Abstract

This paper addresses the problem of durable goods manufacturers in an oligopoly seeking optimal values for three decision variables: product warranty, reliability and price. Each firm seeks a warranty-reliability-price combination that maximizes expected profit subject to quite general constraints on the firm’s decision variables. Warranty serves as a signal of product reliability, which is not observable by consumers. We present a game-theoretic model of warranty-reliability-price competition in such a market and examine Nash equilibria for this game. We show that under fairly general assumptions each firm can optimally set its warranty and reliability independently of price and competitors’ actions. In addition, we show that optimal warranties and reliabilities are complementary, and we explore the impact of different market factors on the optimal warranty and reliability. Finally, we show that optimal warranties are longer and products more reliable when consumers are risk averse. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Warranty policies have been explicitly studied for several decades, with the study of guarantees against random occurrences (in the form of insurance) going back even further. With the recent emphasis on product quality and manufacturer cost cutting, however, the study of warranties has taken on new importance. In particular, the past decade has seen an increase in the use of warranties and product quality as competitive tools, particularly for durable goods such as automobiles and appliances. For example, in 1987 the average duration of the basic warranty on a representative sample of automobiles sold in the US was just under 18 months, while by 1995 that average duration had grown to over 38 months (Gillis, 1987, 1995). During the same period of time, the movement towards increased quality in manufacturing has led to greater reliability in durable goods, and the growth of global manufacturing has brought increased price pressures in most manufacturing industries. This combination of forces has caused firms to seek insights into the selection of optimal levels of warranty and product...
This paper studies the optimal setting of warranty, reliability and price on durable goods while explicitly considering competition among firms in the market. We model the problem as an \( n \)-firm noncooperative game in which each firm selects a warranty, reliability and price from a feasible strategy set that may depend on the actions of other firms in the market. The goal of each firm is to maximize the expected total profit to the firm given the warranties, reliabilities and prices at competing firms. Consumers cannot directly observe product reliability, but they use a firm’s warranty as a signal of reliability. For fairly general constraints on firm warranties, reliabilities and prices we show that the setting of warranty and reliability is separable from the setting of price – i.e., each firm can set its warranty and reliability optimally before deciding what price to charge for its product and without considering how competing firms will set their warranties, reliabilities or prices.

Having established this separability, we examine the optimal warranty and reliability of a firm in detail. In particular, we explore the impact of various market factors – e.g., consumer repair costs, firm repair costs, consumer reliability perceptions, the signaling power of warranty, and unit manufacturing costs – on the setting of warranty and reliability. Finally, we show that if consumers are risk averse, then a firm will offer a more reliable product with a longer warranty than if consumers are risk neutral.

Once optimal warranties and reliabilities have been set, the firms compete through price alone. Detailed treatment of this price problem is beyond the scope of this paper. However, treatments of similar price games appear in Topkis (1979), Milgrom and Roberts (1990), and Vives (1990). With minor modifications, such treatments can be applied to the price game here.

A large portion of the existing warranty literature concentrates on the estimation of repair costs given particular warranty policies and product-reliability assumptions, while stopping short of optimization (e.g., Blischke and Scheuer, 1975, 1981; Mamer, 1982; Nguyen and Murthy, 1984a, b). A number of papers consider warranty optimization (e.g., Glickman and Berger, 1976; Anderson, 1977; Padmanabhan and Rao, 1993; Padmanabhan, 1995) or joint optimization of warranty and reliability (e.g., Nguyen and Murthy, 1988) without addressing competition from other firms. Douglas et al. (1993) investigates the relationship among warranty, quality and price in a purely competitive market using both empirical analysis and a two-person game model involving a single firm and a consumer. Although that model does not explicitly address competition among firms, some of the insights obtained from it bear a resemblance to some of the results in our work.

Models that address the setting of some combination of warranty, reliability and price in a setting that addresses multi-firm competition include Courville and Hausman (1979) and Menezes and Currim (1992). Courville and Hausman (1979) addresses equilibrium warranties and reliabilities from a public policy point of view in both a monopolistic and a purely competitive market. Menezes and Currim (1992) presents a competitive model for setting price and warranty for particular classes of failure-rate, demand and competitor-response functions, and illustrates the use of historical data to estimate required elasticities for those functions.

The work presented here makes a number of contributions. It provides the most detailed and realistic model to date of warranty, reliability and price setting in an oligopoly. In particular, this model explicitly addresses time and product aging, differences in the cost structure between a firm and consumers, signaling of reliability through product warranty, and the impact of reliability choice on manufacturing costs. In addition, this model is used to obtain a number of interesting managerial insights, which are then compared to results from other models and empirical studies.

The remainder of the paper is organized as follows. The game-theoretic model of warranty, reliability and price setting is presented in Section 2. Section 3 contains the separation result allowing firms to set optimal warranties and reliability independently of other firms and market prices, as well as the result establishing the complementarity of warranty and reliability. Section 4
explores the impact of various market factors on the optimal values of warranty and reliability, while Section 5 presents some numerical examples. Finally, Section 6 provides some concluding remarks.

2. Model description

Consider $n$ firms, labeled $1, \ldots, n$, with each firm selling a single product in this market and the products being competing versions of some durable good. Let $w_i$ represent the warranty that firm $i$ offers on its product and let $\pi_i$ denote the product’s selling price. Firm $i$’s warranty $w_i$ represents the length of time after the purchase of firm $i$’s product during which all repairs needed by the product are paid for in full by the firm. The reliability of firm $i$’s product is represented by that product’s failure-rate function $r_i(t)$, often referred to here simply as $r_i$, which indicates the rate at which failures of the product occur when the product’s age is $t$. When no confusion results we use the terms reliability and failure-rate function interchangeably. Of course, an increase in a product’s failure-rate function corresponds to a decrease in the product’s reliability. Let $w = (w_1, \ldots, w_n)$ be the vector of market warranties, $r = (r_1, \ldots, r_n)$ be the vector of market reliabilities and $\pi = (\pi_1, \ldots, \pi_n)$ be the vector of market prices. Also, let $w' \equiv (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$, $r' \equiv (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$ and $\pi' \equiv (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n)$ be the vectors of competitor warranties, reliabilities and prices, respectively.

When the firms in the market offer warranties $w$, reliabilities $r$ and prices $\pi$, each firm $i$ experiences an expected demand $d_i(\pi, r, w)$. Assume that each firm produces to meet demand exactly and that firms are risk neutral. Thus for each unit sold by firm $i$, the firm earns a unit revenue $\pi_i$, and incurs a unit manufacturing cost $m_i(r_i)$ depending on the product’s reliability $r_i$ and an expected unit warranty repair cost $c_i(w_i, r_i)$. (For simplicity we assume that each firm sells its product directly to consumers). Each consumer purchasing firm $i$’s product pays the purchase price $\pi_i$ and incurs the uncertain perceived consumer repair cost $K_i(w_i) - \hat{r}_i(t)$, the cost to a consumer, as perceived by the consumer, of paying for repairs of firm $i$’s product after the warranty expires, plus the inconvenience cost of obtaining repairs whether covered by warranty or not. Although the repair costs that a consumer will incur depend on the true reliability of firm $i$’s product, the consumer’s perception of these costs at the time of purchase do not depend on the true reliability of the product, since this reliability is not directly observable by consumers. As a result, $K_i(w_i)$ does not depend on $r_i$. Instead, consumers take firm $i$’s warranty as a signal of the product’s reliability, so that $K_i(w_i)$ depends on $w_i$ in two ways – through the warranty’s effect on the length of time a consumer is exposed to post-warranty failures, and through consumer perceptions of reliability inferred from the warranty. Assume that consumers are homogeneous with regard to their perceptions of product reliability and define the perceived failure rate $\hat{r}_i(t, w_i)$ to be the failure rate that consumers anticipate for firm $i$’s product at age $t$ given that firm $i$ is offering a warranty of length $w_i$.

The precise manner in which warranty, reliability and price influence the expected demand experienced by firms in the market will clearly have a great impact on the optimal setting of those variables. We assume that firm $i$’s demand is

$$d_i(\pi_1 + k_1(w_1), \ldots, \pi_n + k_n(w_n)),$$

where $k_i(w_i)$ is the certain equivalent of the uncertain perceived consumer repair cost $K_i(w_i) - \hat{r}_i(t)$, i.e., $k_i(w_i)$ is the quantity such that a consumer is indifferent between facing a certain cost of $k_i(w_i)$ or facing the uncertain cost $K_i(w_i)$. (Note that the certain equivalent of consumer repair cost is a generalization of the expected consumer repair cost that handles both risk neutral and risk averse consumers. For a discussion of this and related concepts in preference theory, the reader is referred to Keeney and Raiffa (1976)). Assume that consumers are homogeneous with regard to risk attitude, so that each $k_i(w_i)$ is well defined. This assumption, along with the earlier assumption of homogeneous reliability perceptions on the part of consumers, is reasonable when the product is aimed at a particular market segment of consumers, particularly if consumers have access to consumer publications reporting historical reliability.
records. For brevity, put \( k(w) = (k_i(w_i)) \), so that firm \( i \)'s demand is

\[
d_i(\pi + k(w)),
\]

and assume that \( d_i \) is nonnegative. Of course other attributes such as style, size, performance, etc. will also influence demand for firm \( i \)'s product, but these factors are suppressed in our notation to allow us to focus on the variables of interest.

The assumption that firm \( i \)'s warranty, reliability and price influence that firm’s expected total consumer cost – the sum of the purchase price and the certain equivalent of consumer repair costs throughout the life of the product – can be motivated along two lines. First, this measure accounts for essentially all of the relevant effects that these three variables have on a consumer. The primary impact of price on a consumer’s decision is its direct impact on the purchase cost. (Studies by Wiener (1985) suggest that, while warranty provides valuable information to consumers regarding product reliability, price generally does not. Therefore we assume that price does not serve as a signal of reliability). Product reliability cannot be directly observed, so it has no impact on the purchase decision. Product warranty influences consumer reliability perceptions (through signaling) and the length of the “out-of-warranty” period, both of which are captured by \( K_i(w_i) \). By using the certain equivalent instead of expected cost, we also incorporate consumer risk attitude.

A second reason for assuming that the variables in question influence demand primarily through the certain equivalent of total consumer cost is that similar statistics are used by consumer publications to summarize the impact of these variables. For example, The Complete Car Cost Guide (Levy, 1995) computes the total five-year ownership cost for each vehicle by summing costs due to depreciation, financing, insurance, fees, fuel, maintenance and repairs. By focusing on the full useful life of the vehicle as opposed to just the first five years, and by suppressing costs not directly dependent on price, warranty or reliability (e.g., insurance, fees, and fuel), this total cost measure coincides with the one modeled here.

Assume that the decision of a consumer whether or not to purchase firm \( i \)'s product is stochastically independent of that firm’s subsequent (random) unit warranty repair costs. At the time of purchase, the firm and the consumers have (possibly different) initial estimates of the distributions of the unit warranty repair costs, but neither knows what actual repair costs will occur. This independence assumption implies that firm \( i \)'s expected profit \( \Pi_i(\pi, w, r, \pi) \) is

\[
\Pi_i(w, r, \pi) = d_i(\pi + k(w))[\pi_i - m_i(r_i) - c_i(w_i, r_i)].
\]

Market factors and firm policy decisions may constrain the warranties, reliabilities and prices that can feasibly be offered by the firms in the market. To capture this, we allow a very general set of constraints on these variables, some of which might be vacuous for a given firm in a given market. Assume that each firm \( i \)'s warranty and reliability is restricted to lie in a general feasible warranty set \( W_i \) and feasible reliability set \( R_i \), respectively, each of which is independent of all other decision variables for firm \( i \) and its competitors. Assume that \( R_i \) is a well-ordered set of nonnegative real-valued functions ordered by the usual pointwise ordering of functions. (In practice, this set might be quite limited if design and engineering constraints limit the ability of a firm to alter its product’s reliability). Some examples of such constraint sets might be:

**Warranty constraints.**

Interval \( W_i = [\underline{w}_i, \overline{w}_i] \).

Integer \( W_i = \{0, \ldots, \overline{w}_i\} \).

**Reliability constraints.**

Discrete constant failure rates

\( R_i = \{R_i(t) : r_i(t) = \rho_i \forall t, \rho_i \in \{r_{i1}, \ldots, r_{ik}\}\} \).

Scaled versions of prototype failure-rate function

\( R_i = \{r_i(t) : r_i(t) = \rho_i \cdot z_i(t), \rho_i \in [\underline{\theta}_i, \overline{\theta}_i]\} \).

Many other natural restrictions that firm \( i \) may wish to impose can be captured through constraints of the form

\( M_{ij}(\pi + k(w)) \geq \mu_{ij} \),
where \( M_i(\pi + k(w)) \) is a firm-performance measure – we call such a constraint a firm-performance constraint. Some examples of such constraints might be:

**Firm-performance constraints.**

- Bounded consumer cost: \( C_i \leq \pi_i + k_i(w_i) \leq \bar{C}_i \)
- Minimum market share: \( d_i(\pi + k(w))/\sum_{j=1}^n d_j (\pi + k(w)) \geq \psi_i \)
- Consumer cost leader: \( \pi_i + k_i(w_i) \leq \pi_j + k_j(w_j), j \in J_i \)

Note in particular that the last two constraints above address a firm’s position relative to others in the market. The market-share constraint was of particular interest to a firm being consulted during the construction of this model since it helps avoid a potential drawback of a single-period model. Such a constraint keeps a firm from focusing excessively on short-term profits (perhaps by extracting excessive margins) to the exclusion of building a strong market share. Since many durable goods markets are heavily influenced by customer loyalty, this constraint may cover an important long-term consideration for a firm.

Firm i’s optimal-response problem is to choose its warranty \( w_i \), reliability \( r_i \), and price \( \pi_i \) to maximize its expected profit

\[
\Pi_i(w, r, \pi) = d_i(\pi + k(w))[\pi_i - m_i(r_i) - c_i(w_i, r_i)]
\]

subject to

\[
w_i \in W_i, \quad (2)
\]

\[
r_i \in R_i, \quad (3)
\]

\[
M_{ij}(\pi + k(w)) \geq \mu_{ij} \quad j = 1, \ldots, J_i, \quad (4)
\]

given warranties \( w' \), reliabilities \( r' \), and prices \( \pi' \) of the other firms. The warranty-reliability-price game is the \( n \)-firm noncooperative game in which each firm in the market simultaneously attempts to solve its optimal-response problem.

Our interest here is in studying the properties of a Nash equilibrium for the warranty-reliability-price. A Nash equilibrium is a vector \((w',r',\pi')\) of market prices and warranties such that for each firm \(i\), the triple \((w'_i,r'_i,\pi'_i)\) is an optimal response by firm \(i\) to the firm’s competitors’ warranties \(w'^i\), reliabilities \(r'^i\), and prices \(\pi'^i\).

### 3. Optimal price, warranty and reliability

The first result of this section, Theorem 1, shows that an optimal response for firm \(i\) can be obtained by first setting an optimal warranty-reliability pair independently of all prices and competitors’ warranties and reliabilities. To achieve this optimal response, firm \(i\) chooses its warranty and reliability to minimize its supply cost \(S_i(w_i,r_i)\), defined to be the sum of its expected unit warranty repair cost, unit manufacturing cost and the certain equivalent of consumer repair cost for that firm’s product. This entails solving firm \(i\)’s minimum-supply-cost problem – i.e., choosing the pair \((w^*_i,r^*_i)\) that minimizes

\[
S_i(w_i, r_i) = c_i(w_i, r_i) + k_i(w_i) + m_i(r_i)
\]

subject to Eqs. (2) and (3). Observe that \((w^*_i,r^*_i)\) is independent of the prices \(\pi\) of all firms and the warranties \(w'\) and reliabilities \(r'\) of competing firms.

**Theorem 1.** If firm \(i\) has an optimal response to warranties, reliabilities and prices set by the other firms, then one such optimal response entails setting firm \(i\)’s warranty-reliability combination equal to \((w^*_i,r^*_i)\) independently of those warranties, reliabilities and prices.

**Proof.** By hypothesis there exists \((w_i, r_i, \pi_i) = (\bar{w}_i, \bar{r}_i, \bar{\pi}_i)\) that maximizes Eq. (1) subject to Eqs. (2)–(4) given warranties \(w_j = w_j\), reliabilities \(r_j = \bar{r}_j\) and prices \(\pi_j = \bar{\pi}_j\) of firms other than \(i\). Put \(\bar{a} = \bar{\pi} + k(\bar{w})\). Thus there is no loss in generality assuming the maximum of Eq. (1) subject to Eqs. (2)–(4) satisfies the additional condition

\[
\pi_i + k_i(w_i) = a_i. \quad (6)
\]

But then Eq. (4) becomes \(M_{ij}(a) \geq \mu_{ij}\) for \(j = 1, \ldots, J_i\) and so is trivial. Hence the problem of maximizing Eq. (1) subject to Eqs. (2)–(4) and (6)
reduces to the problem of maximizing
\[ d_i(a) [\pi_i - m_i(r_i) - c_i(w_i, r_i) - c_i(w_i, r_i)], \]
or, since \( d_i(a) \) is a nonnegative constant, maximizing
\[ \pi_i - m_i(r_i) - c_i(w_i, r_i) \tag{1'} \]
subject to Eqs. (2), (3) and (6). Thus on using Eq. (6) to eliminate \( \pi_i \) from (1'), we can take \( (w_i, r_i) = (\bar{w}_i, \bar{r}_i) \) to be any minimizer of Eq. (5) subject to Eqs. (2) and (3) – i.e.,
\[ (\bar{w}_i, \bar{r}_i) = (w^*_i, r^*_i). \]

Once each firm \( i \) selects its optimal warranty-reliability combination \( (w^*_i, r^*_i) \), the warranty-reliability-price game has been reduced to a price game in which the firms compete with each other in terms of price only. Detailed treatment of this price game is beyond the scope of this paper.

Since Theorem 1 is somewhat surprising, it may be useful to provide some insight into why the result holds. First, consider the total cost – shared between a consumer and firm \( i \) – associated with one unit of firm \( i \)'s product. That cost can be expressed as
\[ [\pi_i + k_i(w_i)] + [c_i(w_i, r_i) + m_i(r_i) - \pi_i], \tag{7} \]
where the first bracketed term is the total consumer cost and the second bracketed term is the net firm cost (i.e., the negative of unit profit). Clearly the optimal combination of warranty, reliability and price must minimize Eq. (7) – otherwise another solution could be obtained yielding either higher unit profit for the firm (without increasing the cost to the consumer) or lower cost to the consumer (without sacrificing unit profit for the firm). By cancelling the \( \pi_i \)'s above, it is clear that Eq. (7) is simply \( S_i(w_i, r_i) \), which is independent of firm \( i \)'s price, as well as the warranties, reliabilities and prices offered by the other firms. Since interactions among the firms (through the demand function and constraint Eq. (4)) occur only through the total costs \( \pi_i + k_i(w_i) \) experienced by consumers, firm \( i \) can set its warranty and reliability by minimizing \( S_i(w_i, r_i) \) and then compete directly with other firms entirely through selection of its price \( \pi_i \).

Having established the separability above, we now develop a fundamental relationship between the optimal warranty and reliability obtained from the minimum-supply cost problem. To facilitate this development, let us describe the relevant cost functions in greater detail. Let \( c^*_i(t) \) and \( c_i^*(t) \) be the perceived cost to repair firm \( i \)'s product at age \( t \) (including possible time-value discounting) as perceived by the firm and by the consumer, respectively – i.e., the actual cost per repair and the perceived cost per repair, respectively. (Note that the term “actual cost” is merely for definitional purposes – it is simply the firm’s best estimate of the future cost per repair). Let \( h_i(t) \) be the inconvenience cost to the consumer of a repair on firm \( i \)'s product at age \( t \), regardless of whether the product is under warranty – i.e., the inconvenience cost per repair. Assume throughout that the time to perform a repair is negligible and that repairs are minimal – i.e., a repair restores a product to working condition at the same age that it was at the time of failure. The logic of the latter assumption is that the failure rate of a complex system with relatively reliable components depends much more heavily on the average age of its components than the age of any particular component. These assumptions imply that firm \( i \)'s expected unit warranty repair cost is
\[ c_i(w_i, r_i) = \int_0^{w_i} r_i(t)c_i^*(t) \, dt. \tag{8} \]

The manufacturing cost function \( m_i(t) \) assigns a cost \( m_i(r_i) \) to each failure-rate function \( r_i \). Assume that \( m_i(r_i) \) is nonnegative and nonincreasing in \( r_i \), so that higher reliabilities (lower failure-rate functions) can only be achieved through higher manufacturing costs.

To capture the signaling effect of the warranty on consumers’ perceptions of reliability, assume that the perceived failure-rate function is of the form
\[ r_i^*(t, w_i) = \gamma_i(w_i) \rho_i^*(t), \]
where \( \rho_i^*(t) \) is a prototype failure-rate function representing the basic shape of consumer percep-
tions and $\gamma_i(w_i)$ is the warranty signaling factor. Note that this assumption says that different warranty durations cause consumer perceptions of reliability to be scaled up or down uniformly across the life of the product. Although in reality signaling behavior may be more involved than this, estimating a more complex relationship would probably be intractable. In addition, the basic design and function of many products result in “characteristic” failure patterns, which consumers would have a sense of based on prior experience. As a result, it seems plausible that consumers would adjust their reliability perceptions by simply “scaling” their prior perceptions. Assume that both $p_i^*(t)$ and $\gamma_i(w_i)$ are nonnegative, and $\gamma_i(w_i)$ is nonincreasing in $w_i$ so that a longer warranty signals a more reliable product. This assumption can be justified by a number of empirical studies. For example, in a study of appliances and motor vehicles, Wiener (1985) concludes, “in the markets studied in this article, warranties were accurate signals of product reliability”. Douglas et al. (1993) observed a negative correlation between warranty and reliability in the US automobile market when all manufacturers were included in the analysis. However, when the analysis was controlled for other factors (most notably the cost of running a warranty service network), the authors found that “a more intensive warranty is associated with a higher-quality product”.

The second result of this section states that warranty and reliability are complementary. Let $w^*_i(r_i) \equiv \arg\min_{w_i \in W_i} S_i(w_i, r_i)$ be the optimal warranty for firm $i$ when that firm’s reliability is fixed at $r_i$ and let $r^*_i(w_i) \equiv \arg\min_{r_i \in R_i} S_i(w_i, r_i)$ be the optimal reliability given $w_i$. (In the remainder of the paper we assume that these optimal variable values are unique. If this is not the case, all results can be reformulated in terms of the least or greatest optimal warranty or reliability.) The proofs of the remaining results in Sections 3 and 4 rely on some results in lattice programming. A discussion of the relevant material can be found in Appendix A.

**Proposition 2.** Firm $i$’s optimal warranty increases as the reliability of that firm’s product increases, and firm $i$’s optimal reliability increases as that firm’s warranty increases.

**Proof.** Make the change of variable $q_i \equiv -r_i$, so that the minimum-supply-cost problem is to minimize

$$\int_{a}^{b} -q_i(t)c_i^*(t) \, dt + m_i(-q_i) + k_i(w_i)$$

subject to

$$w_i \in W_i,$$

$$-q_i \in R_i.$$

By the hypotheses of the theorem, the objective function is subadditive in $(w_i,q_i)$. Also, the feasible region defines a sublattice on $W_i \times -R_i$. The result now follows immediately from Theorem A.1 in Appendix A. \[\square\]

The intuition behind this result is straightforward. A more reliable product gives the firm an incentive to increase its warranty since higher reliability reduces the marginal repair cost of increasing the warranty. A symmetric explanation provides the intuition behind the reverse implication. It is interesting to note that Proposition 2 reinforces the assumption that, all else being equal, a longer warranty is a signal of higher reliability.

Before exploring the impacts of various market factors on the optimal choice of warranty and reliability, it may be helpful to say a few words regarding estimation of the functions required to perform this optimization. Recall that the problem facing firm $i$ is to minimize the firm’s supply cost $S_i(w_i, r_i) = c_i(w_i, r_i) + k_i(w_i) + m_i(r_i)$. In order to do this in practice, it would be necessary to estimate the functions appearing in all three terms of $S_i(w_i, r_i)$. To estimate $c_i(w_i, r_i)$, it is clear from Eq. (8) that it is sufficient to estimate $r_i(t)$ and $c_i^*(t)$. Hulting and Robinson (1994) propose a Bayesian method of estimating the reliability of a series system of repairable subsystems, and illustrate the use of the method by estimating the reliability of a vehicle system. Their approach can be directly applied in our context to estimate $r_i(t)$. In addition to deriving an estimate for the reliability of the overall system, the approach described in
Hulting and Robinson (1994) also constructs estimates of the reliability for each subsystem. By using this information to track which types of repairs are more likely at different ages, and combining this with historical estimates of the cost of different types of repairs, the function $c_i^*(t)$ could be estimated.

Based on engineering and manufacturing specifications, combined with historical experience, it should not be difficult for a firm to estimate $m_i(r)$, the unit manufacturing cost of a product with a given level of reliability built in. Therefore the only term remaining to estimate is $k_i(w)$, the certain equivalent of the consumer repair cost. Since this function is inherently subjective (it relies on consumer perceptions of reliability given the warranty $w$, as well as consumer risk attitude), it seems that the only plausible approach to estimating it is to solicit information directly from consumers. One way of doing this would be to ask a series of "willingness to pay" questions – e.g., given a particular product offered with a particular warranty, how much would a consumer be willing to pay in order to guarantee that no failures occurred during a particular portion of the product's life. After gathering this information for disjoint intervals covering the entire useful life of the product (including the portion covered by the warranty, in order to estimate the consumer inconvenience cost of obtaining a repair), the firm could sum the costs to obtain an overall estimate of $k_i(w)$. This process could be repeated for any warranty values that the firm wishes to consider – e.g., for all $w \in W_i$. (Of course, if consumers were able to articulate how much they would be willing to pay to guarantee zero failures throughout the entire life of the product given a warranty $w$, the firm could obtain $k_i(w)$ directly by eliciting this quantity).

4. Market factor effects on optimal warranty and reliability

In this section we use our model and preceding results to investigate the impact of various cost parameters and other market factors on the optimal choice of warranty and reliability. Since the separation result of the previous section allows us to focus attention on a single firm, the subscript $i$ is dropped from all quantities, functions and sets in this section and the next. Due to the difficulty of explicitly expressing $k(w)$ for risk-averse consumers, the majority of the treatment assumes risk-neutral consumers. However, we believe that many of the results would hold under risk aversion as well, since the presence of risk aversion does not change the basic qualitative structure of the model, but merely makes a warranty more valuable to consumers due to its ability to reduce consumer risk. The final result of this section compares the optimal warranty-reliability pair given risk-averse consumers to that for risk-neutral consumers.

4.1. risk-neutral consumers

When consumers are risk neutral, the certain equivalent of consumer repair cost becomes the expected consumer repair cost, and can be expressed in greater detail. To that end recall that $c^i(t)$ is the consumer cost per repair, $b^i(t)$ is the inconvenience cost per repair, and the consumer-perceived reliability is $r^i(t, w) = \gamma(w)\rho^c(t)$. Then letting $L$ be the length of the product's useful life, we have

$$k(w) = \gamma(w) \int_w^L \rho^c(t)c^c(t) \, dt + \gamma(w) \int_0^L \rho^c(t)b(t) \, dt.$$

The next result establishes the relationship between several of the cost functions and the optimal warranty-reliability pair.

**Proposition 3.** A pointwise increase in the consumer cost-per-repair function, inconvenience-cost function, or prototype failure-rate function causes the optimal warranty-reliability pair to increase in both arguments.

**Proof.** Repeating the change of variable as in the proof of Proposition 2, the objective function is subadditive in $(w,q,c^i)$, in $(w,q,b)$ and in $(w,q,\rho^i)$, where $c^i$, $b$ and $\rho^i$ represent the cost-per-repair, inconvenience-cost and prototype perceived-reliability functions, respectively. Also, the feasible region defines a sublattice on $W \times \mathbb{R}$. The result...
now follows immediately from Theorem A.1 in Appendix A. □

If consumer estimates of the functions in Proposition 3 increase, then firm $i$ has two different incentives to offer a longer warranty. First, a longer warranty signals higher reliability, and this serves to mitigate the increased cost perceptions. Second, a longer warranty actually relieves consumers of some repair costs since a consumer is now responsible for repairs over a shorter period of time. Although consumer cost and reliability perceptions have no direct interaction with firm $i$’s actual reliability, the complementarity of warranty and reliability may cause the firm to improve its reliability in conjunction with offering a longer warranty. We should note that in Proposition 3 and the rest of the results in Section 4, the indicated changes in warranty and reliability are weak changes – some parameter changes may result in no change in warranty or reliability.

This result has interesting implications if we consider scenarios that can lead to different consumer cost perceptions. If consumers discount the future to a greater extent than the firm or if there exist independent repair facilities that are cheaper than the firm, consumer cost-per-repair estimates will be lower than the firm’s. Proposition 3 suggests that these situations lead to less reliable products and shorter warranties than if consumer cost estimates matched those of the firm.

The only cost function not addressed in Proposition 3 is the firm’s cost-per-repair function. In contrast to the costs considered in that result, it does not seem easy to predict a priori how the optimal warranty-reliability pair would respond to a higher cost per repair. It seems intuitive that the firm would seek to counter this increased cost by reducing its exposure to warranty repair costs. This could be done by decreasing the length of the warranty or by improving the product's reliability. In fact, if either decision variable is held fixed, it is not difficult to show the optimality of these one-dimensional responses. However, if both variables are free to adjust optimally, Proposition 2 implies that such “initial” responses would tend to counter each other – a shorter warranty would tend to cause lower reliability, while improved reliability would tend to cause a longer warranty. The net effect remains ambiguous. Under some reasonably mild additional assumptions, however, it is possible to obtain more definitive results.

For the following result only, make the following assumptions. The basic shape of the firm’s failure-rate function is essentially determined by the basic design of the product – i.e.,

$$r(t) = \theta \alpha(t),$$

where $\alpha(t)$ is the firm-prototype failure-rate function and $\theta$ is a scaling factor determined by the firm. (Note that constraint (3) can now be expressed as $\theta \in \Theta$ for a set $\Theta$.) Also, the firm’s unit manufacturing cost is of the form

$$m(\theta) = g_1 + g_2 \theta^{g_3},$$

where $g_1 > 0$, $g_2 > 0$ and $g_3 < 0$. Note that this form matches the previous assumption that $m(\cdot)$ is decreasing in the failure rate, and satisfies other intuitive properties: $m(\theta)$ is convex in $\theta$, $m(\theta) \rightarrow \infty$ as $\theta \rightarrow 0$, and $m(\theta) \rightarrow g_1$ (finite) as $\theta \rightarrow \infty$.

Now consider multiplicative changes in the firm cost-per-repair function – i.e., consider $\beta \ c'(t)$ for varying values of $\beta$. Under the above assumptions, Proposition 4 describes the response of the optimal warranty-reliability pair as $\beta$ increases.

**Proposition 4.** If a firm’s cost-per-repair function is expressed as $\beta \ c'(t)$, then $w^*$ decreases and $\beta \ 0^*$ increases as $\beta$ increases.

**Proof.** Make the change of variable $\hat{\theta} \equiv \beta \ 0$, so that the minimum-supply-cost problem is to minimize

$$S(w, \hat{\theta}, \beta) = \hat{\theta} \int_0^w c'(t) \alpha(t) \ dt + g_1$$

$$+ \ g_2 \left( \frac{\hat{\theta}}{\beta} \right)^{g_3} + k(w)$$

subject to

$$w \in W, \quad \hat{\theta} \in \beta \Theta,$$

where $\beta \Theta \equiv \left\{ \hat{\theta} = \beta \ 0: \ 0 \in \Theta \right\}$. It can be shown that the objective function is subadditive in
Proposition 4 says that, all else being equal, a firm with (multiplicatively) higher cost per repair may offer a more or less reliable product, but that any improvement in the optimal reliability will at most offset the increase in costs—i.e., letting \( v^*(\beta) \) be the optimal scaling factor given \( \beta > 1 \), we have

\[
\beta c^f(t) v^*(\beta) \leq c^f(t) v^*(1) \forall t.
\]

In addition, the optimal warranty for that firm will be shorter. In other words, the firm will use warranty and reliability together to mitigate the higher cost to the firm of warranty repairs. It is interesting to note that this result agrees with empirical observations in Douglas et al. (1993). That paper noted that Japanese automobile manufacturers tended to offer more reliable products and shorter warranties than their American counterparts in the US auto market. The authors conjectured (and verified for their model) that this difference was due to the higher cost incurred by Japanese firms to maintain a service network and perform warranty repairs.

The preceding results can also be used to investigate the effect of a common practice in the automobile industry—that of providing free loan vehicles during warranty repairs (or sometimes during any repair). If a firm decides to offer such a service, it would result in a combination of lower inconvenience cost for the consumer and higher cost per repair for the firm. By combining Propositions 3 and 4, we can see that the net effect would be a shorter optimal warranty and a reliability that may be higher or lower, but one that would at most offset the additional cost to the firm of the additional repair cost.

We now turn our attention to the remaining two quantities in firm \( i \)'s supply-cost function: \( m(r) \) and \( \gamma(w) \). We first consider the unit manufacturing cost.

A change in a firm’s unit manufacturing cost may affect not only the overall manufacturing cost for a given level of reliability, but also the marginal cost of increasing/decreasing reliability (i.e., \( m'(\rho) \)), as well as higher-order derivatives of \( m(\rho) \). We address both of these issues by considering both an additive increase, where the unit manufacturing cost is \( \hat{m}(r) \equiv m(r) + a \) (with \( a > 0 \)), and a multiplicative increase, where the cost is \( \hat{m}(r) \equiv a m(r) \) (with \( a > 1 \)).

**Proposition 5.** If a firm’s unit manufacturing cost undergoes an additive increase, the optimal warranty-reliability pair does not change. If the manufacturing cost undergoes a multiplicative increase, the optimal warranty and reliability decrease.

**Proof.** The additive increase merely adds the constant \( a \) to the supply-cost function, which does not affect the optimization. For the multiplicative increase, it can be shown that the objective function is subadditive in \((-w, r, a)\). Also, the feasible region defines a sublattice on \(-W \times R\). Theorem A.1 in Appendix A yields the result.

The second part of Proposition 5 can be seen as similar to a result in Douglas et al. (1993) that addresses the impact of an increase in a firm’s cost of providing quality (reliability). The result there states that such an increase results in a lower quality product, but that the impact on warranty is ambiguous. Due to the complementarity of warranty and reliability established in Proposition 2, we obtain the stronger result.

Finally, consider changes in the warranty-signaling factor \( \gamma(w) \). For the same reasons as those mentioned above for \( m(r) \), we consider both additive and multiplicative increases in \( \gamma(w) \) —i.e., situations where the signaling factor is \( \hat{\gamma}(w) \equiv \gamma(w) + a \) (with \( a > 0 \)) and \( \hat{\gamma}(w) \equiv a \gamma(w) \) (with \( a > 1 \)), respectively.

**Proposition 6.** If a firm’s warranty-signaling factor undergoes an additive or multiplicative increase, the optimal warranty and reliability increase.

**Proof.** For both increases, it can be shown that the objective function is subadditive in \((w, -r, a)\). Also, the feasible region defines a sublattice on \( W \times -R\). Theorem A.1 in Appendix A yields the result.
Proposition 6 states that if any given warranty results in a more pessimistic signal to consumers (i.e., \( \gamma(w) \) is larger, thus implying a lower perceived reliability), the firm should try to overcome this by offering a longer warranty and a more reliable product. This result is similar in flavor to the portion of Proposition 3 that addressed a pointwise increase in the consumer prototype failure-rate function \( \rho^c(t) \). An empirical example that may reflect both of these results can be found in the warranty and reliability efforts of the Chrysler Corporation during the early 1980s. Generally perceived as less reliable in the US automobile market, Chrysler initiated significant increases in its warranty coverage and began significant improvements in the quality of its products.

4.2. Risk-averse consumers

When consumers are risk averse, it seems that a firm would have incentive to reduce consumer risk compared to what would be optimal given risk-neutral consumers. Since product reliability is not observable by consumers, the only way of achieving this is to increase the product’s warranty. Proposition 7 below states that, relative to the case of risk-neutral consumers, a longer warranty is optimal when a firm faces risk-averse consumers. In addition, due to the complementarity of warranty and reliability, the firm will also offer a product with higher reliability.

The result requires the additional assumption that consumers exhibit constant additive risk posture – i.e., for any random payoff \( X \) and constant value \( a \), 
\[
\mathbb{E}(X + a) = \mathbb{E}(X) + a,
\]
where \( \mathbb{E}(\cdot) \) denotes the certain equivalent.

Proposition 7. If consumers are risk averse and exhibit constant additive risk posture, each firm will provide a more reliable product and offer a longer warranty than if consumers were risk neutral.

Proof. See Appendix B. □

This result differs from a result in Courville and Hausman (1979). In that paper, risk-averse consumers caused warranty to be longer but reliability to be lower than in the case of risk neutrality. This is due to the fact that actual reliability served as a signal of reliability in their model, whereas, since actual reliability is not observable by consumers in our model, only warranty serves as a signal of reliability.

5. Numerical Results

This section illustrates the results of the preceding section through a number of numerical examples. A simple base-case example serves as the starting point throughout the section, and variations are generated by modifying different model parameters.

In all cases, the actual and perceived failure-rate functions are scaled versions of a single prototype failure-rate function \( \nu(t) \) – i.e., \( \rho(t) = \theta \nu(t) \) and \( \rho^c(t) = \gamma(w) \nu(t) \). This prototype function is “piecewise-Weibull” – the failure rate is Weibull \( \nu(t) = \lambda \mu(t)^{\mu-1} \) with different parameters \( \lambda \) and \( \mu \) in different regions. The specific parameters are given below.

- \( 0 \leq t \leq 1 \): \( \lambda = 0.618 \), \( \mu = 0.7 \),
- \( 1 < t \leq 4 \): \( \lambda = 0.5 \), \( \mu = 1.0 \),
- \( 4 < t \leq 10 \): \( \lambda = 0.303 \), \( \mu = 1.5 \).

This prototype function over the 10-year useful life of the product is illustrated in Fig. 1.

The firm and consumer cost per repair and the consumer inconvenience cost are simply constant over the life of the product, but are discounted to reflect the time value of money. Specifically

\[
c^\ell(t) = c^\ell e^{-\delta^f t} \quad \text{(Base case: } c^\ell = 250, \delta^f = 0.08),
\]
\[
c^c(t) = c^c e^{-\delta^c t} \quad \text{(Base case: } c^c = 250, \delta^c = 0.08),
\]
\[
b(t) = be^{-\delta^f t} \quad \text{(Base case: } b = 50),
\]
where \( \delta^f \) and \( \delta^c \) are the firm and consumer interest rates for discounting. The unit manufacturing cost is
\[
m(\theta) = 8000 + 1500 \theta^{-2}
\]
and the signaling factor is
\[
\gamma(w) = 0.7 + e^{-0.1w}.
\]
These functions are illustrated in Figs. 2 and 3, respectively. Finally, the feasible warranty and reliability sets are

\[ W = \{1, 2, \ldots, 10\} \]

and

\[ \Theta = [0.5, 5.0]. \]

Table 1a–c show the optimal warranty-reliability pair for different values of consumer cost per repair, consumer inconvenience cost, and consumer interest rate. As indicated by Proposition 3, higher cost levels lead to longer warranties and increased reliability (lower \( h \)).

Table 2 shows the impact of scaling the firm cost per repair by a factor \( \beta \). As stated in Proposition 4, as this cost is scaled up we observe an optimal warranty of equal or shorter duration. However, the precise impact on reliability is ambiguous. As \( \beta \) increases, \( \theta^* \) is not monotone. However, note that \( \beta \theta^* \) is always increasing in \( \beta \) – i.e., as costs scale up, reliability may improve or deteriorate, but if it improves it will at most compensate for the increased cost.

Tables 3 and 4 respectively, illustrate the effects of changing \( m(\theta) \) multiplicatively, and changing \( \gamma(w) \) additively and multiplicatively. Again, these tables reflect the results from Section 4 – Propositions 5 and 6, respectively. Higher manufacturing costs lead to a shorter warranty and lower reliability, while stronger signaling leads to a longer warranty and higher reliability.

Finally, the reader may note that none of the examples in this section have \( 1 < w^* < 4 \). This is an artifact of the simple example chosen – in particular, the flatness of the failure-rate function on \( 1 \leq t \leq 4 \) and the essential symmetry in firm and consumer costs and perceptions.
6. Conclusions

In this paper we have presented a game-theoretic model representing firms in an oligopoly that choose warranties, reliabilities and prices for their version of some durable good. The model is quite general, accommodating nearly any product-failure distribution, risk neutrality and risk aversion on the part of consumers, asymmetry of perceptions between each firm and the consumers, signaling of reliability through warranty, and a wide variety of natural constraints on product warranty, reliability and price. Under the assumptions of the model we have shown that each firm can set its optimal warranty and reliability separately from that firm’s price and from warranties, reliabilities and prices at all competing firms in the market. We also established the complementarity of a firm’s optimal warranty and reliability. Furthermore, we explored the impact on the optimal warranty-reliability pair of several market factors of practical interest. Finally, we showed that risk aversion on

Table 1
Optimal warranty and reliability

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<tr>
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<th>$w^r$</th>
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<tr>
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Table 2
Optimal warranty and reliability for different firm repair costs

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Table 3
Optimal warranty and reliability for scaled unit manufacturing costs

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<td>1.47</td>
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<tr>
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<td>1.60</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>1.74</td>
</tr>
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<td>2.70</td>
</tr>
<tr>
<td>1.2</td>
<td>1</td>
<td>2.78</td>
</tr>
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</table>

Fig. 3. Reliability signaling factor vs. warranty $w$. 
the part of consumers leads to more reliable products and longer warranties.

Given the relative absence of literature addressing the optimal choice of warranty, reliability and prices in a competitive environment, a number of promising research directions remain. Some possibilities include consideration of heterogeneous consumers and market segmentation, multiple time periods, or multiple product lines within a firm that must coordinate warranties and/or manufacturing across these product lines.

Acknowledgements

The author would like to thank two anonymous reviewers for their comments and suggestions, which have resulted in a number of improvements in the paper.

Appendix A. Lattice programming background

The theory of lattice programming was developed by Topkis and Veinott (1973) for subadditive optimization problems. For a more detailed discussion, the reader is referred to Topkis (1978).

For real numbers $x$ and $y$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. For vectors $x, y \in \mathbb{R}^m$, $x \wedge y = (x_1 \wedge y_1, \ldots, x_m \wedge y_m)$ and $x \vee y = (x_1 \vee y_1, \ldots, x_m \vee y_m)$. A subset $L$ of $\mathbb{R}^m$ is a sublattice thereof if $x, y \in L$ implies $x \wedge y \in L$ and $x \vee y \in L$. A subset $L$ of $\mathbb{R}^m$ is a chain if $x, y \in L$ implies $x \leq y$ or $x \geq y$. A point $x \in L$ such that $x \geq y$ for all $y \in L$ (resp., $x \leq y$ for all $y \in L$) is the greatest (resp., least) element of $L$. Every nonempty compact sublattice $L$ of $\mathbb{R}^m$ has a greatest element and a least element.

Suppose $f$ is a $-\infty$ or real-valued function on a sublattice $L$ of $\mathbb{R}^m$. Call $f$ subadditive if $f(x \wedge y) + f(x \vee y) \leq f(x) + f(y)$ for each $x, y \in L$. If $f$ is a $-\infty$ or real-valued (resp., nonnegative real-valued) function on a sublattice $S$ of $\mathbb{R}^m$, $T \subseteq \mathbb{R}^p$, and $f$ is subadditive on $S \times C$ for each chain $C \subseteq T$, we say that $f$ is subadditive on $S \times T$ along chains in $T$.

If $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^p$ and $L \subseteq S \times T$, define the section $L_t$ of $L$ at $t$ by $L_t = \{s \in S: (s, t) \in L\}$ and the projection $\pi_T L$ of $L$ on $T$ by $\pi_T L = \{t \in T: L_t \neq \emptyset\}$.

If $X$ and $Y$ are nonempty sublattices of $\mathbb{R}^m$, write $X \subseteq_a Y$ if $x \in X$ and $y \in Y$ imply that $x \wedge y \in X$ and $x \vee y \in Y$. If $Y \subseteq \mathbb{R}^p$, $\{L_t\}_{t \in Y}$ is a collection of nonempty sublattices of $\mathbb{R}^m$, and $y < y'$ in $Y$ implies $L_y \subseteq_a L_{y'}$ we say $L_y$ is ascending in $y$ on $Y$.

Suppose $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^p$, and $f$ is a real-valued function on $S \times T$. Suppose also that $S_i$ is a nonempty sublattice of $S$ that is ascending in $t$ on $T$. Let $g$ be the projection of $f$ defined by

$$g(t) = \inf_{s \in S_i} f(s, t), \quad t \in T.$$

Let $S_i^0$ denote the optimal-response set at $t$ - i.e.,

$$S_i^0 = \{s \in S: f(s, t) = g(t)\}, \quad t \in T.$$

The following result of Topkis and Veinott (1973) gives conditions on $f$ that guarantee the optimal-response multifunction $S_i^0$ is ascending on $T$ and has increasing least and greatest optimal selections.

Theorem A.1 (Increasing optimal selections). Suppose $S$ is a sublattice of $\mathbb{R}^m$, $T$ is a sublattice of $\mathbb{R}^p$, and $S_i$ is a nonempty sublattice of $S$ that is ascending in $t$ on $T$. Suppose also that $f$ is real-valued and subadditive on $S \times T$ along chains in $T$. Then the optimal response set $S_i^0$ is a sublattice and is ascending on the set $t \in T$ for which $S_i^0$ is

---

Table 4

<table>
<thead>
<tr>
<th>Shift $a$</th>
<th>Scalar $a$</th>
<th>$w^*$</th>
<th>$\theta^*$</th>
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<td></td>
<td></td>
</tr>
<tr>
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<td>1</td>
<td>2.62</td>
<td></td>
</tr>
<tr>
<td>-0.1</td>
<td>1</td>
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<td>0.1</td>
<td>6</td>
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<td></td>
</tr>
<tr>
<td>0.2</td>
<td>8</td>
<td>1.52</td>
<td></td>
</tr>
</tbody>
</table>

Scaled signaling factor |
| 0.8       | 1         | 2.62  |           |
| 0.9       | 1         | 2.62  |           |
| 1.0       | 5         | 1.74  |           |
| 1.1       | 7         | 1.58  |           |
| 1.2       | 9         | 1.47  |           |
nonempty. If also each \( S_t \) is compact and \( f(\cdot, t) \) is upper semicontinuous on \( S_t \) for each \( t \in T \), then \( S_0 \) is nonempty and compact for each \( t \in T \), and \( S_0 \) has a least selection \( \underline{s}(t) \) and greatest selection \( \overline{s}(t) \) which are increasing on \( T \).

In the development above, the restriction of sets to \( \mathbb{R}^k \) was merely for ease of exposition. Identical results hold for more general sublattices, including sets of functions.

**Appendix B. Proof of Proposition 7**

**Proof.** Let \((w^*, r^*)\) and \((w^0, r^0)\) denote the firm’s greatest optimal warranty-reliability pair given risk-neutral consumers and risk-averse consumers, respectively. We wish to show that

\[
(w^*, -r^*) \leq (w^0, -r^0). \tag{A.1}
\]

Note that for fixed \( w \), a change in consumer risk attitude affects the firm’s supply cost \( S(w, r) \) only through \( k(w) \), which is a constant with respect to \( r \). Therefore such a change does not affect \( r^0(w) \), and so if \( w^0 \geq w^* \) then \( r^0 = r(w^0) \leq r(w^*) = r^* \) and Eq. (A.1) holds.

Suppose (8) does not hold – i.e., \( w^0 < w^* \). We will show that this leads to a contradiction. Let \( K(a, b; w) \) denote the (uncertain) repair cost (excluding inconvenience costs) during the time interval \([a, b]\) as perceived by a consumer who observes a warranty \( w \) and must pay for all repairs in the interval. Let \( B(w) \) be the (uncertain) total inconvenience cost perceived by such a consumer and let \( S^a(w, r) \) be the firm’s total supply cost given warranty \( w \) and reliability \( r \) are chosen. Let \( \mathcal{C}(\cdot) \) represent the certain equivalent of a risk-averse consumer. Since such a consumer prefers less risk and lower cost and also exhibits constant additive risk posture,

\[
S^a(w^0, r^0) = \int_0^{w^0} r^0(t) c^f(t) \, dt + m(r^0) + \mathcal{C}(EK(w^0, w^0; w^0)) + K(w^0, L; w^0) + B(w^0)
\]

\[
= \int_0^{w^0} r^0(t) c^f(t) \, dt + m(r^0) + \mathcal{C}(E(\mathcal{K}(w^0, w^0; w^0))) + K(w^0, L; w^0) + B(w^0)
\]

\[
= \int_0^{w^0} r^0(t) c^f(t) \, dt + m(r^0) + \gamma(w^0) \int_0^{w^0} \rho^f(t) c^f(t) \, dt
\]

\[
+ \mathcal{C}(K(w^0, L; w^0) + B(w^0))
\]

Since \((w^*, r^*)\) is optimal for risk-neutral consumers,

\[
\int_0^{w^*} r^0(t) c^f(t) \, dt + m(r^0) + \gamma(w^0) \int_0^{w^*} \rho^f(t) c^f(t) \, dt
\]

\[
+ E[\mathcal{K}(w^*, L; w^0) + B(w^0)]
\]

\[
\geq \int_0^{r^*} r^0(t) c^f(t) \, dt + \int_0^{r^*} r^0(t) c^f(t) \, dt
\]

\[
+ m(r^*) + E[\mathcal{K}(w^*, L; w^*) + B(w^*)]
\]

and therefore

\[
\int_0^{w^*} r^0(t) c^f(t) \, dt + m(r^0) + \gamma(w^0) \int_0^{w^*} \rho^f(t) c^f(t) \, dt
\]

\[
+ \mathcal{C}(K(w^*, L; w^0) + B(w^0))
\]

\[
\geq \int_0^{r^*} r^0(t) c^f(t) \, dt + m(r^*)
\]

\[
+ \mathcal{C}(K(w^*, L; w^*) + B(w^*)) + \{\mathcal{C}(K(w^*, L; w^0) + B(w^0))
\]

\[
+ B(w^0) - E[\mathcal{K}(w^*, L; w^0) + B(w^0)]
\]

\[
- \{\mathcal{C}(K(w^*, L; w^*) + B(w^*)) - E[\mathcal{K}(w^*, L; w^*)
\]

\[
+ B(w^*)]\}.
\]

Since a shorter warranty yields lower reliability perceptions and the gap between the certain equivalent of consumer cost and expected consumer cost is greater for a less reliable product, the difference of the two bracketed terms is nonnegative. Therefore we have

\[
S^a(w^0, r^0) \geq \int_0^{w^0} r^0(t) c^f(t) \, dt + m(r^0)
\]

\[
+ \mathcal{C}(K(w^*, L; w^*) + B(w^*)) = S^a(w^*, r^*),
\]

\[
\int_0^{r^*} r^0(t) c^f(t) \, dt + m(r^*)
\]

\[
+ \mathcal{C}(K(w^*, L; w^*) + B(w^*)) = S^a(w^*, r^*),
\]
which contradicts the choice of \((w^a,r^a)\) as the greatest optimal warranty-reliability pair given risk-averse consumers.

References