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e-companion

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Electronic Companion—"A Soft Robust Model for Optimization Under Ambiguity" by Aharon Ben-Tal, Dimitris Bertsimas, and David B. Brown, *Operations Research*, DOI 10.1287/opre.1100.0821.

Appendix

Proof of Proposition 2.1. We have

$$\min_{\boldsymbol{x}\in X}\max_{\epsilon\in[0,\delta]}\left\{-\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x})\right]-\epsilon\right\} \;=\; \min_{\boldsymbol{x}\in X}\max_{\epsilon\in[0,\delta]}\phi(\boldsymbol{x},\epsilon),$$

where $\phi(\boldsymbol{x},\epsilon)\triangleq -\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x})\right]-\epsilon$. Notice that this function is convex on $\boldsymbol{x}\in X$ for all $\epsilon\in[0,\delta]$ (since f is concave in $\boldsymbol{x}$, and $\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x})\right]$ is the pointwise infimum of a family of concave functions, an operation that preserves concavity). We now show that ϕ is concave on $\epsilon\in[0,\delta]$ for all $\boldsymbol{x}\in X$. To see this, we show that $-\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[Y\right]$ is concave on $\epsilon\in[0,\delta]$ for all $Y\in\mathcal{Y}$. Indeed, consider $\mathcal{Q}(\epsilon_1)$ and $\mathcal{Q}(\epsilon_2)$ for $\epsilon_1,\ \epsilon_2\in[0,\delta]$. Since $Y\in\mathcal{Y}$, the infimum must be finite in both cases. Let $\mathbb{Q}_i\in\mathcal{Q}(\epsilon_i)$ be such that

$$\mathbb{E}_{\mathbb{Q}_i}\left[Y\right] \;\leq\; \inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon_i)}\mathbb{E}_{\mathbb{Q}}\left[Y\right]+\xi,$$

for any $\xi>0$ (since the infimum is attained, such a $\mathbb{Q}_i$ must exist for arbitrarily small ξ). Now, by convexity of the family $\mathbf{Q}$, we have $\lambda\mathbb{Q}_1+(1-\lambda)\mathbb{Q}_2\in\mathcal{Q}(\lambda\epsilon_1+(1-\lambda)\epsilon_2)$ for all $\lambda\in[0,1]$. Therefore,

$$\begin{aligned}\inf_{\mathbb{Q}\in\mathcal{Q}(\lambda\epsilon_1+(1-\lambda)\epsilon_2)}\mathbb{E}_{\mathbb{Q}}\left[Y\right] &\leq \lambda\mathbb{E}_{\mathbb{Q}_1}\left[Y\right]+(1-\lambda)\mathbb{E}_{\mathbb{Q}_2}\left[Y\right]\\ &\leq \lambda\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon_1)}\mathbb{E}_{\mathbb{Q}}\left[Y\right]+(1-\lambda)\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon_2)}\mathbb{E}_{\mathbb{Q}}\left[Y\right]+\xi.\end{aligned}$$

Now taking the limit as $\xi\to 0$, we see that concavity of $-\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[Y\right]$ on $\epsilon\in[0,\delta]$ follows. From this, it is now clear that $\phi(\boldsymbol{x},\epsilon)$ is concave on $\epsilon\in[0,\delta]$ for all $\boldsymbol{x}\in X$.

Since $\phi(\boldsymbol{x},\epsilon)$ is convex on $\boldsymbol{x}\in X$, concave on $\epsilon\in[0,\delta]$, and $[0,\delta]$ is convex and compact, it follows from minimax theory (e.g., Fan [1]) that we can reverse the order of minimization and maximization above. In particular,

$$\begin{aligned}\min_{\boldsymbol{x}\in X}\max_{\epsilon\in[0,\delta]}\left\{-\inf_{\mathbb{Q}\in\mathcal{Q}(\epsilon)}\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x})\right]-\epsilon\right\} &= \min_{\boldsymbol{x}\in X}\max_{\epsilon\in[0,\delta]}\phi(\boldsymbol{x},\epsilon)\\ &= \max_{\epsilon\in[0,\delta]}\min_{\boldsymbol{x}\in X}\phi(\boldsymbol{x},\epsilon)\\ &= \max_{\epsilon\in[0,\delta]}\theta(\epsilon),\end{aligned}$$

so the first part of the claim is established. For the second part of the claim, note that θ is the pointwise infimum of a family of functions that are concave on $\epsilon\in[0,\delta]$ (from above). As before, it is well-known that this operation preserves concavity, so we are done. □

Proof of Proposition 2.2. Fix $\delta > 0$ and denote the optimal robust solution by $\boldsymbol{x}_\delta$. First, we claim that we may assume that $\min_{\omega\in\Omega} f(\boldsymbol{x}_\delta, u(\omega)) < 0$. If this is not the case, then $f(\boldsymbol{x}_\delta) \geq 0$, and monotonicity and normalization imply that $\rho(f(\boldsymbol{x}_\delta)) \leq 0$; thus, noting the representation of soft robustness in Theorem 2.1, $\boldsymbol{x}_\delta$ is trivially soft robust at any $\delta > 0$.

Thus, we assume that $\min_{\omega\in\Omega} f(\boldsymbol{x}_\delta, u(\omega)) < 0$. We know from Theorem 2.1 that $\boldsymbol{x}_\delta$ being robust feasible to the standard robust formulation is equivalent to the optimization problem

$$\begin{array}{ll} \inf_{\mathbb{Q}\in\mathcal{P}} & \mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x}_\delta)\right] \\ \text{subject to} & \alpha^{\mathbf{Q}}(\mathbb{Q}) \leq \delta, \end{array} \tag{11}$$

having nonnegative optimal value, where $\alpha^{\mathbf{Q}}$ is the penalty function inducing the corresponding convex risk measure. Since $\delta > 0$, the Slater condition holds; by assumption (11) is also bounded. Therefore, strong duality holds and we can equate to the Lagrangian dual of (11):

$$\begin{aligned} \max_{\lambda\geq 0}\inf_{\mathbb{Q}\in\mathcal{P}}\left\{\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x}_\delta)\right] + \lambda\alpha^{\mathbf{Q}}(\mathbb{Q}) - \delta\lambda\right\} &= \max_{\lambda\geq 0}\left[-\sup_{\mathbb{Q}\in\mathcal{P}}\left\{-\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x}_\delta)\right] - \lambda\alpha^{\mathbf{Q}}(\mathbb{Q})\right\} - \delta\lambda\right] \\ &= \sup_{\lambda>0}\left[-\lambda\sup_{\mathbb{Q}\in\mathcal{P}}\left\{-\frac{1}{\lambda}\mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x}_\delta)\right] - \alpha^{\mathbf{Q}}(\mathbb{Q})\right\} - \delta\lambda\right] \\ &= \sup_{\lambda>0}\left[-\lambda\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda}\right) - \delta\lambda\right] \\ &= -\inf_{\lambda>0}\left[\lambda\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda}\right) + \delta\lambda\right]. \end{aligned}$$

To justify the fact that we cannot have $\lambda = 0$, as used here, we use the following lemma.

Lemma. *Let ρ be a normalized, convex risk measure generated by a nonnegative, convex function α. Then*

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\rho(\lambda Y) = -y_{\min}, \tag{12}$$

where $y_{\min} = \min_{\omega\in\Omega} Y(\omega)$.

Proof. Since α is nonnegative, we have

$$\begin{aligned} \rho(Y) &= \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{-\mathbb{E}_{\mathbb{Q}}\left[Y\right] - \alpha(\mathbb{Q})\right\} \\ &\leq -y_{\min} + 0 \\ &= -y_{\min}. \end{aligned}$$

Hence, for any $\lambda > 0$,

$$\frac{1}{\lambda}\rho(\lambda Y) \leq -y_{\min}. \tag{13}$$

Now, for any $\epsilon > 0$, consider the convex optimization problem (P_ϵ):

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \{\alpha(\mathbb{Q}) \ : \ \mathbb{E}_{\mathbb{Q}}[Y] \le y_{\min} + \epsilon\}.$$

Clearly, for any $\epsilon > 0$, (P_ϵ) is strictly feasible. The dual may be expressed as:

$$\begin{aligned}
\sup_{\lambda\ge 0}\inf_{\mathbb{Q}\in\mathcal{Q}} \{\alpha(\mathbb{Q}) - \lambda(y_{\min}+\epsilon) + \lambda\mathbb{E}_{\mathbb{Q}}[Y]\} &= \sup_{\lambda\ge 0}\left\{\lambda(-y_{\min}-\epsilon) + \inf_{\mathbb{Q}\in\mathcal{Q}}\{\mathbb{E}_{\mathbb{Q}}[\lambda Y] + \alpha(\mathbb{Q})\}\right\} \\
&= \sup_{\lambda\ge 0}\left\{\lambda\left\{-y_{\min}-\epsilon - \frac{1}{\lambda}\sup_{\mathbb{Q}\in\mathcal{Q}}\{-\mathbb{E}_{\mathbb{Q}}[\lambda Y] - \alpha(\mathbb{Q})\}\right\}\right\} \\
&= \sup_{\lambda\ge 0}\left\{\lambda\left[-y_{\min}-\epsilon-\frac{1}{\lambda}\rho(\lambda Y)\right]\right\},
\end{aligned}$$

which we denote by (D_ϵ). Since (P_ϵ) is feasible for all $\epsilon > 0$, (D_ϵ) is bounded above for all $\epsilon > 0$. Hence, it must be the case that

$$\lim_{\lambda\to 0}\left\{-y_{\min}-\epsilon-\frac{1}{\lambda}\rho(\lambda Y)\right\} \ \le \ 0.$$

Combining this with (13), we have, for any $\epsilon > 0$,

$$-y_{\min}-\epsilon \ \le \ \lim_{\lambda\to 0}\left(\frac{1}{\lambda}\rho(\lambda Y)\right) \ \le \ -y_{\min}.$$

Taking ϵ arbitrarily small, (13) follows. □

From the above lemma, it follows that

$$-\lim_{\lambda\to 0}\frac{1}{\lambda}\rho(\lambda f(\boldsymbol{x}_\delta)) \ = \ \min_{\omega\in\Omega} f(\boldsymbol{x}_\delta, \boldsymbol{u}(\omega)).$$

By assumption, the quantity on the right is strictly negative, and therefore $\lambda = 0$ cannot attain the sup in the dual (since strong duality holds, and the optimal value of (11) must be nonnegative by robust feasibility of $\boldsymbol{x}_\delta$). Thus, there was no loss in replacing the closed interval $\lambda \ge 0$ with the open one $\lambda > 0$, and the above steps are justified.

If follows that if $\boldsymbol{x}_\delta$ is robust feasible at level δ, we have

$$\lambda^\star\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda^\star}\right) + \lambda^\star\delta \ \le \ 0$$

for the $\lambda^\star > 0$ that minimizes the expression in the dual.

First assume $\lambda^\star > 1$, and consider a δ' satisfying $\delta' \le [\lambda^\star/(\lambda^\star-1)]\delta$. For $\boldsymbol{x}_\delta \in X^{\mathbf{Q}}(\delta')$ to hold, there must exist a $\tilde{\lambda} \ge 0$ such that

$$(\tilde{\lambda}+1)\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\tilde{\lambda}+1}\right) + \tilde{\lambda}\delta' \ \le \ 0$$

holds. Let $\tilde{\lambda} = \lambda^\star - 1 > 0$. Then

$$\begin{aligned}(\tilde{\lambda}+1)\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\tilde{\lambda}+1}\right) + \tilde{\lambda}\delta' &= \lambda^\star \rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda^\star}\right) + (\lambda^\star - 1)\delta' \\ &\leq \lambda^\star \rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda^\star}\right) + \lambda^\star\delta \\ &\leq 0,\end{aligned}$$

where we have used $\delta' \leq [\lambda^\star/(\lambda^\star - 1)]\delta$ and the fact that $\boldsymbol{x}_\delta$ is robust feasible at level δ. Therefore, $\boldsymbol{x}_\delta$ is also soft robust feasible at level δ', so $v^S(\delta') \geq v(\delta)$.

If $\lambda^\star \leq 1$, let $\tilde{\lambda} = 0$, and now let δ' be any nonnegative value. Then we have

$$\begin{aligned}(\tilde{\lambda}+1)\rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\tilde{\lambda}+1}\right) + \tilde{\lambda}\delta' &= \rho^{\mathbf{Q}}(f(\boldsymbol{x}_\delta)) \\ &\leq \lambda^\star \rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda^\star}\right) \\ &\leq \lambda^\star \rho^{\mathbf{Q}}\left(\frac{f(\boldsymbol{x}_\delta)}{\lambda^\star}\right) + \lambda^\star\delta \\ &\leq 0,\end{aligned}$$

where we have used the subhomogeneity property of convex risk measures as noted in the proof of Theorem 2.1 (i.e., $w \geq 1$ implies $w\rho(Y/w) \leq \rho(Y)$) and the fact that $\lambda^\star \geq 0$ and $\delta > 0$. This shows that $\boldsymbol{x}_\delta \in X^{\mathbf{Q}}(\delta')$ for any $\delta' \geq 0$, i.e., $\boldsymbol{x}_\delta$ is soft robust feasible at any $\delta' \geq 0$. This completes the proof of Proposition 2.2. □

Proof of Lemma 3.1. For (a), we use the well-known fact that, for any $\beta \in (0,1]$, $\mathrm{CVaR}_\beta(Y) \geq \mathrm{VaR}_\beta(Y)$, where

$$\mathrm{VaR}_\beta(Y) \triangleq \inf\{t \in \mathbb{R} \,:\, \mathbb{P}\{Y + t \geq 0\} \geq 1 - \beta\}$$

is the *value-at-risk at level* β. Moreover, we know that CVaR is an OCE generated in the following way:

$$\mathrm{CVaR}_\beta(Y) = \sup_{\mathbb{Q}\in\mathcal{P}}\left\{-\mathbb{E}_{\mathbb{Q}}[Y] \,:\, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\beta}\right\}.$$

Denote the associated set of measures by $\hat{\mathcal{Q}}(\beta)$. Given that ρ is an OCE with the associated family $\mathbf{Q}$ described earlier, let $\beta(z) \in (0,1]$ be such that $\hat{\mathcal{Q}}(\beta(z)) \subseteq \mathcal{Q}(z)$. Then we have

$$\begin{aligned}\mathrm{CVaR}_{\beta(z)}(Y) - z &= \sup_{\mathbb{Q}\in\hat{\mathcal{Q}}(\beta(z))}\{-\mathbb{E}_{\mathbb{Q}}[Y] - z\} \\ &\leq \sup_{\mathbb{Q}\in\mathcal{Q}(z)}\{-\mathbb{E}_{\mathbb{Q}}[Y] - z\} \\ &\leq 0,\end{aligned}$$

where in the last inequality we are using $\rho(Y) \leq 0$. Thus,

$$\mathrm{VaR}_{\beta(z)}(Y) \;\leq\; \mathrm{CVaR}_{\beta(z)}(Y) \;\leq\; z \;\;\Rightarrow\;\; \mathbb{P}\{Y \geq -z\} \geq 1 - \beta(z),$$

i.e., $\mathbb{P}\{Y < -z\} \leq \beta(z)$. We now seek the smallest such $\beta(z)$. To check whether $\hat{\mathcal{Q}}(\beta) \subseteq \mathcal{Q}(z)$, we need only check the extreme points of the set $\hat{\mathcal{Q}}(\beta)$. These are contained in the set of measures that equal $\mathbb{P}(\omega)/\beta$ for all $\omega \in \hat{\Omega}$, where $\mathbb{P}\left\{\omega \in \hat{\Omega}\right\} = \beta$, and 0 everywhere else. In other words, we have, at such measures $\mathbb{Q}$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) \;=\; \begin{cases} \frac{1}{\beta} & \text{if } \omega \in \hat{\Omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for such a measure, we have

$$\mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \;=\; \int_{\omega\in\Omega} \phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{P}(\omega) \;=\; \int_{\omega\in\hat{\Omega}} \phi\left(\frac{1}{\beta}\right) d\mathbb{P}(\omega) + \int_{\omega\in\Omega\setminus\hat{\Omega}} \phi(0) d\mathbb{P}(\omega) \;=\; (1-\beta)\phi(0) + \beta\phi\left(\frac{1}{\beta}\right).$$

Therefore, we find the smallest such $\beta \in (0,1]$ to get $\beta(z)$:

$$\beta(\epsilon) \;=\; \inf\left\{\beta \in (0,1] \;:\; (1-\beta)\phi(0) + \beta\phi\left(\frac{1}{\beta}\right) \leq z\right\}.$$

For (b), we take any $z \geq -\bar{\epsilon}$ and construct another random variable Y' on the same space given by

$$Y'(\omega) \;=\; \begin{cases} Y(\omega) & \text{if } Y(\omega) > -z \\ -z & \text{otherwise.} \end{cases}$$

Clearly, $Y' \geq Y$, which means $\rho(Y') \leq \rho(Y) \leq 0$. On the other hand, $\mathbb{P}\{Y' \leq -z\} = \mathbb{P}\{Y \leq -z\}$. By a similar argument, consider another random variable Y'' given by

$$Y''(\omega) \;=\; \begin{cases} \bar{\epsilon} & \text{if } Y(\omega) > -z \\ Y(\omega) & \text{otherwise.} \end{cases}$$

Again, we have $Y'' \geq Y$, so $\rho(Y'') \leq \rho(Y) \leq 0$, but $\mathbb{P}\{Y'' \leq -z\} = \mathbb{P}\{Y \leq -z\}$. Combining these observations, we see that it suffices to consider Y with a discrete distribution; in particular, we need only consider Y such that $Y = -z$ with probability p and $Y = \bar{\epsilon}$ with probability $1-p$. We denote such a random variable by Y_z as in the statement. It is now simply a matter of finding the maximum such p such that $\rho(Y_z) \leq 0$ holds, which gives us the result.

For (c), assume that $\mathbb{P}\{Y \leq -z\} = p$. We are considering the problem

$$\sup_{\mathbb{Q}\in\mathcal{P}} \left\{\mathbb{Q}\{Y \leq -z\} \;:\; \mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \leq \Delta\right\}.$$

Notice that since $\Delta > 0$, the Slater condition is satisfied; moreover, the objective is clearly bounded, so strong duality holds. In what follows, we denote by Y' the discrete random variable that takes the value -1 if $Y(\omega) \leq -z$ and 0 otherwise. Then,

$$
\begin{aligned}
\sup_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathbb{Q}\{Y \leq -z\} \ : \ \mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \leq \Delta \right\} &= \min_{\lambda\geq 0}\sup_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathbb{Q}\{Y \leq z\} + \lambda\Delta - \lambda\mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right\} \\
&= \min_{\lambda\geq 0}\sup_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[-Y'\right] + \lambda\Delta - \lambda\mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right\} \\
&= \inf_{\lambda>0} \left\{ \lambda\Delta + \lambda \sup_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\frac{-Y'}{\lambda}\right] - \mathbb{E}_{\mathbb{P}}\left[\phi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right\}\right\} \\
&= \inf_{\lambda>0} \left\{ \lambda\Delta + \lambda\rho\left(\frac{Y'}{\lambda}\right)\right\} \\
&= \inf_{\lambda>0,\ \nu} \left\{ \lambda\Delta + \lambda\left[\nu + p\phi^*\left(\frac{1}{\lambda} - \nu\right) + (1-p)\phi^*(-\nu)\right]\right\},
\end{aligned}
$$

which establishes the equality. Note that we are assuming above that the optimal λ above is strictly positive; if this is not the case, using the lemma from the proof of Proposition 2.2, then the optimal value above is given by $-\min_{\omega\in\Omega} Y'(\omega) = 1$, and the probability bound is vacuous.

Now assume that $\mathbb{P}\{Y \leq -z\} \leq p$; it is not hard to see that the function above in braces is nondecreasing in p. This implies the inequality. □

References

[1] Fan, K. 1953. Minimax theorems. *Proceedings of the National Academy of Science* 39, 42-47.

Asset name	*Symbol*	*Category*
S&P 500 Index	SP500	U.S. equity
Russell Mid-cap Index	RMidC	U.S. equity
Russell 2000 Index	R2000	U.S. equity
MSCI EAFE Index	MSCIEAFE	International equity
MSCI Emerging Market Index	MSCIEmer	International equity
NAREIT Index	NAREIT	Real estate
Lehman Brothers' U.S. Aggregate Index	LBUS	U.S. bond
Lehman Brothers' U.S. Corporate High Yield	LBHY	U.S. corporate bond
Global Governments Bond	GlobGovBnd	International bond
Emerging Markets Bond	EmerMktBnd	International bond
3-month LIBOR	LIBOR	Cash

Table 2: *Descriptions for the various asset classes used in the asset allocation experiment.*

	β			
$\text{CVaR}_\beta(R)$ / $\text{VaR}_\beta(R)$	1.0	.50	.10	.05
SP500	-14.1	-1.30	19.8	24.8
	-48.0	-15.6	12.9	22.4
RMidC	-15.4	-2.00	15.4	21.5
	-50.8	-14.1	11.2	14.9
R2000	-13.3	2.40	19.5	27.0
	-63.8	-13.1	15.3	18.9
MSCIEAFE	-13.9	3.50	20.6	25.7
	-85.8	-12.0	13.1	23.6
MSCIEmer	-16.9	10.0	26.9	31.8
	-79.6	-3.30	20.5	28.3
NAREIT	-13.4	0.200	11.5	22.5
	-54.6	-10.7	5.90	10.1
LBUS	-9.80	-4.50	-1.80	-1.20
	-31.9	-8.70	-2.40	-1.70
LBHY	-11.5	-4.00	1.10	2.00
	-50.9	-8.20	-0.400	1.90
GlobGovBnd	-8.50	1.10	7.60	8.80
	-34.7	-6.70	5.50	7.70
EmerMktBnd	-13.9	-3.70	13.7	13.7
	-38.8	-13.2	12.0	13.4
LIBOR	-6.20	-4.00	-1.70	-1.20
	-14.4	-5.90	-2.10	-1.60

Table 3: *CVaR and VaR of annualized returns for the 11 asset classes from April, 1981 through February, 2006. All numbers expressed in %.*

	$\text{CVaR}_\beta(R)$ (annualized)				$\mathbb{E}[R]$ (annualized)			
β	CVaR_β	R	SR	CR	CVaR_β	R	SR	CR
.1	5.90	4.60	12.1	4.60	6.20	6.20	6.90	6.20
.2	5.70	2.50	6.90	2.50	6.50	6.30	6.90	6.30
.3	5.70	1.60	4.50	1.30	7.20	6.40	7.00	6.40
.4	8.50	1.00	5.50	0.200	7.90	6.60	7.10	6.50
.5	9.40	0.800	2.20	-0.800	10.7	6.90	7.40	6.60
.6	10.4	0.700	1.50	-1.80	12.8	7.50	7.90	6.70
.7	10.1	0.100	0.400	-2.90	12.0	8.80	9.30	6.70
.8	5.50	-0.700	-0.500	-4.00	11.7	11.8	12.0	6.80
.9	-2.00	-3.90	-3.80	-5.20	11.8	12.5	12.5	6.80

Table 4: *Out-of-sample conditional-value-at-risk (left) and out-of-sample expected return (right) for the experiment run with the 4 types of approaches (CVaR, standard robust (R), soft robust (SR), and comprehensive robust (CR)) at different levels of β. All numbers expressed in %.*

	$\mathbb{P}\{\text{Return} < 0\}$				$\mathbb{P}\{\text{Return} < -10\%\}$				$\mathbb{P}\{\text{Return} < -20\%\}$			
β	CVaR_β	R	SR	CR	CVaR_β	R	SR	CR	CVaR_β	R	SR	CR
0.1	0.127	0.095	0.190	0.091	0.020	0.012	0.040	0.020	0.004	0.000	0.020	0.000
0.2	0.187	0.119	0.198	0.119	0.032	0.020	0.040	0.020	0.008	0.004	0.016	0.004
0.3	0.218	0.151	0.210	0.139	0.052	0.024	0.044	0.024	0.016	0.008	0.016	0.004
0.4	0.266	0.190	0.222	0.151	0.115	0.028	0.048	0.028	0.052	0.012	0.016	0.008
0.5	0.282	0.210	0.238	0.167	0.171	0.040	0.071	0.032	0.091	0.016	0.020	0.012
0.6	0.310	0.250	0.258	0.175	0.230	0.083	0.099	0.032	0.155	0.028	0.036	0.012
0.7	0.321	0.266	0.274	0.190	0.266	0.107	0.119	0.032	0.206	0.063	0.079	0.016
0.8	0.357	0.294	0.310	0.190	0.290	0.194	0.210	0.032	0.238	0.099	0.111	0.020
0.9	0.373	0.329	0.329	0.190	0.302	0.266	0.266	0.040	0.262	0.198	0.198	0.020

Table 5: *Probabilities that the out-of-sample monthly returns (annualized) are less than 0 (left), -10% (center), and -20% (right) for the 4 approaches in the asset allocation experiment.*

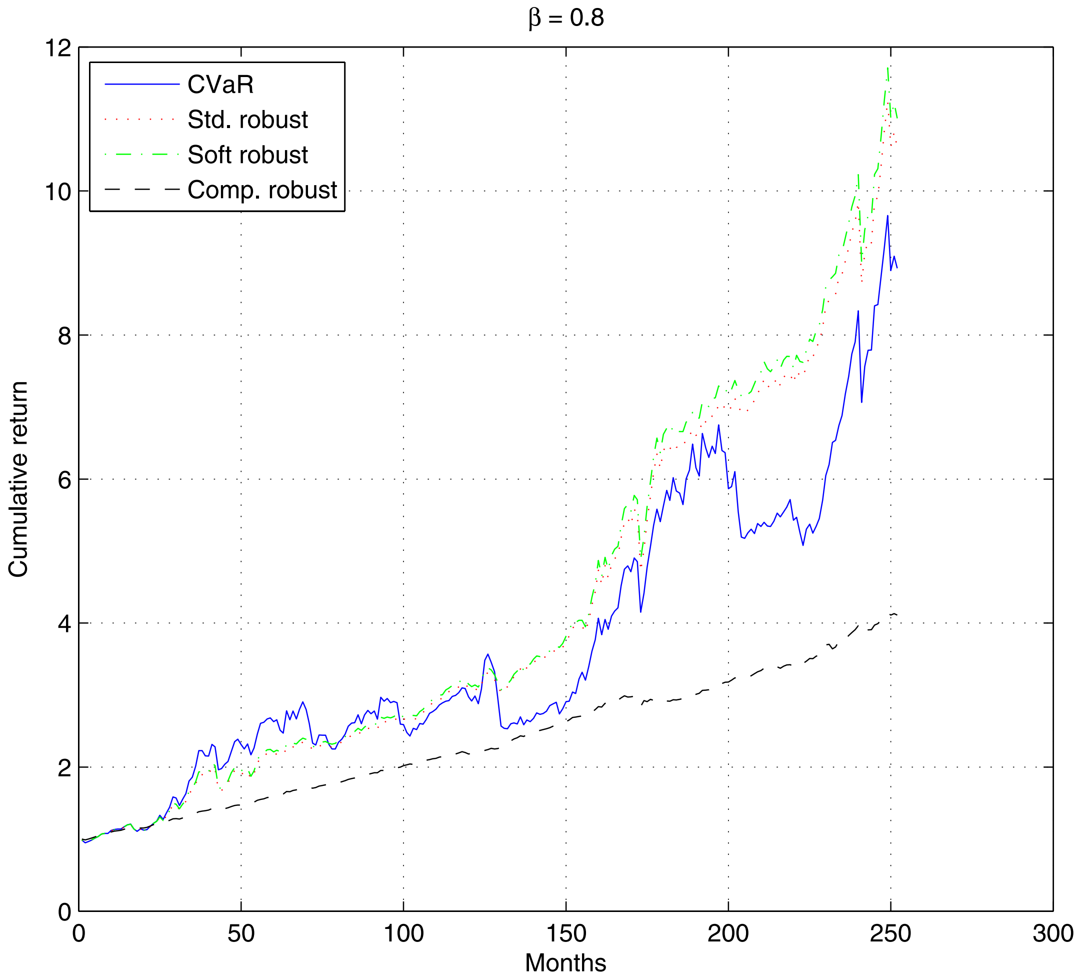

Figure 2: *Out-of-sample cumulative return for the four approaches for the case* $\beta = .8$.