# Satisficing Measures for Analysis of Risky Positions

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In this work we introduce a class of measures for evaluating the quality of financial positions based on their ability to achieve desired financial goals. In the spirit of Simon (Simon, H. A. 1959. Theories of decision-making in economics and behavioral science. *Amer. Econom. Rev.* **49**(3) 253–283), we call these measures *satisficing measures* and show that they are dual to classes of risk measures. This approach has the advantage that aspiration levels, either competing benchmarks or fixed targets, are often much more natural to specify than risk tolerance parameters. In addition, we propose a class of satisficing measures that reward diversification. Finding optimal portfolios for such satisficing measures is computationally tractable. Moreover, this class of satisficing measures has an ambiguity interpretation in terms of robust guarantees on the expected performance because the underlying distribution deviates from the investor's reference distribution. Finally, we show some promising results for our approach compared to traditional methods in a real-world portfolio problem against a competing benchmark.

*Key words*: satisficing; aspiration levels; targets; risk measures; coherent risk measures; convex risk measures; portfolio optimization
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## 1. Introduction

One of the key principles from Simon's (1955) bounded rationality model is that, rather than formulating and solving complicated optimization problems, real-world agents often can choose the first-available actions, which ensure that certain *aspiration levels* will be achieved. In other words, given the computational difficulties in the rational model paradigm, a more sensible (and descriptively accurate) approach may in fact be to view profit not as an *objective* to be maximized, but rather as a *constraint* relative to some given aspiration level. Simon (1959) coined the term *satisficing*[1] to describe this approach.

The focus of this paper is to utilize the concept of aspiration levels from satisficing as a means of quantifying the desirability of investment opportunities with uncertain payoffs. As such, we replace the term "agent" with "investor," and we refer to the investment opportunities as "positions." We restrict ourselves to the scalar case, i.e., the case when each position will realize a payoff representable by a single, real number. Although our primary focus in this paper is on a model useful for financial positions, the framework readily applies to other settings in which risk is an important issue.

In particular, our goal is to provide a framework for measuring the quality of risky positions with respect to their ability to satisfy (i.e., achieve the aspiration level). Aspiration level models based in the spirit of satisficing have been previously explored but the connection is usually drawn via probability measures, i.e., the probability of achieving at least an aspiration level. Here we axiomatize a more general class of measures, called *satisficing measures*, which depend on the investor's aspiration level. Probability measures are one special case of these.

One important advantage of this approach is that aspiration levels are often very natural for investors to specify, whereas traditional models based on risk measures or utility functions depend critically on tolerance parameters, which are often difficult for investors to intuitively grasp and even harder to appropriately assess (certainly, at least, relative to assessing aspiration levels).[2]

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[1] A blend of the words "satisfy" and "suffice."

[2] We concede here that we avoid entirely the issue of ascertaining the "right" aspiration level and assume it is a given primitive; for some recent work that does address this issue, see Bearden and Connolly (2007).

As we have stated, the idea of aspiration levels is not new in the decision theory literature (which often uses the nomenclature "targets"). Some recent work focuses on the probability of achieving an aspiration level as a way of comprehensively and rigorously discussing decision theory without using utility functions (see Castagnoli and LiCalzi 1996, Bordley and LiCalzi 2000). Tsetlin and Winkler (2007) consider the case of using aspiration levels along multiple dimensions within utility functions. From the descriptive standpoint, a number of studies have concluded that aspiration levels play an important role in real-world decision-making behavior. Lanzillotti's (1958) study interviews executives of 20 large companies and concludes that these managers are primarily concerned about target returns on investment. In another study, Payne et al. (1980, 1981) illustrate that managers tend to disregard investment possibilities that are likely to underperform against their target. Simon (1959) also argued that most firms' goals are not maximizing profit but attaining a target profit. In an empirical study by Mao (1970), managers were asked to define what they considered as risk. From their responses, Mao (1970, p. 353) concluded that, "risk is primarily considered to be the prospect of not meeting some target rate of return." The notion of success probability has also been applied widely in the mathematical finance literature (see, for instance, Browne 1999, Müller and Baumann 2006, Föllmer and Leukert 1999).

Of course, one of the drawbacks of maximizing the success probability alone is that it tacitly assumes that the modeler is indifferent to the level of losses and gains. It does not address how catastrophic losses can be (or how exceptional gains can be) when "extreme" (i.e., low probability) events occur. A number of studies have in fact suggested that subjects are not completely insensitive to such magnitude variations, particularly with respect to losses (see, for instance, Payne et al. 1980). More recently, Diecidue and van de Ven (2008, p. 687) have argued that a model that solely maximizes the success probability is "too crude to be normatively or descriptively relevant."

It is certainly relevant to place our work in the context of reference-dependent utility, a framework that arose from the development of prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). This theory was aimed at correcting perceived inadequacies of expected utility theory from a descriptive perspective; for instance, traditional expected utility theory does not account for framing effects and the fact that decision makers may be significantly more sensitive (especially locally) to losses around a reference point than they are to gains of comparable magnitude. These and other criticisms are often borne out in empirical data of real human decisions. Some of the recent work in reference-dependent utility develops reference points (targets in our language) endogenously (e.g., Shalev 2000 in a game theoretic setting) and allows the reference point to be stochastic (e.g., Sudgen 2003). Köszegi and Rabin (2006, 2007) present a framework that allows the reference points to be both stochastic and endogenously determined by preferences. One issue in this setting is that the statistical dependence between prospects and targets is ignored. De Giorgi and Post (2007) extend the framework to account for the (often very strong) dependence between prospects and targets, this has benefits computationally and can be critical from a modeling perspective if the target is, for instance, a stochastic benchmark against which the decision maker is competing.

Our approach is a bit different than the reference-dependent utility literature. For starters, our framework does not rely in any way on the axioms of utility theory. Instead, we start with a very simple set of axioms that depend solely and in simple fashion upon performance relative to a target. Our approach assumes nothing about the stochastic structure of the target (it can certainly be random), but we do take the target as exogenously specified. As discussed above, however, we feel that targets are generally quite natural in many applications, so giving the decision maker this freedom is often sensible practically. We will see by virtue of examples that utility theory models do fall into our framework, but they are not the only ones that do so.

We summarize our primary contributions as follows:

1. We define axiomatically the concept of a satisficing measure, which depends on a position's performance relative to a given aspiration level. These satisficing measures can be thought of as a generalization of the success probability approach. We also show that satisficing measures have a dual connection to risk measures; in particular, we prove a representation theorem, which shows that every satisficing measure can be written in terms of a parametric family of risk measures.

2. In general, a satisficing measure need not reward diversification. When we impose an additional property of quasiconcavity on satisficing measures, however, diversification is rewarded. Moreover, the representation theorem in this case holds over a parametric family of *convex* risk measures introduced by Föllmer and Schied (2002). Most quasiconcave satisficing measures that we explore are *not* insensitive to the magnitude of losses below (or gains above) aspiration levels, which is in stark contrast to probability measures. In addition, the quasiconcavity property is important computationally because it readily admits the use of efficient algorithms for optimization over satisficing measures. This class of satisficing measures has an

alternate interpretation in terms of expected values over *ambiguous* probability distributions, which, interestingly, connects us back to Simon's contention that probabilities are often not known exactly.

3. Finally, when we impose an additional property of scale invariance on a satisficing measure, our representation theorem then ties into to the well-known class of *coherent* risk measures popularized by Artzner et al. (1999). Furthermore, we show that a "separation theorem" holds for portfolio optimization problems over this class of satisficing measures.

Before we proceed, we comment briefly on our choice of nomenclature. The idea of defining a satisficing measure as something over which we can *optimize* may seem to be somewhat of an oxymoron to purists. Indeed, Simon's original presentation of satisficing was a computationally feasible alternative to the potentially onerous approach of optimizing. On the other hand, we are not the first to use the aspiration level idea from satisficing within decision theoretic approaches based on optimization (e.g., Bordley and LiCalzi 2000, Charnes and Cooper 1963). Moreover, the rapid proliferation of massive computational capabilities has radically changed the way decisions are made in many industries, particularly finance; Simon et al. (1986) were obviously quite aware of and interested in the impact of high-speed computing. Our choice of the word satisficing largely reflects a connection to the concept of aspiration levels as a means of quantifying satisfaction with a risky position.

The outline of this paper is as follows. In §2, we motivate and define the axioms of satisficing measures and state our most general representation theorem relating these to risk measures; in §3, we consider imposing additional properties on the satisficing measures and then connect them to convex and coherent risk measures. Finally, §4 considers portfolio optimization with satisficing measures, and §5 explores a simple debt management example. We refer the reader to the e-companion[3] for all proofs as well as for a computational example illustrating satisficing measures in the context of portfolio optimization.

## 2. Defining Satisficing Measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space and let $\mathcal{X}$ be a set of random variables on $\Omega$, i.e., a set of functions $X: \Omega \to \Re$. Each $X \in \mathcal{X}$ represents the payoff (or return) of a different, risky position. Throughout the paper, we will use the notation $X \geq Y$ for $X, Y \in \mathcal{X}$ to represent statewise dominance, i.e., $X(\omega) \geq Y(\omega)$

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[3] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

for all $\omega \in \Omega$. Similarly, $X > Y$ denotes strict statewise dominance ($X(\omega) > Y(\omega)$ for all $\omega \in \Omega$).

We consider the situation in which the investor has an aspiration level $\tau$, which she hopes to achieve via these positions. We assume $\tau$ is a random variable on $\Omega$ as well, which, of course, includes the case of $\tau$ being a constant. This encompasses the cases of both individual investors, who have their own financial goals (often constant targets) to meet, as well as the case of professional managers, who often compete against alternative, risky positions (benchmarks).

Given an uncertain payoff $X \in \mathcal{X}$, we define the *target premium V* to denote the excess payoff above the aspiration level, i.e., $V = X - \tau \in \mathcal{V}$, where $\mathcal{V}$ is also a set of random variables on $\Omega$. Without loss of generality and for notational convenience, we assume $\mathcal{V} = \mathcal{X}$. In other words, we will assume each of the payoffs $X \in \mathcal{X}$ already has the aspiration level embedded within it, and therefore we will suppress the notation $\tau$ in everything that follows. Thus, a position achieves the aspiration level if and only if the realized target premium $X(\omega)$ satisfies $X(\omega) \geq 0$.

In the spirit of satisficing, one measure of satisfaction with respect to a target premium $X \in \mathcal{X}$ is

$$\rho(X) \triangleq \mathbb{P}\{X \geq 0\}. \tag{1}$$

Obviously, such a measure is equivalent to the expected value of a $\{0, 1\}$-utility function $u(x)$, which is one if and only if $x \geq 0$.

We now define a more general notion of satisficing measures, of which probability measures are one example. We are attempting to get the best of both worlds: on one hand, we are exploiting the fact that aspiration levels are simple to understand and (often) natural to specify; on the other hand, by imposing additional properties in the following section, we will circumvent some of the main difficulties with probability measures (e.g., its inability to reward diversification or distinguish magnitudes).

DEFINITION 1. A function $\rho: \mathcal{X} \to [0, \bar{\rho}]$, where $\bar{\rho} \in \{1, \infty\}$, is a *satisficing measure* defined on the target premium if it satisfies the following axioms for all $X, Y \in \mathcal{X}$:

1. *Attainment content*: If $X \geq 0$, then $\rho(X) = \bar{\rho}$.
2. *Nonattainment apathy*: If $X < 0$, then $\rho(X) = 0$.
3. *Monotonicity*: If $X \geq Y$, then $\rho(X) \geq \rho(Y)$.
4. *Gain continuity*: $\lim_{a \downarrow 0} \rho(X + a) = \rho(X)$.

These axioms are rather straightforward to motivate in light of satisficing. Attainment content reflects the fact that satisficing is primarily concerned with achieving the aspiration level; if a position always achieves this, then we are always satisfied with the outcome, and therefore fully "content." On the flip side, a position that *never* reaches the aspiration level does not satisfy us in the slightest (nonattainment

apathy). Monotonicity is quite clear as well: if one position never underperforms another, then we must be at least as satisfied with it. Finally, gain continuity is simply right continuity of $\rho$; if we augment our position with an infinitesimally small but positive amount, in the limit we cannot improve our satisfaction.[4] In other words, we are indifferent to small gains in the payoff, but not necessarily small losses (in particular, if we are right at the aspiration level).

REMARK 1. An immediate criticism of this framework is that, if the target premia are all above (below) zero, then any satisficing measure cannot distinguish among any of the positions. On one hand, if aspiration levels are purely preference-dependent and in no way influenced by the available opportunities, then it is true that the satisficing measures will not be very useful in such circumstances. On the other hand, it is not unreasonable to assume that aspiration levels are often affected by the opportunities available to the investor. In such cases, one could argue that the investor is too pessimistic (optimistic) and should be reassessing her aspiration level. In fact, Simon (1955) provides an example of selling a house and the agent's aspiration level is determined, in part, via an "exploration" phase in which she learns about the climate of her housing market. In short, the satisficing measures we define are most useful when the aspiration levels are set such that positions can fall both above and below this level.

Clearly, $\mathbb{P}\{X \geq 0\}$ is one example of a satisficing measure; $\mathbb{P}\{X > 0\}$, however, violates gain continuity, and is therefore not a satisficing measure.

We also point out that our definition allows for satisficing measures to map to either $[0, 1]$ or all of $\Re_+$. We will give examples of both cases in §2.2.

### 2.1. Quasiconcave and Coherent Satisficing Measures

The probability measure $\mathbb{P}\{X \geq 0\}$ is the most obvious example of a satisficing measure. On one hand, it embodies the concept of satisficing in that it reflects very directly the performance of a position relative to the aspiration level. On the other hand, unfortunately, it suffers a rather critical flaw from an economic perspective in that it does not reflect convex preferences (i.e., does not value diversification).

Clearly, any satisficing measure $\rho$ induces a preference relation $\succeq$ with, for any $X, Y \in \mathscr{X}$, $X \succeq Y$ if and only if $\rho(X) \geq \rho(Y)$. Recall that $\succeq$ is a convex preference if, for all $X, Y, Z \in \mathscr{X}$, the implication $Y \succeq X$, $Z \succeq X \Rightarrow \lambda Y + (1 - \lambda)Z \succeq X \; \forall \lambda \in [0, 1]$ holds. It is clear

that the induced relation $\succeq$ is convex if and only if the function $\rho$ is quasiconcave, i.e., for all $X, Y \in \mathscr{X}$, $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\}$.

For perhaps the simplest example of probability's failure to satisfy quasiconcavity (and hence failure to induce convex preferences), consider a target premium $X$ distributed symmetrically about zero, so $\mathbb{P}\{X \geq 0\} = \frac{1}{2}$. Now consider an alternative position $Y = -X - \epsilon$, where $\epsilon > 0$. If $\epsilon$ is small enough and the distribution of $X$ is continuous, then $\mathbb{P}\{Y \geq 0\}$ will be close to $\frac{1}{2}$ as well. The position $Z = \frac{1}{2}(X + Y)$, however, is $-\epsilon/2$ with probability one, so $\mathbb{P}\{Z \geq 0\} = 0$. Therefore, this satisficing measure says we are worse off by diversifying between these positions as opposed to investing into either individually.

One could object that this is a rather silly example; indeed, if aspiration levels are really all we care about, then the probability measure is "doing its job" just fine. Even in the limit as $\epsilon \to 0^-$, however, the probability measure prefers positions $X$ or $Y$, both of which can have arbitrarily large losses, to the sure position $Z$, even if it is only a fraction of a cent below our target. This does not seem reasonable.

Moving beyond this illustrative example to a general discussion, there are a number of arguments that strongly support quasiconcavity of a satisficing measure. The first is the widely accepted tenet that diversification of positions is almost always a good thing. Indeed, we can go back to Markowitz (1952, p. 77), who boldly declares, "Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim." Since this work, it has become almost a fundamental law of investing that approaches to portfolio choice that do not lead to diversified positions are generally undesirable because of their tendency to expose investors to large positions.

A second and related reason quasiconcavity is a desirable property is for robustness of positions. In particular, given that probability assessments of the real world are inherently inexact, it behooves the investor to split positions up and thereby reduce sensitivity to errors in modeling the real world. In contrast, satisficing measures that are not quasiconcave (like probability measure) can lead us to large positions in which expected performance is highly dependent on the specifics of our model for that position. In some sense, this also links us to Simon's bounded rationality, which posits that models of the real world are largely approximations subject to errors.

Finally, we would ultimately like to exploit the simplicity of satisficing measures and use them as a method for *selection* of positions, and their failure to satisfy quasiconcavity is tantamount to having

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[4] Note that left continuity is critically different and need not hold. In particular, clearly probability measure $\mathbb{P}\{X \geq 0\}$ is a satisficing measure. Now consider the case when $X$ is a constant; right-continuity, but not left-continuity, holds at $X = 0$.

to select over potentially nonconvex spaces of positions. From a computational standpoint, this is in general very difficult when the number of opportunities (or available asset classes in a portfolio setting) is large. Although tractability of an approach cannot by itself form a foundation for an axiomatic theory, it is fair to say that an approach to risk management that is not amenable to rapid computation will be of limited use in practice.

In short, the desire to ensure that satisficing measures recognize diversification appropriately, both for economic and computational reasons, motivates the following.

DEFINITION 2. A function $\rho: \mathscr{X} \to [0, \bar{\rho}]$, where $\bar{\rho} \in \{1, \infty\}$, is a *quasiconcave satisficing measure* (QSM) defined on the target premium if, in addition to Definition 1, it satisfies the following for all $X, Y \in \mathscr{X}$:

1. *Quasiconcavity*: If $\lambda \in [0, 1]$, then $\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\}$.

If, in addition, $\rho$ satisfies

2. *Scale invariance*: If $k > 0$, then $\rho(kX) = \rho(X)$; we say $\rho$ is a *coherent satisficing measure* (CSM).

Note that, in the simple example we gave, a QSM would have to satisfy $\min\{\rho(X), \rho(Y)\} \leq 0$, i.e., either $\rho(X) = 0$ or $\rho(Y) = 0$, which is not the case for probability measure.

The scale invariance axiom arises in situations in which, like with probability measures, we are indifferent to symmetric scalings of the payoffs with respect to our aspiration level. Note that quasiconcavity alone implies, for any $\lambda \in [0, 1]$, $\rho(\lambda X) = \rho(\lambda X + (1 - \lambda)0) \geq \min\{\rho(X), \rho(0)\} = \rho(X)$, since, by definition, $\rho(0) = \bar{\rho}$ for a satisficing measure. For a CSM, this bound is tight. In other words, for a CSM, any decrease in potential losses below the aspiration level is offset by symmetrically increased gains above the aspiration level. We can therefore view CSMs as diversification-rewarding measures that retain the "target-oriented" nature of probability measures. Or, put another way, CSMs are quasiconcave "relaxations" of probability measures.

## 2.2. Examples
We now provide some examples of satisficing measures. We intentionally provide a number of examples here to illustrate how the concepts of QSMs and CSMs cut across a variety of approaches to decision making under uncertainty, including risk measures from finance and ideas from utility theory. Some of these examples require proofs, which we have relegated to the e-companion.

EXAMPLE 1 (PROBABILITY MEASURE). As we have mentioned, $\rho(X) = \mathbb{P}\{X \geq 0\}$ is a satisficing measure. Note that it satisfies all the axioms of coherence except quasiconcavity.

EXAMPLE 2 (SHARPE RATIO). Let $\mathscr{X}$ be a linear space of a finite set of random variables including constants

such that $X \in \mathscr{X}$ is either a constant or a random variable with infinite support. This includes a set of independently distributed random variables with infinite support such as those with normal distributions. The following Sharpe ratio,

$$\rho_S(X) = \sup_{a \in [0, 1)} \left\{ a: -\mathbb{E}(X) + \sqrt{\frac{a}{1-a}} \sigma(X) \leq 0 \right\},$$

(or zero if no such $a$ exists) is a coherent satisficing measure on $\mathscr{X}$. Moreover, $\rho_S(X) \leq \mathbb{P}\{X \geq 0\}$. Note that the function $g(a) = \sqrt{a/(1-a)}$ is an increasing function of $a \in [0, 1)$, hence, whenever $\sigma(X) > 0$ and $\mathbb{E}(X) \geq 0$, $g(\rho_S(X)) = \mathbb{E}(X)/\sigma(X)$, which is the standard definition of the Sharpe ratio (without a risk-free asset). Note that the Sharpe ratio is generally not a coherent satisficing measure on random variables with finite support. For instance, consider the random variable $X \sim U[0, 1]$, which has the Sharpe ratio $\sqrt{3}$ and hence, $\rho_S(X) = 3/4$. However, since $X \geq 0$, the satisficing measure $\rho_S$ violates the axioms of attainment content and monotonicity (since $\rho_S(X) < \rho_S(0) = 1$). This satisficing measure is explored in more detail by Pinar and Tütüncü (2005).

EXAMPLE 3 (OPTIMIZED SHARPE RATIO). Let

$$\mathscr{X} = \{X: \exists (\lambda_0, \ldots, \lambda_n) \in \Re^{n+1}: X = \lambda_0 + \lambda_1 X_1 + \cdots + \lambda_n X_n\},$$

where $X_1, \ldots, X_n$ are random variables defined on $\Omega$ with positive definite covariance, and without loss of generality, zero means. The following optimized Sharpe ratio,

$$\rho_{OS}(X) = \sup_{a \in [0, 1)} \left\{ a: \min_{V \in \mathscr{X}} \left( -\mathbb{E}(X - V) \right. \right.$$
$$\left. \left. + \sqrt{\frac{a}{1-a}} \sigma(X - V): V \geq 0 \right\} \leq 0 \right\},$$

(or zero if no such $a$ exists) on $\mathscr{X}$ is a coherent satisficing measure on $\mathscr{X}$. Moreover,

$$\rho_S(X) \leq \rho_{OS}(X) \leq \mathbb{P}\{X \geq 0\}.$$

Note that we can express $\mu_a(X) = \min_{V \in \mathscr{X}}\{-\mathbb{E}(X - V) + \sqrt{a/(1-a)}\sigma(X - V): V \geq 0\}$ as an optimization problem over an $n$-dimensional convex cone. Moreover, its objective approaches to infinity whenever $\mathbb{E}(V)$ or $\sigma(V)$ approaches infinity. Because the covariance of $X_1, \ldots, X_n$ is positive definite, the optimal solution must be bounded and achieved.

In the same example in which $X \sim U[0, 1]$, we note that for all $a \in (0, 1)$,

$$\mu_a(X) \leq \min_{V \in \mathscr{X}} \left\{ -\mathbb{E}(X - V) \right.$$
$$\left. + \sqrt{\frac{a}{1-a}} \sigma(X - V): V \geq 0, X = V \right\} = 0.$$

Hence, $\rho_{OS}(X) = 1$. Thus, $\rho_{OS}$ provides a better bound to probability measures than $\rho_S$.

EXAMPLE 4 (MAXIMIZED RISK AVERSION). Consider the function $C_a(X) = \sup\{m \in \Re: \mathbb{E}(u_a(X - m)) \geq 0\}$, where $\{u_a: a > 0\}$ is a family of concave, nondecreasing utility functions nonincreasing in $a > 0$. The scalar $a$ is a risk aversion parameter, with larger $a$ reflecting greater risk aversion. $C_a(X)$ is a certainty equivalent and may be interpreted as the *maximum buying price of position $X$*. Then the function $\rho(X) = \sup\{a > 0: C_a(X) \geq 0\}$ (or zero if no such $a$ exists) is a QSM. As an example of this, when $u_a(x) = (1/a)(1 - \exp\{-ax\})$, we have

$$\rho(X) = \sup\left\{a > 0: \ -\frac{1}{a}\ln(\mathbb{E}(\exp(-aX))) \geq 0\right\}$$

(or zero if no such $a$ exists). It maximizes the risk aversion so that its certainty equivalence of $X \in \mathscr{X}$ under an exponential utility achieves the break-even point (this QSM is explored in Chua et al. 2007; interestingly, it is also the reciprocal of the "economic index of riskiness" recently developed by Aumann and Serrano 2008).

EXAMPLE 5 (FAMILY OF CONCAVE UTILITIES). Let $f: \Re \to \Re$ be a continuous concave nondecreasing function with $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Then $\rho(X) = \lim_{\epsilon\downarrow 0} \sup_{a\in\Re_+} \mathbb{E}(f(a(X+\epsilon)))$ is a coherent satisficing measure. Moreover, $0 \leq \rho(X) \leq \mathbb{P}\{X \geq 0\}$. The choice of function, $f(x) = \min\{x, 1\}$ has recently been explored by Chen and Sim (2008).

Most of the examples we have provided in this section are of the form of choosing the largest value of a parameter subject to some sort of risk constraint. In the case of Examples 2 and 3, for instance, we are choosing the largest value of a scalar $a$ such that $-\mathbb{E}(X) + g(a)\sigma(X) \leq 0$, which is a mean-variance risk constraint. In Example 4, we are choosing the largest value of a risk aversion parameter subject to its certainty equivalent under a concave utility function being nonnegative at that level. Although it is not immediately obvious, we can also write Example 5 in a similar form. In fact, it turns out that we can write *any* satisficing measure in the form of the largest value of a parameter subject to a risk constraint, as we will now show.

## 3. Dual Representation of Satisficing via Risk Measures

If we were to choose positions based on higher or maximal values of the satisficing measure $\mathbb{P}\{X \geq 0\}$, intuition suggests that we are somehow embedding risk aversion into our selection process; we are hoping to find portfolios that have a greater likelihood of achieving the aspiration level.

Indeed, probability measures do in fact connect to the classical definition of risk, known as *value-at-risk*. The usual definition of value-at-risk is

$$\mathrm{VaR}_\alpha(X) \triangleq \inf\{t \in \Re: \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\}. \quad (2)$$

This quantity can be interpreted as the smallest amount of capital $t$ necessary to add to $X$ to ensure that the augmented portfolio $X + t$ breaks even with probability at least $1 - \alpha$. Thus, all other things being equal, a portfolio with a *lower* value-at-risk at a pre-specified $\alpha$ is preferred. In addition, value-at-risk is decreasing with $\alpha$, i.e., lower $\alpha$ corresponds to more conservatism.

When the distribution has discontinuities, there are various definitions of VaR that take into account potential limiting conditions at these discontinuities. VaR has received considerable attention among both practitioners and academics alike; see, for instance, Duffie and Pan (1997) for one treatment of value-at-risk.

In fact, probability measures $\rho$ as defined in (1) are simply dual forms of VaR. In particular, it is not hard to see that

$$\mathbb{P}\{X \geq 0\} = \sup\{1 - \alpha: \mathrm{VaR}_\alpha(X) \leq 0\}. \quad (3)$$

Indeed, observe that $\mathrm{VaR}_1(X) = -\infty$ and for $\alpha < 1$, the infimum in problem (2) is achievable since $\mathbb{P}\{X + t \geq 0\}$ is a right continuous, nondecreasing function with respect to $t$. Hence,

$$\begin{aligned}
\sup\{1 &- \alpha: \mathrm{VaR}_\alpha(X) \leq 0\} \\
&= \sup\{1 - \alpha: \ \inf\{t: \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\} \leq 0\} \\
&= \sup\{1 - \alpha: \exists t \leq 0: \mathbb{P}\{t + X \geq 0\} \geq 1 - \alpha\} \\
&= \max\{1 - \alpha: \mathbb{P}\{X \geq 0\} \geq 1 - \alpha\} \\
&= \mathbb{P}\{X \geq 0\} \\
&= \rho(X).
\end{aligned}$$

Note that (3) is a *dual* of (2) in which $\tau$, the aspiration level, is specified and $\alpha$, the tolerance level, is chosen to be as small as possible. VaR, on the flip side, has the tolerance level fixed with the target (or augmenting capital) chosen to be as small as possible. Thus, value-at-risk measures and probability measures are dual forms.

One may ask whether this relationship extends to general satisficing measures; in other words, whether any satisficing measure can be represented in dual form over a family of risk measures. The answer, as we will show, is yes. First, we define formally the concept of a risk measure.

DEFINITION 3. A function $\mu: \mathscr{X} \to \Re$ is a *risk measure* if it satisfies the following for all $X, Y \in \mathscr{X}$:

1. *Monotonicity*: If $X \geq Y$, then $\mu(X) \leq \mu(Y)$.
2. *Translation invariance*: If $c \in \Re$, then $\mu(X + c) = \mu(X) - c$.

The interpretation of risk measures is, like value-at-risk, the smallest amount of capital necessary to augment a position in order to make it "acceptable" according to some standard. As such, the properties

above are clear; if one position never performs worse than another, then it cannot be any riskier. In addition, if we augment our position by a guaranteed amount $c$, then our capital requirement is reduced correspondingly by $c$ as well. See, for instance, Föllmer and Schied (2004) for more on risk measures.[5]

We are now ready for our first result (please see the e-companion for all proofs of main results).

THEOREM 1. *A function* $\rho$: $\mathcal{X} \to [0, \bar{\rho}]$, *where* $\bar{\rho} \in \{1, \infty\}$, *is a satisficing measure if and only if there exists a family of risk measures* $\{\mu_k \colon k \in (0, \bar{\rho}]\}$, *nondecreasing in* $k$, *and* $\mu_0 = -\infty$ *such that*

$$\rho(X) = \sup\{k \in [0, \bar{\rho}]: \mu_k(X) \le 0\}. \quad (4)$$

*Moreover, given* $\rho$, *the corresponding risk measure is*

$$\mu_k(X) = \inf\{a: \rho(X + a) \ge k\}. \quad (5)$$

Theorem 1 states that *every* satisficing measure can be written as the largest value of a parameter subject to a risk constraint at that parameter value over a parametric family of risk measures. Moreover, the proof of the result is constructive, so given a satisficing measure, we can always specify exactly the structure of the corresponding, parametric family of risk measures.

The interpretation is that, rather than specifying risk aversion parameters (e.g., choosing an $\alpha$ for VaR), the satisficing measure approach is always equivalent to fixing an aspiration level, then choosing the *largest* value of the aversion parameter such that the risk of the target premium evaluated at that parameter is acceptable. In other words, satisficing measures quantify how risk averse a position allows us to be while still maintaining the goal of achieving our aspiration level.

### 3.1. Representation of QSMs and CSMs
Of course, without additional, structural properties on the satisficing measure, there are no other properties that the family of risk measures must satisfy. We now examine structural properties of the family of risk measures when we are dealing with QSMs and CSMs, respectively.

To relate these satisficing measures to risk measures, we now formally define a class of risk measures introduced by Artzner et al. (1999) and then generalized by Föllmer and Schied (2002). In particular, we have the following.

DEFINITION 4. *A function* $\mu$: $\mathcal{X} \to \Re$ *is a convex risk measure if, in addition to Definition 3, it satisfies, for*

all $X, Y \in \mathcal{X}$:
1. *Convexity:* If $\lambda \in [0, 1]$, then $\mu(\lambda X + (1 - \lambda)Y) \le \lambda\mu(X) + (1 - \lambda)\mu(Y)$.
If, in addition, we have
2. *Positive homogeneity:* If $\lambda \ge 0$, then $\mu(\lambda X) = \lambda\mu(X)$; we say that $\mu$ is a *coherent risk measure*.

It is known that every coherent risk measure may be written in the form

$$\mu(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(-X) \quad (6)$$

for a family of generating measures $\mathcal{Q}$ (see, for instance, Huber 1981 for a proof of this in a different context). Föllmer and Schied (2002) show more generally that any convex risk measure may be written in the form

$$\mu(X) = \sup_{\mathbb{Q} \ll \mathbb{P}}\{\mathbb{E}_{\mathbb{Q}}(-X) - \alpha(\mathbb{Q})\}, \quad (7)$$

where $\alpha$ is a convex function. It is then not hard to show that a convex risk measure is coherent if and only if the corresponding $\alpha$ in its representation is an indicator function on a subset of measures $\mathcal{Q}$.

We remark that we will assume, without loss of generality, that a finite convex risk measure is normalized such that $\mu(0) = 0$; this implies, for instance, that $\alpha(\mathbb{Q}) \ge 0$ for all $\mathbb{Q} \ll \mathbb{P}$. It is without loss of generality because, when $\mu$ is convex, $\mu(X) - \mu(0)$ is also convex.

As the names of QSMs and CSMs betray, they are intimately related to these classes of risk measures. In fact, one can see that the satisficing measure $\rho(X) = \sup\{k \in [0, \bar{\rho}]: \mu_k(X) \le 0\}$, $\mu_0 = -\infty$ and the $\mu_k$ are a family of convex risk measures nondecreasing in $k$, is indeed quasiconcave; moreover, $\rho$ is a CSM when the $\mu_k$ are also coherent.

Guided by Theorem 1, we may wonder whether every QSM and CSM has such a representation; in fact, this is the case.

THEOREM 2. *A satisficing measure* $\rho$ *is quasiconcave if and only if the family* $\{\mu_k: k \in (0, \bar{\rho}]\}$ *in Theorem 1 is a family of convex risk measures. Similarly, it is coherent if and only if the family is a family of coherent risk measures.*

REMARK 2. We remark that using Theorem 1 in the opposite direction, it is possible to obtain potentially new risk measures starting from reasonable satisficing measures.[6] Consider, for instance, the satisficing measure in Example 5, which provides a recipe for constructing coherent satisficing measures based on appropriate choices of utility functions. From Theorem 2 and using Equation (5), the risk measure

$$\mu_k(X) = \inf\left\{t: \lim_{\epsilon \downarrow 0} \sup_{a \in \Re_+} \mathbb{E}(f(a(X + \epsilon + t))) \ge k\right\} \quad (8)$$

---

[5] These authors use the same definition but refer to risk measures as "monetary measures of risk," which is perhaps more descriptive; for convenience we'll simply use the term "risk measures."

[6] We thank an anonymous referee for suggesting this idea.

is a family of coherent risk measures on $k \in [0, 1]$. To make this expression more concrete, we consider the interesting case when $f(x) = \exp(-x)$, i.e., exponential utility. To avoid excessive analysis, we consider $k > 0$ and assume that the limit in Equation (8) is achievable and that the supremum is attained at some $a > 0$. We have

$$
\begin{aligned}
\mu_k(X) &= \inf\{t\colon \exists a > 0 \colon 1 - \mathbb{E}(\exp(-a(X+t))) \geq k\} \\
&= \inf\{t\colon \exists a > 0 \colon \ln(\mathbb{E}(\exp(-a(X+t)))) - \ln(1-k) \leq 0\} \\
&= \inf\{t\colon \exists a > 0 \colon \ln(\mathbb{E}(\exp(-aX))) - \ln(1-k) \leq at\} \\
&= \inf_{a>0}\left\{\frac{1}{a}\ln(\mathbb{E}(\exp(-aX))) - \frac{1}{a}\ln(1-k)\right\},
\end{aligned}
$$

which is easily verified as a coherent risk measure. In fact, the generating family for this coherent risk measure is $\mathbb{Q} = \{\mathbb{Q} \ll \mathbb{P}\colon \int \ln(d\mathbb{Q}/d\mathbb{P})\, d\mathbb{Q} \leq \ln(k)\}$. This is the coherent risk measure that protects equally against all measures close in relative entropy to the reference distribution $\mathbb{P}$.

It would be interesting to see if a more general connection between utility theory and the theory of convex risk measures (often viewed as very different approaches) could be established; it seems that the satisficing approach offers some clues into these connections.

## 3.2. Optimization over QSMs

In contrast with maximizing the success probability over a convex set of random variables, which is computationally intractable, we show that maximizing a QSM can be reduced to solving a sequence of convex optimization problems. We consider the problem

$$
\rho^* \triangleq \max\{\rho(X)\colon X \in \mathcal{X}\}, \tag{9}
$$

where $\mathcal{X}$ is a convex set of random variables. From a computational perspective, finding a feasible solution in a convex set is relatively easy compared to a nonconvex one. In particular, given a QSM, $\rho$, the set $S(k) = \{X\colon \rho(X) \geq k, X \in \mathcal{X}\}$ is a convex set of random variables. Indeed, if $X \in S(k)$ and $Y \in S(k)$, it is easy to see that $\lambda X + (1-\lambda)Y \in S(k)$ for all $\lambda \in [0, 1]$. Assuming that there exists a computationally efficient method for finding a feasible random variable in the set $S(k)$, and assuming that $\rho^* \in [a, b]$, $a > 0$, and $b < \bar{\rho}$, we propose the following binary search to obtain a solution, $Z \in \mathcal{X}$ satisfying $\rho^* - \zeta \leq \rho(Z) \leq \rho^*$.

ALGORITHM 1 (BINARY SEARCH).

Input: A routine that returns a feasible random variable in the set $S(k)$ or reports infeasible; real, nonnegative numbers $a$, $b$, and $\zeta$.

Output: A random variable $Z$.
1. Set $\gamma_1 := a$ and $\gamma_2 := b$.
2. If $\gamma_2 - \gamma_1 < \zeta$, stop. Output: $Z$.

3. Let $\gamma := (\gamma_1 + \gamma_2)/2$.
4. If $S(\gamma)$ is infeasible, update $\gamma_2 := \gamma$. Otherwise, update $\gamma_1 := \gamma$ and find $Z \in S(\gamma)$.
5. Go to Step 2.

If the satisficing measure is given in its dual form (4), we can evaluate the feasibility of $S(k)$ by solving the following convex optimization problem:

$$
r(k) = \min\{\mu_k(X)\colon X \in \mathcal{X}\}. \tag{10}
$$

Indeed, checking the feasibility of $S(k)$ is the same if checking $r(k) \leq 0$, which is finding the random variable with the smallest risk under the convex risk measure $\mu_k$.

In sum, we can find a $\zeta$-optimal position $X^*$ to (9) in $\log_2((b-a)/\zeta)$ solutions of (10); provided we can solve (10), then, we can also solve (9) to high precision without too much additional difficulty.

## 3.3. An Ambiguity Robust Representation of QSMs

Theorem 2 immediately implies an interesting interpretation for any quasiconcave satisficing measure.

COROLLARY 1. *A satisficing measure $\rho$ is quasiconcave if and only if there exists a family of convex functions $\{\alpha_k\}_{k \geq 0}$, nonincreasing on $k \geq 0$, such that for all $X \in \mathcal{X}$ and all $\mathbb{Q} \ll \mathbb{P}$, $\mathbb{E}_{\mathbb{Q}}(X) \geq -\alpha_{\rho(X)}(\mathbb{Q})$.*

This is a direct implication of Theorem 2 and the representation theorem (7) for convex risk measures.

This interpretation states that $\rho(X)$ can be interpreted as a "robustness level" that guarantees performance in expectation for distributions $\mathbb{Q}$, which differ from our assessed distribution $\mathbb{P}$. We can visualize $\alpha_k$ as a family of functions measuring a scaled notion of distance from $\mathbb{Q}$ to $\mathbb{P}$; as $\rho(X)$ grows, we scale this distance by less, and therefore become closer to achieving our target in expectation even for other distributions $\mathbb{Q}$.

Note that when $\rho$ is a CSM, the $\alpha_k$ will be indicator functions on sets of probability measures. In particular, in this case, there exists a nested family of sets of probability measures $\{\mathbb{Q}(k)\}_{k \geq 0}$, i.e., $\mathbb{Q}(k_1) \subseteq \mathbb{Q}(k_2)$ for $k_2 \geq k_1 \geq 0$, such that, for all $X \in \mathcal{X}$, $\mathbb{E}_{\mathbb{Q}}(X) \geq 0$ $\forall \mathbb{Q} \in \mathbb{Q}(\rho(X))$. In this case, we hit the target as long as the "true" probability measure falls within the set $\mathbb{Q}(\rho(X))$, where $\rho(X)$ can be interpreted as measuring the "size" of this set.

In short, this interpretation states that quasiconcave satisficers are expected value maximizers in an *ambiguous* world in which probabilities are not known exactly and they value positions by how well they perform under other probability models, i.e., how insensitive a position is to changes in the underlying model. This perspective connects directly back to Simon's stance that real-world agents do not know probability

distributions exactly and therefore are making decisions in inherently ambiguous environments.

Put another way, Corollary 1 implies that quasiconcave satisficers are ambiguity averse in the sense that they want to maximize how well they are protected in expectation against worst-case distributions that differ from their underlying reference distribution.

### 3.4. CVaR Measure

In this section, we consider a CSM with some very interesting properties. This satisficing measure is a dual form of the following coherent risk measure, popularized by Rockafellar and Uryasev (2000), among others.

DEFINITION 5. For any $\alpha \in (0, 1]$, the coherent risk measure

$$\text{CVaR}_\alpha(X) = \inf_{\nu \in \Re}\left\{\nu + \frac{1}{\alpha}\mathbb{E}(-\nu - X)^+\right\}$$

is known as the *conditional value-at-risk at level $\alpha$*.

It is well known that when $X$ has a continuous distribution, $\text{CVaR}_\alpha(X) = -\mathbb{E}[X \mid X \leq -\text{VaR}_\alpha(X)]$. Roughly speaking, we can interpret CVaR as the expected value over the lower $\alpha$-tail of the distribution. As it is a coherent risk measure, it induces a CSM, which we now define.

DEFINITION 6. The coherent satisficing measure $\rho: \mathcal{X} \to [0, 1]$

$$\rho_{\text{CVaR}}(X) = \sup\{1 - \alpha: \text{CVaR}_\alpha(X) \leq 0\} \qquad (11)$$

(or zero if no such $\alpha$ exists) is called the *conditional value-at-risk measure* (or *CVaR measure*) on the target premium $X \in \mathcal{X}$.

Given the interpretation of CVaR, we can, roughly speaking, interpret $\rho_{\text{CVaR}}(X)$ as one minus the smallest left quantile such that the expected value of $X$ over this quantile is nonnegative. If the distribution is continuous, this interpretation is exact.

We consider the interesting case when $\mathbb{P}\{X < 0\} > 0$. Noting that $v + (1/\alpha)\mathbb{E}((-X - v)^+) > 0$ for all $v \geq 0$, we can express CVaR measures in the form of an expected utility of a concave function as follows:

$$\begin{aligned}
&\rho_{\text{CVaR}}(X) \\
&= \sup\{1 - \alpha: \text{CVaR}_\alpha(X) \leq 0\} \\
&= \sup\left\{1 - \alpha: \exists v < 0: v + \frac{1}{\alpha}\mathbb{E}((-X - v)^+) \leq 0\right\} \\
&= \sup\{1 - \alpha: \exists v < 0: 1 - \alpha \leq 1 - \mathbb{E}((-X/(-v) + 1)^+)\} \\
&= \sup_{a \geq 0}\{1 - \mathbb{E}((-aX + 1)^+)\} \\
&= \sup_{a \geq 0}\{\mathbb{E}(\min\{aX, 1\})\}.
\end{aligned}$$

As such, it is clear that CVaR measures do not fail to ignore the magnitude of losses, as opposed to probability measures.

We can illustrate this fact with some small examples. Indeed, consider the random variable $X \sim U[-1, 2]$. Clearly, $\mathbb{P}\{X \geq 0\} = \frac{2}{3}$ and a quick calculation shows $\rho_{\text{CVaR}}(X) = \frac{1}{3}$. Now consider the random variable $X' = \frac{1}{3}(2Y + Z)$, where $Y \sim U[0, 2]$ and $Z \sim U[-2, 0]$ and $Y$ and $Z$ are independent; the only difference between $X$ and $X'$ is that the left tail below zero has been spread out from $[-1, 0]$ in $X$ to $[-2, 0]$ in $X'$. Now, $\mathbb{P}\{X' \geq 0\} = \frac{2}{3} = \mathbb{P}\{X \geq 0\}$. On the other hand, we have $\rho_{\text{CVaR}}(X') = \sqrt{2}(\sqrt{2} - 1)/3 < \frac{1}{3} = \rho_{\text{CVaR}}(X)$, so CVaR measure *does* in fact recognize the differences in the left-tail losses, whereas probability measure does not.

A similar example can be used to illustrate that CVaR measure is sensitive to *right-tail* changes as well. Consider the same $X$; now let $X'' = \frac{1}{3}(Y' + 2Z')$, where $Y' \sim U[-1, 0]$, $Z' \sim U[0, 4]$, and $Y'$ and $Z'$ are independent. $X''$ is the same as $X$ with the right-tail (gains) expanded. As before, $\mathbb{P}\{X \geq 0\} = \mathbb{P}\{X'' \geq 0\} = 2/3$, but now we find $\rho_{\text{CVaR}}(X'') = (4 - \sqrt{2})/6 > \frac{1}{3} = \rho_{\text{CVaR}}(X)$.

In fact, it turns out that CVaR measure is a special QSM when it comes to bounding probability measure; we now define the following.

DEFINITION 7. A satisficing measure $\rho$ on $\mathcal{X}$ is *law-invariant* if $\rho(X) = \rho(Y)$ whenever $X$ and $Y$ have the same distribution under $\mathbb{P}$.

This definition extends the notion of law-invariant risk measures (e.g., Föllmer and Schied 2004). Law-invariance simply means that the satisficing measure depends only on the underlying distribution of the random variable.[7]

It is well known that CVaR is the smallest law-invariant convex risk measure which *dominates* VaR for all $X \in \mathcal{X}$ over the space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. By this we mean not only do we have $\text{CVaR}_\alpha(X) \geq \text{VaR}_\alpha(X)$ for all $X \in \mathcal{X}$, but if $\mu$ is any other law-invariant convex risk measure, the implication $\mu(X) \geq \text{VaR}_\alpha(X)$ $\forall X \in \mathcal{X} \Rightarrow \mu(X) \geq \text{CVaR}_\alpha(X)$ $\forall X \in \mathcal{X}$ holds. We now show that this idea extends to satisficing measures, i.e., that CVaR measure is the largest lower bound to probability measure over all law-invariant QSMs.

THEOREM 3. *Consider the case when $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$; for any $X \in \mathcal{X}$, we have $\rho_{\text{CVaR}}(X) \leq \mathbb{P}\{X \geq 0\}$. Moreover, for any law-invariant quasiconcave satisficing measure $\rho: \mathcal{X} \to [0, 1]$ the following implication holds:*

$$\rho(X) \leq \mathbb{P}\{X \geq 0\} \quad \forall X \in \mathcal{X}$$

$$\Rightarrow \quad \rho(X) \leq \rho_{\text{CVaR}}(X) \quad \forall X \in \mathcal{X}. \qquad (12)$$

---

[7] It is worth noting that any law-invariant risk measure preserves first-order stochastic dominance (Föllmer and Schied 2004), which immediately implies that any law-invariant satisficing measure does as well. Every example of a satisficing measure in this paper is law-invariant. In fact, it may be sensible to impose this property as one of the satisficing axioms up front, but we decided against this for the purposes of exposition.

As suggested by our small example above, the gap in this bound can be quite loose; for instance, for a random variable symmetrically distributed around zero, we have $\mathbb{P}\{X \geq 0\} = \frac{1}{2}$, whereas $\rho_{\text{CVaR}}(X) = 0$. Still, Theorem 3 states that, among law-invariant QSMs, we cannot hope to do any better.

Despite the gap in this bound, if $\mathscr{X}$ is a family of normal random variables, then the set of $X \in \mathscr{X}$ that maximize the probability measure is the same as the set of solutions that maximizes the CVaR measure. Indeed, if $X$ is normally distributed and $\mathbb{E}(X) > 0$,

$$\text{CVaR}_\alpha(X) = -\mathbb{E}(X) + \underbrace{\frac{\phi(\Phi^{-1}(\alpha))}{\alpha}}_{\xi(\alpha)} \sigma(X),$$

where $\phi(\cdot)$ is the density of a standard normal, and $\Phi(\cdot)$ is the corresponding cumulative distribution function. Now, noting that $\xi(\alpha)$ is a decreasing function of $\alpha$, we see that any $X \in \mathscr{X}$ that maximizes CVaR measure also maximizes the Sharpe ratio, $\mathbb{E}(X)/\sigma(X)$, as well as the probability measure $\mathbb{P}\{X \geq 0\} = \Phi(\mathbb{E}(X)/\sigma(X))$.

### 3.5. Loss Aversion Characteristics of QSMs
In this section, we briefly discuss some concepts of "loss aversion" and their relationship to QSMs. The classical definition of loss aversion is greater penalties for losses than rewards for gains of the same size. This is loss aversion in the sense of early prospect theory (Kahneman and Tversky 1979); in expected utility approaches, this is equivalent to concavity of the utility function. We do not discuss loss aversion in the sense of a "kink" in the slope of the utility function around the target. Although there are actually many definitions of loss aversion in the decision analysis literature (for a detailed discussion of the many definitions, see Köbberling and Wakker 2005, §6) and we do not provide any formal results, the discussion seems relevant given the large literature on reference-dependent utility and its focus on loss aversion characteristics.

It does seem difficult to make such a comparison, because this classical definition of loss aversion is equivalent to a property of the underlying utility function and is therefore not directly applicable to approaches that are not utility based. Therefore, our discussion can only be meaningful if we are willing to work with other notions of loss aversion.

As an example, and to some extent generalizing the example discussed with CVaR measure in §3.4, we can show that QSMs at least weakly prefer reduced gains to increased losses.

First, consider a risky position $X \in \mathscr{X}$ and now consider scaling the position by some factor $\kappa \geq 1$. For a QSM $\rho$, we will have $\rho(\kappa X) \leq \rho(X)$. To see this, we note that $X = (1/\kappa)\kappa X + ((\kappa - 1)/\kappa)0$, hence, $\rho(X) \geq \min\{\rho(\kappa X), \rho(0)\} = \rho(\kappa X)$. Now, for any position

$X \in \mathscr{X}$ and any $\alpha \geq 1$, let $X^\alpha$ denote the modification of $X$ by scaling its left tail by the factor $\alpha$. Specifically, we have

$$X^\alpha(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \geq 0, \\ \alpha X(\omega) & \text{otherwise.} \end{cases}$$

Now let $X^{\alpha_+}$ be a right-tail modification defined analogously; we have, for any $\alpha \geq 1$, $\rho(X^\alpha) = \rho(\max(X, 0) + \alpha \min(X, 0)) = \rho(\alpha(\alpha^{-1}\max(X, 0) + \min(X, 0))) \leq \rho(X^{\alpha_+^{-1}})$. Therefore, under a QSM, a left-tail scaling by a factor of $\alpha \geq 1$ is less satisfying to a right-tail scaling by a factor of $1/\alpha$. Given the option, an investor who maximizes a QSM would find cutting their upside by a factor of $\alpha$ no worse than increasing their downside by a factor of $\alpha$. In this sense, then, we can view QSMs as more sensitive to losses than gains and can consider this somehow related to more classical definitions of loss aversion.

## 4. Portfolio Optimization Using CSMs
We consider an investor who selects a portfolio return that maximizes her probability of achieving at least the risk-free rate, $r_f$, which is the return she would have had if she puts all her wealth in the risk-free asset. Her portfolio comprises of a nonnegative fraction, $\alpha$, of her wealth in a selected risky asset with random return $X \in \mathscr{X}$ and the remaining portion in the risk-free asset. Hence, her portfolio has random return $\alpha X + (1 - \alpha)r_f$. The fraction $\alpha$ is chosen such that the expected return of her effective portfolio achieves the level $\tau$, with $\tau > r_f$. Clearly, if $\tau \leq r_f$, she would invest only in the risk-free asset. Therefore, it suffices to focus on the set of random variables $\mathscr{X}$ such that $\mathbb{E}(X) > r_f$ for all $X \in \mathscr{X}$. The investor solves the following problem:

$$\max \ \mathbb{P}\{\alpha X + (1 - \alpha)r_f \geq r_f\}$$
$$\text{subject to} \ \mathbb{E}(\alpha X + (1 - \alpha)r_f) = \tau \qquad (13)$$
$$X \in \mathscr{X}, \quad \alpha \geq 0.$$

If the returns are normally distributed, then problem (13) is equivalent to the following:

$$\max \ \Phi\left(\frac{\mathbb{E}(\alpha X) - \alpha r_f}{\sigma(\alpha X)}\right)$$
$$\text{subject to} \ \mathbb{E}(\alpha X + (1 - \alpha)r_f) = \tau$$
$$X \in \mathscr{X}, \quad \alpha \geq 0,$$

where $\Phi(\cdot)$ is the distribution function of a standard normal. Noting that $\Phi(\cdot)$ is an increasing function, this is equivalent to

$$\max \ \frac{\mathbb{E}(\alpha X) - \alpha r_f}{\sigma(\alpha X)}$$
$$\text{subject to} \ \mathbb{E}(\alpha X + (1 - \alpha)r_f) = \tau \qquad (14)$$
$$X \in \mathscr{X}, \quad \alpha \geq 0.$$

Note that $\mathbb{E}(\alpha X) - \alpha r_f = \tau - r_f$, which is a constant; we can therefore reformulate problem (14) as the mean-variance Markowitz portfolio model as follows:

$$\min \quad \sigma^2(\alpha X + (1-\alpha)r_f)$$

$$\text{subject to} \quad \mathbb{E}(\alpha X + (1-\alpha)r_f) = \tau$$

$$X \in \mathscr{X}.$$

Hence, under normal distributions, Markowitz portfolio model is one that maximizes the probability of achieving the risk-free return rate, subject to the desired level of expected return.

Generalizing this idea, we consider an investor who maximizes a CSM, $\rho(\cdot)$ as follows:

$$\max \quad \rho(\alpha X + (1-\alpha)r_f - r_f)$$

$$\text{subject to} \quad \mathbb{E}(\alpha X + (1-\alpha)r_f) = \tau \qquad (15)$$

$$X \in \mathscr{X}, \quad \alpha \geq 0.$$

We now formalize the discussion above.

**Theorem 4.** *Suppose $E(X) > r_f$ for all $X \in \mathscr{X}$. Let $\rho(\cdot)$ be a CSM and suppose the objective of problem (15) is strictly positive for a given $\tau > r_f$. Then the corresponding optimal asset position $X^*$ is also optimal to the following problem:*

$$\max \quad \rho(X - r_f)$$

$$\text{subject to} \quad X \in \mathscr{X}. \qquad (16)$$

Theorem 4 implies that, when optimizing over a CSM with the risk-free aspiration level, we may find a single "tangent" portfolio and then mix it with the risk-free asset according to our desired expected return.

We remark that if the risky assets are normally distributed, the problem

$$\max \quad \mathbb{P}\{X \geq r_f\}$$

$$\text{subject to} \quad X \in \mathscr{X}$$

is equivalent to

$$\max \quad \frac{\mathbb{E}(X) - r_f}{\sigma(X)}$$

$$\text{subject to} \quad X \in \mathscr{X},$$

which is the classical problem of choosing an asset that maximizes the Sharpe ratio.

## 5. Application Example: Debt Management

In this section, we demonstrate the use and applicability of satisficing measures. We encourage the interested reader to also consult §EC.2 of the e-companion

for an application of satisficing measures to portfolio optimization against a competing benchmark.

Consider a firm attempting to manage a future liability $L$ (which may be random) by investing a current amount of capital $C$. For ease of notation, assume $C = \$1$. The firm has at their disposal the option to invest in a risk-free asset with guaranteed return $r_f$ as well as $n$ risky assets with return vector $\tilde{\mathbf{r}}$ (not necessarily independent of $L$). We denote by $y$ the fraction of the capital invested in the risk-free asset and by $x_i$ the fraction of capital in the $i$th risky asset.

Here, the aspiration level for investment is very natural: it is the firm's liability amount $L$, which they want to ensure they cover. On the other hand, although the firm may be risk averse, appropriate tolerance levels could well be unclear. In this situation, a satisficing measure on the net surplus $X = 1 + yr_f + \tilde{\mathbf{r}}'\mathbf{x} - L$ is quite sensible.

Consider the use of an underlying exponential utility function. The satisficing measure for a fixed position $X$ has the form $\rho(X) = \sup\{a > 0: C_a(1 + yr_f + \tilde{\mathbf{r}}'\mathbf{x} - L) \geq 0\}$, where $C_a(X) = -(1/a)\ln \mathbb{E}(\exp(-aX))$ is the certainty equivalent of $X$ under the exponential utility function.

To illustrate this with some near-analytical expressions, we consider the simple case when $L$ is a constant and the covering positions are bonds with independent default probabilities $p_i$ and returns $\gamma_i$, in which $\gamma_i(1 - p_i) - p_i \geq r_f$. We have

$$C_a(X) = C_a(1 + yr_f + \tilde{\mathbf{r}}'\mathbf{x} - L) = 1 + yr_f - L + C_a(\tilde{\mathbf{r}}'\mathbf{x})$$

$$= 1 + yr_f - L - \frac{1}{a}\sum_{i=1}^{n}\ln \mathbb{E}(\exp(-a\tilde{r}_i x_i))$$

$$= 1 + yr_f - L - \frac{1}{a}\sum_{i=1}^{n}\ln\big(p_i \exp(ax_i)$$

$$+ (1 - p_i)\exp(-a\gamma_i x_i)\big).$$

In this case, then, our satisficing measure reduces to

$$\rho(X) = \sup\bigg\{a > 0: 1 + yr_f - L$$

$$\geq \frac{1}{a}\sum_{i=1}^{n}\ln\big(p_i \exp(ax_i)$$

$$+ (1 - p_i)\exp(-a\gamma_i x_i)\big)\bigg\}.$$

If we do not allow short positions, our satisficing problem reduces to the convex optimization problem:

$$\inf\bigg\{b: \sum_{i=1}^{n}f_i(b, x_i) \leq 1 + r_f y - L,$$

$$y + \sum_{i=1}^{n}x_i = 1, \ y \geq 0, \ x_i \geq 0, \ b > 0\bigg\},$$

where $f_i(b, x_i) = b \ln(p_i e^{x_i/b} + (1 - p_i) e^{-\gamma_i x_i/b})$, which is jointly convex in $(b, x_i)$.[8] Therefore, we can find the optimal satisficing position very efficiently.

For a simple example, consider independent and identically distributed (i.i.d.) bonds with default probabilities $p$ and returns $\gamma$; by symmetry and the quasiconcavity of the satisficing measure, it is satisficing optimal for the risky bond investments to be split equally. Hence, $x_i = (1 - y)/n$ and the satisficing measure reduces to

$$\rho(X) = \sup \left\{ a > 0 \colon 1 + y r_f - L \right.$$
$$\geq \frac{n}{a} \ln \big( p \exp(a(1 - y)/n) $$
$$\left. + (1 - p) \exp(-a\gamma(1 - y)/n) \big) \right\}.$$

From our robustness interpretation of satisficing measures, this measure tells us something very concrete. In particular, the risk measure

$$\mu_a(X) = -C_a(X) = (1/a) \ln \mathbb{E}(\exp(-aX))$$

is convex and generated by the penalty function

$$\alpha_a(\mathbb{Q}) = (1/a) \int_{\omega \in \Omega} \ln(d\mathbb{Q}/d\mathbb{P}(\omega)) \, d\mathbb{Q}(\omega),$$

i.e., $a^{-1}$ times the relative entropy. For this example with $n$ i.i.d. bonds, consider that the true distribution $\mathbb{Q}$ differs from our assessed distribution according to $\mathbb{Q}\{\text{default}\} = \epsilon p$ for some $\epsilon \geq 1$. In other words, we are off by a factor of $\epsilon$ in assessing the default probability. Then $\alpha_1(\mathbb{Q}) = n\epsilon p \ln(\epsilon) + n(1 - \epsilon p) \ln((1 - \epsilon p)/(1 - p))$, and the expected value $\mathbb{E}_{\mathbb{Q}}$ of our net surplus $X$ under this actual distribution is bounded below:

$$\mu_{\rho(X)}(X) \leq 0$$
$$\Leftrightarrow \sup_{\mathbb{Q}} \{-\mathbb{E}_{\mathbb{Q}}(X) - \alpha_{\rho(X)}(\mathbb{Q})\} \leq 0$$
$$\Leftrightarrow \mathbb{E}_{\mathbb{Q}}(X) \geq -\frac{n[\epsilon p \ln(\epsilon) + (1 - \epsilon p) \ln((1 - \epsilon p)/(1 - p))]}{\rho(X)};$$

i.e., the firm cannot miss its payments by too much in expectation for distributions with small relative entropy compared to the satisficing measure $\rho(X)$. This satisficing measure, then, offers a worst-case sensitivity guarantee with respect to model mis-specification in the relative entropy sense.

We illustrate a numerical example in Figures 1 and 2. The parameters are $\gamma = 10\%$, $p = 0.01$, and $r_f = 5\%$. In Figure 1, we take $L$ as a constant value, which we vary. For each value of $L$, we compute the

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[8] The optimal satisficing value $\rho^*$ is the reciprocal of the optimal value of this problem, i.e., $\rho^* = 1/b^*$.

**Figure 1** Effect of Liability Size on Satisficing for the Debt Management Example
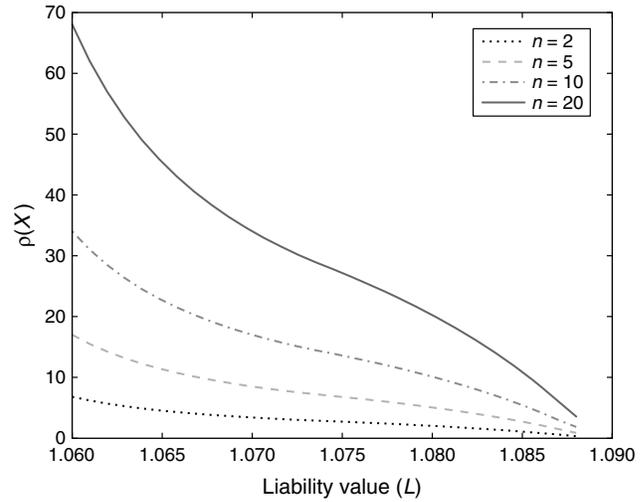
optimal satisficing portfolio and find the optimal satisficing level. We can see the effect of larger $L$ on the optimal satisficing level for various $n$. Clearly, as $L$ gets larger, it becomes more likely that the firm will fall short on its payments. Note that there is a distinct value of $L < \gamma$ at which $\rho$ drops to zero; this occurs when $L$ is large enough such that the firm cannot even achieve $L$ in expectation under the given distribution. This cutoff value for $L$ is given by $L_{\text{thresh}} = 1 + \sum_{i=1}^{n} x_i \mathbb{E}[r_i] = (1 - p)(1 + \gamma)$, which, for this example, we obtain $L_{\text{thresh}} = 1.089$. We also see that satisficing goes up with $n$; this reflects the diversifying effect of increasing the number of available bonds, which thins the left tail of the distribution.

Figure 2 shows how the lower bounds on the expected surplus as the modeling error parameter $\epsilon$

**Figure 2** Impact of Probability Error on Expected Surplus for the Debt Management Example
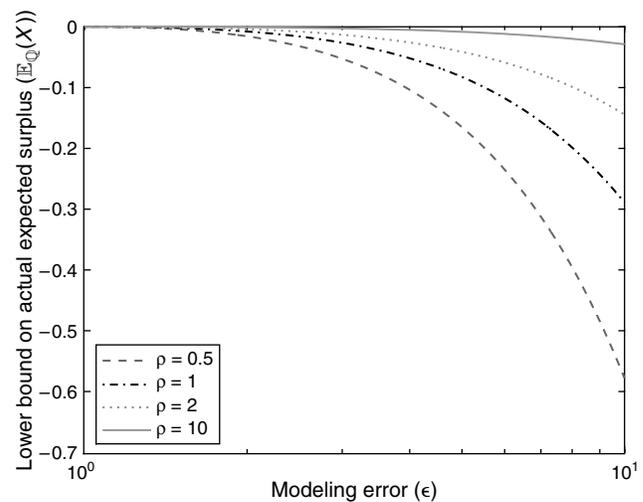
varies ($n = 2$ here). Increasing the satisficing level $\rho$ decreases our sensitivity with respect to such errors.

We can compare this to the much more standard risk management practice of using a utility function and maximizing the certainty equivalent. In such an approach, one would first need to specify a risk tolerance level. There is a significant body of work in the decision analysis literature about what an "appropriate" risk tolerance level for a firm should be (see, e.g., Howard 1988, Smith 2004). Howard finds that a risk tolerance level of about one-sixth of firm equity is commonly assessed in practice. For a company with a debt-to-equity ratio of around one, and using $C = \$1$ as the debt level, this suggests a risk tolerance of around one-sixth of a dollar in this case, or a satisficing level of about $\rho = 6$. Solving the maximum certainty equivalent problem with the above parameters with, say, $n = 20$ bonds and a loss level of $L = 1.07$, such a risk tolerance implies an optimal solution that suggests an investment of zero in the risk-free asset. By contrast, the optimal satisficing portfolio has about 17% in the risk-free asset and a satisficing level of about $\rho^* = 35$. This position performs well relative to the target $L$ and will be about a factor of six less sensitive to probability assessment errors than the position suggested by the maximum certainty equivalent approach with the above parameters and "assessed" risk tolerance.

In addition to finding positions that have favorable robustness properties, satisficing seems like a more natural approach than utility or risk measures when the goal is to meet a target level, as we believe is often the case. Practically speaking, it is unclear why a risk tolerance parameter should exist unilaterally across all decisions for a given decision maker, let alone what the tolerance level should be. Satisficing avoids this specification problem entirely and, using a QSM, computational tractability is preserved (as opposed to using a metric like probability of meeting the target, which is highly nonconvex).

Of course, we made this example highly simple to get some convenient, analytical expressions. In reality, the bonds would not be i.i.d. and would not be necessarily independent of $L$. Although the computation of $\rho(X)$ would be more complicated, the general insights would still hold.

## 6. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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