Secrecy in the first-price auction

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Abstract

This paper endogenizes whether bidders' participation is observable to other bidders in a first-price auction with iid private values. After learning their values, bidders decide whether to participate "publicly", "secretly", or not at all. If public participation is more costly, bidders never participate publicly in any equilibrium. If secrecy entails a small extra cost, all symmetric equilibria exhibit a mixture of secret and public participation, and switching to a second-price format increases expected revenue and expected total welfare among all symmetric equilibria.

1 Introduction

In first-price auctions, the number of those submitting bids is typically assumed to be common knowledge among the bidders at the time of bidding (what I will call "only-public participation") or unobservable at the time of bidding ("only-secret participation"). In

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the benchmark case when bidders are risk-neutral with iid private values and the distribution of the number of bidders is common knowledge, expected revenue and bidders' interim expected surplus are the same in either case. A risk-neutral seller and the bidders therefore have no incentive to influence whether *all* bidders are observed or unobserved prior to the bidding. However, each bidder has an incentive to influence whether *his own* participation is observed, as his own observability can affect others' bids.

This paper endogenizes observability in the first-price auction as the equilibrium outcome of a game in which each bidder chooses whether to make his bidding interest known. First, bidders arrive at the auction and costlessly observe iid private values. Each bidder then simultaneously decides whether to participate "publicly" at cost $c_P \geq 0$, "secretly" at cost $c_S \geq 0$, or not at all. The set of public participants is then commonly observed, after which all participants bid.

Each bidder bids when faced with fewer competitors, in the unique symmetric equilibrium less of a symmetric first-price auction with only-public participation. Once secret participation is possible, however, less *public* competition does not necessarily translate into less *total* competition. Indeed, if public bidders are in equilibrium "weaker" than secret bidders, in the sense of having lower values, it is conceivable that some bidders could bid more when faced with fewer public competitors.

For an intuition why public bidders may differ from secret bidders, consider a sale of real estate in which all bids must be submitted by a deadline. Anticipating that the real-estate agent has an incentive to reveal any interest in the property in order to intensify the bidding competition, the most serious bidders may take steps to conceal their interest until after the deadline. On the other hand, bidders who are only mildly interested in the property have little incentive to conceal their intention to participate, since they expect to lose to any serious competitor whether or not their interest is known. Consequently, public bidders in such an auction may tend to have lower values than those who have chosen to participate secretly.

Might bidders even pay extra in equilibrium to publicize their participation, if doing so convinces others that they have relatively low values? No. Whenever public participation is more costly than secret participation ($c_P > c_S$), there is a unique perfect Bayesian equilibrium in weakly undominated strategies (shorthand "PBE") and no bidder ever participates publicly in this equilibrium (Theorem 3).

On the other hand, when secrecy entails a small extra cost ($c_S > c_P > 0$ and $c_S - c_P \approx 0$), every symmetric PBE exhibits a mix of public and secret participation (Theorem 4). Intuitively, some participation must be public since the lowest-value bidders who participate only win if no one else participates, and hence prefer to participate in whatever way is least costly. On the other hand, some participation must be secret since secret participation, if unexpected, dampens bidding competition. (If bidders do not expect secret participation, then they will assume that all bidders are public bidders. Participating secretly will then induce others to believe that they face fewer competitors and hence bid less.) In this case, disallowing secret participation strictly increases ex post total welfare and expected revenue (Theorem 6).

One way to dissuade bidder secrecy is simply to make secrecy impossible, as when the seller reveals the set of participants prior to the bidding. Such a step is common in practice, e.g. in FCC spectrum auctions[1] and government procurement. Another way to dissuade bidder secrecy is to change the auction format. In a second-price or English auction with private values, each participant will bid his value regardless of whether he is

[1] Another reason why the FCC publicizes the set of participants in spectrum auctions is to help enforce anti-collusion rules. Namely, publicizing the set of participants serves to inform bidders with whom they are lawfully required *not* to communicate about bidding in the auction.

secret or public, and regardless of what he believes about others. These auction formats therefore deter bidders from investing in secrecy, to the ultimate benefit of the seller. A simple example shows that losses due to bidder secrecy can amount to more than 25% of expected revenue.

The rest of the paper is organized as follows. The introduction continues with discussion of some related literature. Section 2 presents the model and preliminaries, as well as a discussion why there may be different costs associated with secret and public participation. Section 3 considers a simple illustrative example. Section 4 characterizes how bidders participate in equilibrium – only publicly, only secretly, or a mixture of both – depending on participation costs (c_P, c_S) and the distribution of bidder values. Section 5 then explores how welfare and revenue vary with these costs. Section 6 offers concluding remarks. Some proofs are in the Appendix.

Related literature. This paper endogenizes whether bidders observe who participates prior to the bidding, in a Samuelson (1985)-style model of the first-price auction with costly participation. Samuelson found that (i) the seller's revenue in the unique symmetric equilibrium[2] depends on the cost of participation, but not on whether participation is only-secret or only-public, and (ii) the seller's revenue in the symmetric equilibrium is decreasing in the cost of participation. When secret and public participation are equally costly, these basic findings continue to hold in my setting, although I show that participation is endogenously secret (Theorem 2). Thus, if bidders sometimes choose to participate publicly, there must be extra costs associated with secrecy. In this case, disallowing secrecy increases welfare and revenue in all symmetric equilibria by dissuading socially wasteful investments in secrecy.

[2]Cao and Tian (2010) showed that asymmetric equilibria can also exist given only-public participation.

Other papers have compared first-price auctions in which participation must be secret or must be public. Most notably, McAfee and McMillan (1987a) shows that expected revenue in a first-price auction with independent private values is higher when participation must be secret than when it must be public, if (i) the number of bidders is random and (ii) bidders are risk-averse with CARA utility. A key difference here is that bidders *choose* whether to make their participation known to others. This distinction is significant if the seller cannot completely control what bidders know about their competition.

Another paper that endogenizes bidders' knowledge of their competition is Hendricks, Onur, and Wiseman (2012). They consider a sequence of eBay auctions in which bidding in any auction reveals a bidder's presence in that and all future auctions, until he wins an item and exits. In equilibrium, bidders wait until the last minute to bid in any given auction, because revealing one's presence lowers others' continuation values in future auctions, thereby inducing others to bid more in the current auction. Similarly, bidders refrain from bidding in earlier auctions, to suppress others' values in later auctions. By contrast, this paper considers a first-price auction in which bidders' participation decisions do not directly affect others' values.

Also somewhat related is the signalling literature, since a bidder's decision whether or not to participate publicly influences others' beliefs about his value.[3] In particular, *secret* participation allows a high-value bidder to pool with others having very low values, who do not participate in the auction at all. Such "non-monotonic signalling" has been observed in several other contexts, such as Feltovich, Harbaugh, and To (2002) and Chung and Eso (2007).

[3] Less closely related is the literature in which bidders signal during the auction, such as Avery (1998) and Horner and Sahuguet (2007), or the literature in which bidders take observable actions prior to acquiring any private information, such as Arozamena and Cantillon (2004).

2 Model and preliminaries

Timing of the game. *Pre-auction phase.* Each of n potential bidders arrives independently at the auction with probability $p \in (0,1]$. Each arrival ("bidder") i costlessly observes iid private value v_i (see comment (a) below). Each bidder then simultaneously decides whether to participate publicly at cost $c_P \geq 0$, participate secretly at cost $c_S \geq 0$, or not participate (see comment (b)). By assumption, the distribution of bidder values has support $[0,1]$ with strictly positive, continuously differentiable p.d.f., and does not depend on the realized set of bidders. Let $F(v) = (1-p) + p\Pr(v_i < v)$ denote each bidder's likelihood of arriving with a value less than v.[4] "Virtual values" $v_i - \frac{1-F(v_i)}{f(v_i)}$ are assumed to be strictly increasing in v_i (see comment (c)).

Bidding phase. All participants observe who has chosen to participate publicly, then submit sealed bids in a first-price auction with zero reserve price. Ties are broken in favor of secret bidders,[5] or randomly among like bidders. If no bidder participates, then the object is not sold and has no subsequent value to the seller.

Strategies. Bidder i's strategy consists of a "participation strategy" specifying probabilities $q_i^P(v)$ and $q_i^S(v)$ of public and secret participation for all $v \in [0,1]$, respectively, and a "bidding strategy" $b_i(v;P)$ specifying bidder i's bid if he participates, as a function of his value and the subset $P \subset \{1,...,n\}$ of bidders who participate publicly. Strategies are *symmetric* if, for all i and all v, (i) $q_i^P(v) = q^P(v)$ and $q_i^S(v) = q^S(v)$ and (ii) $b_i(v;P) = b^P(v;\#(P))$ if $i \in P$ and $b_i(v;P) = b^S(v;\#(P))$ if $i \notin P$. That is, each

[4] If one interprets p as the probability with which each bidder's value is less than the reserve price, then $F(.)$ is the unconditional cdf of bidder values.

[5] Favoring secret bidders ensures that a best response exists off the equilibrium path, for a bidder who has deviated by participating secretly.

participant's bid depends only on his value, the number of public participants $\#(P)$, and whether he himself participates publicly or secretly.

Definition 1 (Only-secret participation)**.** Bidder i's strategy exhibits *only-secret participation* if $E_{v_i}\left[q_i^P(v_i)\right] = 0$.

Definition 2 (Only-public participation)**.** Bidder i's strategy exhibits *only-public participation* if $E_{v_i}\left[q_i^S(v_i)\right] = 0$.

Participation thresholds. In every equilibrium with some public or secret participation, there is a minimal value given which any bidder ever participates publicly or secretly. Such "participation thresholds" will play a central role in the analysis.

Definition 3 (Participation thresholds)**.** Bidder i's *"secret participation threshold"* $\underline{v}_i^S$ and *"public participation threshold"* $\underline{v}_i^P$ are, respectively, the lowest (infimum) values given which he ever participates secretly or publicly:

$$\underline{v}_i^S = \max\left\{v \in [0,1] : \int_0^v q_i^S(v_i)\mathrm{d}v_i = 0\right\}$$
$$\underline{v}_i^P = \max\left\{v \in [0,1] : \int_0^v q_i^P(v_i)\mathrm{d}v_i = 0\right\}.$$

($\underline{v}_i^S = 1$ or $\underline{v}_i^P = 1$, respectively, if bidder i never participates secretly or publicly.) Bidder i's *"participation threshold"* $\underline{v}_i = \min\{\underline{v}_i^S, \underline{v}_i^P\}$ is the lowest value given which he participates. Should bidders adopt symmetric strategies, let $\underline{v}^S, \underline{v}^P, \underline{v}$ be shorthand for their (symmetric) participation thresholds.

Figure 1 illustrates a hypothetical participation strategy for bidder i. According to this strategy, bidder i does not participate when $v_i \in [0, \underline{v}_i^S)$, only participates secretly when $v_i \in (\underline{v}_i^S, \underline{v}_i^P)$, mixes between secret and public participation when $v \in (\underline{v}_i^P, v_i^1)$, only participates publicly when $v \in (v_i^1, v_i^2)$, and only participates secretly when $v_i \in (v_i^2, 1)$. (Bidder i's probability of secret participation, $q_i^S(v_i)$, is denoted by a thick line.)

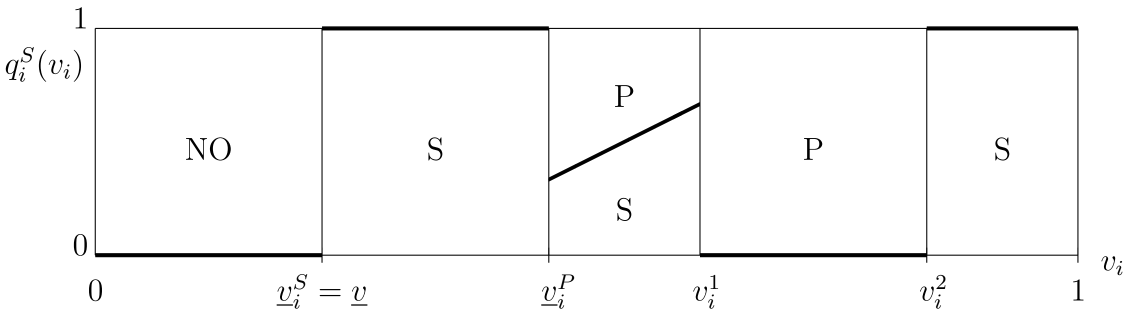

Figure 1: Illustration of a participation strategy for bidder i.

Solution concept. The solution concept is perfect Bayesian equilibrium in weakly undominated strategies (shorthand "PBE").

Comments on the model. (a) Bidders observe their private values before deciding whether to participate. Thus, participation costs here are "costs of bidding" rather than "costs of learning." It would be straightforward to overlay the present model with an earlier stage in which bidders must invest to learn their private values, whether simultaneously as in Levin and Smith (1994) or sequentially as in McAfee and McMillan (1987b).

(b) A crucial feature of the model is that each bidder is uncertain about how many others are participating, since (i) the number of bidders is random (if $p < 1$) and/or (ii) some bidders choose not to participate (if $c_S > 0$ and $c_P > 0$). Without such uncertainty, "secrecy" has no effect on others' beliefs about the number of competitors.

(c) Monotone virtual values is used (only) in the proof of Theorem 6, to ensure that any allocative inefficiencies due to secret participation lower expected revenue.

2.1 Preliminaries: Existence of symmetric PBE

Theorem 1 (Existence of symmetric PBE). *There exists a perfect Bayesian equilibrium in symmetric, weakly undominated strategies ("symmetric PBE").*

Proof. See the Appendix. □

When $c_P > c_S$, there is in fact a unique PBE (Theorem 3). However, more generally, I have not been able to characterize all PBE. Indeed, it remains an open question whether there is a unique symmetric PBE. (The welfare and revenue findings of Section 5 apply to *all* symmetric PBE.) Asymmetric PBE may also exist; see e.g. Cao and Tian (2010) who analyze the case when $p = 1$ and $c_S = \infty$.

2.2 Discussion: costs of secrecy and publicity

Participating in an auction can be costly, for a wide variety of reasons. For instance, Samuelson (1985) noted that "competing firms must bear significant bid-preparation and documentation costs." This paper explores the possibility that there may be additional costs associated with participating secretly or publicly.

Cost of secrecy. Sometimes it is impossible to participate in an auction without the seller's knowledge. For example, in a corporate acquisition, the process of due diligence requires extensive interactions between the target and any potential acquirer. Maintaining secrecy therefore requires convincing the target not to reveal one's participation to other bidders. However, providing a sufficient reward or establishing a sufficient threat to induce such silence could entail substantial cost.

More broadly, a bidder who wants to keep his participation secret must actively suppress all signs of his interest in the good being sold that are visible to other bidders. In particular, maintaining secrecy requires that a bidder inefficiently avoid or delay all observable decisions that signal interest. Such decisions range from the small, such as whether to schedule a private tour of a home for sale (revealing one's potential interest to the seller's agent and thereby to other buyers' agents) or attend a more-anonymous open house, to the large, such as acquiring a complementary asset or not acquiring a substitutable asset.

For example, in the summer of 2010, rumors swirled that both Intel and Apple might be in the market to purchase the wireless division of semiconductor manufacturer Infineon.[6] At that time, Apple faced a critical decision whether and how much to backward integrate into the design and manufacture of smart-phone components. In 2008, Apple had paid $278 million to acquire low-power microchip designer P.A. Semi.[7] This acquisition gave them the capability to design the iPad's A4 microchip in-house, released in January 2010.[8] A similar acquisition of Infineon could allow Apple to design and build its own wireless chips, another key potential differentiator in the smartphone market.

Apple made no public comment about its plans vis-à-vis Infineon, but Intel might still be able to infer Apple's intentions from its other business activities. For instance, consider Apple's decision whether to invest in its relationships with alternative wireless-chip suppliers. (Infineon had been a major supplier for Apple, but a sale to Intel would make them a less desirable partner.) If interested in Infineon, Apple might prefer to avoid investing in alternative supply until after the uncertainty of the Infineon sale is resolved. Otherwise, if uninterested, Apple might prefer to line up alternative supply as soon as possible. If so, *failure* to invest in alternative supply would effectively publicize Apple's intention to bid for Infineon. To keep its bidding intentions secret from Intel, then, Apple would need to make inefficient investments in alternative supply. The losses due to such investments constitute a *cost of secrecy.*

[6] "Why Apple Should Buy Infineon: To Own Mobile And Screw Intel" by Steve Cheney, TechCrunch, July 30, 2010.

[7] "Apple buying microchip designer P.A. Semi" by Scott Hillis, Reuters, April 23, 2008

[8] "A Little Chip Designed by Apple Itself" by Ashlee Vance and Brad Stone, New York Times, February 1, 2010.

Cost of publicity. There could be costs associated with announcing one's participation in an auction, but such costs seem likely to be quite small. For instance, a firm that wanted to convey its interest in a particular target could simply issue a press release. Potentially more substantial are the endogenous effects of publicity on games played *outside* of the auction, including future interactions among the same bidders. Hendricks, Onur, and Wiseman (2012) derive an endogenous cost of public participation in a sequence of eBay auctions with only-public participation. In their model, submitting a bid is costly in early auctions since it induces others to believe that they face more competitors in later auctions.

Similar effects can arise in first-price auctions when secret participation is an option. For example, suppose that only two contractors are qualified to bid for two jobs to be auctioned sequentially, but that each contractor may be unavailable due to a capacity constraint. (If available, each has sufficient capacity to work both jobs. Also, each contractor does not learn its cost for the second job until after the auction for the first.) Suppose further that, in the *first* auction, each contractor has the option to participate secretly at cost $c_S > 0$ or publicly at no cost. Even though public participation is free, there is still a cost of publicity in the first auction. In particular, a contractor who participates publicly in the first auction reveals that it will also participate in the second auction, changing the other contractor's bidding behavior in the second auction. The true *cost of publicity* in the first auction is the expected equilibrium loss that each contractor incurs in the second auction, by having its participation known.[9]

[9] This expected loss is itself endogenous, as it depends on how likely each bidder is to participate publicly in the first auction. At the same time, each bidder's likelihood of participating publicly in the first auction depends on c_P.

3 Illustrative example

This section provides a simple, illustrative example. Suppose that there are two potential bidders, each of whom arrives with probability $p \in (0,1)$, with iid private values $v_i \sim U[0,1]$. Public and secret participation cost $c_P = 0$ and $c_S > 0$, respectively.

Benchmark: second-price auction. In the second-price auction, others will bid their values regardless of who participates publicly. Thus, there is no benefit to secrecy and bidders only participate publicly in all PBE. The seller earns zero revenue unless both bidders participate, in which case she receives revenue equal to the second-highest value. Since public participation is costless, each bidder is sure to participate whenever present. The seller's expected revenue is therefore $REV(p) = p^2 E\left[\min\{v_1, v_2\}\right] = \frac{p^2}{3}$.

A symmetric PBE. I will now construct a symmetric PBE in the first-price auction and compare its expected revenue against that in the second-price auction.

Claim 1. *There exists a symmetric PBE in which each bidder i (i) participates publicly when $v_i \in \left(0, \underline{v}^S\right)$ and (ii) randomizes whether to participate publicly or secretly with $q^S(v_i) = q^S$ and $q^P(v_i) = q^P$ for all $v_i \in (\frac{2}{3}, 1)$, where*

$$\underline{v}^S = \sqrt{\frac{2c_S}{p}}, \tag{1}$$

$$\frac{q^S}{q^P} = \frac{1-p}{p\underline{v}^S} \text{ and } q^S + q^P = 1. \tag{2}$$

In particular, when $p = \frac{1}{2}$ and $c_S = \frac{1}{9}$, $\underline{v}^S = \frac{2}{3}$ and $q^S = \frac{3}{5}$.

Proof. Suppose for the moment that an equilibrium exists with the participation strategies posited in Claim 1. To complete the proof, I will derive subsequent equilibrium bidding strategies in each subgame – when both, one, or none of the bidders participates

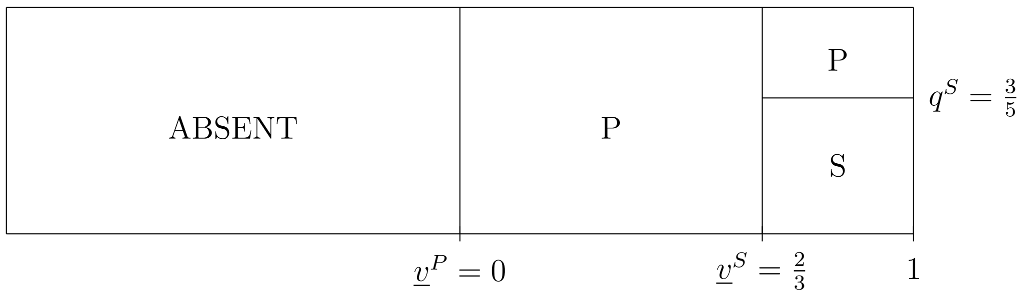

Figure 2: Illustration of participation in a symmetric PBE, when $p = \frac{1}{2}$ and $c_S = \frac{1}{9}$.

publicly – and verify that the specified participation strategies are in fact a best response given such subsequent bidding.

Preliminaries: symmetry above $\underline{v}^S$*.* Recall that bidder values have cdf $F(v) = (1-p)+pv$. Let $F(v|P)$ denote the cdf of bidder values conditional on participating publicly, and let $F(v|S)$ be the corresponding cdf conditional on not participating publicly. Each bidder i participates publicly whenever $v_i \in (0, \underline{v}^S)$ and with probability q^P when $v_i > \underline{v}^S$. The ex ante probability of public participation is $p\left(\underline{v}^S + q^P(1-\underline{v}^S)\right)$, while the conditional cdf $F(v|P) = \frac{\underline{v}^S + q^P(v-\underline{v}^S)}{\underline{v}^S + q^P(1-\underline{v}^S)}$ for all $v > \underline{v}^S$. Similarly, bidder i does *not* participate publicly whenever $v_i < 0$ and with probability q^S when $v_i > \underline{v}^S$, so that the conditional cdf $F(v|S) = \frac{(1-p)+pq^S(v-\underline{v}^S)}{(1-p)+pq^S(1-\underline{v}^S)}$ for all $v > \underline{v}^S$. Since $\frac{q^S}{q^P} = \frac{1-p}{p\underline{v}^S}$, $F(v|P) = F(v|S)$ for all $v > \underline{v}^S$. (However, at values below $\underline{v}^S$, public bidders and others do not have symmetrically distributed values.)

Bidding when both public or neither public. When both bidders participate publicly, subsequent equilibrium bidding proceeds as in a standard first-price auction (with costless, voluntary bidding) in which bidders have iid values distributed according to cdf $F(.|P)$. In particular, each bidder i bids $b(v;\{1,2\}) = E\left[v^{(2)}|v^{(1)} = v, P = \{1,2\}\right]$, the expected second-highest value conditional on (i) his own value being highest and (ii) values being distributed according to cdf $F(.|P)$. For future reference, note that

$b(v;\{1,2\}) = \frac{v}{2}$ for all $v \in (0, \underline{v}^S)$. Similarly, when both bidders participate secretly, each bids $b(v;\emptyset) = E\left[v^{(2)}|v^{(1)} = v, P = \emptyset\right]$, conditioning now that bidders have values distributed according to cdf $F(.|S)$. For future reference, note that $b(\underline{v}^S;\emptyset) = 0$.

Bidding when one bidder is public and one is not. Suppose that bidder 1 participates publicly, while bidder 2 does not. Equilibrium bidding now corresponds to that in a standard first-price auction in which bidders' values are distributed asymmetrically, as follows. Bidder 1's value is distributed on $[0,1]$, with mass $F(\underline{v}^S|P)$ spread uniformly over $[0,\underline{v}^S]$ and the remaining mass spread uniformly over $[\underline{v}^S, 1]$. Bidder 2's value has support $\{-\infty\} \cup [\underline{v}^S, 1]$, with mass $F(\underline{v}^S|S) = F(\underline{v}^S|P)$ at $-\infty$ and the remaining mass again spread uniformly over $[\underline{v}^S, 1]$.

The following is a bidding equilibrium: public bidder 1 bids $b_1(v;\{1\}) = 0$ for all $v \in (0,\underline{v}^S)$ and $b_1(v;\{1\}) = b(v;\emptyset)$ for all $v > \underline{v}^S$; and secret bidder 2 bids $b_2(v;\{1\}) = b(v;\emptyset)$ for all $v > \underline{v}^S$. To see why, note that each bidder's bid is distributed exactly as in the symmetric equilibrium in the subgame when $P = \emptyset$. Thus, each bidder's best response given any value $v > \underline{v}^S$ must be the same as in that subgame. It remains to check that public bidder 1 prefers to bid zero given any value $v_1 \in (0, \underline{v}^S)$. But this follows immediately from monotonicity of his best response, since bidder 1 prefers to bid zero given value $v_1 = \underline{v}^S$. (Recall that $b(\underline{v}^S;\emptyset) = 0$.)

Secret, public, or no participation? Let $S^S(v), S^P(v)$ denote each bidder's *gross* interim expected surplus when secret or public, and when playing a best response to the equilibrium bidding strategies derived above. To complete the proof, it suffices to show that (i) $S^P(v) > S^S(v) - c_S$ for all $v \in (0,\underline{v}^S)$ and (ii) $S^P(v) = S^S(v) - c_S > 0$ for all $v > \underline{v}^S$. (If so, the participation strategy of Claim 1 is a best response given subsequent equilibrium bidding.) Without loss, I will focus on bidder 1.

Case #1: $v_1 \in (0, \underline{v}^S)$. Suppose first that bidder 1 were to participate publicly. If bidder 2 is also public, bidder 1 will bid $\frac{v_1}{2}$ and win with (unconditional) probability pv_1, a contribution of $\frac{p(v_1)^2}{2}$ toward bidder 1's expected payoff. Or, if bidder 2 is not public, bidder 1 will bid zero and win with probability $(1-p)$. Overall, bidder 1's gross interim expected profit is $S^P(v_1) = v_1\left((1-p) + \frac{pv_1}{2}\right)$.

Suppose next that bidder 1 were to participate secretly. Whether bidder 2 participates secretly or publicly, bidder 1's best response in the resulting bidding subgame will be to bid zero, winning when bidder 2 does not arrive or has value less than $\underline{v}^S$, with probability $(1-p)+p\underline{v}^S$. (If bidder 2 has value in $(0, \underline{v}^S)$, he will participate publicly, bid zero, and lose to bidder 1, since ties are broken in favor of secret bidders.) Thus, bidder 1's gross interim expected profit is $S^S(v_1) = v_1\left((1-p)+p\underline{v}^S\right)$. For all $v_1 \le \underline{v}^S = \sqrt{\frac{2c_S}{p}}$, $S^S(v_1) - S^P(v_1) = v_1 p\left(\underline{v}^S - \frac{v_1}{2}\right) \le \frac{p\left(\underline{v}^S\right)^2}{2} = c_S$,[10] with equality only at $v_1 = \underline{v}^S$. Thus, bidder 1 strictly prefers to participate publicly when $v_1 \in \left(0, \underline{v}^S\right)$ and is indifferent between secret and public participation when $v_1 = \underline{v}^S$.

Case #2: $v_1 > \underline{v}^S$. Previously, I showed that (i) $F(v|P) = F(v|S) = v$ for all $v > \underline{v}^S$ and (ii) if $v_1 > \underline{v}^S$, then bidder 1 wins in every bidding subgame – no matter who participates publicly – iff he has the highest value. By the Envelope Theorem, then, $S^S(v) = S^S(\underline{v}^S) + \int_{\underline{v}^S}^{v} x\mathrm{d}x$ and $S^P(v) = S^P(\underline{v}^S) + \int_{\underline{v}^S}^{v} x\mathrm{d}x$ for all $v > \underline{v}^S$. Since $S^S(\underline{v}^S) - S^P(\underline{v}^S) = c_S$ then, $S^S(v_1) - S^P(v_1) = c_S$ and bidder 1 is indifferent between secret and public participation for all $v_1 > \underline{v}^S$. □

Revenue losses due to bidder secrecy. The equilibrium of Claim 1 is allocation-equivalent to the unique symmetric PBE of the second-price auction. (The good is always sold to the bidder with the highest value.) Thus, each bidder's interim expected payoff

[10]FOR REFEREES: Note that $\frac{\mathrm{d}\left[v_1(\underline{v}^S - v_1/2)\right]}{\mathrm{d}v_1} = \underline{v}^S - v_1 > 0$ for all $v_1 < \underline{v}^S$.

is the same as in the second-price auction, and all of the extra costs associated with secret participation translate into lower equilibrium expected revenue. Since each bidder $i = 1, 2$ participates secretly with probability $pq^S(1 - \underline{v}^S)$, where $q^S = \frac{1-p}{(1-p)+p\underline{v}^S}$ and $\underline{v}^S = \sqrt{\frac{2c_S}{p}}$, and incurs extra cost c_S when doing so, this expected revenue loss is

$$LOSS(c_S, p) = 2c_S pq^S(1 - \underline{v}^S) = 2c_S p \frac{1-p}{(1-p) + \sqrt{p2c_S}} \left(1 - \sqrt{\frac{2c_S}{p}}\right). \tag{3}$$

Claim 2. *Suppose that $p = \frac{1}{2}$ and $c_S = \frac{1}{9}$. Expected revenue in the equilibrium of Claim 1 is more than 25% less than in the symmetric PBE of the second-price auction.*

Proof. Expected revenue in the second-price auction $REV(p) = \frac{p^2}{3} = \frac{1}{12}$ when $p = \frac{1}{2}$. However, the loss due to secrecy $LOSS\left(\frac{1}{9}, \frac{1}{2}\right) = \frac{1}{45}$, more than 25% of $\frac{1}{12}$. □

4 Equilibrium secrecy and/or publicity

This section explores when bidders participate publicly or secretly in PBE, depending on the participation costs (c_S, c_P) and the distribution of bidder values. There are two main findings. First, if $c_P > 0$ and $c_P \geq c_S$, all participation is secret in all symmetric PBE (Section 4.1). Second, if $c_P > 0$ and $c_S > c_P$ but the extra cost of secrecy is sufficiently small, every symmetric PBE exhibits a mixture of some public and some secret participation (Section 4.2).

4.1 Only-secret participation

Suppose that secret and public participation are equally costly, $c_S = c_P > 0$. Could a symmetric PBE exist with only-public participation? No. If so, participating secretly would be a profitable deviation, since one's secrecy would induce others to believe that they face one less competitor, and hence bid less. In fact, *all* symmetric PBE in this case exhibit only-secret participation.

Theorem 2. *Suppose $c_S = c_P > 0$. All symmetric PBE exhibit only-secret participation.*

Theorem 2 is a corollary of the following result. (If at most one bidder ever participates publicly in any PBE, then no bidder can ever participate publicly in any *symmetric* PBE.)

Theorem 2'. *Suppose that $c_S = c_P > 0$. In every PBE, at most one bidder participates publicly with positive probability.*[11]

Proof. In the Appendix. □

Proof of Theorem 2 with two bidders. To develop intuition, I offer here a proof of Theorem 2 in the special case when $n = 2$. Suppose that there exists a symmetric PBE in which bidders sometimes participate publicly, i.e. $\underline{v}^P < 1$. I will establish a contradiction by showing that each bidder $i = 1, 2$ strictly prefers to participate secretly given value $v_i = \underline{v}^P$. Without loss, consider bidder 1.

Step #1: What if bidder 2 participates publicly? Suppose first that bidder 2 were to participate publicly. In any bidding equilibrium of the subgame with public bidders $P = \{1, 2\}$, each bidder will always bid at least $\underline{v}^P$ since both bidders' values are sure (on the equilibrium path) to be at least $\underline{v}^P$. Thus, if bidder 1 also participates publicly given value $v_1 = \underline{v}^P$, bidder 1 will never win the object and earn zero gross profit. (Bidder 1's net profit, after accounting for participation costs, is $-c_P$.)

By contrast, if bidder 1 were to participate secretly, bidder 1 would earn positive expected gross payoff given value $v_1 = \underline{v}^P$ in the subgame with public bidders $P = \{2\}$.

[11] I conjecture but have been unable to prove that, when $c_S = c_P > 0$, no PBE exists in which exactly one bidder sometimes participates publicly. If true, all PBE exhibit only-secret participation in this case.

To see why, note that bidder 2 can earn a positive expected gross profit by bidding zero, since bidder 1 chooses not to participate with positive probability. Consequently, bidder 2 must always strictly shade his bid below value. In particular, bidder 2 bids less than $\underline{v}^P$ with positive probability, allowing bidder 1 to earn positive expected gross payoff given $v_1 = \underline{v}^P$.

Step #2: What if bidder 2 does not participate publicly? Suppose next that bidder 2 does not participate publicly. If bidder 1 participates publicly so that $P = \{1\}$, bidder 2 believes that bidder 1's value must be at least $\underline{v}^P$ and hence that bidder 1 will never bid less than $\hat{b} = b_1(\underline{v}^P; \{1\})$. In any best response when participating secretly, then, bidder 2 will (i) bid less than $\hat{b}$ if $v_2 < \hat{b}$ (in order not to win at a price that exceeds his value) and (ii) bid more than $\hat{b}$ if $v_2 > \hat{b}$ (in order to have a chance of winning). In other words, bidder 1 wins the object iff either bidder 2 does not participate or bidder 2 participates secretly but has value $v_2 < \hat{b}$, the same *as if* bidder 2 always bid his full value when participating.

By contrast, if bidder 1 were to participate secretly so that $P = \emptyset$, bidder 2 can earn a positive expected gross profit by bidding zero. Thus, bidder 2 must strictly shade his bid below his value when participating secretly. When playing a best response, then, bidder 1's expected gross payoff must be greater than or equal to his expected gross payoff in the subgame with $P = \{1\}$.

Since secret and public participation are equally costly, we conclude that bidder 1 (i) strictly prefers to participate secretly if bidder 2 participates publicly and (ii) weakly prefers to participate secretly if bidder 2 does not participate publicly, conditional on $v_1 = \underline{v}^P$. Further, since $\underline{v}^P < 1$, bidder 2 participates publicly with positive probability. Thus, bidder 1 strictly prefers to participate secretly at the interim stage when $v_1 = \underline{v}^P$, a contradiction. $\square$

What if public participation is more costly? Suppose now that $c_P > c_S \geq 0$. In this case, I can prove the stronger result that *all* PBE exhibit secret-only participation. Further, there is a unique PBE.

Theorem 3. *Suppose $c_P > c_S \geq 0$. The unique PBE exhibits only-secret participation.*

Proof. Theorem 3 follows from a deeper result. First, a definition is needed.

Definition 4 (Bidder k)**.** Order bidders by their public participation thresholds in a given PBE, $\underline{v}_1^P \leq ... \leq \underline{v}_n^P$, and define $\underline{v}_0^P = 0$. Let "bidder k" be the bidder with the highest public participation threshold, among all who sometimes participate publicly, i.e. $k = \max\{i = 0, 1, ..., n : \underline{v}_i^P < 1\}$. ($k = 0$ iff the PBE exhibits only-secret participation.)

Theorem 3'. *In any PBE in which $k \geq 1$, bidder k's interim expected* gross *payoff given value $v_k = \underline{v}_k^P$ is weakly greater if he participates secretly than if he participates publicly.*

Proof. In the Appendix. □

Part One: All PBE exhibit only-secret participation. When $c_P > c_S$, Theorem 3' implies that all PBE must exhibit only-secret participation. To see why, suppose for the sake of contradiction that $k \geq 1$. By Theorem 3', bidder k's interim expected *net* payoff given value $v_k = \underline{v}_k^P$ is *strictly* greater if he participates secretly than if he participates publicly. By continuity of bidder k's interim expected payoff in v_k, bidder k must therefore participate secretly given any value in a neighborhood of $\underline{v}_k^P$. This contradicts the definition of $\underline{v}_k^P$ as the infimum of values given which bidder k sometimes participates publicly.

Part Two: Uniqueness. By Part One, all participation is secret in all PBE when $c_P > c_S \geq 0$. Any PBE therefore remains an equilibrium if one *disallows* public participation.

In particular, any PBE must be outcome-equivalent to the *unique*[12] equilibrium of a standard first-price auction with iid bidders, only-secret participation, and participation cost $c_S \geq 0$. □

Proof of Theorem 3 with two bidders. To develop intuition, I offer here a proof that all PBE exhibit only-secret participation (Part One above) in the special case when $n = 2$. The argument presented here is distinct from that presented in the Appendix. The Appendix proof of Theorem 3' focuses on the bidder with the *highest* public participation threshold, whereas the argument here focuses on the bidder with the *lowest* public participation threshold. Both approaches are revealing.

Without loss, suppose that bidder 1 has the lowest public participation threshold, $\underline{v}_1^P \leq \underline{v}_2^P$. I will establish a contradiction by showing that bidder 1 strictly prefers to participate secretly, conditional on realized value $v_1 = \underline{v}_1^P$.

Step #1: What if bidder 2 participates publicly? Suppose first that bidder 2 were to participate publicly. (If $\underline{v}_2^P = 1$, this occurs with zero probability.) Since $\underline{v}_2^P \geq \underline{v}_1^P$, bidder 2 is certain to bid at least $\underline{v}_1^P$ in any bidding equilibrium of the subgame with public bidders $P = \{1, 2\}$. So, given value $v_1 = \underline{v}_1^P$, bidder 2 is certain to lose and earns payoff $-c_P$ in this subgame. Of course, participating secretly yields a payoff of at least $-c_S$, strictly better since $c_S < c_P$.

Step #2: What if bidder 2 does not participate publicly? Suppose next that bidder 2 does not participate publicly. If bidder 1 participates publicly so that $P = \{1\}$, repeating the argument in the two-bidder proof of Theorem 2, bidder 1's payoff will be the same *as*

[12]Samuelson (1985) showed that there is a unique symmetric equilibrium, and Lebrun (2006) proves uniqueness when $c_S = 0$. It is straightforward to adapt Lebrun's method when $c_S \geq 0$ to show, first, that all equilibria must be symmetric and, second, that there is a unique symmetric equilibrium. So, there is in fact a unique equilibrium.

if bidder 2 always bid his full value whenever participating secretly. Thus, given value $v_1 = \underline{v}_1^P$, bidder 1's expected payoff in the subgame with $P = \{1\}$ is

$$\Pi_1(\underline{v}_1^P; \{1\}) = \max_{b \geq 0} (\underline{v}_1^P - b) \Pr\left(v_2 < b \text{ or } v_2 < \underline{v}_2^S | 2 \notin P\right) - c_P. \tag{4}$$

By contrast, in the subgame with no public bidders, bidder 1's expected payoff given value $v_1 = \underline{v}_1^P$ is

$$\Pi_1(\underline{v}_1^P; \emptyset) = \max_{b \geq 0} (\underline{v}_1^P - b) \Pr\left(b_2(v_2; \emptyset) < b \text{ or } v_2 < \underline{v}_2^S | 2 \notin P\right) - c_S. \tag{5}$$

$\Pi_1(\underline{v}_1^P; \emptyset) > \Pi_1(\underline{v}_1^P; \{1\})$ for a combination of reasons. First, $b_2(v_2; \emptyset) \leq v_2$ since bidder 2 never bids more than his value. Thus, the maximization term of (4) is greater than or equal to that of (5). Second, secret participation is less costly, so that $-c_P < -c_S$.

In summary, bidder 1 strictly prefers to participate secretly given value $v_1 = \underline{v}_1^P$, regardless of whether or not bidder 2 participates publicly. Thus, bidder 1 strictly prefers to be secret at the interim stage given value $v_1 = \underline{v}_1^P$, a contradiction. □

4.2 Mixture of secret and public participation

This section considers the most interesting case in which secret participation is more costly ($c_S > c_P > 0$). If unexpected, secrecy induces others to believe that they face fewer competitors and hence bid less, making secret participation a profitable deviation should all bidders only participate publicly. Thus, as long as the extra cost of secrecy is small enough, no PBE exists with only-public participation.

Could a PBE exist with only-secret participation? No. By Part Two of the proof of Theorem 3, any such PBE must be outcome-equivalent to the symmetric equilibrium in a more-standard first-price auction in which public participation is not allowed and secret participation costs c_S. In this equilibrium, each bidder participates (secretly) iff his value exceeds the threshold $\underline{v}^* = \sqrt[n]{c_S}$. (A bidder with value $v_i = \underline{v}^*$ bids zero and wins with probability $(\underline{v}^*)^{n-1}$, for gross expected payoff c_S.) However, given such participation, each bidder i can profitably deviate by participation publicly and bidding zero given value v_i slightly below $\underline{v}^*$. (Expected payoff after this deviation is $v_i\,(\underline{v}^*)^{n-1} - c_P \approx c_S - c_P > 0$.)

The following result formalizes this argument and, as a function of c_P and the distribution of bidder values, determines the threshold level for c_S below which some participation must be secret.

Theorem 4. *Suppose that $c_S > c_P > 0$. (i) Every symmetric PBE exhibits some public participation. (ii) As long as $c_P \in (0,1)$, there exists $c_S(c_P) \in (c_P, 1)$ such that all symmetric PBE exhibit some secret and some public participation iff $c_S \in (c_P, c_S(c_P))$.*

Proof. First, suppose for the sake of contradiction that a symmetric PBE exists with only-secret participation. Any such equilibrium must correspond to the unique symmetric equilibrium of a standard first-price auction in which (i) participation costs c_S and each bidder i participates iff $v_i > \underline{v}(c_S)$, (ii) bidders do not observe the number of participants prior to the bidding but, conditional on participation, each bidder's value is drawn according to the cdf $F(v|v > \underline{v}(c_S))$, and (iii) the threshold $\underline{v}(c_S)$ is implicitly determined by the indifference condition $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} = c_S$. In particular, a bidder with value $\underline{v}(c_S)$ bids zero, is sure to lose unless no one else participates, and earns zero net expected profit. But such a bidder could profitably deviate by participating publicly (still bidding zero), for net profit $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} - c_P > 0$. This is a contradiction.

Next, suppose for the moment that a symmetric PBE exists with only-public participation. Any such equilibrium must correspond to the unique symmetric equilibrium of the standard first-price auction in which (i) participation costs c_P and each bidder participates iff $v_i > \underline{v}(c_P)$, (ii) bidders observe the number of participants prior to the bidding where, conditional on participation, each bidder's value is drawn according to the cdf $F(v|v > \underline{v}(c_P))$, and (iii) the threshold $\underline{v}(c_P)$ is implicitly determined by the indifference condition $\underline{v}(c_P)F(\underline{v}(c_P))^{n-1} = c_P$.

In this well-known equilibrium, each bidder's bid is equal to the expected second-highest value for the good, conditional on his own value being the highest. (If there are no other bidders, bidder i bids zero.) In particular, each bidder i bids $b(v_i; m) = E\left[\max_{j=1,...,m} v_j | v_j \in [\underline{v}(c_P), v_i] \text{ for all } j\right]$ when there are m other public bidders.

Without loss, consider bidder n and suppose that only bidders $1, ..., m$ also participate publicly. Let M denote the number of other public bidders, viewed as a random variable. Note that $\Pr(M = m) = (1 - F(\underline{v}(c_P)))^m F(\underline{v}(c_P))^{n-m-1} > 0$ for all $m = 0, ..., n-1$. If bidder n participates publicly and faces m others, his best response to others' strategies will yield expected gross payoff

$$\Pi^P(v_n; m) + c_P = \max_{b \geq 0}(v_n - b) \Pr\left(b > \max_{i=1,...,m} b(v_i; m) | \min_{i=1,...,m} v_i \geq \underline{v}(c_P)\right).$$

By contrast, if bidder n participates secretly and faces the same m opponents, his best response to others' strategies will now yield expected gross payoff

$$\Pi^S(v_n; m) + c_S = \max_{b \geq 0}(v_n - b) \Pr\left(b > \max_{i=1,...,m} b(v_i; m-1) | \min_{i=1,...,m} v_i \geq \underline{v}(c_P)\right).$$

The key difference is that, when bidder n participates secretly, others will bid *as if* there is one fewer bidder. Since $b(v_i; m) > b(v_i; m-1)$ for every $m > 0$ and every $v_i \geq \underline{v}(c_P)$, secret participation yields strictly greater expected gross payoff, $\Pi^S(v_n; m) + c_S > \Pi^P(v_n; m) + c_P$, for all $m \geq 1$. Let $\Delta(v_n; m) = \Pi^S(v_n; m) + c_S - (\Pi^P(v_n; m) + c_P)$ denote this gross-payoff benefit of secrecy. Next, define $\Delta(c_P) = \max_{v_n \in [\underline{v}(c_P), 1]} E_M\left[\Delta(v_n; M)\right]$ and $v^* = \arg\max_{v_n \in [\underline{v}(c_P), 1]} E_M\left[\Delta(v_n; M)\right]$.

Define $c_S(c_P) = c_P + \Delta(c_P)$. $c_S(c_P) > c_P$ since $\Delta(c_P) > 0$. Further, it must be that $c_S(c_P) < v^*$. To see why, note that by definition $c_S(c_P)$ is the secrecy cost that makes bidder-type v^* indifferent between participating secretly or publicly. However, since $v^* > \underline{v}(c_P)$, bidder-type v^* earns positive net profit when participating publicly and hence must also earn positive net profit when participating secretly at cost $c_S(c_P)$. So, $c_S(c_P) < v^*$.

If $c_S > c_S(c_P)$, every bidder prefers not to deviate by participating secretly, so a symmetric PBE exists with only-public participation. By contrast, if $c_S < c_S(c_P)$, each bidder strictly prefers to deviate by participating secretly when his value is in a neighborhood of v^*. So, every symmetric PBE must exhibit some public and some secret participation in this case. This completes the proof. □

4.3 Illustrative example: continued

Consider again the example of Section 3, in which two bidders have iid values $v_i \sim U[0,1]$. However, I will now focus on the case in which both bidders always arrive ($p = 1$) but public participation is costly ($c_P > 0$).

- If $c_P > 0$ and $c_P \geq c_S$, all symmetric PBE exhibit only-secret participation (Theorems 2 and 3).
- If $c_S > c_P > 0$, how bidders participate depends on the extra cost of secrecy, $c_S - c_P$. In particular, all symmetric PBE exhibit some secret and some public participation iff $c_S \in (c_S(c_P), 1)$, where $c_S(c_P) \in (c_P, 1)$ (Theorem 4).

It remains only to compute the threshold secrecy-cost $c_S(c_P)$ of Theorem 4.

Claim 3. *Consider a variation of the example of Section 3, in which $p = 1$ and $c_P \in (0,1)$. The threshold secrecy cost $c_S(c_P) = \frac{1+c_P}{2}$.*

Proof. Any symmetric PBE with only-public participation must correspond to the unique symmetric PBE when secrecy is impossible. Namely, each bidder i (i) participates iff

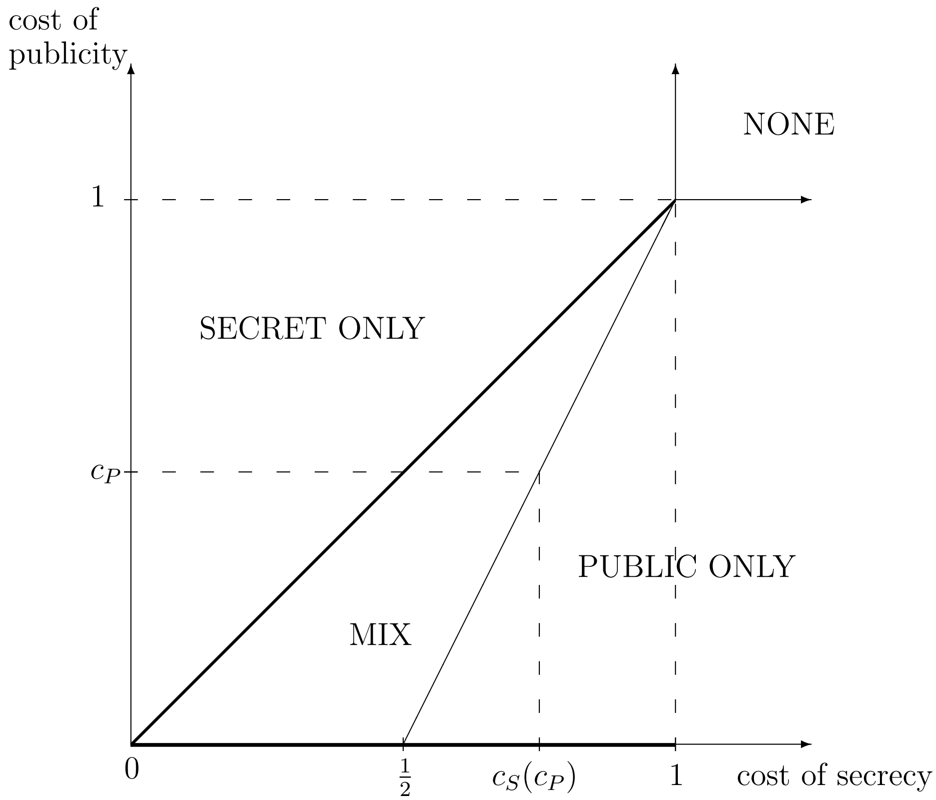

Figure 3: Illustration of how bidders participate in all symmetric PBE.

$v_i > \sqrt{c_P}$, (ii) bids zero if he is the only public bidder, and (iii) bids $b(v_i;\{1,2\}) = \frac{v_i+\sqrt{c_P}}{2}$ if both are public. If $v_j < \sqrt{c_P}$ so that bidder j does not participate, there is no *ex post* benefit to secrecy since bidder i wins at price zero whether he is secret or public. If $v_i > \sqrt{c_P}$, bidding secretly induces bidder j to bid zero, allowing bidder i to win at price zero for gross payoff v_i. By contrast, bidding publicly induces bidder j to bid $b(v_j;\{1,2\}) \geq \sqrt{c_P}$, so that bidder 1's best response is to bid $b(v_i;\{1,2\})$ and earn conditional expected gross payoff $(v_i - b(v_i;\{1,2\}))\Pr(v_j < v_i | v_j \geq \sqrt{c_P}) = \frac{(v_i-\sqrt{c_P})^2}{2(1-\sqrt{c_P})}$. Since bidder j participates with probability $1-\sqrt{c_P}$, bidder i's expected gross benefit from secrecy is $(1-\sqrt{c_P})\left(v_i - \frac{(v_i-\sqrt{c_P})^2}{2(1-\sqrt{c_P})}\right) = v_i - \frac{v_i^2}{2} - \frac{c_P}{2}$.

This secrecy-deviation benefit is increasing in v_i, and hence maximized at $\frac{1-c_P}{2}$ when

$v_i = 1$. Thus, (i) if $c_S \geq c_P + \frac{1-c_P}{2} = \frac{1+c_S}{2}$, all types at least weakly prefer not to deviate by participating secretly while (ii) if $c_S < \frac{1+c_S}{2}$, each bidder i strictly prefers to deviate secretly given values in a neighborhood of $v_i = 1$. This completes the proof. □

5 The cost of secrecy

This section explores how welfare and revenue vary in all symmetric PBE with the cost of secret participation. When secret participation is less costly ($c_S < c_P$), increasing c_S decreases expected bidder surplus and revenue, as one would expect (Theorem 5). However, when secret participation is more costly ($c_S > c_P$), increasing c_S has a non-monotone effect on expected total welfare and revenue. Namely, *either* raising c_S to infinity (Theorem 6) *or* lowering c_S to c_P (corollary to Theorem 6) has the effect of (i) weakly decreasing interim expected bidder surplus, (ii) strictly increasing expected revenue, and (iii) strictly increasing expected total welfare.

Only-secret participation. Suppose first that $c_P > c_S \geq 0$ so that all symmetric PBE exhibit only-secret participation (Theorem 3). In this case, both expected bidder surplus and expected seller revenue are strictly decreasing in c_S. In fact, interim expected bidder surplus and *ex post* seller revenue are each weakly decreasing in c_S.

Theorem 5. *Suppose that $c_P > c_S \geq 0$ so that the unique symmetric PBE exhibits only-secret participation. Interim expected bidder surplus and ex post seller revenue in this equilibrium are strictly decreasing in c_S over the range $(0, c_P)$.*

Proof. Any symmetric PBE with only-secret participation must correspond to the unique symmetric equilibrium of the standard first-price auction in which all participation must be secret. In particular, each bidder participates iff $v_i > \underline{v}(c_S)$, where the threshold $\underline{v}(c_S)$

is determined by $\underline{v}(c_S)F(\underline{v}(c_S))^{n-1} = c_S$. Note that $\underline{v}(c_S)$ is increasing in c_S.

By the Envelope Theorem, each bidder's interim expected surplus can be characterized as the integral of his probability of winning: $\Pi(v_i) = 0$ for all $v_i \leq \underline{v}(c_S)$ and $\Pi(v_i) = \int_{\underline{v}(c_S)}^{v_i} F(x)^{n-1}\mathrm{d}x$ for all $v_i > \underline{v}(c_S)$. Increasing the secret participation cost from c_S to c'_S leads to an increase in the participation threshold from $\underline{v}(c_S)$ to $\underline{v}(c'_S)$ and therefore strictly decreases interim expected surplus for all $v_i > \underline{v}(c_S)$. (If $v_i \leq \underline{v}(c_S)$, bidder i continues to earn zero payoff.)

Since bidders choose not to participate given values less than $\underline{v}(c_S)$, equilibrium bids are the same *as if* in a standard first-price auction with zero participation cost, in which bidders' have iid values with cdf $F(.;c_S)$ defined by $F(v;c_S) = F(\underline{v}(c_S))$ for all $v \leq \underline{v}(c_S)$ and $F(v;c_S) = F(v)$ for all $v > \underline{v}(c_S)$. (That is, each bidder's value has an "atom" at $-\infty$ whose mass $F(\underline{v}(c_S))$ is increasing in c_S.) Consequently, each bidder's equilibrium bid in the unique symmetric equilibrium – equal to the expected second-highest value, conditional on his value being the highest – is strictly decreasing in c_S. We conclude that the seller's *ex post* revenue is weakly decreasing in c_S, and strictly decreasing conditional on $\max_i v_i > \underline{v}(c_S)$. $\square$

Mixture of secret and public participation. Suppose next that $c_S \in (c_P, c_S(c_P))$, so that all symmetric PBE exhibit some public and some secret participation (Theorem 4). Theorem 6 shows that *disallowing secrecy* strictly increases both expected total welfare and expected revenue, when one compares the unique symmetric equilibrium when secrecy is disallowed with *any* symmetric PBE when secrecy is possible.

Theorem 6. *Suppose that $c_P \in (0,1)$ and $c_S \in (c_P, c_S(c_P))$, where $c_S(c_P)$ is the secrecy-cost threshold identified by Theorem 4. Disallowing secrecy (raising c_S to infinity) strictly increases expected total welfare and expected seller revenue among all symmetric PBE.*

Corollary to Theorem 6. *Suppose that* $c_P \in (0,1)$ *and* $c_S \in (c_P, c_S(c_P))$*. Eliminating the extra cost of secrecy (lowering* c_S *to* c_P*) strictly increases expected total welfare and expected seller revenue among all symmetric PBE.*

Proof of the corollary. If the seller were to eliminate the extra cost of secrecy so that $c_S = c_P = c > 0$, Theorem 2 implies that the unique symmetric PBE is outcome-equivalent to the unique symmetric equilibrium in a standard first-price auction with only-secret participation (when $c_S = c$ and $c_P = \infty$). As is well-known, this equilibrium generates the same ex post total welfare, expected seller revenue, and interim expected bidder surplus as the unique symmetric equilibrium when only public participation is possible (when $c_P = c$ and $c_S = \infty$). The corollary then follows immediately from Theorem 6. □

Outline of the proof of Theorem 6. The rest of this section proves Theorem 6. Part One considers the benchmark in which secrecy is disallowed ($c_S = \infty$). The rest considers the case $c_S \in (c_P, c_S(c_P))$. Parts Two-Three establish that all symmetric PBE share the same participation threshold $\underline{v}(c_P)$ as when $c_S = \infty$. This fact is then used to show that both expected total welfare (Part Four) and expected revenue (Part Five) are strictly higher in the unique symmetric PBE when $c_S = \infty$ than in all symmetric PBE when $c_S \in (c_P, c_S(c_P))$.

Part One: Benchmark when secrecy is disallowed. If c_S were raised to infinity, any symmetric PBE must correspond to the unique symmetric equilibrium of the standard first-price auction with only-public participation. Namely, (i) each bidder i participates at cost c_P iff $v_i > \underline{v}(c_P)$, where the participation threshold $\underline{v}(c_P)$ is determined implicitly by $\underline{v}(c_P)F(\underline{v}(c_P))^{n-1} = c_P$, and (ii) whenever anyone participates, the highest-value bidder wins.

Ex post total welfare: Ex post total welfare equals $\max_i v_i - c_P\#\{i : v_i > \underline{v}(c_P)\}$ if $\max_i v_i > \underline{v}(c_P)$, and zero otherwise.

Expected revenue: By standard arguments (see e.g. Bulow and Roberts (1989)), the expected value of the good to the winner minus total expected bidder surplus is equal to the expected maximal virtual value, $E\left[\max_i VV(v_i)\right]$, where $VV(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ for all $v_i > \underline{v}(c_P)$ and we set $VV(v_i) = 0$ for all $v_i < \underline{v}(c_P)$. Thus, the seller's expected revenue $E[R] = E\left[\max_i VV(v_i)\right] - nc_P(1 - F(\underline{v}(c_P)))$. (The expected total costs of participation, $nc_P(1 - F(\underline{v}(c_P)))$, are passed through to the seller in equilibrium.)

Part Two: The lowest-value bidders to participate do so publicly. Consider any symmetric PBE given costs $c_P < 1$ and $c_S \in (c_P, c_S(c_P))$. The first (surprisingly subtle) step of the proof is to show that the lowest-value bidders who participate must sometimes choose to participate publicly, i.e. $\underline{v} = \underline{v}^P$.

Lemma 1. *Suppose that $c_S > c_P > 0$. In every symmetric PBE, $\underline{v}^P = \underline{v}$.*

Proof. See the Appendix □

Discussion of Lemma 1: In standard models with only-secret or only-public participation, any symmetric equilibrium must have the feature that a bidder whose value equals the participation threshold only wins the object if no one else participates. Since no one else participates with probability $F(\underline{v})^{n-1}$, the participation threshold in such models is therefore simply that determined by $\underline{v}F(\underline{v})^{n-1} = c$, where c is the cost of participation.

When bidders sometimes participate publicly and sometimes secretly, however, a bidder with value $\underline{v}$ might potentially outbid higher-value opponents. For instance, suppose that all bidders adopted a symmetric strategy such as that illustrated earlier in Figure 1, in which $\underline{v} < \underline{v}^P$. Even though secrecy is more costly, a bidder (say bidder 1) with value $\underline{v} > 0$ potentially stands to gain by investing in secrecy. For one thing, if there is exactly one other public bidder, being secret will induce that opponent to bid less and thereby potentially allow bidder 1 to win with positive probability. And even if there are no other public bidders, bidder 1 may prefer to avoid "publicizing" that his value is at least $\underline{v}^P$ (especially since his true value is $\underline{v} < \underline{v}^P$), if doing so leads to more aggressive bidding competition from other *secret* bidders.

Of course, to sustain a symmetric PBE with strategies as depicted in Figure 1, each bidder must also at least weakly prefer *not* to participate secretly given the higher value $\underline{v}^P$. The key step of the proof of Lemma 1 is to show that, if $\underline{v}^S < \underline{v}^P$, then the expected benefit of secrecy is strictly greater when $v_1 = \underline{v}^P$ than when $v_1 = \underline{v}^S$. Thus, if bidders are willing to participate secretly given value $v_1 = \underline{v}^S$, they must *strictly* prefer to participate secretly when $v_1 = \underline{v}^P$, a contradiction.

Part Three: The lowest-value bidders to participate bid zero when "alone." Suppose that some bidder (say bidder 1) participates publicly with value $v_1 = \underline{v}^P = \underline{v}$. If any other bidder also participates publicly, both will bid at least $\underline{v}$ since both are sure (on the equilibrium path) to have values greater than or equal to $\underline{v}$. However, what if bidder 1 is the only public bidder? Could *secret* bidders' equilibrium strategies be such that bidder 1's best response is to submit a positive bid, even though he is "alone," or must bidder 1 bid zero as if truly facing no competition? In this step, I will show that he must bid zero.

If bidder 1 participates publicly and faces no public opponent, all others believe that he is certain not to bid less than $\hat{b} = b_1^P(\underline{v}^P; \{1\}) \geq 0$. If $\hat{b} = 0$, we are done. So suppose that $\hat{b} > 0$. Any secret participant having value greater than $\hat{b}$ must bid at least $\hat{b}$, in order to win the good with positive probability. However, since $\underline{v}^S \geq \underline{v}^P$ (by Part One) and $\underline{v}^P > \hat{b}$ (since bidder 1 must shade his bid below his value), any secret participant $i \neq 1$ must have value $v_i > \hat{b}$ *and bid more than* $\hat{b}$. So, in any bidding equilibrium of the subgame with $P = \{1\}$, bidder 1 only wins the good given value $v_1 = \underline{v}^P$ when facing no competition. In any equilibrium, then, bidder 1 must bid zero given value $v_1 = \underline{v}^P$.

Overall, bidder 1's interim expected payoff when participating publicly with value $v_1 = \underline{v}$ equals $(\underline{v} - 0)F(\underline{v})^{n-1} - c_P$. Since bidder 1 must receive an interim expected payoff of zero at the participation threshold $\underline{v}(c_S, c_P)$, this threshold is characterized by

$$\underline{v}(c_S, c_P)F(\underline{v}(c_S, c_P))^{n-1} = c_P. \tag{6}$$

Note that this participation threshold is the same as that derived in Part One, i.e. $\underline{v}(c_S, c_P) = \underline{v}(c_P)$ for all $c_S \in (c_P, \infty)$.

Part Four: Total welfare. The object is sold iff $\max_i v_i > \underline{v}$ whether $c_S = \infty$ or $c_S \in (c_P, c_S(c_P))$, and sold to the highest-value bidder when $c_S = \infty$. Thus, ex post *gross* total welfare is weakly higher when $c_S = \infty$ than when $c_S \in (c_P, c_S(c_P))$. Further, each bidder i participates iff $v_i > \underline{v}$ in either case, but sometimes pays $c_S > c_P$ when $c_S \in (c_P, c_S(c_P))$. Thus, ex post total welfare is weakly higher when $c_S = \infty$ and expected total welfare is strictly higher when $c_S = \infty$ than in any symmetric PBE when $c_S \in (c_P, c_S(c_P))$, by an amount equal to $n(c_S - c_P)E\left[q^S(v_i)\right]$.

Part Five: Revenue. The seller's expected revenue equals the expected virtual value of the winner (or zero if the object is not sold) minus total expected incurred participation costs. Expected incurred participation costs are strictly lower when $c_S = \infty$. Further, since each bidder's virtual value $VV(v_i)$ is increasing in v_i and the highest-value participant always wins when $c_S = \infty$, the winner's expected virtual value is weakly higher when $c_S = \infty$. All together, then, expected revenue is strictly higher when $c_S = \infty$.

6 Concluding remarks

The auction theory literature has long recognized that bidders have an incentive to guard their private information; see e.g. Milgrom and Weber (1982). This paper explores the related idea that bidders have an incentive to hide whether they are interested enough to participate in the auction. When bidders have private values and simultaneously decide whether to hide or reveal their participation in a first-price auction, (i) no bidder will ever pay extra to publicize their interest but (ii) bidders are sometimes willing to pay extra to keep their interest secret. Furthermore, such equilibrium secrecy can translate into significant losses for the seller, providing a novel rationale for the seller to use a second-price or English auction in which bidders do not stand to gain from hiding their participation.

If one relaxes the assumption of *private values*, bidders can have an incentive to pay extra to hide their participation, even in an English auction. Indeed, reputable auction-houses such as Sotheby's frequently offer secret bidding in English auctions – by phone, or by pre-arranged secret signs that allow a bidder to be in the room and bid without anyone else knowing – as a service to favored clients. Further, offering the option of secrecy could potentially increase expected revenue in such auctions, especially if bidders are asymmetric and some bidders are only willing to participate when secret. For instance, a wealthy connoisseur might be unwilling to participate publicly, fearing that dealers in the audience will then want to win and re-sell the art to him on disadvantageous terms. Similarly, an art expert might be unwilling to participate publicly, fearing that wealthy connoisseurs in the audience will then want to win, based solely on his interest in it.

On the other hand, if one relaxes the assumption that bidders must *simultaneously* decide whether to publicize their participation, bidders can have an incentive to pay extra to publicize their participation. Daley, Schwarz, and Sonin (2012) considers a model in which, after receiving a signal about their private values, bidders have an option to burn money and then invest to increase their values (or generate a more informative signal) prior to a second-price auction.[13] Bidders sometimes burn money in equilibrium to deter others from investing, akin to publicizing one's intention to participate to deter others from participating.

Appendix

Proof of Theorem 1

Proof. **Payoffs.**[14] For all $M \subset N = \{1, ..., n\}$, let $\mathcal{B}(M) = \{(P, S) : P, S \subset M, P \cap S = \emptyset\}$ be the set of disjoint pairs of subsets of M. For each profile of realized values $\mathbf{v} = (v_1, ..., v_n)$ and each $(P, S) \in \mathcal{B}(N)$, let $\Pi_i(\mathbf{v}; P, S)$ denote bidder i's *ex post payoff* should the set of bidders P participate publicly, the set of bidders S participate secretly, and bidder i participates one way or the other, i.e. $i \in X \in \{P, S\}$:

$$\begin{aligned}\Pi_i(\mathbf{v}; P, S) &= \frac{v_i - b_i(v_i; P)}{\#\left(\arg\max_{j \in P \cup S} b_j(v_j; P)\right)} - c_X \text{ if } b_i(v_i; P) = \max_{j \in P \cup S} b_j(v_j; P) \\ &= -c_X \text{ if } b_i(v_i; P) < \max_{j \in P \cup S} b_j(v_j; P)\end{aligned}$$

Bidder i's *ex ante payoff* then takes the form

$$\int_{\mathbf{v}} \sum_{(P,S) \in \mathcal{B}(N \setminus i)} \Pi_{j \in P} q_j^P(v_j) \Pi_{k \in S} q_k^S(v_k) \left[q_i^P(v_i)\Pi_i(\mathbf{v}; P \cup i, S) + q_i^S(v_i)\Pi_i(\mathbf{v}; P, S \cup i)\right] \mathrm{d}\mathbf{v}.$$

[13]See also McAdams (2011), who derives welfare comparative statics in a related game in which bidders have the option to publicly pre-commit to participate in a second-price auction.

[14]I thank Phil Reny for suggesting this proof approach.

Equilibrium existence. Bidder i's ex ante payoff may be discontinuous with respect to bidders' strategies (in the usual weak-* topology), but only if bidder i ties in the bidding stage with positive ex ante probability. Consequently, each bidder's ex ante expected payoff is better-reply secure.[15] Given bidder symmetry, then, Corollary 5.3 of Reny (1999) establishes existence of a *symmetric* mixed-strategy Nash equilibrium.

Such a Nash equilibrium might fail to constitute a PBE. To avoid this possibility, consider the following perturbation of the game. Independent of his value, each bidder is forced to participate publicly or secretly, each with probability $\delta > 0$, but otherwise free to choose his participation strategy and to choose his bid. Repeating the previous argument, this perturbed game has a symmetric mixed-strategy Nash equilibrium. Any such equilibrium of the perturbed game is an $\varepsilon(\delta)$-equilibrium of the original unperturbed game, where $\lim_{\delta \to 0} \varepsilon(\delta) = 0$. In particular, since (i) each bidder plays a best response with probability $1 - 2\delta$ and (ii) ex post payoffs are bounded above by 1 (values lie in $[0, 1]$) and bounded below by $-\max\{c_P, c_S\}$ (no bidder ever bids more than his value), $\varepsilon(\delta) = 2\delta(1 + \max\{c_P, c_S\})$. By Remark 3.1 of Reny (1999), a convergent subsequence of symmetric equilibria in the δ-perturbed games therefore converges to a symmetric Nash equilibrium of the original game as $\delta \to 0$.

Along this convergent sequence of equilibria, the subgame with public bidders P is reached with positive probability, for all $P \subset N$. Further, each public bidder's value has full support on $[0, 1]$ (since each is "forced" to participate publicly with probability δ, independent of his value). Thus, in any subsequent bidding equilibrium, no bidder

[15] See Reny (1999) for a definition of better-reply secure payoffs. Reny (1999, pg. 1046) establishes that payoffs in the mixed extension of a standard first-price auction are better-reply secure. This argument extends to the first-price auction here with secret or public participation. The key idea in either case is that, when making a bid (less than his value) that ties with positive probability, a bidder can secure a better-reply by bidding slightly higher.

ever bids more than his value. We conclude that this equilibrium corresponds to a perfect Bayesian equilibrium in symmetric, weakly undominated strategies ("symmetric PBE"). □

Proof of Theorem 2' and Theorem 3'

Proof. Fix any PBE. Let $\Pi_i^P(v_i; X \cup \{i\})$ and $\Pi_i^S(v_i; X)$ denote bidder i's expected equilibrium payoff when participating publicly and secretly, respectively, conditional on value v_i and public opponents X. Let bidder k be the bidder with the highest public participation threshold, among those who sometimes participate publicly: $k = \max\{i = 0, 1, ..., n : \underline{v}_i^P < 1\}$, where $\underline{v}_1^P \leq ... \leq \underline{v}_n^P$ and $\underline{v}_0^P = 0$.

Proof of Theorem 3'. Suppose that $k \geq 1$ and let $X \subset \{1, ..., k-1\}$ denote the set of realized public bidders other than bidder k. (At the interim stage when bidder k decides whether and how to participate, bidder k only knows the distribution of X. However, X is commonly known at the bidding stage.) Recall that bidder k never participates publicly when his value is less than $\underline{v}_k^P$. Since equilibrium bidding strategies are weakly monotone, bidder k never bids less than $\hat{b}(X) = b_k^P(\underline{v}_k^P; X \cup \{k\})$ when public. Thus, any (secret or public) participant $i \neq k$ with value $v_i < \hat{b}(X)$ is certain to lose and any with value $v_i > \hat{b}(X)$ is certain to bid greater than $\hat{b}(X)$ to have a chance of winning.[16]

Bidder k's probability of winning with bid $\hat{b}(X)$ is therefore equal to the probability

[16]Every bidder $i \neq k$ bids *strictly* greater than $\hat{b}(X)$ when $v_i > \hat{b}(X)$. To see why, note that bidding less than $\hat{b}(X)$ is dominated when $v_i > \hat{b}(X)$ since any such bid always loses. Similarly, bidding exactly $\hat{b}(X)$ always loses unless bidder k bids $\hat{b}(X)$ with positive probability. Further, monotonicity of best replies requires that, if bidder i bids exactly $\hat{b}(X)$ given value $\hat{v}_i > \hat{b}(X)$, then he must bid exactly $\hat{b}(X)$ given all values $v_i \in (\hat{b}(X), \hat{v}_i)$. However, this means that bidders i, k tie with positive probability, an impossibility in equilibrium.

that all other bidders either do not participate or have values less than $\hat{b}(X)$. Thus, bidder k's equilibrium expected *gross* payoff $\Pi_k^P(\underline{v}_k^P; X \cup \{k\}) + c_P$ given value $\underline{v}_k^P$ in the subgame with public bidders $P = X \cup \{k\}$ is the same *as if* all other bidders always bid their full values whenever participating:

$$\Pi_k^P(\underline{v}_k^P; X) + c_P = \max_{b \geq \max\{\underline{v}_i^P : i \in X\}} \left(\underline{v}_k^P - b\right) \Pi_{i \in X} F(b | i \in X) \Pi_{j \notin X \cup \{k\}} F(\max\{b, \underline{v}_j^S\} | j \notin X). \tag{7}$$

(Each public bidder $i \in X$ is certain to participate and certain to have value greater than or equal to $\underline{v}_i^P \leq \underline{v}_k^P$. Thus, any bid $b < \max\{\underline{v}_i^P : i \in X\}$ is certain to lose.)

Since no bidder ever bids more than his value in any PBE, bidder k's expected gross payoff $\Pi_k^S(\underline{v}_k^P; X) + c_S$ when secret and bidding a best response to others' equilibrium strategies in the subgame with public bidders $P = X$ must therefore be greater than or equal to the value of the maximization in (7). Finally, $\Pi_k^S(\underline{v}_k^P; X) + c_S \geq \Pi_k^P(\underline{v}_k^P; X) + c_P$ for all $X \subset \{1, ...k-1\}$ implies $E\left[\Pi_k^S(\underline{v}_k^P; X)\right] + c_S \geq E\left[\Pi_k^P(\underline{v}_k^P; X)\right] + c_P$. This completes the proof of Theorem 3'. □

Proof of Theorem 2'. Suppose that $c_P = c_S = c > 0$ and, for the sake of contradiction, that there exists a PBE in which $k \geq 2$. Since participation is costly, each bidder sometimes chooses not to participate. Thus, bidder k must face exactly one public opponent, say $X = \{1\}$, with positive probability. As explained in the proof of Theorem 3', bidder k's expected gross payoff $\Pi_k^P(\underline{v}_k^P; \{1, k\}) + c$ in the subgame with public bidders $P = \{1, k\}$ is that given in (7), the same *as if* bidder 1 and all secret participants always bid their values when participating. Suppose instead that bidder k were to participate secretly given value $v_k = \underline{v}_k^P$, leaving bidder 1 as the only public bidder. Since others $j \neq 1, k$ do not bid more than their values, bidder k's gross expected payoff $\Pi_k^S(\underline{v}_k^P; \{1\}) + c$

when playing a best response is now *at least*

$$\max_{b \geq b_1^P(\underline{v}_1^P;\{1\})} \left(\underline{v}_k^P - b\right) \Pr_{v_1}(b > b_1^P(v_1;\{1\})|1 \in P)\Pi_{j\neq 1,k} F(\max\{b, \underline{v}_j^S\}|j \notin P). \qquad (8)$$

Since $\min\{c_P, c_S\} > 0$, (i) bidders $i \neq 1$ all fail to participate with positive probability and (ii) $\underline{v}_1^P > 0$. Thus, bidder 1 can earn a positive gross expected payoff by bidding zero and hence must always strictly shade his bid below value in the subgame $P = \{1\}$. In particular, (i) $b_1^P(\underline{v}_1^P;\{1\}) < \underline{v}_1^P$ and (ii) $\Pr(b > b_1^P(v_1;\{1\})|1 \in P) > F(b|1 \in P)$ for all $b \geq \underline{v}_1^P$. Thus, the value of the maximization (8) strictly exceeds that of (7).

I have shown that bidder k strictly prefers to participate secretly, conditional on $v_k = \underline{v}_k^P$ and $X = \{1\}$. Since $\Pr(X = \{1\}) > 0$ and bidder k weakly prefers to participate secretly given value $v_k = \underline{v}_k^P$ and any $X \subset \{1, ..., k-1\}$ (by Theorem 3'), bidder k *strictly* prefers to participate secretly at the interium stage given value $v_k = \underline{v}_k^P$, a contradiction. This completes the proof of Theorem 2'. □

Proof of Lemma 1

Proof. Suppose for the sake of contradiction that $\underline{v} < \underline{v}^P$ in some symmetric PBE. By Theorem 4, $\underline{v}^P < 1$ since bidders sometimes participate publicly. Thus, $\underline{v} = \underline{v}^S < \underline{v}^P < 1$. By definition of the thresholds $\underline{v}^S$ and $\underline{v}^P$, each bidder (say bidder 1) must at least weakly prefer to participate secretly given value $v_1 = \underline{v}^S$ and at least weakly prefer to participate publicly given value $v_1 = \underline{v}^P$. That is,

$$E_X\left[\Pi_1^S(\underline{v}^S;X)\right] \geq E_X\left[\Pi_1^P(\underline{v}^S;X\cup\{1\})\right] \text{ and}$$
$$E_X\left[\Pi_1^S(\underline{v}^P;X)\right] \leq E_X\left[\Pi_1^P(\underline{v}^P;X\cup\{1\})\right],$$

where $X \subset \{2, ..., n\}$ is the realized set of other public bidders. I will show that

$$\Pi_1^S(\underline{v}^P;X) - \Pi_1^P(\underline{v}^P;X\cup\{1\}) \geq \Pi_1^S(\underline{v}^S;X) - \Pi_1^P(\underline{v}^S;X\cup\{1\}) \qquad (9)$$

for all $X \subset \{2, ..., n\}$, with strict inequality for some set X that is realized with positive probability. That is, bidder 1's *incremental expected return to secrecy* is strictly higher when his value equals $\underline{v}^P$ than when it equals $\underline{v}^S$. Thus, if bidder 1 weakly prefers to participate secretly given value $v_1 = \underline{v}^S$, he must strictly prefer to participate secretly given value $v_1 = \underline{v}^P$, a contradiction.

Table 1 summarizes the rest of the argument, which is broken down into Steps A-D.

	$\#(X) \geq 2$	$\#(X) = 1$	$\#(X) = 0$
Secret ($v = \underline{v}^P$)	Zero [A]	More [B]	Weakly more [C]
Secret ($v = \underline{v}^S$)	Zero [A]	Less [B]	$\underline{v}F(\underline{v})^{n-1}$ [D]
Public ($v = \underline{v}^P$)	Zero [A]	Zero [A]	Weakly less [C]
Public ($v = \underline{v}^S$)	Zero [A]	Zero [A]	At least $\underline{v}F(\underline{v})^{n-1}$ [D]

Table 1: *Gross* expected payoff in any (supposed) symmetric PBE with $\underline{v}^P > \underline{v}^S$.

Step A: What if there are two or more public participants? Suppose that two or more bidders participate publicly, i.e. *either* $\#(X) \geq 2$ *or* $\#(X) = 1$ and bidder 1 participates publicly. Since every public bidder i has value $v_i \geq \underline{v}^P$ on the equilibrium path, all public bidders bid at least $\underline{v}^P$. So, bidder 1 earns zero *gross* expected payoff given any value $v_1 \leq \underline{v}^P$:

$$\Pi_1^S(v_1; X) + c_S = 0 \text{ for all } v_1 \leq \underline{v}^P \text{ and } X \subset \{2, ..., n\} \text{ such that } \#(X) \geq 2 \quad (10)$$

$$\Pi_1^P(v_1; X \cup \{1\}) + c_P = 0 \text{ for all } v_1 \leq \underline{v}^P \text{ and } X \subset \{2, ..., n\} \text{ such that } \#(X) \geq 1. \quad (11)$$

Step B: What if bidder 1 participates secretly and there is one other public bidder? Suppose now that bidder 1 participates secretly and $X = \{i^*\}$. Clearly, bidder 1's expected payoff is weakly increasing in his value, so that $\Pi_1^S(\underline{v}^P; \{i^*\}) \geq \Pi_1^S(\underline{v}^S; \{i^*\})$. In fact, this inequality is strict. Since all bidders $j \neq i^*$ sometimes fail to participate, bidder i^* can earn positive gross expected payoff (at least $\Delta = \underline{v}^P F(\underline{v}^S)^{n-1}$ since his value is at least $\underline{v}^P$ when public) by bidding zero. Thus, bidder i^* must always shade his bid below value by at least Δ. In particular, bidder i^* bids less than $\underline{v}^P$ with positive probability. So, bidder 1 must earn positive gross expected profit when secret with value $v_1 = \underline{v}^P$, and this expected profit is strictly increasing in v_1 in a neighborhood of $\underline{v}^P$. We conclude that

$$\Pi_1^S(\underline{v}^P; \{i^*\}) > \Pi_1^S(\underline{v}^S; \{i^*\}) \text{ for all } i^* \neq 1. \tag{12}$$

Step C: What if $v_1 = \underline{v}^P$ and there are no other public bidders? Suppose now that $\#(X) = 0$ and $v_1 = \underline{v}^P$. By Theorem 3', if bidder 1 participates publicly, then his expected gross payoff is the same *as if* all other participants always bid their full value and he plays a best response to such strategies. However, since bidders do not bid more than their values, bidder 1 can earn at least this much expected gross payoff when participating secretly given the same value $v_1 = \underline{v}^P$:

$$\Pi_1^P(\underline{v}^P; \{1\}) + c_P \leq \Pi_1^S(\underline{v}^P; \emptyset) + c_S. \tag{13}$$

Step D: What if $v_1 = \underline{v}^S$ and there are no other public bidders? Suppose finally that $\#(X) = 0$ and $v_1 = \underline{v}^S$. If bidder 1 participates secretly, every bidding equilibrium of the subsequent subgame is in strictly monotone, symmetric strategies (see e.g. Lebrun (2006)). Further, a participant with the minimal value $\underline{v}^S$ bids zero and wins the good iff no one else participates, for gross expected payoff $\underline{v}^S F(\underline{v}^S)^{n-1}$. However, if bidder 1 were to participate publicly given value $v_1 = \underline{v}^S$, he could also earn gross expected payoff of $\underline{v}^S F(\underline{v}^S)^{n-1}$ by bidding zero. Thus, when participating publicly and playing a best response to others' strategies in the subgame with public bidders $P = \{1\}$, bidder 1 earns at least as much gross expected payoff:

$$\Pi_1^P(\underline{v}^S; \{1\}) + c_P \geq \Pi_1^S(\underline{v}^S; \emptyset) + c_S = \underline{v}^S F(\underline{v}^S)^{n-1}. \tag{14}$$

Finally, since each bidder sometimes participates publicly and sometimes fails to participate, $X = \{i^*\}$ with positive probability for each $i^* \neq 1$. All together, then, the weak inequalities (10,11,13,14) and the strict inequality (12) imply the desired increasing incremental expected benefit of secrecy (9), with strict inequality for $X = \{i^*\}$. This completes the proof, by contradiction. □

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