij} \cdot d^*_{jj}$ in part 3 of Definition EC5.1. To see that $\varepsilon_i = \varepsilon'_i$ for any $i \in [n]$, note that $x_0 = x_N$ implies that $x_{N,i} = x_i$, i.e., the resources transiting out of spoke $i$ equals the resources transiting into spoke $i$. Based on this, it is easy to show that $\prod_{k : s_k \in S} q_{ij} = \prod_{k : s_k \in S} q_{ij}$ from part 2 of Definition EC5.1 and $\prod_{k : s_k \in S} d_i(x_{k,i}, s_k) = \prod_{k : s_k \in S} d_i(x_{k+1,i}, \tilde{s}_k)$ from part (b) of Proposition EC5.1. Thus $\varepsilon_i = \varepsilon'_i$.

Part (e): Suppose spoke $i$ is connected to hubs $j$ and $j'$ and $x_j \geq 1$. From part (d), the Markov chain is reversible and hence we have

$$\mathbb{P}(x_i, x_s) \cdot q_{ij} \cdot d_i(x_i, j', i) = \mathbb{P}(x_i, x_s + e_i) \cdot q_{ij} \cdot d_i(x_i + 1, i, j),$$

and

$$\mathbb{P}(x_i - e_j + e_{j'}, x_s) \cdot q_{ij} \cdot d_i(x_i, j', i) = \mathbb{P}(x_i - e_j + e_{j'}, x_s + e_i) \cdot q_{ij} \cdot d_i(x_i + 1, i, j').$$

Since the controls $d_i(x_i, j, i)$ and $d_i(x_i, j, i)$ are identical across hubs by part (b) and the request rates satisfy $q_{ij} / q_{jj} = q_{ij} / q_{jj} = c_i > 0$ from Definition EC5.1. \(\mathbb{P}(x_i, x_s) = \mathbb{P}(x_i - e_j + e_{j'}, x_s).\) The general result follows from the fact that the network topology is strongly connected.

**EC5.2 Proof of Proposition EC5.2**

First, for the high multiplicity model, since the expected number of resources in the spokes of the relaxed system is no larger than $m - \delta$, we have $E[X_i(\delta)] \leq \frac{m}{a}$ for all $i \in [n]$. Let random variables $X_0(\delta) = \sum_{j=1}^J X_j(\delta)$ and $\tilde{X}_0(\delta)$ denote the sum of resources in the hubs of the original system and the relaxed system under the stationary distributions of the Lagrangian policy $\pi(\delta)$. Proposition EC5.1(e) implies that conditional on the total number of resources in the hubs, the distribution of resources across the hubs is uniform. Therefore, we have

$$\mathbb{P}(X_j(\delta) = k \mid X_0(\delta) = k) = \frac{(k+J-2)}{(k+J-1)} = \frac{J - 1}{k + J - 1}.$$
Thus,
\[
P[X_j(\delta) = 0] = \sum_{k=0}^{m} P[X_0(\delta) = k] \cdot P[X_j(\delta) = 0 | X_0(\delta) = k] \\
= \sum_{k=0}^{m} P[X_0(\delta) = k] \cdot \frac{J - 1}{k + J - 1} = \mathbb{E}\left[\frac{J - 1}{X_0(\delta) + J - 1}\right] \\
\leq \mathbb{E}\left[1[X_0(\delta) \leq c] + 1[X_0(\delta) > c]\cdot \frac{J - 1}{X_0(\delta) + J - 1}\right] \\
\leq P[X_0(\delta) \leq c] + \frac{J - 1}{c + J - 1},
\]
for any constant \(c \geq 0\), where the last inequality is because \(X_0(\delta) \succeq_{FOSD} \tilde{X}_0(\delta)\) from Lemma 3, dropping the second indicator, and using that \((J - 1)/(x + J - 1)\) is decreasing for \(x \geq 0\).

Let \(\mu = \mathbb{E}\left[\sum_{i=1}^{n} \tilde{X}_i(\delta)\right]\) be the expected number of resources in the spokes of the relaxed system. Since the policy \(\pi(\delta)\) is solved from the perturbed Lagrangian relaxation, we have \(0 < \mu \leq m - \delta\).

For any \(\gamma \in [0, 1]\), set \(c = \gamma \cdot (m - \mu)\) in (EC-69) and we have
\[
P[X_j(\delta) = 0] \leq P[\tilde{X}_0(\delta) \leq \gamma \cdot (m - \mu)] + \frac{J - 1}{\gamma \cdot (m - \mu) + J - 1} \\
\leq P[\tilde{X}_0(\delta) \leq \gamma \cdot (m - \mu)] + \frac{J - 1}{\gamma \cdot \delta + J - 1}. \tag{EC-70}
\]

We can bound the first term in (EC-70) using the concentration inequality developed in Lemma EC1.12 of Appendix EC1.10. Specifically, applying Lemma EC1.12 with \(\lambda = \frac{m - \gamma \cdot (m - \mu)}{\mu}\) and \(b = \frac{1}{1 + m/(2 \cdot n)}\) gives
\[
P[\tilde{X}_0(\delta) \leq \gamma \cdot (m - \mu)] = P\left[\sum_{i=1}^{n} \tilde{X}_i(\delta) \geq m - \gamma \cdot (m - \mu)\right] \\
\leq \exp\left\{ -b \cdot \left(\lambda \mu - \mu + (n + \mu) \cdot \ln\left(1 - \frac{\lambda \mu - \mu}{\lambda \mu + n}\right)\right)\right\} \\
= \exp\left\{ -b \cdot \left(1 - \gamma\right) \cdot (m - \mu) + (n + \mu) \cdot \ln\left(\frac{n + \mu}{m + n - \gamma \cdot (m - \mu)}\right)\right\} \\
= \exp\left\{ b \cdot \left((n + \mu) \cdot \ln\left(1 - \left(1 - \gamma\right) \cdot (m - \mu)\right)\right)\right\} \\
\leq \exp\left\{ b \cdot \left((n + \mu) \cdot \ln\left(\frac{m + n - \gamma \cdot (m - \mu)}{n + \mu}\right) - (1 - \gamma) \cdot (m - \mu)\right)\right\}. \tag{EC-71}
\]
Since \( \ln x \leq \frac{x-1}{\sqrt{x}} \) for \( x \geq 1 \), we have
\[
\begin{align*}
\tau &\leq (n + \mu) \cdot \frac{(1 - \gamma) \cdot (m - \mu)}{n + \mu} \cdot \sqrt{\frac{n + \mu}{m + n - \gamma \cdot (m - \mu)}} - (1 - \gamma) \cdot (m - \mu) \\
&= (1 - \gamma) \cdot (m - \mu) \cdot \left( \sqrt{1 - \frac{(1 - \gamma) \cdot (m - \mu)}{m + n - \gamma \cdot (m - \mu)}} - 1 \right) \\
&\leq -\frac{(1 - \gamma)^2 \cdot (m - \mu)^2}{2 \cdot (m + n - \gamma \cdot (m - \mu))} \\
&\leq -\frac{(1 - \gamma)^2 \cdot \delta^2}{2 \cdot (m + n - \gamma \delta)},
\end{align*}
\]
where the second-to-last inequality is due to \( \sqrt{1-x} - 1 \leq -\frac{x}{2} \) for \( x \leq 1 \). Thus from (EC-71) we have
\[
\mathbb{P}[\bar{X}_0(\delta) \leq \gamma \cdot (m - \mu)] \leq \exp \left( -\frac{b}{2} \cdot \frac{(1 - \gamma)^2 \cdot \delta^2}{m + n - \gamma \delta} \right).
\]
Combining (EC-70) and (EC-72) we have
\[
\mathbb{P}[X_j(\delta) = 0] \leq \exp \left( -\frac{b}{2} \cdot \frac{(1 - \gamma)^2 \cdot \delta^2}{m + n - \gamma \delta} \right) + \frac{J - 1}{\gamma \delta + J - 1}.
\]
Letting \( \gamma = \frac{1}{2} \) gives the desired result.

**EC6 Exponential Relocation Times in a Single Hub Network**

With exponential relocation times, the Lagrangian simplifies greatly as we only need to track the number of resources on each route and in each location. Suppose we have one hub and \( n \) spokes, and the relocation times for requests \((i, 0)\) and \((0, i)\) follow independent exponential distributions with mean values \( \tau_{i0} \) and \( \tau_{0i} \), respectively. Let \( \Lambda = \sum \eta_i \) denote the total request rate.

For each spoke \( i \), problem, we let \( x_i \) denote the number of resources in the spoke and \( x_{0i} \) denote the number of resources in transit from the hub to the spoke; we require that \( x_i + x_{0i} \leq m \) to bound the state space. The resources that are leaving the spoke are irrelevant to the spoke problem. Suppose the current state is \((x_i, x_{0i})\) and no request is fulfilled in the current period. Let \( \rho(x_i, x_{0i}, x_i, x_{0i}) \) denote the transition probability that the next period starts with a state \((\bar{x}_i, \bar{x}_{0i})\).

We have
\[
\rho(x_i, x_{0i}, x_i, x_{0i}) = \begin{cases} 
\frac{(x_{0i})}{(x_i)} \left( \frac{\Lambda}{\Lambda + 1/\tau_{0i}} \right) & \text{if } x_{0i} \leq x_i \text{ and } \bar{x}_i + \bar{x}_{0i} = x_i + x_{0i}, \\
0 & \text{otherwise},
\end{cases}
\]
because each resource on \((0, i)\) will reach spoke \( i \) with probability \( \frac{1/\tau_{0i}}{\Lambda + 1/\tau_{0i}} \) and keep relocating with probability \( \frac{\Lambda}{\Lambda + 1/\tau_{0i}} \) by the end of the current period. The spoke problem is:
\[
h_i^\Lambda = \max_{d_i, (x_i, x_{0i}, i, 0) \in [0, 1]} \sum_{(x_i, x_{0i})} p_i(x_i, x_{0i}) \cdot \left( q_i \cdot r_i \left( d_i(x_i, x_{0i}, i, 0) \right) - \lambda \cdot d_i(x_i, x_{0i}, i, 0) \cdot \Lambda \cdot \tau_{0i} \right)
\]
q_i \cdot r_{0i} \left( d_i(x_i, x_{0i}, 0, i) \right) - \lambda \cdot (x_i + x_{0i}) \right) \right) \right) \\
\text{s.t.} \sum_{(x_i, x_{0i}) \in N^2 : x_i + x_{0i} \leq m} p_i(x_i, x_{0i}) = 1, \\
p_i(x_i, x_{0i}) = \sum_{(\bar{x}_i, \bar{x}_{0i})} p_i(\bar{x}_i, \bar{x}_{0i}) \cdot \left[ q_{0i} \cdot d_i(\bar{x}_i, \bar{x}_{0i}, i, 0) \cdot \rho(\bar{x}_i - 1, \bar{x}_{0i}, x_i, x_{0i}) + q_{0i} \cdot d_i(\bar{x}_i, \bar{x}_{0i}, 0, i) \cdot \rho(\bar{x}_i, \bar{x}_{0i} + 1, x_i, x_{0i}) + \left( 1 - q_{0i} \cdot d_i(\bar{x}_i, \bar{x}_{0i}, i, 0) - q_{0i} \cdot d_i(\bar{x}_i, \bar{x}_{0i}, 0, i) \right) \cdot \rho(\bar{x}_i, \bar{x}_{0i}, x_i, x_{0i}) \right], \\
d_i(x_i, x_{0i}, i, 0) = 0, \forall x_i = 0, \\
d_i(x_i, x_{0i}, 0, i) = 0, \forall x_i + x_{0i} = m, \\
\text{where in the objective, the term } \sum_{(x_i, x_{0i})} \Lambda \cdot q_{0i} \cdot p_i(x_i, x_{0i}) \cdot d_i(x_i, x_{0i}, i, 0) \cdot \tau_{i0} \text{ equals the expected number of resources moving from the spoke to the hub by Little’s law.}

### EC7 More on Numerical Examples

In this section, we plot the stationary distributions of the number of resources in the single hub example (Section 7.1) and consider another synthetic example with two hubs in Section 7.2, and we provide more numerical results for the RideAustin example (Section 7.2) in Section EC7.3.

#### EC7.1 Stationary Distributions in the Single Hub Examples

Figure EC-1 shows the stationary distributions of the number of resources in the hub of the one hub examples in Section 7.1 under the Lagrangian policy $\pi(\delta)$ and the static policy $\pi^s$. Based on the results in Whitt (1984), we actually have analytical expressions for the marginal distributions with the static policy: the probability that there are $x$ resources in any location $i \in [0 : n]$ is $\binom{m+n-1-k}{n-1} / \binom{m+n}{n}$ (note that $n + 1$ locations are in the example), with the mode being that the location has zero resources.

#### EC7.2 Two Hub Examples

In this section, we consider examples with two hubs as illustrated in Figure EC-2; these hubs are not uniformly related as defined in Definition EC5.1. We let the arrival rates be $q_{i1} = \frac{1}{5n}$ and $q_{i2} = \frac{1}{6n}$ for hub 1, and $q_{2i} = \frac{1}{5n}$ and $q_{3i} = \frac{1}{6n}$ for hub 2, for all spokes $i \in [n]$, and let all the other arrival rates be zero; thus without the flow “balancing” constraints captured by the dual variables $\mu$, hub 1 tends to accumulate resources whereas hub 2 tends to lose resources.

For each fixed $n$, we calculate the same quantities as in the one hub examples (Section 7.1), and we additionally calculate: the Lagrangian relaxation upper bound $\bar{V}^R$ omitting the flow balance constraints at the hubs, and the performance $\bar{V}^\pi(\delta)$ of the policy derived from the perturbed problem, with $\delta = \sqrt{n \log n}$, omitting the flow balance constraints at the hubs. Figure EC-3 shows the simulation results for the two-hub case. From Figure EC-3 when there are multiple hubs and the hubs are asymmetric, omitting the flow balance constraints at the hubs leads to a loose upper bound and that the corresponding Lagrangian policy does not perform well.
Figure EC-1: Stationary distributions of resources in the hub for the one hub examples (Section 7.1), with varying number of spokes $n$. (a) is with the Lagrangian policy $\pi(\delta)$ and (b) is with the static policy $\pi^F$.

Figure EC-2: A hub-and-spoke network with 2 hubs (grey) and $n$ symmetric spokes. We only draw the connections between spoke $i$ and the hubs. Edge widths illustrate relative values of request rates.

Figures EC-4 and EC-5 show the stationary distributions under the Lagrangian policy that omits the flow balance constraints of the hubs and the Lagrangian policy that incorporates these constraints, respectively. Since hub 1 tends to accumulate resources whereas hub 2 tends to lose resources, without flow balancing, hub 1 have excessive resources and hub 2 is essentially depleted.

EC7.3 More on the RideAustin Example

Figure EC-6 shows the partition of Austin, Texas with $n = 100$ locations and the ride flow of the city based on the partition. In the ride flow figure, each node represents a location of the city. The radius of a node is proportional to the amount of the requests that leave the location, and the width of an edge is proportional to the size of the requests on the edge. An edge has the same color
Figure EC-3: Simulation results of the two-hub case. (b) is magnified versions of (a), highlighting the performance of our policy and the Lagrangian relaxation upper bound. 95% confidence intervals around $V^\pi\left(\sqrt{n \ln n}\right)$ are plotted with dashed lines in (b).

Figure EC-4: Stationary distributions with the Lagrangian policy that omits the flow balance constraints in the hubs, for the two hub examples (Section EC7.2). (a) is for the total number of resources in the hubs, (b) is for the number of resources in hub one, and (c) is for the number of resources in hub two.

as its origin location.

Figure EC-7 illustrates the locations of hubs obtained from solving (18) with different values of $J$ from one to six. For each value of $J$, the locations of the $J$ hubs are the nodes labelled from one to $J$.

Figure EC-8 demonstrates how the performances of the Lagrangian policy $\pi(\delta)$ and the Lagrangian-based static policy $\pi^S(\delta)$ varies with $\delta$ for each value of $J$. 

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Figure EC-5: Stationary distributions with the Lagrangian policy $\pi(\delta)$ that incorporates the flow balance constraints in the hubs, for the two hub examples (Section EC7.2). (a) is for the total number of resources in the hubs, (b) is for the number of resources in hub one, and (c) is for the number of resources in hub two.

Figure EC-6: (a) The Voronoi diagram of the cluster centers from solving the $k$-center problem with $k = 100$ and the first few centers initialized with $k$-means clustering centers. (b) The ride flow of the city based on data from RideAustin and the partition.

References


Figure EC-7: Locations of hubs obtained from solving (18) with different values of $J$ from one to six. For each value of $J$, the locations of the $J$ hubs are the nodes labelled from one to $J$. (b) is simply a zoom-in of (a).

Figure EC-8: (a) The performance (average revenue per request) $V^\pi(\delta)$ of the Lagrangian policy $\pi(\delta)$ for different values of $\delta$ and $J$. The optimal choice $\delta^\ast$ is roughly 160, 140, 140, 140 and 160 for $J$ from one to six. (b) The performance $V(\pi^S(\delta))$ of the Lagrangian-based static policy $\pi^S(\delta)$ for different values of $\delta$ and $J$. The optimal choice $\delta^\ast_S$ is roughly 20, 20, 40, 40, 40 and 60 for $J$ from one to six. We estimate the values $V^\pi(\delta)$ and $V(\pi^S(\delta))$ with 50 sample paths and for each sample path, we approximate the average revenue with an average of the total revenue of the first $10^6$ requests. 95% confidence intervals are plotted with dashed lines.


