Standard Auctions with Financially Constrained Bidders

Saed Alizamir

Duke University

Authors: Yeon-Koo Che and Ian Gale
Motivation

There are situations where bidders face an increasing marginal cost of expenditure:

- Liquidity constrained buyers: e.g., government auctions
- Rising opportunity cost of diverting resources away from other projects
- Agent’s moral hazard problem
- A device to relax competition: e.g., salary caps in professional sports

Buyer’s willingness to pay is not the minimum of his valuation and budget:

- Revenue equivalence does not hold.
- It may be optimal for the seller to offer a menu of lotteries.
- Optimal auction can be an all-pay auction.

Objective:

- Capture the role played by the size of bidders.
- Study the effect of auction form on the equilibrium behavior of bidders.
- Ranking the standard auctions in expected revenue and social surplus.
Motivation

There are situations where bidders face an increasing marginal cost of expenditure:

- Liquidity constrained buyers: e.g., government auctions
- Rising opportunity cost of diverting resources away from other projects
- Agent’s moral hazard problem
- A device to relax competition: e.g., salary caps in professional sports

Buyer’s willingness to pay is *not* the minimum of his valuation and budget:

- Revenue equivalence does not hold.
- It may be optimal for the seller to offer a menu of lotteries.
- Optimal auction can be an all-pay auction.

Objective:

- Capture the role played by the *size* of bidders.
- Study the effect of auction form on the equilibrium behavior of bidders.
- Ranking the standard auctions in expected revenue and social surplus.
Motivation

There are situations where bidders face an increasing marginal cost of expenditure:

- Liquidity constrained buyers: e.g., government auctions
- Rising opportunity cost of diverting resources away from other projects
- Agent’s moral hazard problem
- A device to relax competition: e.g., salary caps in professional sports

Buyer’s willingness to pay is *not* the minimum of his valuation and budget:

- Revenue equivalence does not hold.
- It may be optimal for the seller to offer a menu of lotteries.
- Optimal auction can be an all-pay auction.

**Objective:**

- Capture the role played by the *size* of bidders.
- Study the effect of auction form on the equilibrium behavior of bidders.
- Ranking the standard auctions in expected revenue and social surplus.
General Framework

- One indivisible item for sale, which values zero to the seller.
- $N$ risk neutral bidders, with random values $v$ and costs $C(x, w)$.
- Private information $(w, v)$, independently drawn from density $f(., .)$.

Assumptions:

1. $C(x, w)$ is strictly increasing and (weakly) convex in $x$.
2. For any $w$ and $x$, $\lim_{w' \to w} C(x, w') = C(x, w)$.
3. For any $w$, $w' > w$, and $x$, we have $C_1(x^+, w') \leq C_1(x^-, w) < \infty$.
4. $C(x, \bar{w}) = x$ for $x \leq \bar{v}$, and $C(x, w) = x$ for $x \leq v$.

Example:

$$C(x, w) = \begin{cases} x & \text{if } x < w; \\ x + R(x - w) & \text{if } x \geq w, \end{cases}$$

with $R(0) = 0$, and $R(y)$ is non-decreasing, convex, and Lipschitz continuous.
General Framework

- One indivisible item for sale, which values zero to the seller.
- $N$ risk neutral bidders, with random values $v$ and costs $C(x, w)$.
- Private information $(w, v)$, independently drawn from density $f(., .)$.

Assumptions:

1. $C(x, w)$ is strictly increasing and (weakly) convex in $x$.
2. For any $w$ and $x$, $\lim_{w' \to w} C(x, w') = C(x, w)$.
3. For any $w$, $w' > w$, and $x$, we have $C_1(x^+, w') \leq C_1(x^-, w) < \infty$.
4. $C(x, \bar{w}) = x$ for $x \leq \bar{v}$, and $C(x, w) = x$ for $x \leq v$.

Example:

$$C(x, w) = \begin{cases} x & \text{if } x < w; \\ x + R(x - w) & \text{if } x \geq w, \end{cases}$$

with $R(0) = 0$, and $R(y)$ is non-decreasing, convex, and Lipschitz continuous.
General Framework

- One indivisible item for sale, which values zero to the seller.
- N risk neutral bidders, with random values \( v \) and costs \( C(x, w) \).
- Private information \((w, v)\), independently drawn from density \( f(.,.)\).

Assumptions:
1. \( C(x, w) \) is strictly increasing and (weakly) convex in \( x \).
2. For any \( w \) and \( x \), \( \lim_{w' \to w} C(x, w') = C(x, w) \).
3. For any \( w, w' > w \), and \( x \), we have \( C_1(x^+, w') \leq C_1(x^-, w) < \infty \).
4. \( C(x, \bar{w}) = x \) for \( x \leq \bar{v} \), and \( C(x, w) = x \) for \( x \leq v \).

Example:

\[
C(x, w) = \begin{cases} 
  x & \text{if } x < w; \\
  x + R(x - w) & \text{if } x \geq w,
\end{cases}
\]

with \( R(0) = 0 \), and \( R(y) \) is non-decreasing, convex, and Lipschitz continuous.
Auction Properties

Consider a family, $\mathcal{F}$, of auction rules which satisfy:

1. The highest bidder receives the object, and the rules apply identically to all bidders.
2. There exist a symmetric pure strategy Bayes-Nash equilibrium, in which $b = B(w, v)$.
3. The equilibrium bid $B(w, v)$ is increasing in $(w, v)$, and strictly increasing if both $w$ and $v$ rise.
4. For any equilibrium bid, there exists an unconstrained type with budget $\bar{w}$ that makes the same bid.

Definition

Auction $A$ satisfies the single-crossing property with respect to auction $B$ if, for any type $(w, v)$ that is active in both auctions,

$$\Omega^-_A(w, v) \subseteq \Omega^-_B(w, v) \text{ and } \Omega^+_A(w, v) \supseteq \Omega^+_B(w, v)$$
Auction Properties

Consider a family, $\mathcal{F}$, of auction rules which satisfy:

1. The highest bidder receives the object, and the rules apply identically to all bidders.
2. There exist a symmetric pure strategy Bayes-Nash equilibrium, in which $b = B(w, v)$.
3. The equilibrium bid $B(w, v)$ is increasing in $(w, v)$, and strictly increasing if both $w$ and $v$ rise.
4. For any equilibrium bid, there exists an unconstrained type with budget $\bar{w}$ that makes the same bid.

Definition

Auction $A$ satisfies the single-crossing property with respect to auction $B$ if, for any type $(w, v)$ that is active in both auctions,

$$\Omega^-_A(w, v) \subseteq \Omega^-_B(w, v) \text{ and } \Omega^+_A(w, v) \supseteq \Omega^+_B(w, v)$$
If $\hat{v}_A \leq \hat{v}_B$ and auction $A$ satisfies the single-crossing property with respect to auction $B$, then $A$ yields higher social surplus than $B$. Auction $A$ yields strictly higher expected social surplus than auction $B$ if the single-crossing property holds strictly.

Illustration of Single-Crossing Property
Let $\bar{\Omega}_M(v) = \Omega_M(\bar{w}, v)$, and define

$$F_M(v) = \int_{\bar{\Omega}_M(v)} f(w, s) dw ds$$

Then, the expected revenue under auction $M$ is

$$R_M = N \int_{\hat{v}_M}^{\bar{v}} t_M(v) dF_M(v)$$

$$= \bar{v} - \hat{v}_M F_M(\hat{v}_M) N - \int_{\hat{v}_M}^{\bar{v}} F_M^{(2)}(v) dv$$

Notice the similarity to the expected revenue under the standard auctions without budget constraints! But ...
Let $\bar{\Omega}_M(v) = \Omega_M(\bar{w}, v)$, and define

$$F_M(v) = \int_{\bar{\Omega}_M(v)} f(w, s) dw ds$$

Then, the expected revenue under auction $M$ is

$$R_M = N \int_{\hat{\nu}_M}^{\bar{\nu}} \hat{t}_M(v) dF_M(v)$$

$$= \bar{\nu} - \hat{\nu}_M F_M(\hat{\nu}_M) N - \int_{\hat{\nu}_M}^{\bar{\nu}} F_M^{(2)}(v) dv$$

Notice the similarity to the expected revenue under the standard auctions without budget constraints! But ...
If \( \hat{v}_A = \hat{v}_B = \hat{v} \) and \( F_A(v) \leq F_B(v) \) for all \( v \geq \hat{v} \), then auction A yields (weakly) higher expected revenue than auction B. The ranking is strict if there exists an interval of \( v \) on which \( F_A(v) < F_B(v) \).

Corollary

If auction A satisfies the single-crossing property with respect to auction B, then auction A yields (weakly) higher expected revenue than auction B. The ranking is strict if the single-crossing property holds strictly.
Theorem

If $\hat{v}_A = \hat{v}_B = \hat{v}$ and $F_A(v) \leq F_B(v)$ for all $v \geq \hat{v}$, then auction $A$ yields (weakly) higher expected revenue than auction $B$. The ranking is strict if there exists an interval of $v$ on which $F_A(v) < F_B(v)$.

Corollary

If auction $A$ satisfies the single-crossing property with respect to auction $B$, then auction $A$ yields (weakly) higher expected revenue than auction $B$. The ranking is strict if the single-crossing property holds strictly.
Auctions With Budget-Constrained Bidders

Consider absolute limits on bidders’ budget:

\[ C(x, w) = \begin{cases} 
  x & \text{if } x \leq w; \\
  \infty & \text{if } x > w,
\end{cases} \]

where \( \nu \leq w < \bar{\nu} \leq \bar{w} \). It is a dominated strategy to bid above one’s budget.

**Second-Price Auction:**
- Only buyers with \( \min \{w_i, \nu_i\} \geq r_s \) will participate.
- It is a dominant strategy for a participating buyer to bid \( \min \{w_i, \nu_i\} \).

**First-Price Auction:**
- Buyer \( i \) participates if and only if \( \min \{w_i, \nu_i\} \geq r_f \).
- The equilibrium bid is of the form \( B_f(w, \nu) = \min \{w, b_f(\nu)\} \).
- Only consider bidders of type \((\bar{w}, \nu)\), with valuations drawn from \( F_f(.) \). Then, \( b_f(\nu) \) satisfies

\[
b_f(\nu) = \nu - \frac{\int_{r_f}^{\nu} F_f(s)^{N-1} \, ds}{F_f(\nu)^{N-1}} \quad \text{for } \nu \geq r_f
\]
Consider absolute limits on bidders’ budget:

\[ C(x, w) = \begin{cases} 
  x & \text{if } x \leq w; \\ 
  \infty & \text{if } x > w,
\end{cases} \]

where \( v \leq w < \bar{v} \leq \bar{w} \). It is a dominated strategy to bid above one’s budget.

**Second-Price Auction:**
- Only buyers with \( \min \{ w_i, v_i \} \geq r_s \) will participate.
- It is a dominant strategy for a participating buyer to bid \( \min \{ w_i, v_i \} \).

**First-Price Auction:**
- Buyer \( i \) participates if and only if \( \min \{ w_i, v_i \} \geq r_f \).
- The equilibrium bid is of the form \( B_f(w, v) = \min \{ w, b_f(v) \} \).
- Only consider bidders of type \( (\bar{w}, v) \), with valuations drawn from \( F_f(.) \). Then, \( b_f(v) \) satisfies

\[
 b_f(v) = v - \frac{\int_r^v F_f(s)^{N-1} ds}{F_f(v)^{N-1}} \quad \text{for } v \geq r_f.
\]
Auctions With Budget-Constrained Bidders

Consider absolute limits on bidders’ budget:

\[
C(x, w) = \begin{cases} 
  x & \text{if } x \leq w; \\
  \infty & \text{if } x > w,
\end{cases}
\]

where \( v \leq w < \tilde{v} \leq \tilde{w} \). It is a dominated strategy to bid above one’s budget.

**Second-Price Auction:**
- Only buyers with \( \min \{ w_i, v_i \} \geq r_s \) will participate.
- It is a dominant strategy for a participating buyer to bid \( \min \{ w_i, v_i \} \).

**First-Price Auction:**
- Buyer \( i \) participates if and only if \( \min \{ w_i, v_i \} \geq r_f \).
- The equilibrium bid is of the form \( B_f(w, v) = \min \{ w, b_f(v) \} \).
- Only consider bidders of type \((\tilde{w}, v)\), with valuations drawn from \( F_f(.) \).
  Then, \( b_f(v) \) satisfies

\[
b_f(v) = v - \int_{r_f}^v \frac{F_f(s)^{N-1}}{F_f(v)^{N-1}} ds \quad \text{for } v \geq r_f
\]
Revenue And Social Surplus Comparisons

Proposition

Given a second-price auction with reserve price \( r_s \in [v, \bar{v}) \), a first-price auction with the same reserve price yields (weakly) higher social surplus, strictly higher expected social surplus, and strictly higher expected revenues.

Intuition ...

Proposition

A revenue-maximizing seller chooses a weakly lower reserve price in a first-price auction than in a second-price auction. Given the optimal reserve prices, the rankings of the above proposition hold.
Proposition

Given a second-price auction with reserve price $r_s \in [v, \bar{v})$, a first-price auction with the same reserve price yields (weakly) higher social surplus, strictly higher expected social surplus, and strictly higher expected revenues.

Intuition ...

Proposition

A revenue-maximizing seller chooses a weakly lower reserve price in a first-price auction than in a second-price auction. Given the optimal reserve prices, the rankings of the above proposition hold.
Availability Of Credit

- In both auction forms, bidder participates if and only if $v \geq C(r, w)$.
- In the second-price auction, it is dominant strategy to bid the highest $b$ satisfying $v \geq C(b, w)$.
- The social surplus comparison is not possible in general.

**Proposition**

Given a second-price auction with reserve price $r_s \in [v, \bar{v})$, a first-price auction with the same reserve price yields higher expected revenue than the second-price auction. The ranking is strict if $C(b, w)$ is strictly convex at some equilibrium bid $b$, for some $w$.

- Analogy to the case of risk-averse bidders.
- The use of entry fees.
In both auction forms, bidder participates if and only if $v \geq C(r, w)$.

In the second-price auction, it is dominant strategy to bid the highest $b$ satisfying $v \geq C(b, w)$.

The social surplus comparison is not possible in general.

**Proposition**

*Given a second-price auction with reserve price $r_s \in [v, \bar{v})$, a first-price auction with the same reserve price yields higher expected revenue than the second-price auction. The ranking is strict if $C(b, w)$ is strictly convex at some equilibrium bid $b$, for some $w$.*

- Analogy to the case of risk-averse bidders.
- The use of entry fees.
In both auction forms, bidder participates if and only if $v \geq C(r, w)$.

In the second-price auction, it is dominant strategy to bid the highest $b$ satisfying $v \geq C(b, w)$.

The social surplus comparison is not possible in general.

\begin{quote}
**Proposition**

*Given a second-price auction with reserve price $r_s \in [\underline{v}, \overline{v})$, a first-price auction with the same reserve price yields higher expected revenue than the second-price auction. The ranking is strict if $C(b, w)$ is strictly convex at some equilibrium bid $b$, for some $w$."
\end{quote}

- Analogy to the case of risk-averse bidders.
- The use of entry fees.
THANK YOU!