Optimal Auctions

Basic setup. \( n \) risk-neutral buyers are interested in one unit of a perishable good. Each buyer \( i = 1, \ldots, n \) has private value \( v_i \in [0, \infty) \). \( v_i \) are assumed to be independently distributed, with pdfs \( f_i(v) \) and cdfs \( F_i(v) \). Participation is voluntary in the sense that each buyer must get non-negative interim utility.

Technical assumption: For this note, assume that virtual values are monotone in values, i.e. \( \psi_i(v_i) = v_i - \frac{1 - F_i(v)}{f_i(v)} \) is strictly increasing in \( v_i \) for all \( i \), and that bidder values are iid with \( f_i(.) = f(.) \) and \( F_i(.) = F(.) \). We shall relax these conditions somewhat in class.

Optimal auctions. Consider the problem of a seller who wants to maximize revenue and who is able to commit to a direct revelation mechanism. To maximize expected revenue, the seller would like to sell the object to whichever buyer has the highest virtual value or, if every buyer has a negative virtual value, not sell the object at all. As long as virtual values are increasing in value, as assumed, this means selling to the buyer with the highest value above an established cut-off value \( v^* \), where \( v^* \) is defined implicitly by \( \psi_i^{-1}(v^*) = 0 \). Also, obviously, all types less than \( v^* \) should be left with zero surplus to maximize revenue, so that they make zero payments.

More broadly, Bulow and Roberts (1989) have a beautiful interpretation of optimal auction design by analogy with optimal monopoly pricing, where “virtual value” takes the place of “marginal revenue” in the analysis.

Uniform Example. \( n \) buyers have iid values \( U[0, 1] \). \( \psi_i(v_i) = 2v_i - 1 \) for all \( i \). The first-price auction with reserve price \( r = 1/2 \) is an optimal mechanism, as is the second-price auction with reserve price \( r = 1/2 \). The expected revenue can be computed in several ways:

1. Compute revenue in second-price auction. Bidding truthfully is the equilibrium strategy in the second-price auction, in which case the winner pays the second-highest realized value (call it \( v^{(2)} \)) or the reserve price, whichever is lower. Expected revenue is therefore \( E[\max\{r, v^{(2)}\}] \). In the efficient case, when \( r = 0 \), this is just \( \frac{n-1}{n+1} \).

2. Compute expected virtual value of winner. When the object is sold to a bidder having value \( v \), the surplus created can be thought of as being
split with $\psi(v)$ going to the seller and $v - \psi(v)$ going to the buyer. Expected revenue is therefore

$$n \int_1^r F(v')^{n-1} \left( v' - \frac{1 - F(v')}{f(v')} \right) dv' = n \int_1^r (v')^{n-1} (2v' - 1) dv'$$

$$= \left[ \frac{2n(v')^{n+1}}{n+1} - (v')^n \right]_r = \frac{n-1}{n+1} + r^n \left( 1 - 2r \frac{n}{n+1} \right)$$

When $r = 0$, this reduces to $\frac{n-1}{n+1}$. In the optimal auction, $r = 1/2$, this is $\frac{n-1}{n+1} + \frac{r^n}{n+1}$. (These formulae apply as well when $r = 1$, in which case $\frac{r^n}{n+1}$ is a monopolist’s expected revenue.)

Some observations on this example. (A) The reserve price is the same regardless of the number of bidders. (B) The reserve price is the same as the profit-maximizing monopoly price in a world in which a monopolist faces a single buyer whose willingness to pay is distributed $U[0, 1]$. Similar observations continue to hold regardless of the distribution of bidder values. Interested students: think about why.

**Reserve price or Extra bidder?** Bulow and Klemperer (1996) raises an interesting question. Would a seller prefer to attract another buyer with no reserve price, or acquire the ability to set an optimal reserve price without changing the set of buyers? In the uniform example, this amounts to comparing expected revenue terms $\frac{n-1}{n+1} \geq \frac{n-2}{n} + \left( \frac{1}{2} \right)^{n-1}$. Multiplying both sides by $n(n+1)$ yields $n^2 - n \geq n^2 - n - 2 + (n+1)(1/2)^{n-1}$. Since $(n+1)(1/2)^{n-1} < 2$ for all $n$, we conclude that for the uniform case that the seller would always prefer the additional buyer. Bulow and Klemperer (1996) establish this result more generally.

**Readings.**
