

# Empirical Work on Auctions of Multiple Objects

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## Abstract

Abundant data has led to new opportunities for empirical auctions research in recent years, with much of the newest work on auctions of multiple objects, including: (i) auctions of ranked objects (such as sponsored-search ads), (ii) auctions of identical objects (such as Treasury bonds), and (iii) auctions of dissimilar objects (such as FCC spectrum licenses). This paper surveys recent developments in the empirical analysis of such auctions.

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# 1 Introduction

Auctions are ancient economic institutions,<sup>1</sup> but economists' influence over real-world auction design is relatively new. In 1992, Milton Friedman's long-time advocacy of the uniform-price auction to sell government securities influenced the U.S. Treasury to experiment with and ultimately switch to that auction format. Then, in 1994, the Federal Communications Commission held its first spectrum auction based on designs proposed and developed by Preston McAfee, Paul Milgrom, and Robert Wilson, among others.

Auctions have since become a popular policy tool for governments and businesses to buy, sell, or trade all manner of scarce resources, including: automobile license plates (Chen and Zhao (2013)); business procurement (discussed later); electricity procurement (discussed later); emissions rights (Lopomo et al (2011)); financial securities (discussed later); import quota licenses (McCorriston (1996)); sponsored-search advertisements (discussed later); spectrum rights (discussed later); water rights (Hartwell and Aylward (2007)); and many more.

For single-object auctions, a rich theoretical literature provides conditions under which the revenue performance of commonly-used auction mechanisms can be unambiguously ranked. For instance, in Milgrom and Weber (1982)'s "mineral-rights model," the English auction is revenue-superior to the second-price auction, which is itself revenue-superior to the first-price auction. Unfortunately, few such clear-cut theoretical results exist in the multiple-object auction literature. For instance, Ausubel et al (2014) show that the two most commonly-used multi-unit auctions (the "uniform-price auction" and "pay-as-bid auction") cannot be ranked on either efficiency or revenue grounds. The

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<sup>1</sup>Cassady (1967) documents how auctions were held in ancient Rome "in what was called an atrium auctionarium, where samples of the goods were probably displayed prior to the sale", as well as in ancient Babylon and ancient China.

question of which auction format to adopt in any given application is therefore one that can often only be answered empirically.

The purpose of this survey is to highlight recent developments in the empirical analysis of data generated by multiple-object auctions. Natural or designed experiments are the most straightforward way to compare the performance of different auction mechanisms but, in most application areas,<sup>2</sup> such experiments are rarely if ever conducted. Moreover, some important questions, such as the efficiency of the allocation achieved by an auction, cannot be answered coherently without first specifying an internally-consistent economic model. Most papers in the recent literature have therefore taken a “structural” approach to analyze multiple-object auction data.

The aim of such structural empirical analysis is to recover the structural parameters, e.g. the joint distribution of bidders’ valuations, of an economic model that rationalizes observed bidder behavior. Of course, this structural approach is only useful from a policy evaluation and formulation perspective to the extent that it is based on an accurate model of the game that is being played and of bidder behavior in that game. An attractive feature of auctions is that they have clearly specified rules, largely eliminating the concern that the game itself has been misspecified. Moreover, because the participants in many multiple-object auction markets are highly experienced, profit-oriented professionals with significant payoffs at stake (who sometimes employ auction theorists to assist with their bidding decisions), the standard behavioral assumption<sup>3</sup> that bids are chosen optimally

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<sup>2</sup>In Internet applications, especially advertising, auction-design experimentation is relatively common. See e.g. Ostrovsky and Schwarz (2009), discussed more later, who used randomized controlled trials to evaluate Yahoo!’s reserve prices in its sponsored-search auctions.

<sup>3</sup>Important exceptions exist in which the “standard” behavioral assumptions are not imposed. For example, Haile and Tamer’s (2003) incomplete model of bidding encompasses non-equilibrium behavior, and Bajari and Hortaçsu (2005), Gillen (2015), and Fox and Bajari (2013) have investigated the use

to maximize bidders' expected payoff (i.e. that bids are generated in Bayesian Nash equilibrium) may be reasonable in these applications.

To a casual observer, the empirical literature on multiple-object auctions can appear fragmented. For instance, Google's sponsored-search auctions may seem to have little in common with the FCC's spectrum auctions or the U.S. Treasury's bond auctions. In fact, there are deep thematic connections among these disparate applications. Our goal in this review is to establish these connections for readers, while also highlighting the unique empirical challenges and opportunities that arise in several specific multiple-object auction environments that have been studied in the literature. Bearing that in mind, we focus on relatively few papers that we feel best illustrate the growth and remaining challenges facing the field, and urge readers to consult other recent surveys (e.g. Hortaçsu (2011), Bajo-Buenestado (2014)) for additional references.

Section 2 begins with a basic review and discussion of the main empirical approaches that have been developed to study single-object auctions. The rest of the paper then surveys progress in each of the three main sorts of multiple-object auction environments that have been studied in the literature: auctions of rankable objects when bidders want at most one object (Section 3); auctions of identical objects (or "units") when bidders have multi-unit demand, also known as "multi-unit auctions" (Section 4); and auctions of dissimilar objects when bidders have multi-object demand, also known as "package auctions" (Section 5).

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of alternative solution concepts such as quantal response equilibrium, cognitive hierarchy models, or stability to study auction data.

## 2 Background: Auctions of a Single Object

Recent advances in the empirical study of multiple-object auctions build on methods developed to estimate bidder values in single-object auctions, especially the penetrating insight of Guerre, Perrigne and Vuong (2000) (GPV). This section provides some background on this methodological foundation, focusing on the first-price auction and on ascending-price auctions in which bidders are risk-neutral with i.i.d. private values. For more in-depth discussion of the empirical single-object auction literature, please see the excellent treatments of this topic in, for instance, Athey and Haile (2007), Hendricks and Porter (2007), Paarsch and Hong (2006), and Hong and Shum (2003).

**First-price auction.** Given i.i.d. private values drawn from c.d.f.  $F(\cdot)$  and p.d.f.  $f(\cdot)$ , the first-price auction has a unique equilibrium (Lebrun (2006)), in which each bidder adopts a symmetric bidding strategy:

$$b^{FP}(v_i) = v_i - \frac{\int_0^{v_i} F^{N-1}(x)dx}{F^{N-1}(v_i)}. \quad (1)$$

Let  $G_{-i}^{FP}(\cdot)$  and  $g_{-i}^{FP}(\cdot)$  denote the c.d.f. and p.d.f. of the distribution of  $b_{-i} = \max_{j \neq i} b_j$  when all bidders adopt equilibrium bidding strategies.

**Proposition 1.** *Suppose that bids are generated in the unique equilibrium of the first-price auction. Bidder  $i$ 's value when bidding  $b_i$  is  $v_i = v^{FP}(b_i)$ :*

$$v^{FP}(b_i) = b_i + \frac{G_{-i}^{FP}(b_i)}{g_{-i}^{FP}(b_i)}. \quad (2)$$

*Proof.* Let  $\Pi^{FP}(v_i, b_i) = (v_i - b_i)G_{-i}^{FP}(b_i)$  be bidder  $i$ 's interim expected surplus in the first-price auction. Condition (2) follows immediately from the first-order condition of bidder  $i$ 's optimization problem, i.e. that  $d[\Pi(v_i, b_i)]/db_i = (v_i - b_i)g_{-i}^{FP}(b_i) - G_{-i}^{FP}(b_i) = 0$ . □

**Estimation issues.** Condition (2) allows bidder values to be estimated from bid data, in much the same way that monopoly cost can be estimated from price and demand data. When monopoly cost and buyer demand are each stochastic and independently distributed, Rosse (1970) shows how to estimate the distribution of monopoly cost using a simple two-stage process: first, estimate the distribution of demand by observing realized demand at each price; then, estimate monopoly cost using the first-order condition associated with profit-maximization given that distribution of demand, i.e. by inverting the standard monopoly markup formula.

Given bids that are independently distributed in a first-price auction, GPV (2000) showed how to estimate the distribution of bidder values using a similar two-stage process: first, estimate the distribution  $G_{-i}^{FP}(\cdot)$  of the highest bid submitted by other bidders; then, estimate bidder value using condition (2). Since it captures how much bidder  $i$  should optimally shade his bid given the distribution of others' bids, condition (2) is referred to as bidder  $i$ 's "markdown equation."

More specifically, GPV propose a nonparametric estimation method that estimates  $G_{-i}^{FP}(\cdot)$  and  $g_{-i}^{FP}(b_i)$  utilizing kernel methods. The method can be utilized to account for observed auction-level heterogeneity, though a fully nonparametric approach requires the auction-level covariates to be few in number due to the "curse-of-dimensionality problem." For this reason, many authors in this literature adopt a semi-parametric approach, where the auction-level covariates have a linear index structure in determining the mean of the valuations, with the residual allowed to have a nonparametric specification. In this framework, bids are first regressed on auction covariates and the residuals are utilized in the nonparametric specification.<sup>4</sup> Another possibility is to use a parametric bid distri-

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<sup>4</sup>Another important challenge faced in the nonparametric approach is the fact that estimates are quite sensitive to the tails of the bid density  $g_{-i}^{FP}(b_i)$ , which leads to an asymptotic bias. To address

bution/density for estimation, with some empirical motivation as to the appropriateness of the parametric restriction; see e.g. Athey, Levin, Seira (2011).<sup>5</sup>

The GPV insight has been extended in many different ways to incorporate features of the economic environment. Notable extensions include the case of affiliated private values (Li, Perrigne, Vuong (2002)), unobserved auction-level heterogeneity (Li, Perrigne, Vuong (2000), Krasnokutskaya (2011), Hu, McAdams and Shum (2013)), interdependent valuations (Somaini (2013)) and dynamic interlinkages between auctions (Jofre-Bonet and Pesendorfer (2003)). Some of these approaches extend in a straightforward way to multiple-object auction environments, see e.g. our discussion of unobserved heterogeneity in Section 3, but it’s worth noting that this is still an open research frontier.

**Ascending-price auctions: button auction.** The simplest sort of ascending-price auction is the so-called “button auction,” in which the price rises continuously and each bidder decides when to permanently exit the bidding, with the winning bidder paying the price at which the last other bidder dropped out. Given private values, each bidder’s weakly dominant strategy is the “truthful strategy” of remaining in the bidding until the price reaches his value. When such truthful strategies are played, the distribution of bidder values can be identified from the distribution of observed drop-out prices in a straightforward way, even if all that one observes is the final price paid by the winner.

**Proposition 2.** *Suppose that bidders adopt truthful strategies in the button auction. The distribution of bidder values is identified from the distribution of the final price.*

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this, GPV propose a trimming procedure to prevent undue influence from tail observations.

<sup>5</sup>The GPV approach to estimating first-price auctions has been very popular as alternative approaches, such as maximum-likelihood methods, are computationally intensive and have non-standard asymptotic properties. See Donald and Paarsch (1993), or, for Bayesian approaches following Hirano and Porter (2003), Bajari (1998) and Sareen (2000).

*Proof.* Let  $G_p^B(\cdot)$  denote the c.d.f. of the final price in a button auction. Given truthful bidding, the final price is equal to the second-highest of  $N$  i.i.d. private values drawn from c.d.f.  $F(\cdot)$ . In particular,  $G_p^B(b) = NF(b)^{N-1}(1 - F(b)) + F(b)^N$ , from which the c.d.f. of bidder values can be directly identified.  $\square$

**Ascending-price auctions: open-outcry auction.** Button auctions are used in some real-world applications, such as in Japanese fish markets, but many ascending-price auctions are more free-wheeling affairs in which bidders have the option to “jump the bid” with any offer exceeding the current high bid plus a minimum bid increment  $\epsilon > 0$ , and those who are not active early in the auction may nonetheless have high values. What can we infer about bidder values from observed bidding activity in such auctions, in which the “rules of the game” may not be entirely clear, perhaps not even to those participating?

Haile and Tamer (2003, hereafter “HT”) address this challenge by identifying plausible a priori restrictions on the data-generating process that can then be used to *bound* what bidder values could have been to generate observed bidding behavior. In particular, in the context of open-outcry auctions, HT assume that (i) no bidder  $i$  ever calls out a bid greater than its own value and (ii) no losing bidder ever allows the auction to conclude when it could have kept the auction going with a bid less than its own value. For ease of reference, we will call any bidder strategy satisfying these conditions a “no-regret strategy.”

**Proposition 3.** *Suppose that bidders adopt no-regret strategies in an open-outcry auction. Upper and lower bounds on the distribution of bidder values can be identified from the distribution of the final price.*

*Proof.* Let  $G_p^{OO}(\cdot)$  denote the c.d.f. of the final price  $p$  in an open-outcry auction. When

all bidders adopt no-regret strategies, the highest bidder value  $v^{(1)}$  must be greater than or equal to  $p$ , while the second-highest bidder value  $v^{(2)}$  must be less than or equal to  $p + \epsilon$ .  $v^{(1)} \geq p$  implies  $F(b)^N \leq G_p^{OO}(b)$ , providing a lower bound on  $F(\cdot)$ . Second,  $v^{(2)} < p + \epsilon$  implies  $NF(b + \epsilon)(1 - F(b + \epsilon))^{N-1} + F(b + \epsilon)^N \geq G_p^{OO}(b)$ , providing an upper bound on  $F(\cdot)$ . Note that, when  $\epsilon = 0$ , the upper bound on  $F(\cdot)$  derived here coincides with the distribution of bidder values (derived in the proof of Proposition 2) that would be point-identified from the same distribution of final prices, if generated in a button auction under truthful bidding.  $\square$

**First-price vs open-outcry auctions.** Estimation approaches for the first-price auction based on Proposition 1 are very different from those for ascending-price auctions based on Propositions 2-3, and with good reason. The first-price auction is a sealed-bidding game in which bidders face uncertainty about what price they need to bid to win, while in ascending-price auctions bidders can observe the progress of the auction and make choices, often up until the very last moment, to influence the realized outcome. This difference reflects a deep distinction between auctions in which bidders play a best response to the distribution of others' bids and those in which bidders can be interpreted as making an optimal choice given others' actual bidding strategies.

The first-price auction with i.i.d. private values is an example of the first sort, in which bidders choose bids to maximize their expected payoffs, with first-order condition (2) identifying bidder values. As we will see, this “first-order condition approach” has been applied to a wide variety of multiple-object auction environments, from sponsored search auctions (Section 3) to Treasury bond sales (Section 4) and multi-object procurement (Section 5).

The open-outcry auction is an example of the second sort. Even though the space of

possible strategies is complex and it is difficult to characterize best response strategies, we can interpret each bid that bidder  $i$  makes (or does not make) as a choice that influences potential auction outcomes and thereby reveals bidder preference for some outcomes over others. In particular, choosing not to bid and allowing the auction to end at price  $p$  guarantees that bidder  $i$  wins nothing and pays nothing, whereas announcing a higher price at least keeps open the possibility of winning the object at that higher price. If bidder  $i$  has allowed the auction to end at price  $p$  and was playing a best response, then, bidder  $i$ 's value must be less than  $p$ . As we will see, variations on this “revealed preference approach” have been applied to multiple-object auction environments as well, from sponsored search (Section 3) to FCC spectrum auctions (Section 5).

### **3 Auctions of rankable objects**

One of the most exciting recent multi-object auction applications is in the context of Internet advertising, revenues from which has reached \$13.3 billion in the first quarter of 2015, and is forecasted to continue its steady growth (PriceWaterhouseCoopers (2015)). As in traditional media advertising, Internet search engines or content providers preserve space on their content page for advertisers, and advertisers bid for the most desirable spots. This setting naturally invokes a multi-object auction model in which bidders have single-object demand and the items for sale (ad spots on the content page) can be ranked in terms of quality.

#### **3.1 Empirical content of rankable-object auction models**

Following the leading models in this literature (e.g. Varian (2007), Edelman, Ostrovsky and Schwarz (2007)), suppose that  $K$  ranked objects are for sale to  $N$  bidders, each of

whom has private values  $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,K})$  where  $v_{i,1} \geq \dots \geq v_{i,K}$  and  $\mathbf{v}_i$  are i.i.d. across bidders. To simplify the exposition, we will mostly focus on the special case in which  $v_{i,k} = \alpha_k v_i$ , where  $\alpha_1 \geq \dots \geq \alpha_K$  are constants known to the econometrician.

We will consider the two most commonly-studied ranked-object auction formats, the generalized first-price auction and the generalized second-price auction, as well as the Ausubel-Milgrom auction, an ascending-price format with several desirable theoretical properties.

### 3.1.1 Generalized first-price auction

In 1997, Overture (since acquired by Yahoo!) became the first search engine to use an auction to sell advertisements to appear next to its search results, adopting what is referred to as the “generalized first-price auction (GFPA).”

The GFPA is a sealed-bid auction in which each bidder submits a single bid  $b_i$ , and the  $k$ -th highest bidder is assigned object  $k$  at price  $\alpha_k b_i$ . When only one object is sold, the GFPA reduces to a standard first-price auction. Let  $G_{-i;k}^{GFP}(\cdot)$  and  $g_{-i;k}^{GFP}(\cdot)$  be the c.d.f. and p.d.f. of the  $k$ -th highest bid submitted by others than bidder  $i$ . Equilibrium markdown equations akin to condition (2) in the first-price auction identify bidder values from the distribution of bids.

**Proposition 4.** *Suppose that bids are generated in equilibrium of the generalized first-price auction. Bidder  $i$ 's value when bidding  $b_i$  is  $v_i = v^{GFP}(b_i)$ :*

$$v^{GFP}(b_i) = b_i + \frac{\sum_{k=1}^K (\alpha_k - \alpha_{k+1}) G_{-i;k}^{GFP}(b_i)}{\sum_{k=1}^K (\alpha_k - \alpha_{k+1}) g_{-i;k}^{GFP}(b_i)} \quad (3)$$

*Proof.* Bidders' interim expected surplus in the generalized first-price auction takes the

form:

$$\begin{aligned}
\Pi^{GFP}(v_i, b_i) &= (v_i - b_i) \sum_{k=1}^K \alpha_k \Pr(\text{win object } k \text{ when bidding } b_i) \\
&= (v_i - b_i) \left( \alpha_1 G_{-i,1}^{GFP}(b_i) + \sum_{k=2}^K \alpha_k (G_{-i,k}^{GFP}(b_i) - G_{-i,k-1}^{GFP}(b_i)) \right) \\
&= (v_i - b_i) \sum_{k=1}^K (\alpha_k - \alpha_{k+1}) G_{-i,k}^{GFP}(b_i) \tag{4}
\end{aligned}$$

(3) now follows from (4) by first-order condition  $d[\Pi^{GFP}(v^{GFP}(b_i), b_i)]/db_i = 0$ .  $\square$

**Unobserved heterogeneity.** First-order condition (3) identifies the distribution of  $v_i$  in the GFPA much as (2) identifies bidder values in the first-price auction. A key assumption when estimating the distributions  $G_{-i,k}^{GFP}(\cdot)$  is that bids are drawn i.i.d. across auctions, conditional on observable auction-level characteristics. In practice, of course, auctions may differ in ways that are unobserved by the econometrician. For instance, bidders' values in a sponsored search auction on the keyword "pizza" may be drawn from a different distribution on nights with major sporting events. If the econometrician does not control for such variation, estimates based on (3) will be inconsistent.

Fortunately, techniques developed to address this concern in first-price auctions can be easily extended to the GFPA. Suppose that one is willing to assume that the unobserved "state" of the auction is one-dimensional and has a multiplicative effect on bidder values, i.e. that bidder  $i$ 's value in auction  $t$  takes the form  $v_i^t = z^t w_{i,t}$  where  $(w_{i,t}, z^t)$  is i.i.d. across auctions and  $w_{i,t}$  and  $z^t$  are themselves independent. (Given this assumption on values, equilibrium bids take the same multiplicative form.) Li, Perrigne, and Vuong (2000) identifies the distribution of  $(z^t, w_{i,t})$  when neither bidders nor the econometrician observe  $z^t$  ("conditionally i.i.d. private values"), while Krasnokutskaya (2011) identifies the distribution of  $(z^t, w_{i,t})$  in the case when bidders observe  $z^t$  but the econometrician

does not (“unobserved heterogeneity”). The key idea of this literature is to view bids as measurements of the unobserved state, with measurement error due to the unobserved realization of bidder values. These papers leverage classical measurement error results in Kotlarski (1966) and Li and Vuong (1998), for which two measurements (i.e. two bids) are required to identify the joint distribution of the underlying state and of bidder values.

Hu, McAdams, and Shum (2013, hereafter HMS) extend these results to settings in which the state impacts the distribution of bids in a non-separable way, leveraging non-classical measurement error results in Hu (2008). HMS replace the assumption that the distribution of bids is multiplicative in the state with a weaker assumption that some functional of the distribution of bids is monotone in the state, at the cost of requiring an additional measurement, i.e. three bids rather than two. This allows one not only to handle richer ways in which the state can impact bidder values, but other sorts of auction heterogeneity that affect equilibrium bids without necessarily affecting bidder values.

In the context of sponsored search auctions, unobserved heterogeneity could be important for a number of reasons. For instance, returning to our pizza-keyword example, even if pizza restaurants’ value for clicks did not change during major sporting events, consumers might be relatively more likely to click on more prominently-placed ads, giving bidders an incentive to bid more aggressively. If so and equilibrium bids are in fact drawn from a higher distribution during sporting events, one could apply HMS to identify (i) the fraction of auctions in the observed sample which occurred during major sporting events, and (ii) the distribution of bidder values conditional on their being a major sporting event or not. Among other things, this would allow one to test whether in fact pizza restaurants’ value for clicks is the same during major sporting events as at other times, without needing to know when such events took place.

### 3.1.2 Generalized second-price auction

Since 2002, Google has used a “generalized second-price auction (GSPA)” to sell advertisements that appear next to its search results. By 2012, Google hosted over 5.1 billion searches per day; the corresponding sponsored search auctions generated \$43.7 billion for Google that year, 95% of its total revenue.<sup>6</sup>

The GSPA is a sealed-bid auction in which each bidder submits a single bid  $b_i$ , and the  $k$ -th highest bidder is assigned object  $k$  at price  $\alpha_k$  times the  $(k+1)$ -highest bid submitted in the auction. The GSPA reduces to a standard second-price auction when only one object is sold but, when  $K > 1$ , truthful bidding is not a best response. Markdown equations akin to (2) are once again needed to identify bidder values from observed bids.

**Proposition 5.** *Suppose that bids are generated in an equilibrium of the generalized second-price auction. Bidder  $i$ 's value when bidding  $b_i$  is  $v_i = v_i^{GSP}(b_i)$ :*

$$v_i^{GSP}(b_i) = b_i + \frac{\sum_{k=1}^K g_{-i,k}^{GSP}(b_i) \alpha_{k+1} E[b_i - b_{-i,k+1} | b_{-i,k} = b_i]}{\sum_{k=1}^K g_{-i,k}^{GSP}(b_i) (\alpha_k - \alpha_{k+1})} \quad (5)$$

*Proof.* Raising bid  $b_i$  on the margin affects bidder  $i$ 's realized payoff only when bidder  $i$  is “tied” to win an object, i.e. when  $b_i = b_{-i,k}$  for some  $k = 1, \dots, K$ , in which case there are two effects: (i) bidder  $i$  wins object  $k$  instead of  $k+1$  for extra value  $(\alpha_k - \alpha_{k+1})v_i$  and (ii) bidder  $i$  pays  $\alpha_k b_i$  instead of  $\alpha_{k+1} b_{-i,k+1}$ . Overall, then, bidder  $i$  can only be indifferent to marginally raising his bid given value  $v_i^{GSP}(b_i)$  such that

$$\sum_{k=1}^K g_{-i,k}^{GSP}(b_i) ((\alpha_k - \alpha_{k+1})(v_i^{GSP}(b_i) - b_i) - \alpha_{k+1}(b_i - E[b_{-i,k+1} | b_{-i,k} = b_i])) = 0$$

which can be re-arranged to yield (5). □

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<sup>6</sup>[www.statisticbrain.com/google-searches](http://www.statisticbrain.com/google-searches), accessed October 24, 2013.

### 3.1.3 Simultaneous ascending-bid auction

In a simultaneous ascending-price auction, the price of each object rises continuously whenever two or more bidders are actively bidding on it and the auction ends once only one bidder remains active on each object.<sup>7</sup> (Someone who becomes inactive on an object may become active on it later, as the prices of other objects change.) Milgrom (2000) shows that, as long as bidders choose throughout the auction to be active on whichever object(s) give them the most surplus at current prices, the final prices and final allocation will coincide with the (efficient) Vickrey-auction prices and allocation. Moreover, such “naive bidding” constitutes an equilibrium of the game whenever the objects being sold are substitutes, as they always are when bidders have single-object demand. We focus in this section on the case when bidders want at most one object, and later in Section 5.3 on the case when bidders may want packages.

What can we infer about bidder values  $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,K})$  from auction outcomes, i.e. that bidder  $i_k$  won object  $k$  at price  $p_k$ ? If we are willing to assume that the naive-bidding equilibrium is played, revealed preference implies that each bidder must get more surplus from the assigned object than any other object at the final prices. Consequently, for all  $k = 1, \dots, K$ , (i) bidder  $i = i_k$  must have values satisfying  $v_{i,k} - p_k \geq \max_{k'} \{v_{i,k'} - p_{k'}\}$  and  $v_{i,k} - p_k \geq 0$  and (ii) any bidder  $i \notin \{i_1, \dots, i_K\}$  must have values satisfying  $v_{i,k} - p_k \leq 0$  for all  $k$ . These inequality conditions bound the distribution of

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<sup>7</sup> Ausubel and Milgrom (2002) extend the simultaneous ascending-price auction to a more general environment in which bidders are able to bid on all possible subsets of the objects for sale. Kagel, Lien, and Milgrom (2010) have critiqued the “Ausubel-Milgrom auction” in this package-bidding context, on the basis of experimental findings that bidders do not adopt naive bidding strategies in practice and that “aggressive bidding tactics” can lead to substantial inefficiencies. See also Bulow, Levin, and Milgrom (2009), who describe how a telecommunications company saved over a billion dollars in an FCC spectrum auction by *not* adopting a naive-bidding strategy in an ascending-bid package auction.

bidder values, providing a partial identification. However, if one is willing to assume that  $v_{i,k} = \alpha_k v_i$ , the distribution of bidder values is actually point-identified. (To the best of our knowledge, this is a new result. See Fox and Bajari (2013), Appendix B, for a related discussion.)

**Proposition 6.** *Suppose that bids are generated in the naive-bidding equilibrium of the Ausubel-Milgrom auction, and that bidders have i.i.d. private values  $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,K})$  such that  $v_{i,k} = \alpha_k v_i$  where  $\alpha_1, \dots, \alpha_K$  are constants. Then the distribution of  $\mathbf{v}_i$  is point-identified from the distribution of final prices  $\mathbf{p} = (p_1, \dots, p_K)$ .*

*Proof.* When bidder values take the form  $v_{i,k} = \alpha_k v_i$ , naive bidding results in an assortative matching in which higher-quality objects are assigned to higher-value bidders, namely,  $v_{i_k,k} = \alpha_k v^{(k)}$  where  $v^{(k)}$  is the  $k$ -th highest bidder value. Moreover, for all  $k = 2, \dots, K$ , bidder  $i_k$  must be indifferent between receiving object  $k$  at price  $p_k$  or receiving the next-better object  $k - 1$  at price  $p_{k-1}$ , i.e.  $(\alpha_{k-1} - \alpha_k)v^{(k)} = p_{k-1} - p_k$ , while the highest-value bidder who does not win is indifferent between receiving object  $K$  at price  $p_K$  or nothing at all, i.e.  $\alpha_K v^{(K+1)} = p_K$ .

Note that  $\alpha_K v^{(K+1)} = p_K$  is identified from  $\mathbf{p}$  and, for notational convenience, normalize  $\alpha_K = 1$ . Given our assumption that  $v_i$  are i.i.d., the distribution of  $v_i$  and hence of every order statistic  $(v^{(1)}, \dots, v^{(N)})$  is identified from the distribution of  $v^{(K+1)}$ . Next, observe that  $\alpha_{k-1} - \alpha_k = \frac{p_{k-1} - p_k}{v^{(k)}}$  for all  $k = 2, \dots, K$ , identifying the constants  $\alpha_1, \dots, \alpha_{K-1}$  given the normalization  $\alpha_K = 1$ . Overall, then, the distribution of  $\mathbf{v}_i = (\alpha_1 v_i, \dots, \alpha_K v_i)$  is identified. □

### 3.2 Applications and empirical challenges

The most well-studied ranked-object auctions are those used in sponsored search.<sup>8</sup> The sponsored advertisements placed next to the search results on sites like Google and Bing are assigned in so-called “sponsored-search auctions.” Sponsored-search ads are ranked objects, as all advertisers naturally prefer to be placed in a higher, more prominent slot. If ads placed in the  $k$ -th position are clicked with probability  $\alpha_k$  and each click is worth  $v_i$  to bidder  $i$ , then bidder  $i$ ’s values take the form  $v_{i,k} = \alpha_k v_i$ .<sup>9</sup>

Ostrovsky and Schwarz (2009) (OS) analyzed a sample consisting of bids in generalized second-price auctions associated with all Yahoo! search auctions for over 400,000 keywords over several weeks in 2008, using the markdown equation (5) to estimate the distribution of bidder values for each keyword. Using the estimated distribution of bidder values, they then compute the optimal reserve price for each auction. The optimal reserve price was found to exceed the actual reserve price for 90% of the keywords considered. (Yahoo! set a uniform reserve price of 10 cents for each keyword.)

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<sup>8</sup> Ranked-object auctions have been proposed in a variety of other applications, such as the allocation of priority of service in on-demand computing (Haviv and van der Waal (1997)) and the assignment of airport landing and takeoff runway-slots (Ball, Donohue, and Hoffman (2006)). For more on the theory of priority-of-service auctions, also known as “queueing auctions,” see Glazer and Hassin (1986), Lui (1985), Kittsteiner and Moldavanu (2005), and Afeche and Mendelson (2004).

<sup>9</sup> Athey and Ellison (2011) develops a directed-search model that endogenizes click-through rates, in which consumers decide which ads to click until they find an ad that meets their need. The position of an ad serves in equilibrium as a credible signal that the ad is more likely to meet the consumer’s need, leading consumers to click first (and hence most frequently) on ads in higher positions. An emerging consensus in the empirical literature, however, is that the value per click can depend on ad placement (Borgers et al (2013)) in ways that vary across bidders (Goldman and Rao (2014), McDevitt (2014)) and/or that higher-placed ads may sometimes be less likely to be clicked (Hsieh et al (2015), Jerath et al (2011), Amaldoss et al (2015)).

OS then ran an experiment to test their “theoretically optimal” reserve prices. To do this, they used 438,632 keywords as the treatment group, to which the “optimal” reserve prices were assigned, and 23,016 keywords as the control group, which used the 10 cent uniform reserve price. The effect of the optimal reserve price treatment was assessed using a difference-in-difference strategy. While the optimal reserve prices increased average revenue for the treatment group compared to the control group by 13%, OS find the result to be marginally statistically significant, due to the large heterogeneity in search volume across keywords. Computing instead the average revenue per search, OS find that the new reserve prices led to an increase of 2.7%, with very high statistical significance ( $p < 0.0001$ ).

Ostrovsky and Schwarz (2009) highlights the way in which structural empirical analysis naturally complements randomized-controlled experimentation over different auction designs. OS could have used a purely experimental approach to determine the optimal reserve price for each keyword, but such an approach would have involved running many experiments across different reserve price levels to pin down the “empirically-optimal” reserve price. By first conducting a structural analysis on existing non-experimental data to “approximate” the optimal reserve, OS could then design their subsequent experiments to focus on reserve price levels that the structural analysis suggested were likely to raise revenue for Yahoo!<sup>10</sup>

A key assumption underlying Ostrovsky and Schwarz (2009)’s structural analysis is that the distribution of bids is statistically independent across auctions (conditional on

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<sup>10</sup>In a 2008 conference call with investors, Yahoo! President Sue Decker remarked that Yahoo! had adopted a new method for setting reserve prices in its auctions and that this so-called “Market Reserve Pricing,” based on Ostrovsky and Schwarz’s research, was “probably the most significant [factor] in terms of its impact in the quarter. We had a full quarter impact of that in Q3, but we still have the benefit of rolling that around the world.”

observable auction covariates), allowing them to estimate the distribution of bids in any given auction from the empirical distribution of bids across all auctions. In the context of sponsored search, however, each bidder’s willingness to pay for placement in a particular keyword search will naturally be positively correlated across auctions, as the value of clicks changes slowly over time. Moreover, each bidder can observe how often its own bid has won in prior auctions. For this reason, most theoretical and empirical work has modeled sponsored-search auctions as being complete-information games in which bidders all know each others’ values, rather than auctions with i.i.d. private values.

Lahaie (2006), Varian (2007), and Edelman, Ostovsky, and Schwarz (2007) developed the theory of equilibrium bidding in the GSPA under complete information. Suppose that bids  $b_1 \geq b_2 \dots \geq b_N$  are made, so that bidder  $i = 1, \dots, K$  wins object  $i$ . In equilibrium, bidders must prefer not to deviate to win another object. Note that bidder  $i$  can win any lower-ranked object  $j > i$  with a bid slightly higher than  $b_{j+1}$  but, to win any higher-ranked object  $j < i$ , he needs to outbid the current winner of object  $j$  with a bid above  $b_j$ . For winning bids  $(b_1, \dots, b_K)$  to arise in pure-strategy equilibrium, then, it must be that, for each  $i = 1, \dots, K$ ,

$$v_{i,i} - \alpha_i b_{i+1} \geq v_{i,j} - \alpha_j b_{j+1} \text{ for all } j > i \tag{6}$$

$$\geq v_{i,j} - \alpha_j b_j \text{ for all } j < i \tag{7}$$

Conversely, when bidders’ values satisfy these inequality conditions, a pure-strategy equilibrium exists that generates these bids.

The multitude of equilibria in this context presents an obvious challenge for theoretical and empirical research. To address this concern, much of the literature has focused on an equilibrium selection in which a stronger version of inequality (7) holds:

$$v_{i,i} - \alpha_i b_{i+1} \geq v_{i,j} - \alpha_j b_{j+1} \text{ for all } j < i \tag{8}$$

(Equilibria satisfying (6,8) are referred to as “symmetric equilibria.”<sup>11</sup>)

Hsieh, Shum, and Yang (2015, hereafter HSY) analyze GSPA sponsored-search auctions conducted by “the largest online marketplace in China” for one month in 2010, under an assumption that bidders have complete information and play a symmetric equilibrium. HSY leverage the inequality conditions (6,8) to estimate bidder values, then conduct a counterfactual analysis of how equilibrium outcomes would change with the introduction of “bid scoring,” in which ads that are more likely to be clicked are scored more favorably. A key finding is that “while scoring the auction grants price discounts to popular merchants ... scoring also heightens the bid competition, thus leading to higher prices for top positions. Overall, we do not find large effects on platform revenue and sorting patterns from shifting to a scoring rule.”

Borgers, et al (2013, hereafter BCPP) analyze GSPA data from the search-engine Overture during one month in 2004, under the assumption that bids are generated in a pure-strategy Nash equilibrium but not necessarily a symmetric equilibrium, leveraging the inequality conditions (6,7). Key empirical findings include: advertisers’ willingness to pay for clicks is decreasing in ad position; an alternative auction format, such as the Ausubel-Milgrom auction, could have avoided significant welfare losses; and a fully rent-extracting mechanism would have raised 49% more revenue.

BCPP also develop a novel approach to test the joint hypothesis that (i) bids are generated in equilibrium and (ii) bidder  $i$ ’s values are constant across some sample of auctions. If so, bidder  $i$ ’s values must satisfy the inequality conditions (6,7) derived from realized bids in each of these auctions. If the set of bidder values satisfying all of these

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<sup>11</sup> The terminology here is somewhat inapt, as bidders’ strategies need not be symmetric in a “symmetric equilibrium” nor need they be asymmetric in an “asymmetric equilibrium.” (The only “symmetry” of these equilibria is in the inequality constraints (6,8), which take the same form for lower- and higher-ranked objects.)

inequality constraints is empty, we can then conclude that *either* bidder  $i$ 's values were changing from auction to auction within the sample *or* bidder  $i$  was not always playing a best response. BCPP find that violations of this joint hypothesis arise for bidders submitting numerous bids, with a sample-length period of 1.34 days (in which 4.7 bids are made) being on average enough to reach a violation.

The fact that bidder values are persistent but do change over time is just one of several aspects of sponsored-search auctions that create novel conceptual challenges. Another is that, while bidders can observe information about auction outcomes, search queries may arrive too quickly for bidders to then adjust their bids prior to the next auction.<sup>12</sup> In a significant recent development, Athey and Nekipelov (2012, hereafter AN) show that introducing some of these complexities into the theoretical model that one brings to the data can strengthen (and even simplify) the resulting empirical analysis.

AN consider a model in which search queries arrive more quickly than advertisers can change their bids, but in which bidders face uncertainty about the “quality scores” that the search engine uses when ranking bids. They provide a structural econometric model that point-identifies bidder values as long as there is sufficient uncertainty in the environment, along with sufficient conditions for existence and uniqueness of equilibrium. Applying this methodology to auctions of several search terms, AN reach striking conclusions about bidder profitability in these auctions. “Bidders earn substantial profits: values per click are between 40 and 90 percent higher than prices for low-ranked bidders, but are between 90 and 270 percent higher for high-ranked bidders.”

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<sup>12</sup> Yet another complication is that bids are often actually for bundles of searches. For instance, an advertiser who wins the search-term “Paris” will get searchers interested in everything from the European capital (Paris, France) to the reality-TV star (Paris Hilton) and the 1984 cult classic (Paris, Texas).

## 4 Auctions of identical objects (“multi-unit auctions”)

The most well-studied multiple-object auctions in the empirical literature are those known as “multi-unit auctions” in which  $Q$  identical units of a good are bought, sold, or traded. Multi-unit auctions are used in Treasury bill and bond sales, electricity procurement, emission-permit trading, and monetary infusions by central banks, among many other applications.<sup>13</sup> For analytical convenience, we focus on the case of perfectly divisible goods.<sup>14</sup>

### 4.1 Empirical content of multi-unit auction models

A “bid” in a multi-unit auction is a non-increasing demand schedule (or a non-decreasing supply schedule in the case of a procurement auction). The auctioneer aggregates these demand curves to determine the market-clearing price and assigns bidders their market-clearing quantities. Bidder payments depend on the rules of the auction, the two most popular being the *pay-as-bid auction*, also known as the “discriminatory” auction, and the *uniform price auction*, discussed more below.<sup>15</sup>

We will focus on an i.i.d. private-values version of Wilson (1979)’s framework. Each

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<sup>13</sup>For other recent surveys of these auctions, please see Hortaçsu (2011) and Bajo-Buenestado (2014).

<sup>14</sup>In most applications, quantity is discrete, each bidder submits a set of price-quantity pairs that defines a step-function demand curve and, when relevant, additional rules select among multiple market-clearing prices or among multiple market-clearing allocations. We abstract from such details for now, but will return later to discuss some of the novel empirical challenges that arise when bidders submit step-function bids.

<sup>15</sup>Bartolini and Cottarelli (1994) and Brenner et al. (2009) survey Treasury-bond applications of the pay-as-bid and uniform-price auction. Mazon and Nunez (1999) report that Spain uses a hybrid mechanism where winning bidders pay their bid if it is lower than the weighted average of winning bids, while other winning bids pay the market clearing price.

bidder  $i$  has a marginal valuation function  $v_i(\cdot; s_i)$  that is increasing in  $s_i \in \mathbb{R}$  and bidders' signals are drawn i.i.d. across bidders and auctions. (Bidder  $i$ 's total value for quantity  $q$  is  $\int_0^q v_i(x; s_i) dx$ .) In a Bayesian-Nash equilibrium (or simply "equilibrium"), bidders' strategies are bid functions,  $y_i(\cdot; s_i)$ , mapping each bidder's realized values into a submitted demand curve. Let  $b_i(q; s_i) = y_i^{-1}(q; s_i)$  be the corresponding inverse demand.

Others' bids and the realized supply determine bidder  $i$ 's residual supply,  $RS_i(p, s_{-i}) = Q - \sum_{j \neq i} y_j(p, s_j)$ , whose intersection with bidder  $i$ 's bid  $y_i(p, s_i)$  sets the market clearing price,  $p^c$ . For simplicity, suppose that bidders all submit strictly decreasing and differentiable demand schedules.<sup>16</sup> Given such bids, there is a unique market-clearing price  $p^c$  and a unique market-clearing allocation, in which bidder  $i$  receives quantity  $y_i(p; s_i)$ . Define

$$H_i(p, q) = \Pr\{RS_i(p, s_{-i}) \geq q\}. \quad (9)$$

$H_i(p, q)$  is the probability that bidder  $i$  wins (at least) quantity  $q$  when submitting a bid  $y_i(\cdot)$  such that  $y_i(p) = q$  (or, equivalently, a bid  $b_i(\cdot)$  such that  $b_i(q) = p$ ). Alternatively, since bidder  $i$  wins quantity  $q$  if and only if the market-clearing price  $p^c \leq b_i(q)$ ,  $H_i(b_i(q), q)$  is the probability that  $p^c \leq b_i(q)$  when bidder  $i$  has bid  $b_i(\cdot)$ . Note that, as long as all bids are strictly decreasing and differentiable, the partial derivatives  $\frac{\partial H_i(p, q)}{\partial p}$  and  $\frac{\partial H_i(p, q)}{\partial q}$  exist.

#### 4.1.1 Pay-as-bid auction.

In the pay-as-bid auction, bidder  $i$  pays the price that he bids on all quantity won, for total payment  $\int_0^{y_i(p^c, s_i)} b_i(q) dq$ . Expected payoffs in the pay-as-bid auction can be

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<sup>16</sup>Equilibrium bidding strategies may not be strictly decreasing or differentiable, in which case bidder values in the pay-as-bid and uniform-price auctions are only partially identified (McAdams (2008)).

expressed most conveniently in terms of inverse demand  $b_i(\cdot)$ :

$$\Pi_i^{PAB}(b_i(\cdot); s_i) = \int_0^Q (v_i(q; s_i) - b_i(q)) H_i(b_i(q), q) dq \quad (10)$$

The integrand in (10) is the expected payoff that bidder  $i$  enjoys from winning a  $q$ -th unit at price  $b_i(q)$ , which happens with probability  $H_i(b_i(q), q)$  when bidder  $i$  bids  $b_i(\cdot)$ , for all quantities  $q \in [0, Q]$ . Note that, because bidder  $i$ 's expected payoffs take this additively-separable form,  $b_i(\cdot)$  can only maximize bidder  $i$ 's overall expected payoff if each unit-bid  $b_i(q)$  maximizes  $(v_i(q; s_i) - b_i(q)) H_i(b_i(q), q)$ , yielding the system of first-order conditions (11) very similar to the first-order condition (2) in the first-price auction.

**Proposition 7.** *Suppose that bids are generated in an equilibrium of the pay-as-bid auction in which all bids are strictly decreasing<sup>17</sup> and differentiable. The following system of first-order conditions point-identifies bidder  $i$ 's marginal value  $v_i(q)$  from unit-bid  $b_i(q)$  for all quantities  $q$  that bidder  $i$  sometimes wins:*

$$v_i^{PAB}(q; s_i) = b_i(q) + \frac{H_i(b_i(q), q)}{\frac{\partial H_i(b_i(q), q)}{\partial p}} \quad (11)$$

*Proof.* The proof is omitted to save space. □

**Properties of equilibrium.** When bidders are risk-neutral with i.i.d. private values and there is no supply uncertainty, a monotone pure-strategy equilibrium (MPSE) exists in the pay-as-bid auction (Reny (1999)) and all mixed-strategy equilibria are outcome-equivalent to a MPSE (McAdams (2006)). If bidders are not risk-neutral, a monotone pure strategy equilibrium need not exist (McAdams (2007b)) but, if bidders exhibit constant absolute risk aversion, a pure-strategy equilibrium does (Reny (2011)). However, it remains an open question in these models whether equilibrium is unique. The best

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<sup>17</sup>McAdams (2008) extends this result to a setting in which equilibrium bids are not assumed to be strictly decreasing but may be subject to “strategic ironing” as in Woodward (2014).

uniqueness results currently available apply only to the case when bidders do not have private information and supply is random; see e.g. Pycia and Woodward (2015), which provides sufficient conditions on the distribution of supply given which the pay-as-bid auction has a unique equilibrium.

**Step function bids and partial identification.** The above discussion assumed that bidders submit strictly decreasing and differentiable demand schedules. In practice, however, bidders typically submit step-function bids (with fewer steps than they are allowed to use) in which they bid the same price over a range of quantities. This feature of bidding was noted in several papers including Hortaçsu (2000), McAdams (2008), Hortaçsu and McAdams (2010), Kastl (2011), and Cassola, Hortaçsu and Kastl (2013). These papers develop identification strategies based on deviations from best-response, where the deviations are constrained to be step-function bids. Such (constrained) deviations typically lead to partial identification of the underlying willingness-to-pay, leading to bounds on the rationalizing marginal valuations. McAdams (2008), in particular, obtains tight bounds for the pay-as-bid auction. Hortaçsu (2000) and Hortaçsu and McAdams (2010) discuss estimation of bounds on marginal valuations in the pay-as-bid auction context.

**Estimation issues.** If bidder  $i$  is “small” in the sense of having negligible impact on the market-clearing price, the distributions  $H_i(\cdot, q)$  will be (approximately) the same for all quantities that bidder  $i$  may win in the auction, and equal to the c.d.f. of the realized market-clearing price. In this case,  $H_i(\cdot, q)$  can be estimated from the empirical distribution of the market-clearing price, without needing to observe others’ bids, from which bidder  $i$ ’s values can be estimated using the first-order conditions (11). On the other hand, if the distributions  $H_i(\cdot, q)$  differ across quantities that bidder  $i$  could win in the auction, it is essential to observe others’ bids, to construct the empirical distribution

of residual supply from which estimates of  $H_i(\cdot, \cdot)$  and  $\frac{\partial H_i(\cdot, \cdot)}{\partial p}$  can be derived.

Two sorts of approaches have been developed to estimate the equilibrium distribution of residual supply, depending on the data that is available and the appropriateness of relevant economic assumptions. The first approach, following GPV (2000) and developed in Hortaçsu and McAdams (2010), is to exploit a panel dataset in which bids from multiple auctions are collected and the econometrician assumes that (conditional on auction-specific observables) bidder values are drawn from the same distribution and bidders play the same equilibrium in all auctions in the panel.

The second approach, proposed in Hortaçsu (2000), uses the empirical distribution of individual bids to “simulate” the distribution of residual supply from panel data. Alternatively, if one only has access to a short panel but there are numerous bids submitted in each auction, one can impose a symmetry assumption on bidders’ valuations within each given auction. Then, assuming a symmetric equilibrium is being played, one can use the empirical distribution of individual bids to “resample” the distribution of the residual supply (by, e.g., resampling with replacement from the empirical distribution of individual bids). Cassola, Hortaçsu and Kastl (2013) show that bidder-value estimates derived using this resampling procedure are consistent as the number of bidders grows large. Even when a rich panel is available, the Hortaçsu (2000) and Cassola, Hortaçsu and Kastl (2013) method can nonetheless be useful if one is concerned about unobserved heterogeneity. See also Hortaçsu and McAdams (2010), who discuss how to deal with unobserved heterogeneity due to different equilibria being played in different auctions in the sample.

### 4.1.2 Uniform-price auction.

In the uniform-price auction, bidder  $i$  pays the market-clearing price on all quantity, for total payment  $p^c y_i(p^c, s_i)$ . Expected payoffs in the uniform-price auction can be expressed most conveniently in terms of demand  $y_i(\cdot)$ :<sup>18</sup>

$$\Pi_i^{UPA}(y_i(\cdot); s_i) = \int_0^\infty \frac{dH_i(p, y_i(p))}{dp} \int_0^{y_i(p)} (v_i(q, s_i) - p) dq dp. \quad (12)$$

The inner integral in (12) is bidder  $i$ 's payoff from winning quantity  $y_i(p)$  at price  $p$  when the realized market-clearing price  $p^c = p$  which, when bidder  $i$  submits bid  $y_i(\cdot)$ , has density equal to the total derivative  $\frac{dH_i(p, y_i(p))}{dp}$ . (Intuitively, (12) can be viewed as integrating bidder  $i$ 's expected payoff along his submitted demand curve, with respect to the the distribution of the realized market-clearing outcome along that curve.) Once again, then, it is possible to “separate” bidder  $i$ 's best-response problem and, by considering each price-quantity pair along bidder  $i$ 's submitted demand curve separately, characterize a system of first-order conditions that point-identifies bidder values on all quantities that a bidder wins with positive probability.

**Proposition 8.** *Suppose that bids are generated in a Bayesian Nash equilibrium of the uniform-price auction in which all bids are strictly decreasing and differentiable. The following system of first-order conditions point-identifies bidder  $i$ 's marginal value  $v_i(q)$  when bidding  $b_i(\cdot)$  for all quantities  $q$  that bidder  $i$  sometimes wins:*

$$v_i^{UPA}(q) = b_i(q) - q \frac{\frac{\partial H_i(b_i(q), q)}{\partial q}}{\frac{\partial H_i(b_i(q), q)}{\partial p}}. \quad (13)$$

*Proof.* The proof is omitted to save space. □

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<sup>18</sup>Expected payoffs in the uniform-price auction can also be expressed as an integral of terms that depend only on  $(b_i(q), v_i(q))$ , as in equation (11) for the pay-as-bid auction. See equation (24) in McAdams (2008).

Equation (13) implies pointwise markdown, i.e. the bid curve is weakly below the marginal valuation curve at all quantities. However, if bidders are constrained to submit step functions with a limited number of steps, it is possible that the bid price may be above the bidder’s marginal valuation for some quantities. Kastl (2011) demonstrates this with the illustrative example of a price-taking bidder with downward sloping demand who is constrained to submit a single step as her bid function. Therefore, modelling the step function nature of bids in uniform price auctions becomes very important for practical applications. We refer the reader to Kastl (2011, 2012) for existence results and a derivation of optimality conditions with (constrained) step function bids.

**Properties of equilibrium.** The best available existence results in the uniform-price auction are very similar to those for the pay-as-bid auction, namely: when bidders are risk-neutral with i.i.d. private values, a monotone pure-strategy equilibrium (MPSE) exists and all mixed-strategy equilibria are outcome-equivalent to a MPSE (McAdams (2006)); if bidders are not risk-neutral, a monotone pure strategy equilibrium need not exist (McAdams (2007b)) but a pure-strategy equilibrium does (Reny (2011)).

The main difference between the auction formats is that the uniform-price auction is well-known to have multiple equilibria in many settings, including common-value environments (Wilson (1979), Kyle (1989)), private-value environments (Back and Zender (1993)), and complete-information settings. Indeed, even in the simplest possible model when bidders are all known to have the same marginal value  $v$  for all units, multiple “collusive-seeming equilibria” exist in which arbitrarily low prices can be realized, in addition to the competitive equilibrium in which the realized price equals  $v$ ; see Back and Zender (1993, pg. 736).<sup>19</sup> Klemperer and Meyer (1989) have established uniqueness in

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<sup>19</sup>A number of papers have explored the conditions under which such low-price equilibria can exist; see e.g. Kremer and Nyborg (2004), LiCalzi and Pavan (2005), Back and Zender (2001), and McAdams

at least one case, when bidders do not have private information and supply is random with full support.

**Partial identification and testing the equilibrium hypothesis.** Bidders in the uniform-price auction have an incentive to bid their true marginal value for the first unit (i.e. at  $q = 0$ ) but then shade their bids below true value on all subsequent units. As in the pay-as-bid auction, in practice, however, bidders typically submit step-function bids (with fewer steps than they are allowed to use) in which they bid the same price  $b_i(0)$  over a range of quantities  $[0, \hat{q}]$ . The only way for such step-function bids to arise in equilibrium would be if  $v_i(0) < b_i(0)$  and  $v_i(\hat{q}) > b_i(0)$ , since otherwise bidder  $i$  would have preferred to raise her price on quantity  $q = 0$  or lower her price on  $q = \hat{q}$ . In particular, it must be that  $v_i(0) < v_i(\hat{q})$ , as would be the case if (say) bidder  $i$  must incur a fixed cost of consumption when winning any quantity.

What marginal values could bidder  $i$  have, to rationalize such a bid? One possibility is the (unique) marginal value schedule that solves the system (13), given which bidder  $i$  is indifferent to raising or lowering any of her unit-bids. However, since bidder  $i$  is constrained in what sort of deviations are possible – only non-increasing inverse demand schedules are permitted, so bidder  $i$  cannot lower her bid on quantity 0 without also raising it on all quantities in  $(0, \hat{q})$ , and vice versa – such a step-function bid could still be a best response when  $v_i(0)$  is lower than the solution to (13) and/or when  $v_i(\hat{q})$  is higher than that solution. McAdams (2008) strengthens this intuition by characterizing tight bounds on marginal values for all quantities  $q \in [0, \hat{q}]$  within each step.

In some applications, of course, the assumption that bidders have non-increasing marginal values (“NIMV”) is very natural. In these cases, the most natural conclusions

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(2007a).

when observing bids that cannot be rationalized by NIMV are that observed bids are not generated in equilibrium, or that the auction game has somehow been misspecified. This observation has prompted a number of authors to develop more refined tests of the best-response hypothesis in multi-unit auctions, to propose alternative assumptions about the auction game, and to develop alternative solution concepts with which to rationalize auction data.

McAdams (2008) develops tests of the joint hypothesis of equilibrium bidding and NIMV, for both the pay-as-bid and uniform-price auctions, which are implemented by Chapman, McAdams, and Paarsch (2007) (hereafter “CMP”) in an application to the Bank of Canada’s pay-as-bid auctions of cash reserves. Although many bids were not consistent with best response, nearly all bids were “close” to being a best response, in the sense of being an  $\epsilon$ -best response given some profile of non-increasing marginal values and some small  $\epsilon$ . To estimate bidder values, CMP propose assuming that the data is generated in  $\epsilon$ -equilibrium; under this assumption, bidder values are then partially identified. Kastl (2011) develops an alternative approach, assuming that observed bids in a uniform-price auction are a best response but that bidders must incur an extra cost for every step that they include in their submitted step-function bids. Kastl finds that fairly small step-bid costs can rationalize the data, suggesting again that observed bids are “close” to being consistent with equilibrium of the game without bidding costs.

A more direct approach to test the best-response hypothesis is to exploit “outside” information on bidders’ actual marginal valuations (or costs) and check whether bidders’ observed bids are in fact best responses. One such study, Hortacısu and Puller (2008) on electricity auctions in Texas, will be discussed in detail in Section 4.2.2.

These ideas and empirical methods have been put to use in a number of applied settings, each of which has posed interesting challenges to putting the methodological

ideas to work. We now review some of these applications.

## 4.2 Applications and empirical challenges

### 4.2.1 Financial securities auctions

The design of Treasury securities auctions has been subject to scrutiny at least since Friedman (1960), who suggested the uniform price auction as a superior alternative to the then used pay-as-bid format. However, in settings when bidders have private information,<sup>20</sup> the ranking of the pay-as-bid and uniform-price auction is ambiguous, both in terms of revenue and efficiency (Ausubel et al. (2014)), making this market-design issue an empirical question.

Several researchers have addressed this question using data from either pay-as-bid or uniform-price auctions, using variations on the methods summarized earlier. Hortaçsu (2000), Hortaçsu and McAdams (2010), Kang and Puller (2008) study pay-as-bid auctions (in Turkey and South Korea respectively). Assuming an independent private values framework, these authors utilize variations on the first-order conditions in equation (11) to recover bidders' marginal valuations and then conduct counterfactual analyses to assess what would have happened if other auction formats had been used instead. These studies find small revenue gains from switching auction format. For example, Hortaçsu and McAdams (2010) are not able to find statistically significant revenue gains from switching auction formats, and that the efficiency gain from switching to a Vickrey auction would be at most 0.02% (2 basis points) of the total surplus. A similar strategy is pursued in Kastl (2011) and Hortaçsu, Kastl, and Zhang (2015) to study uniform-price auctions

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<sup>20</sup>In a model without private information about bidder values but stochastic supply, Pycia and Woodward (2015) show that the pay-as-bid auction generates weakly greater expected revenue than the uniform-price auction.

used by the Czech and U.S. Treasuries.

The private-values assumption in these studies is typically motivated by the fact that bidders' demand for the (typically short-term) securities sold in the studied auctions are driven by collateral or liquidity needs, and not so much for resale purposes. That said, data-driven tests of the private vs. common value hypothesis are sometimes possible. For instance, Hortaçsu and Kastl (2012) (hereafter "HK") examine data on how some bidders (securities dealers or "observers") adjusted their bids upon observing other bidders' (non-dealer customers or "observed bidders") bids in a pay-as-bid auction (these bid adjustments were first documented in Hortaçsu and Sareen (2005)). To build intuition for how this allowed HK to test the private-values hypothesis, consider a simpler second-price auction. If the observer has private values, observing another bidder's bid will not change her (weakly) dominant strategy of bidding her private value (as long as the observed bid is below her value). However, if the observer has common/interdependent values, the observer will attempt to "invert" the information contained in the observed bid to form a new expectation of the value of the auctioned object and revise her bid accordingly. HK's pay-as-bid setting is slightly more complicated in that the observer bidder has an incentive to revise her bid even in a private-value environment, but HK account for this under the null hypothesis of private values and check to see if observed bid revisions differ from what would be predicted under private values. Their main finding is that, in fact, private values cannot be rejected.

Given the private-values assumption, bidders' estimated marginal valuations can be used to assess the efficiency of the auction mechanism, as well as counterfactual revenue in alternative auction formats such as the Vickrey auction. This is achieved by calculating surplus under the efficient allocation, when bidders with the highest willingness-to-pay are allocated the securities, and by calculating the auctioneer's revenue when bidders bid

truthfully in the Vickrey auction format. Empirical estimates of efficiency and revenue losses (compared to the Vickrey counterfactual) in the above studies are typically not very large.

Another application of the empirical methods discussed above is to dissociate changes in bidder preferences from strategic behavior in important regulatory settings. For example, Cassola, Hortaçsu and Kastl (2013) study the European Central Bank's weekly refinancing auctions of short-term (weekly) repo loans, the main conduit of monetary policy in the Eurozone. In August 2007, there was a sudden and dramatic increase in banks' bids for ECB loans, reflecting funding shortages in the inter-bank market. However, at the same time, the dispersion of bids across banks also increased, suggesting heterogeneity in funding needs across banks. While some of the observed bid increases may reflect a true and sudden shift in the underlying willingness-to-pay for short-term funding, some banks may have started bidding higher just to remain competitive in the auctions. If so, the strategic nature of bids may have masked the true heterogeneity of funding troubles across Eurozone banks. Cassola, Hortaçsu and Kastl (2013) explore this issue using a version of the first-order condition (11) to estimate banks' underlying willingness-to-pay for ECB loans, accounting for their strategic bid-shading incentives. An interesting result of this exercise is that bidders' estimated willingness-to-pay were better predictors of banks' balance sheet troubles at the end of 2007 than their bids were, demonstrating the value of dissociating preferences from strategic behavior.

#### **4.2.2 Electricity auctions.**

An important application of multi-unit auctions is in wholesale electricity markets. Starting in the late 1980s, many electricity markets around the world transitioned from production and distribution by vertically integrated regulated monopolies to a partially

deregulated system with independent power producers competing to supply electricity to a common power grid. Most of these markets use a two-sided uniform-price auction, in which power plant operators submit supply bids and power distributors submit demand bids. The market is cleared, subject to transmission and other engineering constraints.<sup>21</sup>

Electricity markets have proved to be fertile ground for studying behavior in multi-unit auctions, as most markets clear quite frequently (typically hourly or even more frequently in some settings), and detailed bidding data for several markets are publicly available for research. Along with bidding data, there is typically good information on the type of generation technology used by each plant (e.g. nuclear, coal, natural gas, wind, solar) and fuel costs. Based on the available cost/technology information, a number of early and influential papers studied whether power generators strategically misreport their marginal costs and/or withhold capacity to exercise market power, and whether these strategic actions lead to departure from efficient allocation of generation resources. For example, Wolfram (1998) studies bidding behavior of power generators in the UK electricity market, and finds evidence for strategic withholding and markup behavior by power generators, where plants predicted to be marginal in the auction submit bids that help to increase the inframarginal profits of the generation company's portfolio. Borenstein, Bushnell and Wolak (2002) utilize cost and other operational information for the California electricity market to construct a competitive benchmark, against which the "strategic" outcome of the auction is contrasted. They find that the majority of the

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<sup>21</sup>When transmission or other engineering constraints are binding, alternative market-clearing mechanisms are used. A popular solution is to divide the power grid into "zones," and market-clearing is done at the zonal level. Another solution is "nodal" pricing, where each supplier and demander has its own power injection/reception node, and the net quantity and price at each node is determined through a linear program that minimizes total surplus (based on demand and cost bids) subject to transmission constraints and Kirchoff's laws governing the flow of electricity across the grid.

large increase in wholesale electricity expenditures from summer 1999 to summer 2000 (in what has been termed the “California electricity crisis”) was due to the exercise of market power by strategic players.

Subsequent efforts have taken advantage of detailed bidding data to explore additional sorts of research questions, including how to recover components of generation costs that were not easily observable in the data, and how to test the hypothesis that power generators are choosing their bids optimally. Wolak (2003) and Reguant (2014) are examples of the first line of work. Wolak (2003) utilizes an analog of the uniform price first order condition (13) to estimate the marginal cost structure that rationalizes observed bids by a generator. (Uncertainty in Wolak (2003)’s setting is due to shocks to aggregate demand, as in Klemperer and Meyer (1989), not private information about bidder values.)

Marginal costs are not sufficient to describe the behavior of most fossil fuel generation plant operators, as it is costly to stop and start-up these generators (since the plant has to heat-up to a particular temperature range to generate at peak operating efficiency). Reguant (2014) focuses on estimating the start-up costs of generators. To do this, she utilizes a unique feature of the Spanish electricity market, in which generators submit bids for 24 hours in advance and can specify minimum revenue requirements across these hours. The optimality conditions that rationalize these “complementary” bids, along with the optimality of hourly “spot” bids, allows Reguant to recover both the marginal costs and the start-up costs of the generators.

Hortaçsu and Puller (2008) explores whether best-response behavior is an appropriate assumption to impose on bidders participating in multi-unit auction mechanisms. To do this, the authors use engineering data on power plants and fuel costs to reconstruct the marginal costs of electricity production in every hour for each bidding generator in Texas electricity auctions. Using these marginal cost estimates, “best-response” bids (using info

available to bidders at the time of bidding) are calculated and compared to actual bids. The authors report a large amount of variation across bidders in terms of achieving “best-response” behavior. While some bidders were able to reap almost all of the achievable best-response profits, others were far less successful. While some of this variation was ascribable to learning, most of it appeared to be explainable by scale. In particular, companies with the most money at stake in these auctions appear to best-respond almost perfectly, whereas smaller generators were more likely not to play a best response. This suggests, perhaps not surprisingly, that large generators tend to be more capable bidders. Another finding of the paper is that the amount of productive/allocational inefficiencies caused by market power exercise by optimizing bidders appears to be less than the loss of efficiency due to deviations from optimal best-response behavior.

## **5 Auctions of dissimilar objects (“package auctions”)**

In many applications, the seller is interested in auctioning a set of non-identical objects, and buyers may be interested in purchasing subsets (or “packages”) of these objects. Most of the empirical literature on such package auctions focuses on sealed-bid formats such as the “pay-as-bid package auction.” We consider the pay-as-bid package auction, and some of its many applications in public and private procurement, in Sections 5.1 and 5.2. Another emerging strand of work studies open-outcry package auctions such as the simultaneous ascending-bid auctions used by the FCC to sell spectrum rights. We discuss this empirical literature, which is still in relative infancy, and its connection to the literature on “matching mechanisms” in Section 5.3.

## 5.1 Empirical content of pay-as-bid package auction models

The most commonly-studied sealed-bid package auction is the “pay-as-bid package auction,” in which  $K$  dissimilar objects are sold and bids can be submitted on potentially any subset (or “package”) of objects  $P \subset \{1, \dots, K\}$ . In particular, each bidder’s “bid” is a profile of package-bids  $\mathbf{b}_i = (b_{i,P})_{P \in \mathcal{P}}$  where  $\mathcal{P}$  is the set of all subsets of  $\{1, \dots, K\}$ , the auctioneer chooses a single (non-overlapping and perhaps empty) package each bidder will win so as to maximize total revenue, and each bidder pays its bid for the package that it wins.

In some applications, such as Cantillon and Pesendorfer (2007)’s bus-route procurement discussed in Section 5.2, the rules of the auction constrain what sort of package bids can be submitted. For now, however, assume only that  $b_{i,P} \geq 0$  for all  $P$ . Also, as elsewhere in this review, we will focus for simplicity on the case in which bidders have i.i.d. private values. In particular, suppose that bidders have private values  $v_{i,P}$  for each package  $P \subset \{1, \dots, K\}$ , where  $\mathbf{v}_i = (v_{i,P} : P \subset \{1, \dots, K\})$  are i.i.d. across bidders.

**Proposition 9.** *Suppose that bids are generated in an equilibrium of the pay-as-bid package auction, and that bidder  $i$  wins every possible package with positive probability.<sup>22</sup> Bidder  $i$ ’s value when bidding  $\mathbf{b}_i$  is  $\mathbf{v}_i = \mathbf{v}_i^{COM}(\mathbf{b}_i)$ , uniquely determined by the following system of equations:*

$$\sum_{P' \subset \mathcal{P}} (v_{i,P'}^{COM}(\mathbf{b}_i) - b_{i,P'}) \frac{dq_{i,P'}(\mathbf{b}_i)}{db_{i,P}} - q_{i,P}(\mathbf{b}_i) = 0 \text{ for all packages } P, \quad (14)$$

where  $q_{i,P}(\mathbf{b}_i)$  is bidder  $i$ ’s probability of winning package  $P$  when bidding  $\mathbf{b}_i$ .

*Proof.* For bid  $\mathbf{b}_i$  to be a best response, bidder  $i$  must be indifferent on the margin to raising (or lowering) his bid on each package, as captured by the first-order conditions

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<sup>22</sup>If bidder  $i$  never wins some package, there is no way to point-identify his value  $v_{i,P}$  for that package.

(14). To parse these first-order conditions, note that raising package-bid  $b_{i,P}$  has three possible effects: first, if bidder  $i$  continues to win some other package  $P'$ , bidder  $i$  also continues to pay the same price  $b_{i,P'}$  for that package, in which case nothing changes; second, if bidder  $i$  continues to win package  $P$ , which happens with probability  $q_{i,P}(\mathbf{b}_i)$ , he must pay more for it; and finally, bidder  $i$  may win package  $P$  instead of some other (possibly empty) package  $P'$ , in which case he get a different set of objects and pays  $b_{i,P}$  instead of  $b_{i,P'}$ . (To avoid confusion, note that the summation in (14) is over all subsets of  $\mathcal{P}$ , including the package  $P$  whose price is being increased, and that  $\frac{dq_{i,P}(\mathbf{b}_i)}{db_{i,P}} = -\sum_{P' \neq P \subset \mathcal{P}} \frac{dq_{i,P'}(\mathbf{b}_i)}{db_{i,P}}$ .)

Cantillon and Pesendorfer (2007 hereafter “CP”) show that, whenever bids are generated in equilibrium and bidder  $i$  wins all packages with positive probability, the system (14) has a unique solution (see CP Lemma 3). So, given our maintained hypothesis that bids arise in equilibrium, bid  $\mathbf{b}_i$  must be made given values  $\mathbf{v}_i = \mathbf{v}_i^{COM}(\mathbf{b}_i)$ .  $\square$

## 5.2 Applications and empirical challenges

In 2006, Proctor & Gamble Vice President for Global Purchases Rick Hughes and other co-authors reported in Sandholm et al (2006) how P&G had begun using a pay-as-bid package auction to meet some of its complex procurement needs, spending about \$3 billion each year in such auctions and saving an estimated \$300 million per year relative to what P&G had previously been paying. The company that designed P&G’s auction later announced in September 2013 that it had “approximately 100 customers across the globe including leaders in the retail, consumer packaged goods, restaurant and food and beverage manufacturing industries” (SciQuest (2013)). Clearly, sealed-bid multi-object auctions are big business. Most of the published empirical work on multi-object procurement, however, has focused on the public sector.

We have already alluded to the work of Cantillon and Pesendorfer (2006, 2007, hereafter “CP”)’s , who pioneered the econometric analysis of sealed-bid multi-object auctions. In their empirical application, CP analyzed the auctioning of London bus routes. In 2006, London was served by about 800 bus routes carrying more than 3.5 million passengers each day, a market worth roughly \$900 million. Since the mid-1990s, the contracts for bus services have been tendered using a pay-as-bid combinatorial auction, in which suppliers bid for packages of routes.

CP’s key finding is that bidders’ cost of providing a package of bus routes appears to exhibit decreasing returns to scale, suggesting that “possible cost savings emerging from the sharing of spare buses and bus servicing are outweighed by cost increases due to the limited garage size.” While services for hundreds of bus routes were procured within CP’s London dataset, in most auctions only a few routes were tendered at the same time. The relatively small number of objects being sold made it computationally feasible to estimate bidder costs using the first-order condition approach discussed above.

**Constraints on permissible bids.** A complication in CP’s London bus-route application is that bidders were not allowed to submit package-bids reflecting diseconomies of scale. In particular, anyone bidding  $c_{i,1}$  for route 1 and  $c_{i,2}$  for route 2 cannot bid more than  $c_{i,1} + c_{i,2}$  for the package of both routes. In an auction in which bidders are buyers, this restriction translates as the “superadditivity constraint” that  $b_{i,1} + b_{i,2} \leq b_{i,\{1,2\}}$ .

What can be inferred about bidder values when bidder  $i$  submits a bid with this constraint binding? Some deviations are still feasible in which bidder  $i$  only changes one package-bid: lowering  $b_{i,1}$ , lowering  $b_{i,2}$ , and raising  $b_{i,\{1,2\}}$ . For these deviations to be unprofitable, an inequality version of the first-order conditions (14) must hold:

$$\frac{d\Pi_i(\mathbf{v}_i, \mathbf{b}_i)}{db_{i,1}} \geq 0, \frac{d\Pi_i(\mathbf{v}_i, \mathbf{b}_i)}{db_{i,2}} \geq 0, \text{ and } \frac{d\Pi_i(\mathbf{v}_i, \mathbf{b}_i)}{db_{i,\{1,2\}}} \leq 0, \quad (15)$$

where  $\Pi_i(\mathbf{v}_i, \mathbf{b}_i)$  is bidder  $i$ 's interim expected payoff when bidding  $\mathbf{b}_i$  given values  $\mathbf{v}_i$ . In addition, other deviations remain feasible in which two package-bids are changed at the same time: raising  $b_{i,1}$  and lowering  $b_{i,2}$  by the same amount (or vice versa); and raising  $b_{i,1}$  (or  $b_{i,2}$ ) while raising  $b_{i,\{1,2\}}$  by the same amount. For such deviations to be unprofitable, it must be that

$$\frac{d\Pi_i(v_i, b_i)}{db_{i,1}} = \frac{d\Pi_i(v_i, b_i)}{db_{i,2}} = -\frac{d\Pi_i(v_i, b_i)}{db_{i,\{1,2\}}}. \quad (16)$$

Note that the inequalities (15,16) only partially identify bidder values.<sup>23</sup>

**The “curse of dimensionality”?** A different and more fundamental challenge is that the number of packages grows exponentially with the number of objects being sold, making it impractical for bidders to submit distinct package-bids on all packages. As a result, many packages may receive few if any bids, making it difficult to reliably estimate the derivatives of bidders’ probabilities of winning each possible package without imposing additional restrictions.

Kim, Olivares, and Weintraub (2014, hereafter KOW), in their study of public procurement of school meals in Chile, address this challenge by imposing a novel restriction that bidders shade their bids below true value by the same amount on all packages having the same characteristics. In particular, KOW assume that bids  $\mathbf{b}_i = (b_{i,P} : P \subset \{1, \dots, K\})$  take the linear form  $\mathbf{b}_i = \mathbf{v}_i + \mathbf{W}\theta_i$ , where  $\mathbf{W}$  is a known matrix of package characteristics and bidders choose  $\theta_i \in \mathbb{R}^d$  as a function of  $\mathbf{v}_i$ , where  $d$  is the num-

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<sup>23</sup> Geometrically speaking, the partially-identified set is a ray originating at  $\mathbf{v}_i^{COM}(\mathbf{b}_i)$ , the unique solution to (14), in the direction of higher  $v_{i,1}$ , higher  $v_{i,2}$ , and lower  $v_{i,\{1,2\}}$ . What this means is that, if  $\mathbf{v}_i^{COM}(\mathbf{b}_i)$  exhibits decreasing returns, i.e.  $v_{i,1}^{COM}(\mathbf{b}_i) + v_{i,2}^{COM}(\mathbf{b}_i) < v_{i,\{1,2\}}^{COM}(\mathbf{b}_i)$ , then so do all the values in the partially identified set. On the other hand, if  $v_i^{COM}(\mathbf{b}_i)$  exhibits increasing returns, some values in the partially identified set may nonetheless exhibit decreasing returns.

ber of package characteristics, to maximize their expected payoff within this restricted bid-space.

The main advantage of this approach is that, by reducing the dimensionality of each bidder’s decision,<sup>24</sup> it reduces the dimensionality of the associated estimation problem, making it possible to conduct meaningful counter-factual analysis in extraordinarily complex auction environments. For instance, in KOW’s empirical application to the provision of school meals in Chile (a half-billion dollar business that feeds 2.5 million children), KOW are able to estimate that “bidders’ cost synergies are economically significant (about 5%) and that the current [combinatorial auction] mechanism achieves high allocative efficiency (about 98%) and reasonable margins for the bidders (about 5%).” See also Olivares et al (2012).

The key assumption underlying KOW’s approach is that bidders are restricted (or act as if they are restricted) to submit bids specifying a constant markup on all packages having identical characteristics. In this framework, bidders are assumed to solve a simpler problem than the full-fledged optimization problem they could be solving. This is an interesting and promising idea that deserves further attention. In past work, researchers have responded to bidders’ failure to submit optimal bids by imposing alternative assumptions, such as that bidders must incur extra costs when submitting more complex bids (Kastl (2011)) or that bidders are in fact playing  $\epsilon$ -best response (Chapman, McAdams, and Paarsch (2006) (2007)). Restricting bidders’ strategy space and focusing on equilibrium play in that restricted game is a tractable alternative approach to interpret auction data that is not generated by equilibrium play in the unrestricted

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<sup>24</sup>The dimension-reduction associated with KOW’s approach is dramatic. In the usual model in which bidders’ strategies are unrestricted, each bidder’s strategy specifies  $2^K - 1$  real-valued functions  $b_{i,P}(\mathbf{v}_i)$  whereas, in KOW’s model, each bidder’s strategy specifies  $d$  constants.

game.

### 5.3 New directions in open-outcry package auctions

In 1994, the Federal Communications Commission introduced the first auction of radio spectrum. Since then, auctions have emerged as the dominant method by which radio spectrum rights are sold around the globe. Until recently, econometricians have lacked the tools to analyze the revenue and/or efficiency performance of these auctions using the same sort of structural empirical methods that we have seen deployed in so many other settings. In the last few years, however, a new line of research has made meaningful progress with an empirical approach (leveraging so-called “pairwise stability”) inspired by the literature on matching mechanisms.

**Simultaneous ascending-bid auction.** Real-world open outcry auctions differ in a wide variety of details but, for concreteness, we will focus on the “simultaneous ascending-bid auction (SAA),” a multi-round open-outcry auction that has been used to sell wireless spectrum licenses in several countries and received the most attention in the empirical literature. Package bids are not allowed in the SAA, but bidders can bid on multiple objects at the same time and acquire any given package by outbidding others on every object in that package. In particular, “standing high bidders” for each object are determined after each round, at which point minimum bid-increments are announced for each object in the next round, and the auction continues until no one calls out a price that meets the minimum bid on any object.<sup>25</sup>

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<sup>25</sup>Simultaneous ascending-bid auctions have a host of other rules as well, such as “activity rules” that restrict how many objects bidders can bid on in each round, as a function of their bidding activity in previous rounds. We abstract from such details to keep the present discussion as simple and focused as possible.

A substantial theoretical literature has studied the simultaneous ascending-bid auction. Much like the Ausubel-Milgrom clock auction (discussed in Section 3), the SAA has a number of desirable theoretical properties, especially if bidders adopt “straightforward bidding strategies” of always bidding on whichever object offers the most surplus at current prices and the objects for sale are “mutual substitutes”; see Milgrom (2000). However, many equilibria exist in which bidders do not adopt straightforward bidding strategies, including “low-price equilibria” in which bidders exploit the dynamic-bidding process to suppress prices below competitive levels; see Engelbrecht-Wiggans and Kahn (2005) and Brusco and Lopomo (2002). Moreover, such equilibria may lead to an allocationally inefficient outcome.

Fox and Bajari (2013, hereafter FB) develop a novel revealed-preference approach to estimate bidder values in the FCC’s 1995-1996 C block spectrum auction. (This auction divided the continental U.S. into 480 geographically distinct licenses, and 255 bidders participated in the SAA. The auction closed with winning bids totalling \$10.1 billion.) FB’s approach is based on the assumption that the final allocation of licenses is *pairwise stable*, meaning that no two bidders can increase their joint surplus by exchanging two licenses.<sup>26</sup> Equivalently, the sum of any two bidders’ (incremental) values for any two licenses that they receive must exceed the sum of their values were they to exchange those licenses. Assuming a parametric specification for bidders’ valuations over packages of licenses, pairwise stability generates inequality restrictions on the parameters of the econometric model.

The inequality restrictions that arise from pairwise stability provide the basis for a computationally convenient estimation approach that follows the Fox (2010a) “matching

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<sup>26</sup>Pairwise stability is consistent with equilibrium play, arising in the “low-price equilibria” studied by Engelbrecht-Wiggans and Kahn (2005).

game estimation” method, and is a generalization of the semi-parametric maximum score estimators of Manski (1975) and the maximum rank correlation estimator of Han (1987). As the asymptotic distribution of estimated parameters in these estimation approaches are difficult to compute in practice, FB use a subsampling approach for inference.

An interesting aspect of FB’s estimation approach is that the inequalities implied by pairwise stability do not necessarily incorporate data on bids or final prices, but instead relies only on information on the final allocations, i.e. which bidder ended up winning each license. This departs dramatically from other estimation approaches in the auction literature, where bids are the primary source of information. Utilizing data only from allocations comes with important limitations, especially in terms of pinning down the scale of bidder valuations, but, as FB show, can be informative about key structural parameters.

This “matching game approach to estimation” allows FB to perform meaningful inferences about bidder values and also to perform counterfactual exercises. For instance, in their FCC spectrum auction application, FB’s estimated parameter values imply that licenses have strong complementarities, which in turn suggests that the FCC’s decision to split the country into hundreds of separate zones may have led to an unnecessarily inefficient outcome. Indeed, based on their parameter estimates, FB conclude that efficiency would have been 48% higher had the FCC awarded four large, regional licenses to the four bidders possessing the highest valuations for these licenses.

FB’s estimation approach is similar in spirit to prior work on matching games by Choo and Siow (2006) and Fox (2010b), who identify transferable-utility marriage models from data on final allocations without needing to observe transfers between partners. Akkus, Cookson, and Hortaçsu (2015) generalizes Fox (2010b)’s approach to include transfer prices in a merger context, leading to inequalities very similar to those developed by

Haile and Tamer (2003). Similarly incorporating bid and price data in the empirical analysis of open-outcry multiple-object auction is an interesting open area for future work in empirical multiple-object auctions.

## 6 Concluding remarks

This paper has surveyed two broad strands of the empirical auctions literature. The first of these strands of the literature focuses on sealed-bid auctions and leverages “the first-order condition approach” to estimate how much observed bids are shaded below bidders’ true values. This approach, pioneered by Guerre, Perrigne, and Vuong (2000) for first-price auctions of a single object, has now been applied to study applications as diverse as Treasury bond sales, electricity procurement auctions, and a nationwide program to procure school meals. The second strand of the literature focuses on open-outcry auctions and leverages “the revealed-preference approach” to derive bounds on bidders’ values based on revealed preference from the choices that bidders make during the auction. This approach, pioneered by Haile and Tamer (2003) for open-outcry auctions of a single object, has also been deployed in a wide variety of applications, from sponsored-search auctions to FCC spectrum auctions.

There are a number of other promising areas of auction econometrics research that we have excluded from this review, including auctions with resale, auctions with collusion, competing auctioneers, and entry in auctions. Moreover, by focusing on settings where multiple objects are sold through a single auction mechanism, we have excluded a number of interesting and economically important environments in which objects are sold in multiple auction mechanisms, either simultaneously, sequentially, or in overlapping

auction events.<sup>27</sup> Certainly, there is room for plenty more work in the years to come.

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<sup>27</sup>Some notable recent contributions in these literatures include Gentry, Komarova, and Schiraldi (2016) on simultaneous auctions, applied to highway procurement auctions, and Lamy (2016) and Kong (2016) on sequential auctions, applied to tobacco auctions and oil-and-gas lease auctions.

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