Theory and Methodology

Purchasing demand information in a stochastic-demand inventory system

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Abstract

This paper studies a periodic-review, stochastic-demand inventory system in which the manager has the opportunity each period to purchase information about demand in the upcoming period before deciding how much product to order. We analyze the information-purchase and product-replenishment decisions for both perfect and imperfect demand information. Under perfect information, we provide a characterization of the optimal policy for both finite and infinite horizon problems, and also establish useful managerial insights into the behavior of the system. We show that future demand information becomes less valuable at higher inventory levels, and more valuable when longer horizons remain. When the initial inventory is zero, solving the perfect-information problem reduces to computing a single quantity, for which we provide a closed form expression. As a result, this problem is shown to be equivalent to one in which the manager purchases perfect information over the entire horizon with a single lump-sum payment at the beginning of the horizon. Our analytical and numerical results demonstrate that most of the insights from the perfect-information scenario carry over to the imperfect-information case. © 1997 Elsevier Science B.V.

1. Introduction

Information about future demand is a critical factor in the proper management of an inventory system. In a recent study at Hewlett-Packard, it was estimated that demand uncertainty accounted for approximately 60% of the inventory investment in their manufacturing and distribution system (Davis, 1993). Lowering demand uncertainty could provide dramatic payoffs in the form of reduced inventory investments. However, the vast majority of the stochastic-demand inventory literature does not explicitly address the costs and benefits of reducing demand uncertainty. Most of the work in this area implicitly assumes that demand information has been obtained at some point in the past, using historical experience and/or judgment to estimate a demand distribution. Little attention has been paid to dynamic information-gathering approaches, for example, purchasing additional information about future demand every time a product replenishment decision is made.

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One possible reason for the lack of attention paid to purchasing demand information may be that purchasing such information frequently may have been too costly or time consuming. Now, with the widespread availability of computerized information systems and communication networks, both the cost and the time required to purchase future demand information is being reduced. For example, with the tremendous growth of on-line information services available through the Internet, such information may be obtained within hours or even minutes, with the cost of obtaining the information consisting of a fixed fee, or in some cases the cost of the on-line time. However, it is not clear whether and under what circumstances information should be purchased. Thus information purchase decisions could become an important addition to the inventory manager’s responsibilities.

In this paper, we study the information purchase decision in the context of a periodic-review, stochastic-demand inventory system. The manager has the opportunity to purchase information about the upcoming period’s demand before the product-replenishment decision is made. To motivate the problem studied here, consider the following real world example. A retailer of automobile spare parts decides at the beginning of every month how much of each part to purchase to meet the upcoming month’s demand. For simplicity, restrict attention to one particular part. Based on historical information the dealer has estimated an a-priori demand distribution for the part. To provide better information on the upcoming month’s demand, the dealer can purchase vehicle registration data, giving the model and year of each car registered during the previous month in the geographical area. This data is used in a prediction model to obtain a more informative posterior demand distribution for the upcoming month’s demand. The questions arise: (i) should the dealer purchase the vehicle data at the price in question? and (ii) having made the information decision, how many parts should the dealer order?

Although a considerable amount of research exists on information acquisition problems, most of it lies outside the field of inventory management. For example, the Moore and Whinston (1986, 1987) sequential information acquisition model has inspired work in several related disciplines including those in deductive and inductive decision making, for example, Hall et al. (1986), Moore et al. (1990) and Mookerjee and Dos Santos (1993). These models study sequential acquisition of information followed by a final action, for example, several clinical tests conducted by a physician followed by the treatment. In the context of inventory systems, this would be analogous to a news-vendor model in which there are several pieces of information that could be purchased before deciding how much product to order. Hausman and Peterson (1972) study an inventory problem in which demand forecasts are revised as time passes. There are no costs associated with such revisions, so the decision of whether or not to purchase demand information is not addressed.

Our model differs from the Moore and Whinston models (1986, 1987) in that it considers a multi-period model in which information acquisition and product replenishment decisions alternate and interact. We also extend the work of Hausman and Peterson (1972) by introducing a cost of acquiring information and explicitly modeling the decision of whether or not to purchase information.

The primary contributions of this paper are the analytical and numerical insights obtained for the information-acquisition problem in the context of the inventory system studied. We restrict our attention to a single product, single facility inventory system with linear ordering, holding and shortage costs. There is negligible lead time for demand information and product delivery. Under the restrictive assumption that future demand information is perfect we fully characterize the optimal information-purchase and product-replenishment policy for both finite and infinite horizons. We show that the information becomes less attractive at higher inventory levels and more attractive when longer horizons remain. When the initial inventory level is zero, solving both the finite and infinite horizon problems reduces to computing a single quantity, for which we derive a closed-form expression. Analytical and numerical studies show that most of the structure from the perfect-information scenario carries over to the more realistic case in which the information about future demand is imperfect.

In the next section the inventory model is defined and notation and assumptions are specified. Section 3 considers the case in which information about upcoming demand is perfect, while Section 4 addresses the imperfect-information case. Section 5 presents some concluding remarks.
2. The model

Consider a facility that maintains inventory of a single product that is subject to random demand. At the beginning of each period, the on-hand stock of the product is observed and two decisions are made. First, the manager decides whether or not to purchase information regarding product demand for the upcoming period. If this information is purchased, the information comes in the form of a signal of demand for the upcoming period, and this signal is observed immediately. In the case of perfect information, the signal represents the true demand that will be experienced. In the case of imperfect information, the signal identifies a particular posterior demand distribution, for example, one with lower variance than the manager's prior distribution of demand. Second, after the information (if purchased) is observed, the manager must decide how much product to order. Assume that the product arrives immediately and can be used to satisfy demand in the upcoming period. After the ordered quantity arrives, the facility experiences random demand for the product, with demands in successive periods being independent and having identical prior distributions. If at the end of a period demand for the product exceeds the on-hand supply, the excess demand is backordered. The facility incurs a cost for each unit of the product ordered, and a unit holding (backorder) cost for each unit held (backordered) at the end of each period. We will use the following notation.

2.1. Notation and definitions

- \( n \): Number of periods remaining in finite-horizon problem.
- \( c \): Unit ordering cost.
- \( h \): Unit holding cost.
- \( p \): Unit backorder cost.
- \( M \): Cost of obtaining information for next period’s demand.
- \( \alpha \): Discount factor.
- \( x_n \): Inventory at the beginning of the period, with \( n \) periods remaining.
- \( y_n \): Inventory after order received but before demand, with \( n \) periods remaining.
- \( y^*_n(x) \): Optimal initial stock after order received but before demand, when starting with inventory \( x \) with \( n \) periods remaining.
- \( D \): Random variable representing one-period demand.
- \( z \): Particular value of one-period demand.
- \( f(z) \): Demand density function.
- \( F(z) \): Demand cumulative distribution function.
- \( T \): Random variable representing signal of demand for the upcoming period.
- \( t \): Particular value of signal.
- \( \psi(t) \): Signal density function.
- \( \phi(z|T=t) \): Posterior density function for demand given signal value \( t \).
- \( L(y) \): One-period expected holding and shortage cost function.

When considering an infinite horizon or when there is no ambiguity as to the period, we drop the index \( n \) from the above quantities.

Assume that at the end of the horizon any remaining stock can be salvaged, with the manager receiving the full cost \( c \) per unit, while any backorders remaining can be filled with an emergency order at a cost of \( c \) per unit. In order to ensure that it is more economical to fill backorders than to allow them to build up, assume that \( p > (1 - \alpha)c \).

We address the problem of finding the optimal information-purchasing and inventory-management policy for
both finite-horizon and infinite-horizon problems. We will use the following notation to represent the relevant
cost functions.

\[ K_n^B(x) = \text{Minimum } n\text{-period expected present cost starting at } x, \text{ assuming we buy information in the current period.} \]

\[ K_n^{DB}(x) = \text{Minimum } n\text{-period expected present cost starting at } x, \text{ assuming we do not buy information in the current period.} \]

\[ C_n(x) = \text{Minimum } n\text{-period expected present cost starting at } x. \]

The infinite-horizon versions of the above quantities are obtained by dropping the subscript \( n \).

The inventory functional equations that define these cost functions are given below.

**Inventory functional equations:**

(a) Do not buy information:

\[ K_n^{DB}(x) = \min_{y \geq x} \left\{ c(y-x) + L(y) + \alpha E C_{n-1}(y-D) \right\}, \quad (1) \]

where \( C_0(x) = -cx \).

(b) Buy information:

(i) Perfect information:

\[ K_n^B(x) = M + \int_0^\infty \left\{ \min_{y \geq x} \left[ c(y-x) + \phi(y-z) + p(z-y) + \alpha C_{n-1}(y-z) \right] f(z) \, dz \right\} \psi(t) \, dt, \]

where \( C_0(x) = -cx \) and \( \nu^+ = \max(0, \nu) \).

(ii) Imperfect information:

\[ K_n^B(x) = M + \int_0^\infty \left\{ \min_{y \geq x} \left[ c(y-x) + \int_0^y (y-z) \phi(z|T=t) \, dz \right. \\
+ \left. p \int_y^\infty (z-y) \phi(z|T=t) \, dz + \alpha \int_0^\infty C_{n-1}(y-z) \phi(z|T=t) \, dz \right] \psi(t) \, dt \right\}, \quad (2) \]

where \( C_0(x) = -cx \).

**Information decision:**

\[ C_n(x) = \min \{ K_n^B(x), K_n^{DB}(x) \}. \quad (3) \]

For the infinite-horizon problem the equations are identical, except that the subscripts are dropped and \( C_0(x) \) is
no longer defined.

There are some similarities between the model presented here and the classic dynamic inventory model with
uniform setup costs (referred to here as the \( (s, S) \) model). The two models are similar in that a fixed cost is
charged whenever a particular action is taken, either placing an order in the \( (s, S) \) model, or acquiring
information in our model. However, the models are different in some very important ways.

In the model studied here, there exists the opportunity for gathering information about the current period’s
demand by paying an amount \( M \). If such information is purchased, a demand prediction is observed before the
ordering decision is made. This information structure is represented by the fact that the integral (over possible
demand values) is outside the minimization in the recursion for \( K_n^B(x) \), i.e., different ordering decisions can be
made after observing different demand predictions. This information structure does not exist in the classic \( (s, S) \)
model. This structural difference results in different dynamics exhibited by the two systems. In the model studied here, we find that it is sometimes optimal to acquire information (incurring a fixed charge), but depending on the signal observed, place no order. Therefore a fixed charge might be incurred even if no order is placed. This can never occur in the classic \((s, S)\) model.

3. Perfect information

We begin our analysis by considering the case in which the information regarding upcoming demand is known to be perfectly accurate. Although this information assumption is very strong, it may serve as a reasonable model for a limited number of realistic problems. In addition, the perfect-information model provides valuable insights into the information-purchasing decision given more general information models.

3.1. Product-replenishment decision

If the manager purchases perfect information when there are \(n\) periods remaining, the optimal product-replenishment decision is quite simple and matches one's intuition. If demand is greater than the on-hand stock, it is optimal to bring inventory up to exactly meet demand. If demand is smaller than current inventory, it is optimal to order nothing. The optimality of this policy, along with a useful technical result, are stated in Lemma 1 below.

Lemma 1. For every \(n\), if perfect information is purchased and demand \(z\) is predicted, the optimal ordering policy is to order up to \(y^*(x) = \max\{x, z\}\). Also, for every \(n\), \(C_n'(x) = -c\) for \(x \leq 0\) and \(C_n'(x) \geq -c\) for all \(x\).

If the manager does not purchase perfect information when there are \(n\) periods remaining, then the product-replenishment decision is represented by Eq. (1). Unfortunately, the minimand in that equation is not convex in general, since \(C_n(x) = \min\{K_n^B(x), K_n^B(x)\}\) may not be convex even if \(K_n^B(x)\) and \(K_n^B(x)\) are. As a result, the usual arguments establishing the optimality of a base-stock policy do not hold.

However, the structure of the problem is clearly sufficient to establish the optimality of the more general policy below:

\[
\begin{align*}
\text{Order up to } S_{n1}^{DB} & \text{ if } x \in I_{n1}, \\
\text{order up to } S_{n2}^{DB} & \text{ if } x \in I_{n2}, \\
\vdots \\
\text{order up to } S_{nk}^{DB} & \text{ if } x \in I_{nk}, \\
\text{order nothing if } x \in (\bigcup_{j=1}^k I_{nj})^c, \text{ where the } I_{nj}'s \text{ are disjoint intervals on the real line.}
\end{align*}
\]

Although the solution to the product-replenishment decision does not seem to have a simple form in general, we can gain some insight into the solution by comparing it to the optimal base-stock policy for the problem without the information-purchase option. To this end, define

\[
DB_n(x) = \text{Minimum } n\text{-period expected present cost starting at } x, \text{ for the problem in which there is no information-purchase option.}
\]

The solution to the problem represented by \(DB_n(x)\) is the familiar base-stock policy, with

\[
S = F^{-1}\left(\frac{p - (1 - \alpha)c}{p + h}\right) > 0
\]

being the optimal base-stock level in every period.

The following result establishes a relationship between this base-stock policy and the optimal product-replenishment policy for our problem.
Lemma 2. \( C_n(x) \geq DB'_n(x) \) for all \( x \). Also, when information is not purchased with \( n \) periods remaining, the optimal order-up-to quantity \( y^*_n(x) \) has the form

\[
y^*_n(x) = \begin{cases} 
S & \text{if } x < S, \\
= x & \text{if } x \geq S.
\end{cases}
\]

If perfect information is not purchased, it is never optimal to order a positive quantity that causes the inventory level to exceed the base-stock level under the classical base-stock policy. One by-product of this is that the inventory level is always bounded above by \( S \), except perhaps at the very beginning of the horizon if the initial inventory is large. This fact will be useful in the proof of Proposition 3 below.

The following two results establish that the finite-horizon cost functions converge to their infinite-horizon counterparts, and that Lemma 1 holds for the infinite horizon as well.

Proposition 3. The \( n \)-period cost functions \( K_{DB}^n(x), K_B^n(x), \) and \( C_n(x) \) converge to their infinite-horizon counterparts, i.e.,

\[
K_{DB}^n(x) = \lim_{n \to \infty} K_{DB}^n(x), \quad K_B^n(x) = \lim_{n \to \infty} K_B^n(x) \quad \text{and} \quad C^n(x) = \lim_{n \to \infty} C_n(x).
\]

Corollary 4. The following are true for the infinite-horizon problem:

(i) If perfect information is purchased, the optimal ordering policy is to order up to \( y^*(x) = \max\{x, z\} \).

(ii) \( C'(x) \) for \( x < 0 \) and \( C'(x) \geq -c \) for all \( x \).

3.2. Information-purchase decision

The information-purchase decision, as represented by Eq. (3), simply involves a comparison between \( K_{DB}^n(x) \) and \( K_B^n(x) \) or their infinite-horizon counterparts. In analyzing this decision, it is quite convenient to work with the worth of perfect information when the initial inventory level is \( x \). Define the worth functions for the finite- and infinite-horizon problems, respectively, as

\[
W_n(x) = K_{DB}^n(x) - K_B^n(x) \quad \text{and} \quad W(x) = K_{DB}(x) - K_B(x),
\]

i.e., the amount by which the cost of the don’t-buy option exceeds that of the buy option. When this quantity is positive (nonpositive) it is optimal to buy (not buy) perfect information. Note that the worth function differs from the traditional ‘value of information’ concept by including the cost \( M \) of the information. As a result, the worth function may be negative, whereas value of information is always nonnegative. Since the information-purchase option occurs repeatedly in our problem, the above approach is much more tractable.

The following result establishes the basic structure of the worth function and the optimal information-purchase decision.

Proposition 5. The worth functions \( W_n(x) \) and \( W(x) \) are constant on \( x \leq 0 \) and decreasing in \( x \) on \( x > 0 \). With \( n \) periods remaining, there exists a critical inventory level \( \bar{x}_n \in (-\infty) \cup (0, +\infty) \) such that the optimal information-purchase policy is:

- if \( x < \bar{x}_n \), purchase information,
- if \( x \geq \bar{x}_n \), do not purchase information.
In the infinite-horizon problem there exists a critical inventory level \( \bar{x} = \lim_{n \to \infty} \bar{x}_n \) such that the optimal information-purchase policy is:

- if \( x < \bar{x} \), purchase information,
- if \( x \geq \bar{x} \), do not purchase information.

The preceding result states that perfect information is less valuable when the initial inventory in a period is greater. This is not surprising, since the value of information arises from its ability to affect the ordering decision. The higher the initial inventory level, the more likely the product-replenishment action will be the same (i.e., order nothing) regardless of the information observed, thereby making the information less valuable at these higher levels.

Proposition 5 also provides an additional intuitive interpretation of Lemma 2. The second part of Lemma 2 says that, when the option of purchasing information exists in the future, it is optimal to bring inventory up to a lower level than if no such information option exists. In light of Proposition 5, we can explain this by the fact that the future information option is less valuable at higher inventory levels. As a result, we keep inventory lower to enable us to take advantage of future opportunities to purchase information.

The next result establishes some interesting behavior of the worth functions \( W_n(x) \), as well as the optimal order quantities \( y^*_n(x) \) when information is not purchased, as the number of periods remaining in the horizon increases.

**Proposition 6.** The following are true:

(i) \( W_n(x) \) is nondecreasing in \( n \) for all \( x \).

(ii) \( y^*_n(x) \geq y^*_{n-1}(x) \) for all \( x \).

(iii) If \( W_1(0) \leq 0 \), then \( W(0) = W_n(0) = W_1(0) \leq 0 \) for all \( n \).

Advance demand information helps the manager reduce costs in two ways. First, it helps reduce immediate holding and/or shortage costs at the end of the period for which the information is known. Second, the information helps the manager start the following period in a better position, i.e., with a more desirable initial inventory level. The first benefit does not change based on the number of periods remaining in the horizon, but the latter becomes more valuable when there is a longer horizon over which to take advantage of the superior starting position. As a result, the worth of perfect information increases as the horizon length increases.

Part (ii) of Proposition 6 states that, given that information is not purchased in a given period, the optimal order-up-to level is smaller when there are more periods remaining in the horizon. This is a stronger result than Lemma 2, which simply compared the order-up-to quantity to the optimal base-stock level for the classical base-stock problem. The intuition behind this result is that, since information is more valuable when the horizon is longer, there is more incentive to keep inventory levels low to take advantage of the opportunity to purchase information. Combining Lemma 2 and Proposition 6 yields

\[
y^*_n(x) \leq y^*_{n-1}(x) \leq \cdots \leq y^*_1(x) = S.
\]

It is also interesting to note that even though the end-of-horizon conditions allow for salvage and emergency orders, the optimal order-up-to quantity is not necessarily stationary as it is in the classical base-stock problem. The introduction of the opportunity to purchase information fundamentally changes the form of the solution.

If \( W_1(0) \leq 0 \), then the final period of the horizon is identical to the final period in the classical base-stock problem. Due to the salvage value and emergency order assumptions, the beginning of this final period 'looks' exactly like the end of the horizon in terms of its impact on earlier decisions, for example, those made with two periods remaining. In the classical base-stock problem this results in every period having the same optimal base-stock level. In our problem, it results in information having the same worth in all periods, i.e., \( W(0) = W_n(0) = W_1(0) \leq 0 \) for all \( n \).
Propositions 5 and 6 provide us with the information we need to develop a nearly complete characterization of the optimal information-purchase and product-replenishment policies for both the finite- and infinite-horizon problems. We now state this characterization.

**Corollary 7.** If $W_l(0) \leq 0$, then it is never optimal to purchase perfect information in the infinite-horizon problem or in any finite-horizon problem. In this event, the optimal product-replenishment policy is the standard base-stock policy with stationary base-stock level $S$. If $W_l(0) > 0$, then the optimal information-purchase policy has the following characteristics:

(i) If the initial inventory level is $x \leq 0$, then it is optimal to purchase information in every period for both infinite- and finite-horizon problems.

(ii) For the infinite-horizon problem, if the initial inventory level is $x > 0$, then with probability 1 there exists some finite $j$ such that it is optimal to not purchase perfect information in periods $i = 1, \ldots, j$ and to purchase perfect information in periods $i = j + 1, j + 2, \ldots$.

Let us take a moment to describe how the inventory system might evolve under the optimal policy described above. The behavior of the system when $W_l(0) \leq 0$ is simple and quite familiar. If $W_l(0) > 0$ and $x \leq 0$ when the system begins operation, then it is optimal to buy perfect information about the upcoming period's demand. After observing a prediction of $z$ units of demand, the manager orders exactly $z - x$ units, so the next period begins with zero initial inventory. Since $W(0) \geq W_l(0) \geq W_r(0) > 0$ this behavior repeats every period.

Now consider an infinite-horizon problem with $W_l(0) > 0$ and initial inventory $x > 0$. If $x \leq \bar{x}$, then it is optimal to purchase information and order up to $\max\{x, z\}$. As a result, the initial inventory for the next period will be at most $\bar{x}$. This situation perpetuates itself, with initial inventory eventually dropping to zero and staying there. If $x > \bar{x}$, it is not optimal to purchase perfect information, so the manager brings the inventory level before demand up to some optimal level, say $y_1$. Demand for the period is then experienced. If demand is large enough so that the initial inventory level for the following period is below $\bar{x}$, then it is optimal to purchase perfect inventory in the next and all succeeding periods. If not, the manager again places an order without purchasing demand information. After this order, however, the inventory level will be at most $y_1$. This follows since the only change from the preceding period is that the initial inventory level is lower, and the only impact this has on the problem is to place a smaller lower bound $y \geq x$ on the order-up-to level $y$. Since demand is nonnegative, the initial inventory is $x \leq y_1$, so $y_1$ is feasible in this period as well, and as a result it is not optimal to build inventory past $y_1$. The system may experience several periods in which it is not optimal to purchase perfect information. In each such period, the inventory level after ordering but before demand will be at most $y_1$. Thus there is some positive probability each period that the inventory level will drop below $\bar{x}$, at which point the system follows the pattern described above.

For many cases, Corollary 7 reduces the solution of the perfect-information problem to that of computing $W_l(0)$. This quantity can be easily obtained from its definition, yielding

$$W_l(0) = (1 - \alpha) c(S - E(D)) + L(S) - M.$$ 

The only cases for which Corollary 7 does not thoroughly characterize the optimal policy are those with $W_l(0) > 0$ and initial inventory $x > 0$. For an infinite-horizon problem, any reasonable policy will result in inventory dropping to zero after some finite period of time. As a result, the manager could follow a reasonable heuristic, for example, order nothing until inventory is reduced to zero, or follow the information-purchase policy defined by $W_l(x)$ (which is easy to compute) and order up to $\bar{x}$ if information is not purchased, etc. For a finite-horizon problem, however, it is possible for the optimal information-purchase decision to be quite complex, as illustrated in Fig. 1.

Suppose the initial inventory with four periods remaining is $x_4$, with $0 < x_4 < \bar{x}_4$. Information is purchased, and demand $z_4 < x_4 - \bar{x}_3$ is predicted, so that $x_3 > \bar{x}_3$. Information is not purchased, so an order is placed and demand is experienced resulting in an initial inventory of $x_2 < \bar{x}_2$ with two periods remaining. Information is
purchased, and demand \( z_2 < x_2 - \bar{x}_1 \) is predicted, so that \( x_1 > \bar{x}_1 \) and information is not purchased in the final period. Therefore the information-purchase pattern for the last four periods of this problem would be Buy – Don’t Buy – Buy – Don’t Buy. As long as \( \bar{x}_n > \bar{x}_{n-1} \), this pattern could clearly be extended to \( n \) periods.

In the next section we explore the behavior of our model across a wide variety of numerical examples for both perfect and imperfect information. Our experience there suggests that the quantities in question converge in a relatively small number of periods. This suggests that a dynamic programming approach would be feasible for solving relatively short finite-horizon problems, while longer problems can be effectively approximated by an infinite-horizon problem.

3.3. One-time information purchase and the worth of information

Before leaving the perfect-information model, we first establish some connections between our problem and a related problem that we will refer to as the Lump-Sum problem. In the Lump-Sum problem, the inventory manager has the option of paying a single lump sum at the beginning of the horizon. In return, at the beginning of each period the manager receives free perfect information regarding demand for the upcoming period. This Lump-Sum problem might represent the decision of whether or not to purchase an information system that can be used to predict demand one period in advance. If the system is purchased, the information is used in every period, with orders being placed to exactly meet demand. If the system is not purchased, the problem reduces to the standard base-stock problem.

It is easy to show that if the initial inventory at the beginning of the horizon is zero, then our problem is equivalent to the Lump-Sum problem. This result is suggested by Corollary 7, which establishes the optimality of an ‘all-or-nothing’ information-purchase policy for both the finite- and infinite-horizon problems. The finite-horizon problem with a period-by-period information cost \( n \) is equivalent to the Lump-Sum problem with a lump-sum cost of \( Q = M \cdot \sum_{i=0}^{n-1} \alpha^i \). If it is optimal to purchase one period’s demand information with \( n \) periods remaining, then it is optimal to purchase the information in each of the following \( n-1 \) periods, yielding a total information cost of \( M \cdot \sum_{i=0}^{n-1} \alpha^i \). In that case it is also optimal to pay for all of that information up front at the same cost. Conversely, if it is optimal to pay a lump sum of \( Q = M \cdot \sum_{i=d}^{n-1} \alpha^i \) up front to receive demand information for every period, then it is also optimal to pay \( M \) to receive that information in each period. The same holds true for the infinite-horizon problem, with the lump-sum cost for the equivalent problem being \( Q = M/(1 - \alpha) \) in that event.

4. Imperfect information

The preceding section provides significant insight into the operation of an inventory system when the opportunity exists to purchase perfect information about future demand. Of course, in many real situations the information that can be obtained about future demand is not perfect. This section addresses such cases.

As can be seen from Eq. (2), imperfect information makes the model significantly more complex than was the case for perfect information. This complexity makes analytical treatment of the model quite difficult, and as a result much of this section will focus on numerical analysis of a wide variety of sample problems. Before introducing the numerical work, however, we present a result that is similar in flavor to Proposition 6 for the perfect-information model. The result relies on the assumption that the worth function \( W_n(x) \) is constant on
$x \leq 0$ and nonincreasing on $x > 0$ for all periods $n$. This property was shown to hold under perfect information, but due to the complexity of the imperfect-information model and the generality of the different probability distributions involved, it seems quite difficult to prove under imperfect information. However, the property seems quite reasonable in light of the intuition gained from the perfect-information model. Also, in our subsequent discussion of numerical trials, we note that the property holds for every one of the 324 test problems considered.

**Proposition 8.** For the imperfect-information model, assume that $W_n(x)$ is constant on $x \leq 0$ and nonincreasing on $x > 0$ for all periods $n$. If $W(0) < 0$, then

$$W_n(x) \leq W_n(0) \leq W(0) < 0 \quad \text{for all } x \text{ and all } n,$$

and

$$W(x) \leq W(0) \leq W(0) < 0 \quad \text{for all } x.$$

As a result, in every period of both the finite- and infinite-horizon problems, it is optimal to not purchase information. The optimal product-replenishment policy is the standard base-stock policy with stationary base-stock level $S$.

If we introduce superscripts P and I to denote perfect and imperfect information, respectively, then clearly $W^P(0) \geq W^I(0)$, so if $W^I(0) < 0$, then $W^I(0) < 0$. As a result, the easier to calculate $W^P(0)$ can be used as an initial screen for the imperfect-information problem as well.

At first glance, Proposition 8 seems to be a weaker result than the combination of Proposition 6 and Corollary 7 for perfect information, in that it does not characterize the optimal policy when $W(0) > 0$ and $x \leq 0$. However, a closer look reveals that this case is fundamentally more difficult to handle when demand information is imperfect. With perfect information the system repeatedly restarts with zero inventory, but this is not true when information is imperfect. Some safety stock will always be held, so that experiencing a demand that is on the small end of the posterior distribution results in positive initial inventory for the next period. As a result, the case of $W(0) > 0$ and $x \leq 0$ really cannot be distinguished from the case of $W(0) > 0$ and $x > 0$ when information is imperfect. Since the latter case leads to potentially complex policies even when information is perfect, it is not surprising that the same occurs with imperfect information.

### 4.1. Numerical trials

In order to develop methods for solving the case above, as well as to explore other properties of the imperfect-information model, we study a wide variety of numerical examples. For all of our numerical trials demand was a discrete random variable ranging from 0 to 20, with the prior distribution approximating a truncated Normal distribution. Cost parameters considered were $c = 1$, $h = 0.01, 0.05$ and 0.10, $p = 0.11, 0.15$ and 0.30, $M = 0.1, 0.2$ and 0.4, and $\alpha = 0.99, 0.95$ and 0.90, yielding 81 different combinations of cost parameters.

To describe the treatment of imperfect information, we begin by presenting a family of distributions describing the error represented by the imperfect signal. The distributions we use are triangular, with each distribution being completely characterized by the maximum possible error size. Let $k$ be the maximum

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1 The numerical experiments were also carried out using a Uniform prior distribution. The findings were similar to those obtained from a Normal prior distribution.
possible error size and let \( \mu(\nu) \) be the probability of an error of size \( \nu \), where \( \nu \) may be positive or negative. Then the error distribution is given by

\[
\begin{align*}
\mu(0) &= \frac{1}{k + 1}, \\
\mu(1) &= \mu(-1) = \frac{1 - 1/(k + 1)}{k + 1}, \\
\mu(2) &= \mu(-2) = \frac{1 - 2/(k + 1)}{k + 1}, \\
&\vdots \\
\mu(k) &= \mu(-k) = \frac{1 - k/(k + 1)}{k + 1}.
\end{align*}
\]

The error distribution is used to define the posterior demand distribution given a particular signal. Since the prior distribution for demand only assigns positive probability to integer demand values from 0 to 20, we take care to define the posterior distribution to share this attribute. Given a signal \( t \) such that \( k \leq t \leq 20 - k \), the posterior demand distribution is

\[
\phi(z | T = t) = \mu(t - z).
\]
For signals $t < k$ or $t > 20 - k$, the above relationship would assign positive probability to demand values outside of the possible range. For such a signal we start with the probability assignment given in Eq. (4) for values of $z$ that fall between 0 and 20. Then any probability that would be assigned to values of $z$ outside of that range is spread evenly among those demand levels with positive probability. Figs. 2–4 illustrate this method of computing the error distribution and the posterior demand distributions given two different signals when the maximum error is $k = 4$.

Finally, the signal distribution $\psi(t)$ is computed so that the prior demand distribution is matched, i.e.,

$$f(z) = \sum_t \psi(t) \cdot \phi(z | T = t).$$

For each of the 81 cost-parameter combinations, we generated problems with maximum error sizes $k = 1, 2, 3$ and 4, for a total of 324 problems. We also generated the 81 problems with $k = 0$, corresponding to perfect information. We used dynamic programming to solve each of these for a horizon length of 20 periods. In all 324 imperfect-information cases considered, the following properties of the perfect-information model were observed to hold.

### 4.2. Observations from numerical trials

(A) $W_n(x)$ was decreasing in $x$ and constant on $x \leq 0$ for every $n$.

(B) $W_n(x)$ was increasing in $n$.

(C) $W_n(x)$ converged to $W(x)$ by $n = 3$.

(D) The optimal ordering policy for the 'don't-buy' option was of the base-stock type, and the base-stock level $S_n$ converged to some $S_0$ by $n = 3$.

As can be seen from the above observations, all of the basic qualitative behavior seems to carry over from the perfect-information system to the imperfect-information system. In particular, observation (A) provides the necessary support for the assumption made in Proposition 8.

The numerical trials also provide some additional insight that is useful in handling those cases for which we have not established a complete characterization of the optimal policy – $W_n(0) > 0$ and $x < 0$ for finite-horizon problems under perfect information, and $W_n(0) > 0$ under imperfect information. Since the worth functions and order-up-to levels converged in a relatively small number of steps, solution of the problems through dynamic programming may be a feasible approach. Once the dynamic program has been solved by backward recursion for $N$ iterations so that $W_N(0)$ is sufficiently close to $W(0)$ (and therefore $\bar{x}_N$ is sufficiently close to $\bar{x}$), the
manager can follow the optimal information-purchase and product-replenishment decisions computed in iteration $N$ for all periods $n \geq N$, and follow the policies computed by the dynamic program for periods $n < N$.

### 4.3. Worth of information vs. information accuracy

The final issue we address is the rate at which the worth of information decreases as the accuracy of that information deteriorates. To make the study of this issue manageable, we focus on $W(0)$ as a proxy for the entire worth function $W(\cdot)$. In order to track the variation in $W(0)$ as information accuracy changes, we modify our notation slightly. Define $W_i(0, k)$ to be the one-period worth of information given zero initial inventory under an information scenario identified by a maximum error of $k$, for $k = 0, 1, 2, 3$ and 4.

The behavior of $W_i(0, k)$ was quite similar for all of the 81 cost-parameter scenarios studied. For this reason, we simply illustrate $W_i(0, k)$ for four cases that represent some of the extreme values of the parameter set. The scenarios we present are:

(i) $p = 0.3, h = 0.1, M = 0.4$;
(ii) $p = 0.3, h = 0.1, M = 0.1$;
(iii) $p = 0.11, h = 0.01, M = 0.4$;
(iv) $p = 0.11, h = 0.01, M = 0.1$.

![Fig. 5. Four examples of $W_i(0, k)$ using $\alpha = 0.95$.](image-url)
In all four cases we use $\alpha = 0.95$. Graphs of $W_i(0, k)$ for these cases are shown in Fig. 5.
Clearly $W_i(0, k)$ behaves in the way that one would expect. $W_i(0, k)$ decreases in $k$ and seems to be convex, i.e., the immediate drop-off in information worth as the information moves away from perfection is greater than it is as the information moves further from perfection. However, this effect does not seem to be as strong as might be expected, i.e., $W_i(0, k)$ is not far from being linear in $k$. This observation suggests a method for using $W_i(0, 0)$ (which is easy to compute) to approximate $W_i(0, k)$ for $k \geq 1$ (which is somewhat more complicated to compute). Clearly completely useless information would have a worth of $-M$. Define $\hat{k}$ to be the size of the maximum error that would yield essentially useless information. (In our numerical examples, since demand is between 0 and 20, a reasonable value would be $\hat{k} = 10$.) Then if $W_i(0, k)$ is approximately linear in $k$, an approximation for $W_i(0, k)$ would be

$$\tilde{W}_i(0, k) = W_i(0, 0) - \left(\frac{k}{\hat{k}}\right) \cdot (W_i(0, 0) + M).$$

This approximation could provide the manager with an additional screen to apply to the imperfect-information problem. Recall that the manager can use $W_i(0, 0)$, which is easy to compute, to initially test whether to consider the information purchasing opportunity. If $W_i(0, 0) \leq 0$, clearly $W_i(0, k) \leq 0$ for all values of $k$. However, if $W_i(0, 0) > 0$, it may or may not be wise to purchase imperfect information. In this event, the manager could compute $\tilde{W}_i(0, k)$. If $\tilde{W}_i(0, k) \leq 0$, it suggests that $W_i(0, k) \leq 0$, and therefore it is never optimal to purchase the information. Otherwise, if $\tilde{W}_i(0, k) > 0$, the manager would need to solve the dynamic program to determine the optimal policy.

5. Conclusions

This paper has presented a model for evaluating the opportunity to purchase future-demand information in a random-demand, periodic-review inventory system. We characterized the form of the optimal information-purchase decision for the perfect-information case and established a simple method of solving such problems. In addition, we developed valuable insights into the behavior of the worth of demand information at various inventory levels and with different horizon lengths. We demonstrated that many of these insights carry over to the imperfect-information scenario, and we used numerical studies to explore the deterioration in the information-worth function as information moves further away from perfection.

The work presented in this paper is a first step toward understanding the interaction between information gathering and inventory management. A number of extensions suggest themselves. Such extensions might include consideration of a positive leadtime in the delivery of information or product, obtaining demand information for a block of periods instead of a single period, or obtaining demand information for a number of products simultaneously with a single fee. Although we expect that many of the basic insights gained here will carry over to these more complicated scenarios, it would be interesting to see what additional insights could be obtained for each of those specific situations.

Appendix A. Proofs of results

Proof of Lemma 1. Define

$$G_n^B(y) = c \cdot (y-x) + h \cdot (y-z)^+ + p \cdot (z-y)^+ + \alpha C_{n-1}(y-z).$$
Then
\[ G_{1}^{*}(y) = \begin{cases} 
(1 - \alpha)c - p < 0 & \text{for } y < z, \\
(1 - \alpha)c + h > 0 & \text{for } y > z, 
\end{cases} \]
so clearly \( y^{*}(x) = \max\{x, z\} \) is optimal for the 1-period buy-information case. As a result, for \( x \leq 0 \),
\[ K_{1}^{P}(x) = M + c \int_{0}^{\infty} (z - x)f(z) \, dz = M + c \cdot (E(D) - x), \]
so
\[ K_{1}^{P}(x) = -c. \]
For \( x \geq 0 \),
\[ K_{1}^{P}(x) = M + (h - \alpha c) \int_{0}^{x} (x - z)f(z) \, dz + c \int_{x}^{\infty} (z - x)f(z) \, dz, \]
so
\[ K_{1}^{P}(x) = (h - \alpha c)f(x) - c(1 - F(x)) \geq -c. \]
Now if we do not purchase the perfect information, define
\[ G_{n}^{DB}(y) = c \cdot (y - x) + L(y) + \alpha \int_{0}^{\infty} C_{n-1}(y - z)f(z) \, dz. \]
For \( n = 1 \), this is convex in \( y \), so there exists a base-stock level
\[ S_{1}^{DB} = F^{-1}\left(\frac{p - (1 - \alpha)c}{p + h}\right) > 0, \]
such that the optimal ordering policy is to order up to \( S_{1}^{DB} \) when possible and to order nothing otherwise. Therefore, for \( x \leq S_{1}^{DB} \),
\[ K_{1}^{DB}(x) = c \cdot (S_{1}^{DB} - x) + L(S_{1}^{DB}) - \alpha c \int_{0}^{S_{1}^{DB}} (S_{1}^{DB} - z)f(z) \, dz, \]
so
\[ K_{1}^{DB}(x) = -c. \]
For \( x > S_{1}^{DB} \),
\[ K_{1}^{DB}(x) = (h - \alpha c) \int_{0}^{x} (x - z)f(z) \, dz + (p + \alpha c) \int_{x}^{\infty} (z - x)f(z) \, dz, \]
so
\[ K_{1}^{DB}(x) = (h + p)F(x) - (p + \alpha c). \]
Since \( x > S_{1}^{DB} \), we have \( F(x) \geq \frac{p - (1 - \alpha)c}{p + h} \), so \( K_{1}^{DB}(x) \geq -c. \) Since either \( C_{1}(x) = K_{1}^{P}(x) \) or \( C_{1}(x) = K_{1}^{DB}(x) \), this establishes the result for \( n = 1. \)
Assume that the result holds for \( n - 1. \) Then for \( y > z \),
\[ G_{n}^{P}(y) = c + h + \alpha C_{n-1}(y - z) \geq (1 - \alpha)c + h > 0, \]
and for \( y < z \),
\[
G_n^B(y) = c - p + \alpha C_{n-1}(y - z) = (1 - \alpha) c - p < 0.
\]

So clearly \( y^*(x) = \max\{x, z\} \) is optimal for the \( n \) - period buy-information case. As a result, for \( x \leq 0 \),
\[
K_n^B(x) = M + c \cdot (E(D) - x) + \alpha C_{n-1}(0),
\]
so
\[
K_n^B(x) = -c.
\]

For \( x \geq 0 \),
\[
K_n^B(x) = M + \int_0^x h(x-z)f(z) \, dz + \int_x^\infty c(z-x)f(z) \, dz + \alpha \int_x^\infty C_{n-1}(x-z)f(z) \, dz
\]
\[+ \alpha (1 - F(x)) C_{n-1}(0),\]
so
\[
K_n^B(x) = (h + c) F(x) - c + \alpha \int_0^x C_{n-1}(x-z)f(z) \, dz \geq hF(x) + (1 - \alpha) cF(x) - c \geq -c.
\]

If we do not purchase the perfect information, the optimal policy follows the form outlined immediately following Lemma 1. Define \( P_n^{DB} = \bigcup_{j=1}^k I_{nj} \) to be the set where a positive order is placed. For \( x \in P_n^{DB} \),
\[
K_n^{DB}(x) = c \cdot (S_{nj}^{DB} - x) + L(S_{nj}^{DB}) + \alpha \int_0^\infty C_{n-1}(S_{nj}^{DB} - z)f(z) \, dz,
\]
for some \( j \), so
\[
K_n^{DB}(x) = -c.
\]

It is easy to establish that, since \( p > (1 - \alpha)c \), it follows that \( (-\infty, 0] \subseteq I_1 \). As a result we have \( K_n^{DB}(x) = -c \) for \( x \leq 0 \). For \( x \in (P_n^{DB})^C \),
\[
K_n^{DB}(x) = h \int_0^x (x-z)f(z) \, dz + p \int_x^\infty (z-x)f(z) \, dz + \alpha \int_0^\infty C_{n-1}(x-z)f(z) \, dz,
\]
so
\[
K_n^{DB}(x) = (h + p) F(x) - p + \alpha \int_0^\infty C_{n-1}(x-z)f(z) \, dz.
\]

Since \( x \in (P_n^{DB})^C \), we have
\[
G_n^{DB}(x) = c + (h + p) F(x) - p + \alpha \int_0^\infty C_{n-1}(x-z)f(z) \, dz \geq 0,
\]
so \( K_n^{DB}(x) \geq -c. \)

Proof of Lemma 2. Recall that
\[
K_n^{DB}(x) = DB_1(x) = \begin{cases} 
    c \cdot (S - x) + (h - \alpha c) \int_0^x (S - z)f(z) \, dz + (p + \alpha c) \int_x^\infty (z - S)f(z) \, dz, & x < S, \\
    (h - \alpha c) \int_0^x (x - z)f(z) \, dz + (p + \alpha c) \int_x^S (z - x)f(z) \, dz, & x \geq S.
\end{cases}
\]
Clearly for \( x \geq S \) it is not optimal to order in the don’t-buy scenario. To obtain \( C'_n(x) \geq DB'_n(x) \), we only need to consider the case in which \( C_n(x) = K^B_n(x) \). Lemma 1 implies that

\[
K^B_1(x) = (h + \alpha c)F(x) - c(1 - F(x)),
\]

so since \( p > (1 - \alpha)c \),

\[
C'_1(x) = (h + \alpha c)F(x) - c(1 - F(x)) \geq (h + p)F(x) - (p + \alpha c) = DB'_1(x).
\]

Now suppose the result holds for periods 1, \ldots, \( n - 1 \). Recall that

\[
G^B_n(y) = cy + h\int_y^{y} (y - z)f(z) \, dz + p\int_y^{\infty} (z - y)f(z) \, dz + \alpha\int_0^{\infty} C_{n-1}(y - z)f(z) \, dz,
\]

and define the objective function for the \( n \)-period never-buy problem to be

\[
DG_n(y) = cy + h\int_y^{\infty} (y - z)f(z) \, dz + p\int_y^{\infty} (z - y)f(z) \, dz + \alpha\int_0^{\infty} DB_{n-1}(y - z)f(z) \, dz.
\]

By the inductive hypothesis, clearly \( G^B_n(y) \geq DG_n(y) \). Since \( DG'_n(y) \geq 0 \) for \( x \geq S \), it is never optimal to bring inventory above \( S \) when demand information is not purchased with \( n \) periods remaining.

If \( x < S \), then \( DB'_n(x) = -c < C'_n(x) \). Suppose instead that \( x \geq S \) and \( C_n(x) = K^B_n(x) \). Then

\[
C'_n(x) = K^B_n(x) = hF(x) - p(1 - F(x)) + \alpha\int_0^{\infty} C_{n-1}(x - z)f(z) \, dz \\
\geq hF(x) - p(1 - F(x)) + \alpha\int_0^{\infty} DB_{n-1}(x - z)f(z) \, dz = DB'_n(x).
\]

Finally, suppose that \( x \geq S \) and \( C'_n(x) = K^B_n(x) \) so that

\[
C'_n(x) = hF(x) - c(1 - F(x)) + \alpha\int_0^{x} C_{n-1}(x - z)f(z) \, dz
\]

Then

\[
C'_n(x) - DB'_n(x) = (p - c)(1 - F(x)) + \alpha\int_x^{\infty} [C_{n-1}(x - z) - DB_{n-1}(x - z)] f(z) \, dz
\]

\[-\alpha\int_x^{\infty} DB'_{n-1}(x - z)f(z) \, dz \geq 0\]

by the inductive hypothesis and the fact that \( DB'_{n-1}(z) = -c \) for \( z < 0 \).

**Proof of Proposition 3.** Since \( C_0(x) = 0 \) for all \( x \), clearly \( K^DB_n(x) \), \( K^B_n(x) \) and \( C_n(x) \) are increasing in \( n \) for fixed \( x \). Note that \( K^DB_n(x) \) is bounded above by \( DB_n(x) \), which is finite for all \( x \). Therefore there exists a function, say \( J^DB_n(x) \), such that \( J^DB_n(x) \rightarrow \lim_{n \rightarrow \infty} K^DB_n(x) \). A similar argument establishes \( K^B_n(x) \) that is bounded above by

\[
M/(1 - \alpha) + DB_n(x),
\]

so there exists a function, say \( J^B_n(x) \), such that \( J^B_n(x) = \lim_{n \rightarrow \infty} K^B_n(x) \). Define \( J(x) = \min\{J^DB(x), J^B(x)\} \) so that \( J(x) = \lim_{n \rightarrow \infty} C_n(x) \). It remains to show that \( J^DB(x), J^B(x), \) and \( J(x) \) satisfy the inventory functional equations (3)-(5).
Eq. (4) follows immediately from the Monotone Convergence Theorem:

\[
J^B(x) = \lim_{n \to \infty} J^B_n(x) = M + h \int_0^X (x - z) f(z) \, dz + \int_0^\infty f(z) \, dz
\]

\[
= M + h \int_0^X (x - z) f(z) \, dz + \int_0^\infty f(z) \, dz + \int_0^X C(x - z) f(z) \, dz + (1 - F(x)) \lim_{n \to \infty} C_n(0)
\]

For Eq. (3) we show that inequality holds in both directions. First, note that

\[
J^D B(x) = \lim_{n \to \infty} J^D B_n(x)
\]

\[
= \lim_{n \to \infty} \min_{y \geq x} \left\{ c(y - x) + h \int_0^Y (y - z) f(z) \, dz + p \int_0^\infty (z - y) f(z) \, dz + \int_0^\infty C_n(y - z) f(z) \, dz \right\}
\]

\[
\leq \lim_{n \to \infty} \min_{y \geq x} \left\{ c(y - x) + h \int_0^Y (y - z) f(z) \, dz + p \int_0^\infty (z - y) f(z) \, dz + \int_0^\infty C(y - z) f(z) \, dz \right\}
\]

\[
= \min_{y \geq x} \left\{ c(y - x) + h \int_0^Y (y - z) f(z) \, dz + p \int_0^\infty (z - y) f(z) \, dz + \int_0^\infty C(y - z) f(z) \, dz \right\}.
\]

For the other inequality, note that since \(K^D B_n(x)\) converges monotonely to \(J^D B(x)\), the Monotone Convergence Theorem implies that \(G^D B_n(y)\) converges monotonely to

\[
G^D B(y) = c \cdot (y - x) + L(y) + \alpha EC(y - D).
\]

Since \(G^D B_n(y)\) and \(G^D B(y)\) are continuous everywhere, \(G^D B_n(y)\) converges uniformly to \(G^D B(y)\) on the compact set

\[
Y(u) = \{ y : y \geq u \text{ and } y \leq S \},
\]

where \(S\) is the optimal base-stock level for the infinite-horizon problem without the option of purchasing information. As a result of Lemma 2, we can impose the additional constraint \(y \leq S\) on the don't-buy scenario without loss of generality. Now

\[
J^D B(x) \geq K^D B_n(x) = \min_{y \geq x} \{ G^D B_n(y) \} - c \cdot x.
\]

For any \(\varepsilon > 0\) there exists \(N\) such that for all \(n \geq N\), \(0 \leq G^D B(y) - G^D B_n(y) < \varepsilon\) for all \(y\) in \(Y(x)\). Let \(\nu_B^n\) be a minimizer of \(G^D B_n(y)\) and \(\nu^{DB}\) be a minimizer of \(G^D B(y)\) on \(Y(x)\). Then

\[
0 \leq G^D B(\nu^{DB}_n) - G^D B(\nu^{DB}) < \varepsilon,
\]

and clearly

\[
G^D B(\nu^{DB}) - G^D B(\nu^{DB}_n) \leq 0 \text{ and } G^D B(\nu^{DB}) - G^D B(\nu^{DB}_n) \geq 0,
\]

so

\[
0 \leq G^D B(\nu^{DB}) - G^D B(\nu^{DB}_n) < \varepsilon,
\]

i.e.,

\[
\min_{y \in Y(x)} \{ G^D B(y) \} - c \cdot x < \min_{y \in Y(x)} \{ G^D B_n(y) \} - c \cdot x + \varepsilon,
\]
for all \( n \geq N \). Taking the limit of the right-hand side as \( n \to \infty \) and then as \( \varepsilon \downarrow 0 \) yields

\[
\min_{y \in Y(x)} \{G^D_B(y)\} \leq \lim_{n \to \infty} \min_{y \in Y(x)} \{G^D_n(y)\},
\]
i.e.,

\[
J^D_B(x) \geq \min_{y \geq x} \left\{ c(y-x) + h \int_0^y (y-z) f(z) \, dz + p \int_0^\infty (z-y) f(z) \, dz + \int_0^\infty C(y-z) f(z) \, dz \right\}.
\]

Therefore \( K^D_B(x) = J^D_B(x) = \lim_{n \to \infty} K_n^D_B(x) \) and \( C(x) = J(x) = \lim_{n \to \infty} C_n(x) \). □

**Proof of Corollary 4.** (i) By replacing \( C_n(x) \) with \( C(x) \) and \( G_n^B(y) \) with

\[
G^B(y) = c(y-x) + h(1-y)(y-z)^+ + p(z-y)^+ + \alpha C(y-z)
\]
in the proof of Lemma 1, the arguments used there establish this result.

(ii) If \( C'(x) \) exists, then \( C'(x) = \lim_{n \to \infty} C_n'(x) \), so the result holds immediately. □

**Proof of Proposition 5.** For \( x \leq 0 \), \( W_n'(x) = K_n^D_B'(x) - K_n^B'(x) = 0 \). For \( x > 0 \),

\[
K_n^B'(x) = (h + c)F(x) + c + \alpha \int_0^x C_n^{-1}(x-z) f(z) \, dz.
\]

If \( x \in P_n^D_B \), \( K_n^D_B(x) = -c \). Then since the proof of Lemma 1 established that \( K_n^B(x) \geq -c \), we have \( W_n'(x) \leq 0 \). Now if \( x \in (P_n^D_B)^c \),

\[
K_n^B'(x) = (h + p)F(x) + p + \alpha \int_0^\infty C_n^{-1}(x-z) f(z) \, dz,
\]

so

\[
W_n'(x) = (c - p)(1 - F(x)) + \alpha \int_0^\infty C_n^{-1}(x-z) f(z) \, dz = (c-p)(1-F(x))\alpha c(1-F(x))
\]

\[= (1-\alpha)c(1-F(x)) - p(1-F(x)) \leq 0.\]

Since \( W_n(x) \) is constant on \( x \leq 0 \) and decreasing everywhere, if \( W_n(0) \leq 0 \), then \( W_n(x) \leq 0 \) for all \( x \), and therefore \( \overline{x}_n = \infty \). If \( W_n(0) > 0 \), then \( W_n(x) > 0 \) for all \( x < \overline{x}_n \) and \( W_n(x) \leq 0 \) for all \( x \geq \overline{x}_n \), for some \( \overline{x}_n \in (0, +\infty) \).

We immediately obtain the desired behavior of \( W(x) \) since \( W(x) = \lim_{n \to \infty} W_n(x) \) and the result holds for each \( W_n(x) \). The rest of the result is obtained by replacing \( W_n(x) \) with \( W(x) \) in the arguments above. □

**Proof of Proposition 6.** (i) and (ii): by induction. First note that

\[
G_2^B(y) - G_1^B(y) = \alpha c + \alpha \int_0^\infty C_1(y-z) f(z) \, dz \geq \alpha c - \alpha c = 0,
\]

so \( y_2^* = y_1^* \). To simplify notation, let \( S_2 = y_1^* \). Then

\[
W_2(x) - W_1(x) = \left[ K_2^D(x) - K_1^D(x) \right] - \left[ K_2^B(x) - K_1^B(x) \right]
\]

\[\geq \alpha c S_2 F(S_2) - \alpha c \int_0^{S_2} f(z) \, dz + \alpha \int_0^x C_1(S_2 - z) f(z) \, dz + \alpha C_1(0)(1-F(S_2))
\]

\[- \alpha c x F(x) + \alpha c \int_0^x z f(z) \, dz - \alpha \int_0^x C_1(x-z) f(z) \, dz - \alpha C_1(0)(1-F(x)).\]
Some tedious but straightforward algebraic manipulations of the above expression establish that $W_2(x) - W_1(x) \geq 0$.

To complete the first step of the induction we also prove a technical result that is useful in the inductive step of the proof, i.e., $C_2(x) - C_1(x)$ is increasing in $x$. Since $W_2(x) \geq W_1(x)$ we cannot have $C_2(x) - C_1(x) = K_2^{DB}(x) - K_1^{DB}(x)$, so the three possibilities for $C_2(x) - C_1(x)$ are: (a) $K_2^{DB}(x) - K_1^{DB}(x)$, (b) $K_2^{DB}(x) - K_1^{DB}(x)$, and (c) $K_2^{DB}(x) - K_1^{DB}(x)$.

Case (a): If it is optimal to order a positive quantity when starting at $x$ in both periods, then $K_2^{DB}(x) - K_1^{DB}(x)$ is constant in $x$. If it is optimal to order nothing in period 2, but to order a positive quantity in period 1, then

$$K_2^{DB}(x) - K_1^{DB}(x) = G_2^{DB}(x) + (\text{constant in } x) \geq 0.$$  

If it is optimal to order nothing in both periods, then

$$K_2^{DB}(x) - K_1^{DB}(x) = ac + \alpha \int_0^\infty C'(x-z) f(z) \, dz \geq ac - ac = 0.$$  

Since $y^*_2(x) \leq y^*_1(x)$, it can never be optimal to order a positive quantity in period 2 but order nothing in period 1.

Case (b): If it is optimal to order nothing in period 1, then

$$K_2^{DB}(x) - K_1^{DB}(x) = \alpha c + (p - c)(1 - F(x)) + \alpha \int_0^x C'(x-z) f(z) \, dz \geq [p - (1 - \alpha)c](1 - F(x)) \geq 0.$$  

If it is optimal to order a positive quantity in period 1, then

$$K_2^{DB}(x) - K_1^{DB}(x) = (h + c)f(x) + \alpha \int_0^x C'(x-z) f(z) \, dz \geq [h + (1 - \alpha)c] F(x) \geq 0.$$  

Case (c):

$$K_2^{DB}(x) - K_1^{DB}(x) = acF(x) + \alpha \int_0^x C'(x-z) f(z) \, dz \geq 0.$$  

Using the inductive hypothesis, similar arguments extend these results to period $n$.

(iii) Since $W_2(0) \leq 0$, $C_1(0) = K_1^{DB}(0)$ and as a result $K_2^{DB}(0) = M + c \cdot E(D) + \alpha K_1^{DB}(0)$ and $W_2(0) = K_2^{DB}(0) - \alpha K_1^{DB}(0) - M - c \cdot E(D)$. Simple algebra yields

$$K_2^{DB}(0) - \alpha K_1^{DB}(0) = K_1^{DB}(0),$$  

which in turn implies that

$$W_2(0) = K_1^{DB}(0) - M - c \cdot E(D) = W_1(0).$$  

By assuming that $W_{n-1}(0) = W_i(0)$ and $K_n^{DB}(0) = \alpha K_{n-1}^{DB}(0) = K_1^{DB}(0)$, similar arguments yield the result by induction. Since $W(0) = \lim_{n \to \infty} W_n(0)$ the final piece follows immediately.  

**Proof of Corollary 7.** By Propositions 5 and 6, $W(0) \leq 0$ implies that $W(x) \leq 0$ for all $x$ and $W_n(x) \leq 0$ for all $x$ and all $n$, so it is never optimal to purchase perfect information with either an infinite or finite horizon. The cost and demand structures of the problem immediately establish the optimality of the standard base-stock policy. Suppose instead that $W(0) > 0$.

(i) If $x \leq 0$, then $W(x) = W(0) \geq W_i(0) > 0$ and $W_n(x) = W_n(0) \geq W_i(0) > 0$, so it is optimal to purchase perfect information in that period. As a result, it is optimal to order up to $y^*(x) = z$, so that the initial inventory in the succeeding period is $x = 0$. Clearly this situation perpetuates.
(ii) In the infinite-horizon problem, if \( x > 0 \), there are two cases to consider: (a) \( x > \bar{x} \) and (b) \( x > \bar{x} \). Clearly in case (a) it is optimal to purchase perfect information in that period. It suffices to show that the system begins every period with initial inventory \( x \leq \bar{x} \). Starting any period at \( x \leq \bar{x} \) and purchasing perfect information, it is optimal to order up to \( y^*(x) = \max(x, z) \), so that the initial inventory in the succeeding period is \( y^*(x) - z = \max(x - z, 0) \leq x \leq \bar{x} \).

For case (b), let \( x_k \) be the initial inventory \( k \) periods from now and define \( P_k = \) probability that it is optimal to not purchase perfect information \( k \) periods from now, i.e.,

\[
P_k = \Pr(x_k > \bar{x}).
\]

Since \( x > \bar{x} \) we have \( P_0 = 1 \), and from (i) above we have \( P_k = 0 = P_{k+1} \). We show by induction that

\[
\lim_{k \to \infty} P_k = 1.
\]

From the initial inventory level \( x_0 = x \) it is optimal to order up to some \( y_0 \) in the initial period. Clearly \( y_0 \) is such that, for any \( \varepsilon > 0 \), \( F(y_0 - \varepsilon) < 1 \). Otherwise the cost could be reduced by ordering \( \varepsilon \) less product. Define \( y = F(y_0 - \bar{x}) \). Clearly \( x_1 = y_0 - D \leq y_0 \) and

\[
P_1 = \Pr(x_1 > \bar{x}) = \Pr(D < y_0 - \bar{x}) = \gamma < 1.
\]

Now suppose that \( x_{k-1} \leq y_0 \) and \( P_{k-1} \leq \gamma^{k-1} \). We show that \( x_k \leq y_0 \) and \( P_k \leq \gamma^k \). If \( x_{k-1} \leq x \), then it is optimal to purchase perfect information in that period, so \( P_k = P_{k-1} = 0 < \gamma^k \) and \( x_k = \max(x_{k-1} - z, 0) \leq x_{k-1} \leq y_0 \). Suppose instead that \( x_{k-1} > \bar{x} \). Since the only impact the initial inventory level has on the optimal ordering decision is that it provides a lower bound for the feasible order-up-to region, \( x_{k-1} \leq y_0 \) implies that the optimal order-up-to level from \( x_{k-1} \) is at most \( y_0 \). Therefore \( x_k \leq y_0 \) and \( P_k \leq \gamma \cdot P_{k-1} \leq \gamma^k \). \( \square \)

**Proof of Proposition 8.** Under the stated hypotheses, \( W_i(x) \leq W_i(0) < 0 \) for all \( x \). Suppose that \( W_i(x) < 0 \) for \( k = 1, 2, \ldots, n - 1 \), so that

\[
K^D_n(x) = DB_n(x) = (1 - \alpha^n) cS + L(S) \sum_{i=0}^{n-1} \alpha^i + cE(D) \sum_{i=1}^{n} \alpha^i
\]

and

\[
K^D_1(x) = \alpha K^D_{n-1}(x) = (1 - \alpha) cS + L(S) + \alpha cE(D) = K^D_1(0).
\]

Since \( C_{n-1}(x) = K^D_{n-1}(x) \) for all \( x \),

\[
K^D_n(0) = M + \int_0^{\infty} \min_{y \geq 0} \left\{ cy + h \int_0^{y} (y - z) \phi(z|T = t) \, dz \right. \\
+ \left. p \int_{y}^{\infty} (z - y) \phi(z|T = t) \, dz + \alpha \int_0^{\infty} K^D_{n-1}(y - z) \phi(z|T = t) \, dz \right\} \Psi(t) \, dt.
\]

As a result,

\[
K_n(0) = -M - \int_0^{\infty} \min_{y \geq 0} \left\{ cy + h \int_0^{y} (y - z) \phi(z|T = t) \, dz + p \int_{y}^{\infty} (z - y) \phi(z|T = t) \, dz \\
+ \alpha \int_0^{\infty} \left[ \alpha K^D_{n-1}(y - z) - K^D_n \right] \phi(z|T = t) \, dz \right\} \Psi(t) \, dt.
\]
Since $K_{n-1}^{DB}(x) = DB_{n-1}(x)$, it follows that $K_{n-1}^{DB}(x) - c$ and $K_{n-1}^{DB}(x) \geq K_{n-1}^{DB}(0) - cx$. As a result,

$$W_n(0) \leq -M - \int_0^\infty \min_{y \geq 0} \left\{ cy + h \int_y^\infty (y-z) \phi(z|T=t) \, dz + p \int_y^\infty (z-y) \phi(z|T=t) \, dz \right\} \psi(t) \, dt$$

$$= K_n^{DB}(0) - K_n^{DB}(0) - M$$

$$- \int_0^\infty \min_{y \geq 0} \left\{ cy + h \int_y^\infty (y-z) \phi(z|T=t) \, dz + p \int_y^\infty (z-y) \phi(z|T=t) \, dz \right\} \psi(t) \, dt$$

$$= K_1^{DB}(0) - K_1^{DB}(0) = W_1(0) < 0.$$

The remaining conclusions follow immediately. □

References


Mookerjee, V., and Dos Santos, B. (1993), "Inductive expert system design: Maximizing system value", Information Systems Research 4/2, 111–140.

