Using Product Rotation to Induce Purchases from Strategic Consumers

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Abstract

Dynamic product rotation is perceived as a useful lever to increase sales. The effect over individual customers is, however, unclear: more choice in the future may induce customers to postpone a purchase if the current offer is not sufficiently appealing, hoping to buy a better product in the future; however, visiting a store to learn about a new product may be costly, thereby diminishing the value of product updates. We propose and analyze a model of strategic customer behavior in the face of a rotating product offering. We find that the customers’ visit and purchase decisions follow a relatively simple structure: a customer should visit the store only when a new product has been introduced and purchase this product if the value it provides is higher than a threshold. We then use this structure to examine the retailer’s product rotation strategy. Interestingly, the choice of products offered to strategic consumers may be higher or lower than that offered to myopic consumers (who buy the first item that fits their needs). This is because myopic consumers are satisfied with relatively fewer changes, while strategic consumers tend to become more demanding as the number of future products increases. The behavior of strategic customers sometimes deters the retailer from changing the product offers too frequently, while other times it provides an incentive for more variety. We also examine the effects of customer heterogeneity, customer discounting and revenue derived from store traffic.

1. Introduction

Assortment planning is a key element of successful retail operations. Assortment decisions include the variants to carry within a product category, store localization, and decisions regarding the time to discontinue a product or introduce a new one. Static assortments are relatively well understood, with a broad literature and successful practical implementations, see Kök et al. (2008). In contrast, industries with short life cycles, such as apparel, require retailers to update their product offering often. In particular, fast-fashion retailers such as Zara or H&M strategically update their assortments periodically to induce frequent visits to their stores (Caro and Martínez-de-Albéniz 2009). Indeed, these retailers do not release new products twice a year as most traditional players do in the Spring-Summer and Fall-Winter collections. They rather manage products individually
and refresh their categories constantly: for instance, Zara designs two thirds of the 8,000 annual SKUs during the season (Caro 2011). These in-season introductions are well-known by consumers and sometimes even advertised, e.g., H&M initiated in 2004 a series of in-season collaborations with high-street designers like Maison Martin Margiela and celebrities like Madonna. The main objective of this strategy is to keep customers interested in the stores all along the season, while traditional competitors receive most of the attention either at the beginning of each 6-month season (when the new collection is released) or at the end of the season (when deep discounts apply). In fact, in recent years, everyone in the industry has started to pay increasingly more attention to in-season product introductions. Some have started to imitate the pace of fast-fashion retailers: Chanel and Dior now launch eight collections a year (El País 2008), Esprit twelve (Esprit 2011).

When consumers have a large budget and can afford multiple purchases during the season, updating the assortment can be seen as a way of offering more variety to consumers. Thus, customers are not satiated by previous consumption and will spend their budget at the retailers that offer something new (Caro and Martínez-de-Albéniz 2012). This typically applies to relatively inexpensive items, so consumers can afford more than one purchase during a single season.

However, when the budget is limited (relative to the cost of the items), the effect is less clear. For example, consider a customer that wants to purchase one (relatively costly) coat between September and December. Periodic changes in the assortment will force this consumer to decide whether to make the purchase during a visit, or forgo the items currently in stock with the hope that a future product will be a better fit with respect to her needs. Hence, while the shopping decision with an unlimited budget can be decomposed as a series of static decisions repeated multiple times, a limited budget forces the customer to incorporate the dynamics of changes in the product selection in its choice process. In other words, with a limited budget, dynamic product introductions involve inter-temporal substitution in consumer choice.

The objective of this paper is to provide a model of decision-making for designing dynamic changes in the product offering when customers are forward-looking and may substitute consumption across periods. For this purpose, we examine how a retailer should manage its store’s product rotation dynamics to influence consumers into buying with the highest possible likelihood, which of course requires them to visit the store as frequently as possible. To that end, we model a retailer’s product refreshment problem in the presence of strategic consumers. The retailer commits to a product rotation policy prior to the beginning of the selling season. Customers face a clear trade-off: purchase the product currently on display or wait for a potentially higher-valued item in the future. A customer’s valuation for the product is realized upon visiting the store. Because of the nature of seasonal products, customers discount future purchases in their decision process. In addition, customers may incur search costs, which means that the refreshed offer needs to be sufficiently
attractive to make the store worth visiting. We compare the optimal policy when customers are strategic to the one appropriate for myopic consumers – i.e., those who buy a product as soon as it meets the customer’s minimum acceptable standards for the product.

By appropriately defining the state space and recursion for a customer’s decision process, we find that the optimal visit and purchase strategy has a relatively simple structure, characterized by a sequence of period-dependent threshold values. This structure allows us to show that, if the retailer operates under a deterministic product rotation policy (i.e., the customer has certainty over the periods in which the product is refreshed), visits to the store only occur in periods in which the retailer refreshes the assortment. Moreover, for a given product rotation policy, myopic customers make their purchase earlier in the selling season than strategic customers. For both types of customers, a higher frequency of product introductions increases the probability of purchase.

We next study the retailer’s optimal product rotation policy by first focusing on the case of a homogeneous set of customers. After showing that the retailer profit is concave in the number of product changes, we compare the optimal number of product introductions for the cases of myopic and strategic customers. We observe that the marginal value of bringing an extra product to the store is usually higher for myopic customers, but also quickly falls to zero as the number of products increases further. As a result, if the cost of product changes is small, then the number of products offered is higher when the firm faces strategic customers, while this comparison reverses if the cost is high. We then show how to solve for the optimal product rotation policy when customers are heterogeneous. In that setting, it is not only the frequency of product introductions, but also their timing that are relevant to the firm’s optimal product rotation strategy. Finally, we study how our results extend to incorporate customer discounting and ancillary revenue from customer visits to the store (i.e., store traffic).

Our work contributes to the literature in several dimensions. First, the scarce literature on dynamic assortments usually assumes myopic preferences (no inter-temporal choice), so we provide a model to incorporate inter-temporal substitution. This is done through the use of a stopping-time dynamic program on the customer side, from which we derive the purchase probabilities over time. Once these are known, we are able to extract insights on the optimal pace of assortment changes. It has been shown that it is optimal to restrict the availability of inventory when strategic customers decide between buying a product during the selling season or waiting for a purchase at a discounted price. This scarcity reduces the incentive to wait for a lower price. In contrast, while more frequent product introductions encourage procrastination, our results suggest that a firm should rotate its product line more frequently to increase the probability of a purchase when customers are strategic (provided that the cost of such changes is not too high). Moreover, when customers are heterogeneous with respect to their arrival to the system and patience, we find that
it is generally optimal to spread out the introduction of new products to capture a higher portion of
the market. More patient customers, in that customers are willing to remain in the system longer – are beneficial to the firm (in terms of firm profit), but the relation between customer patience and the optimal frequency of product changes is non-monotonic.

The rest of the paper is organized as follows. In §2, we review the related literature. We then present the customer choice model in §3 and analyze in §4 the optimal dynamic assortment policy. We provide in §5 some extensions of our model and conclude in §6. All proofs are included in Appendix A. In Appendix B, we explore the optimality of deterministic rotation policies.

2. Literature Review

There are two main streams of literature relevant to this paper. On one hand, this paper is related to the literature on dynamic assortment planning, as we model how customers choose to purchase a product out of many possibilities over time; on the other hand, there is a growing body of work studying settings that involve strategic consumers.

Kök et al. (2008) provides an excellent review of the static assortment planning literature. While this area is quite well studied, there has been limited work on dynamic assortment planning. In economics, Dagsvik (2002) shows that, under an inter-temporal version of the Independence from Irrelevant Alternatives (IIA) axiom, the demand needs to have an attraction form that depends on the existing choice at each moment of time, and that the utilities are extremal processes. In the operations literature, four types of approaches have been considered. First, there is work studying the product replacement process: Lim and Tang (2006) study how to optimize product roll-overs; Druehl et al. (2009) analyze the optimal pace of replacement. Second, some research has been done on stockout-based substitution. Although the assortment may be static in these settings, stock-outs lead to smaller assortments over time. Along this line, the emphasis is on deriving the optimal starting inventory levels (Honhon et al. 2010), or on dynamically rationing inventory to drive demand (Bernstein et al. 2010). Third, several papers discuss how to learn about demand through dynamic variations in the assortment, resulting in an exploration-exploitation trade-off. Caro and Gallien (2007) assume that product demand is unknown but can be learned over time: following a dynamic index policy (under which the assortment contains the products with the highest index) is close to optimal. Rusmevichientong et al. (2010) include a capacity constraint and design an algorithm for the dynamic problem, where parameters are estimated in parallel with revenue generation. Sauré and Zeevi (2013) focus on the asymptotic performance of such algorithms. Farias and Madan (2011) include irrevocability in the analysis, i.e., a product can only be introduced once, and design a heuristic that performs well. Alptekinoglu et al. (2011)
use a locational model with unknown demand distributions that can be discovered by varying the assortment over time. All these papers assume that the demand parameters are stationary (and, in some cases, are learned over time). Fourth, some other papers recognize that demand parameters (e.g., product attractiveness in the Multinomial Logit formulation) change over time. Caldentey and Caro (2010) assume that these parameters follow a stochastic process, which they call the “vogue”. Caro and Martínez-de-Albéniz (2012) use a satiation model where consumers progressively move away from stores that do not refresh their assortments often enough. Caro et al. (2012) study how to release products from a fixed set into stores over multiple periods, taking into account that product attractiveness decays once in the store. Çınar and Martínez-de-Albéniz (2013) propose a closed-loop model with similar decay characteristics, but where products can be added in each period at a cost: they show that it is best to introduce products up to a target attractiveness level in each period and category. Note that all these papers consider customers that may be strategic so as to substitute across products within a period, but do not look into inter-temporal substitution, as we do in this paper.

A recent stream of work examines how firms’ operational decisions are affected by the presence of strategic consumers. Strategic consumers are forward looking and time their purchasing decision by anticipating changes in price or product value. A number of papers in this area study settings in which customers strategically decide whether and when to purchase a product, as the product’s price changes over time and there is uncertainty about its availability at the discounted price. Examples include Su and Zhang (2008), Aviv and Pazgal (2008), Liu and van Ryzin (2008), Osadchiy and Vulcano (2010), Cachon and Swinney (2009, 2011), Cachon and Feldman (2010), Tereyağoğlu and Veeraraghavan (2012), and Parlaktürk (2012). In these papers, the customer’s decision is based on the product’s price and stocking level. In contrast, the decision to wait in our setting is driven by the desire to find a product that better matches the customer’s preference. Along these lines, other papers consider settings in which the value of the product is unknown and delaying the purchase may help to reduce this uncertainty. DeGraba (1995) considers a setting in which customers learn their valuation for a product over time, but the firm prefers selling to customers before their uncertainty is resolved. This is achieved by setting the product’s price accordingly. Alexandrov and Lariviere (2012) examine the role of reservations when customers are a priori uncertain about their valuation for the product. Swinney (2011) examines the value of quick response when customers’ valuation for the product is uncertain (but the uncertainty can be resolved if customers delay their purchase) and the size of the market is uncertain to the firm. Lobel et al. (2013) propose a model for optimizing product launches when strategic customers purchase repeatedly over time and their utility depends on the product’s technology level, which increases over time. They show that a combination of short and long cycles is optimal. Besbes and Lobel (2012) study a firm’s optimal
pricing policy when customers are strategic and differ in terms of their valuation for the product and their willingness to wait before purchasing the product (or leaving the system). They show that a cyclic pricing policy is optimal.

Finally, our work also has similarities with traditional search problems, e.g., Weitzman (1979). In particular, we consider a stopping time problem – see Bertsekas (2000) for a review of this class of problems. A novel aspect of our model is that it combines visit and purchasing decisions; also, it requires the purchase to be decided upon discovery of the value of the item, or otherwise the item may be removed by the retailer. Some experiments have been conducted on this type of decisions: Meyer (1997) shows that individuals tend to search less than the theoretically optimal amount when they are confronted with larger assortments.

3. A Model for Consumer Choice under Dynamic Assortments

3.1 Product Dynamics

A retailer carries a product that is refreshed several times during the season. The products are sold throughout a selling season consisting of $T$ periods $t = 1, \ldots, T$.

At the beginning of each period $t$, the retailer may replace the current product with another one. This event occurs with probability $\alpha_t$, and is independent of any future developments. Thus, the retailer uses a product rotation policy characterized by the vector of probabilities $\{\alpha_1, \ldots, \alpha_T\}$. The retailer selects the product rotation policy to maximize its profit over the entire selling season, and commits to this policy throughout. In general, one could allow the retailer to use a randomized policy, although we focus our analysis on the situation in which the assortment policy is deterministic, i.e., $\alpha_t = 0$ or 1. There are several reasons for this. First, this is optimal when customers are myopic. We numerically explore the use of randomized policies when customers are strategic and find that these policies are, indeed, suboptimal for the retailer. (The result for myopic customers and numerical study for strategic customers is relegated to Appendix B.) Second, it simplifies the analysis significantly. Third, retailers do plan product introductions in advance in a deterministic fashion. Even fast fashion retailers, that introduce products during the season using the latest trend information, tend to reserve sourcing budget, production capacity and store space in advance. Thus, while the product type may vary, the fact that a product will be introduced is known a priori.

When considering deterministic policies, we denote by $t_1 = 1, t_2, \ldots, t_n$ the periods in which the product is refreshed, where $n$ is the total number of products offered.
3.2 Strategic Customers’ Decision Process

We consider a customer interested in buying an item from the retailer. Because the item is expensive, she purchases only one such item during the selling season. She “discovers” this need or interest at a time \( t = 1 \) and is willing to explore different product offers until time \( T \). Customers are heterogeneous – i.e., customers may arrive and leave at different times during the selling season and may be characterized by different consumption parameters. We first focus our analysis on the case of a homogeneous customer type and then discuss the general problem when customers are heterogeneous in §4.4. Within the selling window of search, the retailer displays \( n \) different products. Hence, from the customer’s perspective, the number and timing of product changes is given and known in advance.

Although we focus on the case of a single product, which is renewed multiple times during the selling season, the results extend to an assortment of product variants that either stays on display or is refreshed simultaneously. This is usually the case in retail settings in which trends change fast. Indeed, fashion retailers such as Zara plan relatively small production quantities so as to make sure that an item never stays too long in the assortment (4-8 weeks is typical). In this case, when a new assortment is deployed, the old items tend to have low inventory levels, insufficient for appropriate display (e.g., the broken assortment effect identified in Caro et al. 2010). These remaining items are then removed from the store and returned to a distribution center for use during the discount season.

Given that the entire assortment is refreshed simultaneously, information about the maximum valuation from the assortment is sufficient. Given an assortment of size \( S \) and individual product valuations \( V_{t,SKU}, SKU = 1, \ldots, S \), we would have that \( V_t := \max_{SKU=1,\ldots,S} V_{t,SKU} \). Therefore, the focus on a single product can easily extend to the case of multiple products that are jointly renewed by the retailer as a collection of fashion items.

In each period \( t = 1, \ldots, T \), the customer may visit the store, in which case a search cost \( k \) is incurred. If she decides to go shopping, her realized valuation for the existing product, which was uncertain prior to the visit, is revealed upon observation of the items. This implies that the customer’s valuation for the product remains unknown until the customer visits the store. We model the customer’s valuation as a random variable \( V_t \) drawn from a distribution with c.d.f. \( F_t \) if the product is first seen in period \( t \). In order to keep the analysis simple, we assume valuations to be stationary over time, i.e., \( F_t = F \). To incorporate possible customer impatience (decreased value of the item over time), we include a discount factor on customer utility, \( \beta \leq 1 \) per period.

At the beginning of period \( t + 1 \), the retailer may replace the product with another one – and this event is denoted by the binary variable \( \alpha_{t+1} \) – in which case another draw of the customer’s valuation takes place. Otherwise, the same product is kept in place and the valuation remains the
same, i.e., the valuation of the product in $t+1$ is the same as that in period $t$.

The customer is aware of whether the assortment has been changed or not in every period. Based on this information, she needs to decide (i) whether to visit the store to learn about the value of the product, incurring a search cost $k$; and (ii) if the visit takes place, whether or not to make a purchase. If a purchase is made, then the customer leaves permanently, and otherwise she can decide to shop at a later period. If no product has been bought during the season, then the customer receives an exogenous valuation $v_{out}$ at the end of the horizon – this represents the utility associated with an outside option. The sequence of events is summarized in Figure 1.

![Figure 1: Sequence of events](image)

The customer problem can be formulated as a stopping problem using dynamic programming. In each period, the state consists of a binary variable indicating whether or not the customer has knowledge of the valuation of the item in the store and, if so, an additional variable denoting that value. We thus let $J_t(v)$ be the profit-to-go function given that the customer is at the store in period $t$ and observes a current realized valuation $v$. We let $U_{t}^{old}(v)$ be the profit-to-go function in period $t$ prior to deciding whether to make a visit to the store or not, where $v$ is the valuation that was last observed on a visit to the store, which has not been changed since then. Finally, let $U_{t}^{new}$ be the profit-to-go function in period $t$ prior to deciding whether to make a visit or not, when there is no knowledge about the current product’s valuation (e.g., because the product has changed since the last visit). Bellman’s equation can be written as $U_{T+1}^{old} = U_{T+1}^{new} = v_{out}$ for the last period, and then following a recursion that depends on the current state of the system, i.e., new (new product introduced – no knowledge of current product’s valuation) or old (no assortment rotation – known
product valuation $v$, which is itself a state variable). The recursion is given by:

$$ U_t^{new} = \max \{ EJ_t(V) - k, \beta U_{t+1}^{new} \} \quad (1) $$

if the state is new, and

$$ U_t^{old}(v) = \max \{ J_t(v) - k, \beta U_{t+1}(v) \}, \quad (2) $$

if the state is old with current product valuation $v$, where

$$ J_t(v) = \max \{ v, \beta U_{t+1}(v) \}, \quad (3) $$

and where $U_{t+1}(v) = \alpha_{t+1} U_{t+1}^{new} + (1 - \alpha_{t+1}) U_{t+1}^{old}(v)$. Given that at the beginning of the selling season a new product is always presented ($\alpha_1 = 1$), the consumer utility at that time is independent of the initial valuation and hence can be denoted by $U_1^{new}$.

Two observations are in order. First, using Jensen’s inequality, we have that

$$ E[ U_{t}^{old}(V) ] = E[ \max \{ J_t(V) - k, \beta U_{t+1}(V) \} ] \geq \max \{ EJ_t(V) - k, \beta EU_{t+1}(V) \}. $$

Given that $U_{t+1}(v) = \alpha_{t+1} U_{t+1}^{new} + (1 - \alpha_{t+1}) U_{t+1}^{old}(v)$ and that $U_{T+1}^{old} = U_{T+1}^{new} = v_{out}$, it follows by induction that $E[U_{t}^{old}(V)] \geq U_1^{new}$ for all $t$. This is because the customer is better off having information about the value of the item before visiting the store rather than after the visit. Next, observe that in (2), if the valuation is known, it is always optimal for the customer to buy the product if she visits the store because not doing so would imply that $-k + \beta U_{t+1}(v) > \beta U_{t+1}(v)$. As a result, we can rewrite (2) as

$$ U_t^{old}(v) = \max \{ v - k, \beta U_{t+1}(v) \}. \quad (4) $$

Before proceeding with the analysis of the customer’s problem, we define $\bar{v}$ as the unique solution to

$$ E(V - \bar{v})^+ = k. \quad (5) $$

We introduce the following assumption: $\beta v_{out} < \bar{v}$, i.e.,

$$ E(V - \beta v_{out})^+ > k. \quad (6) $$

This condition guarantees that the expected surplus derived from visiting the store at some point during the selling season dominates the associated search cost, i.e., the expected value derived from purchasing a product over that of the outside option is no smaller than the cost of a visit to the store. Without this condition, one should eliminate the last periods from consideration, thereby reducing the actual value of the outside option until (6) is valid.
The recursion provided by (1), (3) and (4) has some structure. It is simple to establish that $J_t$, $U_t^{old}$ and $U_t$ are increasing and convex in $v$, for all $t$ (they are maxima of convex functions, which can be proved by induction). It can also be shown that the slope of $U_t$ is no greater than one. The structure of the optimal policy follows, and is presented in the next result.

**Theorem 1** For every period $t$, there exists a threshold $v_t$ such that the optimal policy is:

- If the valuation is known to be equal to $v$, then visit the store if and only if $v \geq v_t + k$.
- Otherwise, visit the store if and only if $EJ_t(V) - k \geq \beta U_t^{new}$. If the customer visits the store and observes a valuation of $v$, then purchase if and only if $v \geq v_t$.

The threshold $v_t$ is defined as the unique solution to $v = \beta U_{t+1}(v)$. Moreover, for all $t$, $v_t \leq \bar{v}$, and when $\beta = 1$, $v_t$ is non-increasing in $t$.

The theorem hence allows us to compute the values of $U_t^{new}$ and $U_t^{old}(v)$ recursively. In particular, note that $U_t(v)$ can be written as a piecewise-linear convex function of $v$. Although the thresholds $v_t$ cannot be computed in closed-form, some additional structure can be established when $\beta = 1$, i.e., when there is no discount, namely that $v_t$’s are non-increasing. This means that it is more likely that a customer buys the product at a visit to the store later in the selling season. When the customer discounts a future purchase, this property is not necessarily true because a customer may prefer to settle with a mediocre product early in the season if the discount is steep.

One additional property can be extracted when $\alpha_t = 0$. In that case, although the value of the product is known, the discount factor may urge the customer to visit the store and make the purchase before further discounting erodes the value of the product. As we show next, this is never optimal as the product could been purchased when the value of the product was first revealed.

**Lemma 1** There exists an optimal policy under which store visits only occur in periods in which there is a positive probability that the product will change.

Moreover, under a deterministic product renewal policy, stronger results can be shown and the customer visit and purchase process can be simplified. Specifically, if $\alpha_t = 1$, then $U_t(v) = U_t^{new}$; if $\alpha_t = 0$, then we can run the recursion with $U_t(v) = \beta U_{t+1}(v)$ because no visit takes place in that period (Lemma 1). Let $t + d$ be the next period in which the product changes. Then, $U_t(v) = U_t^{new} = \max\{E \max\{V, \beta^d U_{t+d}^{new}\} - k, \beta^d U_{t+d}^{new}\}$. Hence, letting $\tilde{U}_{t+d}^{new} = \beta^d U_{t+d}^{new}$, we have

$$\tilde{U}_t^{new} = \tilde{U}_{t+d}^{new} + \beta^t \left[ E \left( V - \frac{\tilde{U}_{t+d}^{new}}{\beta^t} \right) ^+ - k \right] ^+.$$  \(\text{(7)}\)
This recursion implies that $\tilde{U}^\text{new}_t = \beta^t U^\text{new}_t$ (for the periods where the product has changed) is non-increasing over time. This monotonicity, together with assumption (6), allows us to completely characterize the optimal visit pattern.

**Lemma 2** Under a deterministic product rotation policy (i.e., when $\alpha_t \in \{0, 1\}$ for all $t$), it is optimal to visit the store in period $t$ if and only if the product has changed, i.e., $\alpha_t = 1$.

Indeed, when $\alpha_t \in \{0, 1\}$, the customer is able to forecast assortment changes with certainty, and hence there is no need to visit the store to hedge against the uncertainty of not being able to do so in the future. Consider a period $t$ with a new product. Hence, we can characterize $\tilde{U}^\text{new}_t$ in closed-form and thus the thresholds $v_t$. Next is our main result regarding strategic customer behavior.

**Theorem 2** Let $t_1 < \ldots < t_n$ be the $n$ time-periods in which a new product is introduced. Thus, in period $t_i$, there have been $i$ product changes (including that of period $t_i$). Then, letting $\beta_i = \beta^i$, $i = 1, \ldots, n$, we have that $\tilde{U}^\text{new}_{t_i} = w_i$ where the sequence $\{w_i\}_{i=1}^n$ is defined recursively, as follows:

$$w_{n+1} = \beta^{T+1}v_{\text{out}}$$

and

$$w_i = w_{i+1} + \beta_i \left[ E\left(V - \frac{w_{i+1}}{\beta_i}\right)^+ - k \right].$$

Moreover, $w_1 \geq \cdots \geq w_n \geq w_{n+1}$ and the thresholds are given by $v_{t_i} = w_{i+1}/\beta^i$. The thresholds follow the recursion $v_{t_{i-1}} = E\max\{V, \beta^{t_i-t_{i-1}}v_{t_i}\} - k$, with initial condition $v_{t_n} = \beta^{T+1-t_n}v_{\text{out}}$.

Interestingly, when $\beta = 1$, it follows from Theorem 2 that the thresholds $v_{t_i}$ approach $\bar{v}$ as $n$ grows large. We next formalize this result, which is useful to characterize the asymptotic purchasing behavior of a strategic consumer (see §3.4).

**Corollary 1** Assume $\beta = 1$. For any fixed $i$, $v_{t_i} \rightarrow \bar{v}$ as $n \rightarrow \infty$.

### 3.3 Behavior of a Myopic Customer

In contrast to the optimization performed by a forward-looking customer, a myopic customer treats each period as if it was the last. That is, a myopic customer visits the store and purchases the product if the benefit associated with those decisions dominates the value obtained by the outside option (which is realized at the end of the selling season). Then, in period $t$, a myopic consumer faces the following optimization problem:

$$U^\text{new}_t = \max \left\{ E_V J_t(V) - k, \beta^{T-t+1}v_{\text{out}} \right\}$$
if the state is new, and
\[ U_t^{old}(v) = \max\{v - k, \beta^{T-t+1}v_{out}\}, \]
if the state is old with current product valuation \( v \), where
\[ J_t(v) = \max\{v, \beta^{T-t+1}v_{out}\}. \]

Suppose first that the myopic customer started a period with knowledge about the product (i.e., old state) with valuation \( v \). Then,
\[ U_t^{old}(v) = \max\{v - k, \beta^{T-t+1}v_{out}\} \]
which means that the customer would purchase the product if \( v \geq \beta^{T-t+1}v_{out} + k \). If this condition held, then the myopic customer would have already purchased the product in her last visit to the store in which the realized valuation of the product would have also been equal to \( v \), as \( v \geq \beta^{T-t+1}v_{out} + k \) for all \( s \leq t \). Therefore, a myopic customer only visits the store if the product is new (i.e., right after a new product is introduced). Under the state new, the customer solves
\[ U_t^{new} = \max\{EJ_t(V) - k, \beta^{T-t+1}v_{out}\}. \]
That is, the myopic customer visits the store if and only if the product is new and \( E[\max\{V, \beta^{T-t+1}v_{out}\}] \geq \beta^{T-t+1}v_{out} + k \). The latter inequality always holds, following the condition in (6). Thus, a myopic customer visits the store every time a new product is introduced and buys the product if her realized valuation \( v \geq \beta^{T-t+1}v_{out} \), i.e., the purchasing threshold for a myopic consumer is \( v_{\text{myopic}}^t := \beta^{T-t+1}v_{out} \). It follows that \( v_{t_n} = w_n + 1/\beta_n \geq w_{N+1}/\beta_n = \beta^{T-t_n+1}v_{out} = v_{t_n}^{\text{myopic}} \). That is, the purchasing thresholds for strategic customers are no smaller than those for myopic customers. The next result compares the timing of purchase of myopic customers and strategic customers.

**Theorem 3** Myopic customers make their purchase earlier in the selling season than strategic customers.

Figure 2 below compares the probability of postponing the purchase for myopic and strategic customers, illustrating the result from Theorem 3 – myopic customers purchase earlier than strategic customers. We provide a formal calculation of these probabilities in the next section.

### 3.4 Obtaining Purchase Probabilities

So far, we have assumed that the product replacement events \( A_t \) are exogenous. However, these depend on how the retailer manages the assortment during the selling season.

To see how the retailer can influence customer behavior through \( \alpha_2, \ldots, \alpha_T \) (recall that \( \alpha_1 = 1 \) as the product is always new at the beginning of the horizon), we need to understand what fraction of customers \( p_t \) will make a purchase in period \( t \). To calculate this probability, let us define \( B_t \) as
Figure 2: Probability of not purchasing after a certain period, when $T = 20$, $\alpha_t = 1$ for all $t$, $\beta = 1$, $v_{out} = 0.6$ and $k = 0.02$.

the event in which the customer purchases in period $t$ ($B$ as in buying) and let $W_t = \bar{B}_t$ be the complementary event ($W$ as in waiting). Then,

$$p_t = P(\cap_{\tau=1}^{t-1} W_{\tau}) P(B_t | \cap_{\tau=1}^{t-1} W_{\tau}),$$

i.e., the probability of purchase in period $t$ is equal to the probability of not having purchased in periods 1 through $t - 1$ multiplied by the conditional probability of purchasing at $t$ given that a purchase did not occur in periods 1 through $t - 1$. We denote $q_t = P(B_t | \cap_{\tau=1}^{t-1} W_{\tau})$. Note that the purchase event $B_t$ implies that the customer visited the store in period $t$ and experienced a valuation higher than the corresponding threshold. Any other outcome is represented by the event $W_t$.

Interestingly, the probability of not having purchased up to $t - 1$ is equal to $\prod_{\tau=1}^{t-1} (1 - q_{\tau})$. Indeed, $p_1 = P(B_1) = q_1$ and $P(W_1) = 1 - q_1$. Using induction, suppose that $P(\cap_{\tau=1}^{t-1} W_{\tau}) = \prod_{\tau=1}^{t-1} (1 - q_{\tau})$ and $p_l = q_l \prod_{\tau=1}^{t-1} (1 - q_{\tau})$ for $l = 2, ..., t$. Then, we have that $p_t = q_t \prod_{\tau=1}^{t-1} (1 - q_{\tau})$ and $P(\cap_{\tau=1}^{t} W_{\tau}) = 1 - \sum_{t=1}^{t} P(\cap_{\tau=1}^{t-1} W_{\tau} \cap B_t) = 1 - \sum_{t=1}^{t} p_t = 1 - q_1 - \sum_{t=2}^{t} \left[ q_t \prod_{\tau=1}^{t-1} (1 - q_{\tau}) \right] = \prod_{\tau=1}^{t} (1 - q_{\tau})$.

We next proceed to the computation of the purchase probabilities. Let $t_1 < \cdots < t_n$ be the $n$ time periods in which a new product is introduced. Using Theorem 1 and Lemma 2, we have that

$$p_{t_i} = F(v_{t_i}) \prod_{j=1}^{i-1} F(v_{t_j}) \text{ for } i = 1, \cdots, n.$$ 

Thus,

$$SALES_{n}^{strategic} = \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} F(v_{t_j}) - \prod_{j=1}^{i} F(v_{t_j}) \right) = 1 - \prod_{i=1}^{n} F(v_{t_i})$$

is the probability that a strategic customer will purchase a product during the selling season, given that the retailer has rotated its product offering $n - 1$ times during the selling season. One can
verify that an increase in the frequency of product rotation increases the probability of selling a unit to a strategic customer, i.e., \( SALES_{n}^{\text{strategic}} \) increases with \( n \).

A myopic consumer visits the store in each period in which a new product is introduced and purchases the product in period \( t \) if her realized valuation is no smaller than \( \beta^{T-t+1}v_{\text{out}} \). Therefore, her purchase probability during the selling season is

\[
SALES_{n}^{\text{myopic}} = 1 - \prod_{i=1}^{n} F\left(\beta^{T-t_{i}+1}v_{\text{out}}\right),
\]

which is also increasing with \( n \), i.e., an increase in the frequency of product rotation increases the overall sales probability.

When \( \beta = 1 \), the expressions for expected sales can be further simplified. In particular, \( SALES_{n}^{\text{myopic}} = 1 - \left( F(v_{\text{out}}) \right)^{n} \), which is the CDF of a geometric distribution. Similarly, when \( n \) is large and \( F(\bar{v}) < 1 \), it follows from Corollary 1 that \( SALES_{n}^{\text{strat}} = 1 - \prod_{i=1}^{n} F(v_{t_{i}}) \) is also approximately geometric, with rate \( F(\bar{v}) \). We have from (6) that \( \bar{v} \geq v_{\text{out}} \) and, as a result, the response of a strategic customer to an increased number of product introductions \( n \), in terms of the customer’s purchasing behavior, is less pronounced than that of a myopic customer. For instance, with the parameters of Figure 2, we have \( F(v_{\text{out}}) = 0.6 \) and \( F(\bar{v}) = 0.8 \), and the graph exhibits the geometric decay pattern.

### 4. Choosing the Right Product Rotation Policy

Consider now the retailer’s problem. We focus on the case with \( \beta = 1 \), so as to make \( F(v_{t}) \) only depend on \( i \) (see Theorem 2). We discuss the general situation with \( \beta < 1 \) in §5.1. The retailer thus solves

\[
\max_{\alpha_{2}, \ldots, \alpha_{T-1} = 0,1} \pi_{R}^{i} := \sum_{t=1}^{T} r_{t} p_{t}(\alpha_{1}, \ldots, \alpha_{T-1}) - c \sum_{t=2}^{T} \alpha_{t}, \quad \text{for } i = \text{myopic, strat},
\]

where \( r_{t} \) is the margin associated with a purchase in period \( t \) and \( c \) is the cost of introducing a new product. We consider first the case of stationary revenue: \( r_{t} = r \). (In §5.2, we study the case in which \( r_{t} \) increases over time, reflecting the fact that the retailer may capture additional revenue by making customers visit the store periodically, thus generating traffic.)

Under a deterministic rotation policy, when \( r_{t} = r \), it turns out that maximizing over \( \alpha_{t} \) is equivalent to maximizing over the number of products \( n \) to be introduced throughout the selling season. That is, the retailer maximizes the expected revenue from sales minus the cost of renewing the assortment, given by

\[
\pi_{R}^{i}(n) = r SALES_{n}^{i} - c(n-1) \quad \text{for } i = \text{myopic, strat},
\]
Lemma 3 Suppose that $r_t = r$. Then, there exists $n < T$ such that it is optimal for the retailer to introduce products in periods $1, \ldots, n$, i.e., $\alpha_1 = \ldots = \alpha_n = 1$ and $\alpha_{n+1} = \ldots = \alpha_T = 0$.

This result allows us to concentrate on the optimal number of products to be introduced, rather than on their timing. Indeed, the timing decision becomes straightforward once the number has been determined.

4.1 Product Rotation with Myopic Consumers

We first examine the retailer’s problem for the case with myopic customers. Let $\gamma = F(v_{out})$. In this case, $n_{myopic}^R(n) = r(1 - \gamma^n) - c(n - 1)$, which is concave in $n$. Hence, at optimality, we set $n_{myopic}^R$ to be the first integer such that $(1 - \gamma)\gamma^n \leq c/r$, i.e.,

$$n_{myopic}^R = \left\lceil \ln \left( \frac{c/r}{1 - \gamma} \right) / \ln (\gamma) \right\rceil.$$

4.2 Product Rotation with Strategic Consumers

In a setting with strategic consumers, we have that $SALES_{strategic}^n = 1 - \prod_{i=1}^n F(v_t)$. This is increasing with the number of product rotations, while the associated cost increases with $n$. From Lemma 3, new products are introduced in consecutive periods. Recall the sequence $\{w_i\}_{i=1}^n$ introduced in Theorem 2 – when $\beta = 1$, this sequence is defined as $w_{n+1} = v_{out}$ and $w_i = E \max\{V, w_{i+1}\} - k$ for all $i$. It follows from $w_1 \geq v_1 > w_2 \geq \ldots \geq v_{n-1} = w_n \geq v_n = w_{n+1} = v_{out}$ that $\sigma_{strat}^R(n)$ is concave in $n$. Let $n_{strat}^R$ be its maximizer. That is, $n_{strat}^R$ is the first integer value to satisfy $\sigma_{strat}^R(n+1) - \sigma_{strat}^R(n) = \bar{F}(v_t) \prod_{i=2}^{n+1} F(v_t) \leq c/r$.

Figure 3 illustrates the change in optimal profit and optimal number of product changes $n_{strat}^R$ as a function of the search cost $k$ and outside option $v_{out}$. The graphs on the left show that, while higher search costs decrease the optimal number of product changes offered in the store, they also increase retailer profit. In other words, higher search costs make the strategic customer more similar to the myopic customer in that the customer has a higher incentive to buy the product earlier rather than latter – when the search cost is sufficiently high, the purchase occurs whenever the product’s realized value is higher than that of the outside option (as is the case for myopic customers). Similarly, the graphs on the right of Figure 3 show that a lower value of the outside option makes strategic customers more similar to myopic customers, thereby increasing retailer profit again.
4.3 Comparison of Myopic and Strategic Customers

To compare the optimal number of products that the retailer would choose against myopic or strategic consumers, recall that $SALES_n$ is geometric for myopic customers, and approximately so for strategic ones, with different rates, as discussed at the end of §3.4. This means that for relatively low $n$, $SALES_n^{myopic}$ is much more sensitive to $n$ compared to $SALES_n^{strat}$; and the opposite is true for high $n$. This suggests that the optimal number of products is not necessarily ordered in the same way for all values of the parameters. This approximate comparison also holds true for the exact values, as formalized in the following result.

**Theorem 4** Assume that $\beta = 1$. There exist thresholds $1 \leq \hat{n}_1 \leq \hat{n}_2$ independent of $c$ such that:

- If $n^{myopic} \leq \hat{n}_1$, then $n^{strat} \leq n^{myopic}$;
- If $n^{strat} \geq \hat{n}_2$, then $n^{strat} \geq n^{myopic}$.

The retailer’s optimal quantities $n^{myopic}$ and $n^{strat}$ are decreasing in the product refreshment cost $c$, while the thresholds $\hat{n}_1$ and $\hat{n}_2$ are independent of $c$. This implies that for sufficiently large $c$, $n^{myopic} \leq \hat{n}_1$ which, in turn, implies that $n^{strat} \leq n^{myopic}$. In other words, when product
changes are expensive, a retailer should rotate its product less frequently if faced with strategic
customers than with myopic customers. In that case, additional changes are too costly for strategic
customers who are more insensitive to the new products being introduced, so the marginal return
of one additional product change is low relative to its cost. Indeed, strategic consumers become
more ‘picky’ (i.e., they tend to postpone their purchase) when the number of product changes is
high. In contrast, the number of product changes is higher for strategic customers when the cost $c$
is relatively smaller. As $c \to 0$, both $n_{\text{strat}}, n_{\text{myopic}} \to \infty$. Because $\hat{n}_2$ is independent of $c$, we have
that $n_{\text{strat}} \geq \hat{n}_2$ for sufficiently small $c$, implying that $n_{\text{strat}} \geq n_{\text{myopic}}$. In other words, when the
cost of refreshing the product offering is low, the retailer must introduce products more frequently
when facing strategic consumers to increase the probability of selling its products. This result is
illustrated in Figure 4.

![Figure 4: Optimal number of products offered as a function of the cost per change $c$, with $T = 20, \beta = 1, v_{\text{out}} = 0.8, k = 0, V$ uniform in $[0, 1]$.](image)

4.4 Customer Heterogeneity

The analysis so far has focused on the case of a homogeneous set of customers. We now explore
a more general setting with different customer types $j = 1, \ldots, J$. Each customer class is defined
by a frequency within the entire population $\theta_j$, an arrival time $A_j \geq 1$, departure time $D_j$, cost
of search $k_j$, and valuation $F_j$. In particular, given a sequence of $n$ product introductions, each
customer type $j$ will effectively see $n_j \leq n$ of them. This description of customer heterogeneity is
broad, and it includes, for example, the case of random arrivals and random stays in the system.
If arrivals occur uniformly over the selling season $[1, T]$ (e.g., arrivals follow a stationary Poisson
process), and customers remain in the system a random number of periods that follows a geometric distribution with rate \( \mu \), then the proportion of customers with \( A_j = a \) and \( D_j = d \), \( 1 \leq a \leq d \leq T \), is 
\[
\theta_j = \frac{1}{T} \left( 1 - (1 - \mu)^{d-a} \right). \tag{10}
\]
(The time window in which customers remain in the system can be regarded as the customers’ patience, as in Besbes and Lobel 2012.)

In Lemma 3, we have shown that the timing of product introduction is irrelevant when customers are homogeneous. However, when customers are heterogeneous, the timing of product introduction is important, even when \( \beta = 1 \). Because the profit associated with one customer type only depends on \( n_j \), the number of products seen during the customer’s stay, and it is concave in \( n_j \), we can formulate the retailer’s problem in the case of different customer types in a tractable way. Namely, we define \( n_j = 1 + \sum_{t=A_j+1}^{D_j} \alpha_t \) and write the retailer’s revenue from class \( j \) as 
\[
R_j(n_j) = r \times \text{SALES}_j^n, \quad \text{for } i = \text{myopic, strat}. \tag{9}
\]
(We omit the superscript \( i \) when the result applies to both myopic and strategic customers.) The optimization thus becomes

\[
\begin{align*}
\max_{\alpha_2, \ldots, \alpha_T \in \{0,1\}} & \quad \sum_{j=1}^{J} \theta_j R_j(n_j) - c(n - 1) \\
\text{s.t.} \quad & \quad n = 1 + \sum_{t=2}^{T} \alpha_t \\
& \quad n_j = 1 + \sum_{t=A_j+1}^{D_j} \alpha_t \text{ for } j = 1, \ldots, J.
\end{align*}
\]

This is an integer maximization program with a concave objective function and linear constraints. To solve this integer program, consider a continuous extension of \( R_j(n_j) \), defined as follows: for \( x_j \in \mathbb{R} \), let \( \hat{R}_j(x_j) = R_j(\lfloor x_j \rfloor) + (x_j - \lfloor x_j \rfloor) \left( R_j(\lfloor x_j \rfloor + 1) - R_j(\lfloor x_j \rfloor) \right) \). Consider the linear relaxation:

\[
\begin{align*}
\max_{\alpha_2, \ldots, \alpha_T \in [0,1]} & \quad \sum_{j=1}^{J} \theta_j \hat{R}_j(x_j) - c(x - 1) \\
\text{s.t.} \quad & \quad x = 1 + \sum_{t=2}^{T} \alpha_t \\
& \quad x_j = 1 + \sum_{t=A_j+1}^{D_j} \alpha_t \text{ for } j = 1, \ldots, J.
\end{align*}
\]

Interestingly, as we show next, such linear relaxation yields an integer solution, making it quite easy to solve.

**Lemma 4** The extremes of the polytope \( P = \{ \alpha_t | 0 \leq \alpha_t \leq 1, n_j \leq 1 + \sum_{t=A_j+1}^{D_j} \alpha_t \leq n_j + 1, j = 1, \ldots, J \} \) are integral. As a result, the optimal solution to the linear relaxation in (10) is integer.
The result in Lemma 4 allows us to efficiently calculate the retailer’s optimal product rotation policy following a simple greedy algorithm. In each step, we move from a given vector \((\alpha_2, \ldots, \alpha_T)\) to the neighboring vector (where each coordinate is increased or decreased by one unit) that results in the highest profit. This is akin to using the simplex method on the hypercube \(0 \leq \alpha_t \leq 1\). This approach proves to be very effective to quickly find the optimal solution. We next derive insights regarding the optimal product rotation policy when customers are strategic and heterogeneous.

When customers are heterogeneous, the expected number of customers in the system affects the retailer’s optimal product rotation decision. Moreover, the customers’ arrival pattern affects the optimal timing of product rotation. Consider a scenario with \(A_j = 1\) for all customer types (while their departure times are random). Suppose there is a solution \(\{\alpha^*\}\) with \(\alpha^*_k = 0\) and \(\alpha^*_{k+l} = 1\) for some \(l \geq 1\). Consider an alternate solution \(\{\hat{\alpha}^*\}\) with \(\hat{\alpha}^*_k = 1\) and \(\hat{\alpha}^*_{k+l} = 0\) and \(\hat{\alpha}^*_i = \alpha^*_i\) otherwise. The total number of product changes is the same as under \(\{\alpha^*\}\), so the associated cost is the same. However, the number of product changes for all customer types with \(D_j < k + l\) is one unit higher than under \(\{\alpha^*\}\), while the number of product changes for the other customer types does not change. Thus, the revenue associated with the customer types with \(D_j < k + l\) is higher under \(\{\hat{\alpha}^*\}\) than under \(\{\alpha^*\}\). Therefore, in such setting, it is best to concentrate all product introductions at the beginning of the season. Conversely, consider a setting in which customers arrive randomly throughout the selling season, but they all remain in the system (i.e., keep looking for a product that matches their taste) until the end of the selling horizon. In such setting, one can similarly prove that it is optimal to concentrate all product introductions towards the end of the selling season.

Figure 5 plots the optimal number of product changes and corresponding profit in two settings with customer heterogeneity. In the example shown in the graphs on the left, all customers arrive at the beginning of the selling season, but they remain in the system for a duration that is geometrically distributed with an expected length of stay (or window of patience) equal to \(1/\mu\), either for strategic or myopic customers. As shown earlier, all product changes occur at the beginning of the season in this case, and the number of product introductions (as well as the retailer’s profit) increases with \(1/\mu\). The graphs on the right in Figure 5 correspond to a setting in which customers arrive uniformly over the selling season, following a stationary Poisson process, and their stay in the system is geometrically distributed with mean \(1/\mu\). In this case, it is best to spread product introductions throughout the season. We observe again that the optimal number of changes may be higher for strategic or myopic customers, depending on the system parameters (see graph on the upper-right). In addition, more patient customers lead to an increase in the number of product changes at first, because customers stay longer in the system and hence more changes increase their purchase probability; however, when the expected stay in the system is already long, further increases of \(1/\mu\)
induce the retailer to spread out more the introduction of new products, so the optimal frequency of product changes may decrease. That is, the relation between the level of patience and number of product introductions is non-monotonic. We conclude this section by showing that longer stays in the system (i.e., more patient customers) always lead to higher retailer profit.

**Lemma 5** For any setting with heterogeneous customers, the firm’s optimal profit increases when customers stay longer in the system.

![Graphs showing optimal number of products as a function of the expected length of stay $1/\mu$.](image)

Figure 5: Optimal number of products as a function of the expected length of stay $1/\mu$. In the graphs on the left, all customers arrive at $t = 1$. In the graphs on the right, customers arrive uniformly over $[1, T]$. Other parameters: $T = 20, \beta = 1, v_{out} = 0.1, k = 0.1, c = 0.002, V$ uniform in $[0, 1]$. In the graphs below, timing of introductions when $1/\mu = 4$.

### 5. Extensions

#### 5.1 Timing Issues with $\beta < 1$

When strategic customers discount their utility, i.e., $\beta < 1$, the timing of product updates matters and Lemma 3 does not apply. One may think that introducing products early is always beneficial when there is a discount factor on the consumer side, as consumers have an incentive to buy earlier.
and hence capture higher utilities. However, the value of the outside option is also discounted, so customers are more ‘picky’ and reluctant to buy. As a result, it is not clear what the optimal timing of product rotation is for the retailer. We first show that if the discount is not too high, the retailer’s profit function continues to be concave.

**Lemma 6** There exists $\beta_0$ such that if $\beta \geq \beta_0$, then $\pi^{\text{strat}}_R(n)$ is concave in $n$.

This result guarantees that the retailer’s profit is concave and therefore the comparison between $n^{\text{strat}}$ and $n^{\text{myopic}}$ established in Theorem 4 holds. However, the optimal timing of product rotation depends on the extent of customer discounting, the optimal number of product changes, and the distribution of customer valuation, even in a setting with homogeneous customers.

Suppose that the retailer rotates the product $n$ times throughout the selling season, with $t_1 = 1$. The recursion to compute the thresholds follows from Theorem 2:

$$v_t = \beta^{T-t+1} v_{\text{out}}$$

and

$$v_t = E \max \{V, \beta^{i-t} v_{t+i} \} - k \text{ for } i = 1, \ldots, n - 1.$$

In general, retailer expected sales are optimized by choosing non-consecutive product refreshment times: for instance, for $n = 3$, uniform valuation in $[0,1]$, $T = 5$, $\beta = 0.8$, $k = 0.1$, and $v_{\text{out}} = 0.4$, it is optimal to set $t_1 = 1, t_2 = 2, t_3 = 4$.

### 5.2 Increasing Revenue

So far, we have focused on stationary margins, i.e., $r_t = r$. This implies that revenue is proportional to the overall purchase probability of the customer.

In some circumstances, $r_t$ may vary over time, even though product prices may be the same from the consumer perspective. Indeed, the retailer may capture additional revenue in every visit that the customer makes to the store. For instance, customers may buy other items while shopping for the main product; or there may be a value directly generated from traffic, such as advertising revenue in online retailing. We focus here on the case where the retailer receives a revenue $r$ when the customer purchases the main item, and also collects $\hat{r} \geq 0$ in every visit. We can define the expected number of visits for strategic customers, as follows:

$$\text{VISITS}^{\text{strategic}}_n = \sum_{i=1}^{n} i \left( \prod_{j=1}^{i-1} F(v_{t_j}) - \prod_{j=1}^{i} F(v_{t_j}) \right) + n \prod_{j=1}^{n} F(v_{t_j}),$$

and for myopic customers:

$$\text{VISITS}^{\text{myopic}}_n = \sum_{i=1}^{n} i \left( \prod_{j=1}^{i-1} F(\beta^{T-t_j+1} v_{\text{out}}) - \prod_{j=1}^{i} F(\beta^{T-t_j+1} v_{\text{out}}) \right) + n \prod_{j=1}^{n} F(\beta^{T-t_j+1} v_{\text{out}}).$$

21
Note that the last term in these sums corresponds to the fraction of customer that does not purchase
the product, but nevertheless visits the store \( n \) times.

The revenue of the retailer is therefore \( r_{\text{SALES}}_n + \hat{r}_{\text{VISITS}}_n \). A simple transformation shows
that the revenue in period \( t \) is given by \( r_t = r + \hat{r}i \). When all the products are introduced in the
earlier periods, we thus have that \( r_t = r + \hat{r}t \) increases over time. We focus on the case of \( \beta = 1 \)
for tractability. In §4.1-4.2, we have shown that \( \text{SALES}_n \) is concave in \( n \). We show here that \( \text{VISITS}_n \) is also concave.

**Theorem 5** Assume \( \beta = 1 \). \( \text{VISITS}^\text{myopic}_n \) is concave in \( n \). Moreover, when
\( \frac{F(x+E(V-x)^+-k)}{F(x)} \) is non-decreasing in \( x \) (e.g., when the distribution of \( V \) is uniform, exponential, or normal ), then \( \text{VISITS}^\text{strat}_n \) is also concave in \( n \).

Thus, finding the optimal number of product refreshments is easy, since we are maximizing a
concave function \( r_{\text{SALES}}_n + \hat{r}_{\text{VISITS}}_n \). Figure 6 displays the optimal solution for a setting with
\( T = 20 \). When traffic revenue \( \hat{r} \) is high (vertical axis), the optimal solution is to set \( n^\text{strat} = 10 \),
at which point the additional traffic revenue does not recover the cost \( c \). Interestingly, the optimal
solution is more sensitive to \( \hat{r} \) than to \( r \), which suggests that the impact of \( n \) on the number of visits
may be more significant than on expected sales. However, in any practical setting, \( \hat{r} \) is an order of
magnitude lower than \( r \) and therefore the impact of \( n \) on the probability of purchase prevails.

![Optimal number of products](image)

**Figure 6:** Optimal number of refreshed products for strategic customers, as a function of sales
margin \( r \) and traffic margin \( \hat{r} \). \( T = 20, \beta = 1, v_{out} = 0.3, k = 0.1, c = 0.01, V \) uniform in \([0, 1]\).

Another observation from Figure 6 is that the regions where \( n^\text{strat} \) is constant seem to have linear
boundaries. This is indeed the case for myopic customers where \( r_{\text{SALES}}^\text{myopic}_n + \hat{r}_{\text{VISITS}}^\text{myopic}_n = [r + \hat{r}/(1 - F(v_{out})^n)][1 - F(v_{out})^n] \). Similarly, for strategic customers, we know that for large \( n \),
\( \text{SALES}^\text{strat}_n \) and \( \text{VISITS}^\text{strat}_n \) are approximately geometric, with parameter \( F(\bar{v}) \). As a result,
\( rSALES_{n}^{strat} + \bar{r}VISITS_{n}^{strat} \) is approximately equal to \( [r + \bar{r}/(1 - F(\bar{v}))][1 - F(\bar{v})^n] \). This provides a rationale for the linear shape of the boundaries when customers are strategic.

6. Conclusion

In this paper, we explore a model of decision-making for determining product rotation strategies in settings in which customers are forward-looking. The practice of periodically refreshing the product offering within a selling season has been adopted in several industries, notably by fashion apparel retailers (e.g., Zara, H&M, Esprit, etc.) In our model, a retailer commits to a product rotation policy prior to the selling season. Product valuation is realized upon visiting the store. On a visit to the store, customers decide whether to purchase the current product or wait for a potentially higher-valued item in the future. Customers may incur search costs for visits to the store.

In this context, we model and solve for a customer’s dynamic visit and purchasing decisions, and find that the optimal policy has a relatively simple structure, characterized by a sequence of period-dependent threshold values. Armed with an understanding of the customer’s decision process, we next examine the retailer’s optimal product rotation policy. We find that a deterministic rotation policy is optimal. The retailer trades-off the cost of rotating its product line with its associated benefits: the probability of purchase increases with the number of product introductions, even if the valuation of future options is uncertain and customers incur a search cost for visiting the store (and realizing the value for the product). We find that if the cost of product rotation is relatively small, then the firm should rotate its product offering more frequently when facing strategic customers, while the opposite occurs when the cost is high. In addition to the optimal number of product introductions, we also explore the optimal timing of such product changes. The timing is irrelevant when customers are homogeneous in terms of the distribution of valuation, arrival to the system, patience, etc. However, when all customers start looking for products at the beginning of the selling season, it is optimal to concentrate the product changes also in the first few periods of the season. When customers arrive randomly throughout the season, but remain in the system until the end of the season, then it is optimal to relegate the rotation of products to the last few periods of the selling season. When customers arrive uniformly throughout the selling season, the retailer is better off by spreading out the release of new products as well. In general, we show that the firm benefits from serving a market in which customers are more patient (i.e., remain in the system for longer periods of time). However, a more patient customer base may increase or decrease the frequency of product changes. We find that the customer’s discount factor may also impact the timing of product introductions. Finally, we explore the firm’s optimal policy when customer traffic generates revenue in addition to sales of the main product.
References


Appendix A – Proofs

Theorem 1

Proof. The existence of the threshold follows because $U_t(v)$ is increasing convex and with slope no greater than one, which is easily shown by induction on $t$. (The fact that $U_t(v)$ is increasing with slope no greater than one ensures that each threshold $v_t$ is uniquely defined.)

We now prove that $v_t \leq \bar{v}$ together with $\beta U_{t+1}^{new} \leq \bar{v}$ by induction. This is true for $t = T$, because from (6), $\beta U_{T+1}^{new} = \beta v_{out} \leq \bar{v}$ and $v_T = \beta v_{out} \leq \bar{v}$. Assume the result is true for $t + 1 \leq T$ and let us show that it holds for $t$. First,

$$U_{t+1}^{new} = \max\{E \max\{V, \beta U_{t+1}(V)\} - k, \beta U_{t+2}^{new}\}$$

If the maximum is achieved at $\beta U_{t+2}^{new} \leq \bar{v}$, then $\beta U_{t+1}^{new} \leq \bar{v}$. Otherwise,

$$E \max\{V, \beta U_{t+1}(V)\} - k \leq E \max\{V, v_{t+1}\} - k$$

$$= v_{t+1} + E(V - v_{t+1}) - E(V - \bar{v})^+$$

$$\leq v_{t+1} + \bar{v} - v_{t+1}$$

$$= \bar{v}.$$ 

This proves that $\beta U_{t+1}^{new} \leq \bar{v}$ always. Moreover,

$$\beta U_{t+1}(\bar{v}) = \alpha_{t+1} \beta U_{t+1}^{new} + (1 - \alpha_{t+1}) \beta U_{t+1}^{old}(\bar{v})$$

$$\leq \alpha_{t+1} \bar{v} + (1 - \alpha_{t+1}) \beta \max\{\bar{v} - k, \beta U_{t+2}(\bar{v})\}$$

from the induction property

$$\leq \alpha_{t+1} \bar{v} + (1 - \alpha_{t+1}) \beta \bar{v}$$

because $\bar{v} \geq v_{t+1}$ implies $\bar{v} \geq \beta U_{t+2}(\bar{v})$

$$\leq \alpha_{t+1} \bar{v} + (1 - \alpha_{t+1}) \bar{v} \leq \bar{v}.$$ 

Hence $\beta U_{t+1}(\bar{v}) \leq \bar{v}$, which itself implies that $v_t \leq \bar{v}$.

Finally, when $\beta = 1$, we prove that $v_t \geq v_{t+1}$ by induction. We show this by proving that, in each period, $v_t \geq v_{t+1}$ and $U_{t+1}^{new} \geq v_t$. This is true for $t = T$ since $v_T = v_{T+1} = v_{out}$. Assume it is true for $t$, and let us show it for $t - 1$. First, we have that $U_t^{old}(v_t) = \max\{v_t - k, v_t\} = v_t$ and $U_t^{new} \geq U_t^{new} \geq v_t$. This implies that $U_t(v_t) \geq v_t$ and hence, since $U_t(\cdot)$ is non-decreasing with slope no greater than one, $v_{t-1} \geq v_t$. Second, we have that $U_t(v_{t-1}) = \alpha_t U_t^{old}(v_{t-1}) + (1 - \alpha_t) U_t^{new} = v_{t-1}$. Since $v_{t-1} \geq v_t$, $U_t^{old}(v_{t-1}) = \max\{v_{t-1} - k, U_{t+1}(v_{t-1})\} \leq v_{t-1}$. Hence, $U_t^{new} \geq v_{t-1}$. 


Lemma 1

Proof. Take $t$ such that $\alpha_t = 0$, i.e., the product between periods $t - 1$ and $t$ does not change. Then, by its definition, $U_t(v) = \alpha_t U_t^{new} + (1 - \alpha_t) U_t^{old}(v) = U_t^{old}(v)$. This implies that $U_t^{old}(v) = \max\{v - k, \beta U_{t+1}(v)\} = U_t(v)$ and $U_t^{new} = \max\{E \max\{V, \beta U_{t+1}(V)\} - k, \beta U_{t+1}^{new}\}$. 

26
Consider two possible scenarios for period $t - 1$.

Suppose first that the customer knew the product’s valuation in period $t - 1$. Then, in period $t - 1$, the customer decided whether or not to visit the store by solving $U^\text{old}_{t-1}(v) = \max\{v - k, \beta U_t(v)\} = \max\{v - k, \beta \max\{v - k, \beta U_{t+1}(v)\}\} = \max\{v - k, \beta^2 U_{t+1}(v)\}$. Clearly, the maximum will never be achieved by visiting and buying in period $t$, which would generate a profit of $\beta(v - k)$.

Suppose now that the customer did not know the product’s valuation in period $t - 1$. In that case, the customer would have visited the store in period $t - 1$ if and only if $\beta U^\text{new}_t \leq E J_{t-1}(V) - k = E \max\{V, \beta U_t(V)\} - k = E \max\{V, \beta(V - k), \beta^2 U_{t+1}(V)\} - k = E \max\{V, \beta^2 U_{t+1}(V)\} - k$. Again, the maximum cannot be reached by visiting and buying in period $t$, yielding a profit of $\beta(V - k)$. Finally, the remaining case involves a situation in which the customer did not visit the store in period $t - 1$, but then visited the store in period $t$. We claim that this is again not possible. In this scenario, it must be that $\beta U^\text{new}_t > E \max\{V, \beta^2 U_{t+1}(V)\} - k$; and because the product was not changed in period $t$, the state in that period is new, so $U^\text{new}_t = E \max\{V, \beta U_{t+1}(V)\} - k$. This means that

$$E \max\{\beta V, \beta^2 U_{t+1}(V)\} - \beta k > E \max\{V, \beta^2 U_{t+1}(V)\} - k$$

which yields

$$k > \frac{E \max\{V, \beta^2 U_{t+1}(V)\} - E \max\{\beta V, \beta^2 U_{t+1}(V)\}}{1 - \beta} \geq E (V - v_t)^+. \tag{7}$$

The last inequality follows by considering each possible sample path. If $V \leq \beta^2 U_{t+1}(V)$, then $V \leq v_t$ so $E (V - v_t)^+ = 0$ and the inequality holds. If $\beta^2 U_{t+1}(V) \leq V \leq \beta U_{t+1}(V)$, then again $V \leq v_t$ so $E (V - v_t)^+ = 0$. Finally, when $V \geq \beta U_{t+1}(V)$, the left-hand side of the inequality is equal to $V \geq E (V - v_t)^+$. As a result, $k = E (V - \bar{v})^+ > E (V - v_t)^+$, implying that $v_t \geq \bar{v}$. This contradicts the result in Theorem 1. □

**Lemma 2**

**Proof.** We already know from Lemma 1 that if the customer visits the store in period $t$ then it cannot be that $\alpha_t = 0$, so it must be that $\alpha_t = 1$ in view of the deterministic product rotation policy. Next, we prove that the customer is actually better off visiting the store when the product changes.

Equation (7) implies that $\tilde{U}^\text{new}_t$ is non-increasing over time (for the periods in which new products are introduced). Let $t_1 < \cdots < t_N$ be the $N$ periods in which a new product is introduced. Let $\beta_n = \beta^n$, $w_{N+1} = \beta^{T+1} v_{\text{out}}$ and $w_n = w_{n+1} + \beta_n [E(V - w_{n+1}/\beta_n)^+ - k]^+$. It first follows that $w_n$ is equal to $\tilde{U}^\text{new}_{tn}$. 27
We show that \( w_{n+1}/\beta_n < \bar{v} \) for all \( n = 1, \ldots, N \) by backward induction. Indeed, this is true for \( n = N \) because \( w_{N+1}/\beta_N = \beta^{N+1} v_{out} \leq \beta v_{out} < \bar{v} \), where the last inequality follows from (6). Assume that it is valid for \( n \), i.e., \( w_{n+1}/\beta_n < \bar{v} \). If \( w_n = w_{n+1} \), then the inequality holds because \( \beta_{n-1} = \beta^{n-1} > \beta^n = \beta_n \) as \( t_{n-1} < t_n \), so \( w_n/\beta_{n-1} = w_{n+1}/\beta_n - w_{n+1}/\beta_{n-1} < w_{n+1}/\beta_n < \bar{v} \) by induction. Otherwise, \( w_n = w_{n+1} + \beta_n [E(V - w_{n+1}/\beta_n)^+ - k] \) so \( w_n/\beta_{n+1} = [E \max(V, w_{n+1}/\beta_n) - k]/\beta_n < [E \max(V, \bar{v}) - k]/\beta_n = \bar{v}/\beta_{n+1} < \bar{v} \). Thus, \( w_{n+1}/\beta_n < \bar{v} \) for all \( n \). This implies that \( E(V - w_{n+1}/\beta_n)^+ > E(V - \bar{v})^+ = k \) for all \( n \). That is, the customer will visit the store as many times as the firm introduces a new product. This proves the result.

Note that if the condition in (6) does not hold, then \( w_{N+1}/\beta_N \geq \bar{v} \), implying that \( E(V - w_{N+1}/\beta_N)^+ \leq E(V - \bar{v})^+ = k \). This, in turn, implies that \( w_N = w_{N+1} \) and, repeating the argument, we have that \( w_n = w_{N+1} \) for all \( n \). In that case, the customer never visits the store. ■

**Theorem 2**

**Proof.** From the proof of Lemma 2, we have that \( w_{N+1} = \beta^{N+1} v_{out} \) and \( w_n = w_{n+1} + \beta_n [E(V - w_{n+1}/\beta_n)^+ - k] \). Recall also from that proof that \( w_n \) is equal to \( \tilde{U}_n^{new} \) when there have been \( n \) product changes between the beginning of the selling season until period \( t_n \) (included). Hence, \( v_{t_n} \) is the unique solution to \( v = \beta U_{t_n+1}(v) = \beta^{t_n+1-t_n} U_{t_n+1}^{new} = \tilde{U}_{t_n+1}^{new}/\beta_n = w_{n+1}/\beta_n \). ■

**Theorem 3**

**Proof.** Based on the discussion prior to the statement of Theorem 3, a myopic customer visits the store every time that a new product is introduced. Consider a situation in which the product is changed between periods \( t - 1 \) and \( t \). Both a myopic and a strategic customer would visit the store in period \( t \). Because \( v_t \geq v_t^{myopic} \), a purchase from the strategic customer implies a purchase from the myopic customer. This implies the result. ■

**Lemma 3**

**Proof.** Consider a deterministic policy such that \( \sum_{i=1}^{T-1} \alpha_i = n \). Retailer costs are thus fixed to \( c(n-1) \). On the other hand, retailer sales for the myopic customer are equal to \( 1 - \prod_{i=1}^{n} F(\beta^{t_i+1} v_{out}) \) so they are clearly maximized by concentrating product changes at the beginning of the selling season. For a strategic customer, sales are equal to \( 1 - \prod_{i=1}^{n} F(v_{t_i}) \). Given the recursion (8), we have that

\[
v_{t_{i+1}} = \frac{w_i}{\beta_{t_i+1}} = \frac{v_{t_i} \beta_t}{\beta_{t_i+1}} + E(V - v_{t_i})^+ - k.
\]
When \( \beta = 1 \), \( v_t \) only depends on \( i \), so sales only depend on \( n \). ■

**Theorem 4**

**Proof.** Because \( \beta = 1 \), we have that \( v_t = w_{i+1} \) for \( i = 1, \ldots, n \), \( w_1 \geq \cdots \geq w_n \geq w_{n+1} = v_{\text{out}} \), and \( v_{\text{out}} \) for all \( t \). We denote by \( w_i^n \) the threshold associated to the \( i \)th product rotation when there are a total of \( n \) product changes throughout the season (so, for example, \( w_2^{n+1} = w_1^n \)). To compare the optimal values \( n_{\text{myopic}} \) and \( n_{\text{strat}} \), we first note that

\[
\begin{align*}
  n_{\text{myopic}} &\leq n_{\text{strat}} & \text{if} & \quad (1 - \gamma)n - 1 \leq \tilde{F}(w_1^n) \prod_{i=2}^n F(w_i^n) \text{ at } n = n_{\text{strat}} \\
  n_{\text{myopic}} &\geq n_{\text{strat}} & \text{if} & \quad \tilde{F}(w_1^n) \prod_{i=2}^n F(w_i^n) \leq (1 - \gamma)n - 1 \text{ at } n = n_{\text{myopic}},
\end{align*}
\]

where \( \gamma = F(v_{\text{out}}) \). We define

\[
P_n \overset{\text{def}}{=} \frac{\tilde{F}(w_1^n) \prod_{i=2}^n F(w_i^n)}{\gamma^{n-1}}.
\]

Then,

\[
P_{n+1} = \frac{\tilde{F}(w_1^{n+1}) \prod_{i=2}^{n+1} F(w_i^{n+1})}{\gamma^n} = P_n \times \frac{F(w_1^n) \tilde{F}(w_1^{n+1})}{\gamma F(w_1^n)}.
\]

First, note that when \( n = 1 \), \( P_1 = \tilde{F}(w_1^1) \leq 1 - \gamma \). Consider now the factor

\[
\frac{F(w_1^n) \tilde{F}(w_1^{n+1})}{\gamma F(w_1^n)}.
\]

Because \( w_i^n = E \max\{V, w_i^{n+1}\} - k \), we have that \( w_1^n, w_i^{n+1} \rightarrow \tilde{v} > v_{\text{out}} \) as \( n \rightarrow \infty \). Therefore,

\[
\frac{F(w_1^n) \tilde{F}(w_1^{n+1})}{\gamma F(w_1^n)} \rightarrow \frac{F(\tilde{v})}{F(v_{\text{out}})} > 1 \text{ as } n \rightarrow \infty.
\]

It follows that there exists \( \hat{n} \) such that the factor in (11) is greater than 1 for all \( n \geq \hat{n} \). This implies that \( P_n \) is increasing for \( n \geq \hat{n} \). Moreover,

\[
\frac{P_{n+1}}{P_n} = \frac{F(w_1^n) \tilde{F}(w_1^{n+1})}{\gamma F(w_1^n)} \rightarrow \frac{F(\tilde{v})}{F(v_{\text{out}})} > 1 \text{ as } n \rightarrow \infty.
\]

Then, we must have that \( P_n \) is unbounded and therefore \( P_n \rightarrow \infty \) as \( n \rightarrow \infty \). Thus, there exist \( 1 \leq \hat{n}_1 \overset{\text{def}}{=} \max\{n : P_m \leq 1 - F(v_{\text{out}}) \text{ for all } 0 \leq m \leq n\} \) and \( \hat{n}_2 \overset{\text{def}}{=} \min\{n : P_m > 1 - F(v_{\text{out}}) \text{ for all } m \geq n\} \). The result follows from the definitions of \( P_n, \hat{n}_1 \) and \( \hat{n}_2 \). ■
Lemma 4

Proof. The argument is as follows. In each region where $x_j$ remains in the interval $[n_j, n_j + 1]$, the objective is linear, and as a result, within each of those regions, there is an extreme optimum. Moreover, As a result, we know that the optimal product timing decision can be found by solving (10). the extremes of the polytope $P = \{\alpha_t | 0 \leq \alpha_t \leq 1, n_j \leq 1+\sum_{i=A_j+1}^{D_j} \alpha_i \leq n_j+1\}$ turn out to be integral because the matrix $M \in \mathbb{R}^{(T-1) \times J}$ with entries $M_{ij} = 1_{A_j+i\leq D_j}$ is a totally unimodular matrix. Indeed, consider $S$ a square sub-matrix of $M$, and let us show that the determinant is 1, 0 or −1. The rows of $S$ are made of zeros, then ones, and then zeros again. Since permutating the rows of $S$ only changes the sign of the determinant, we can assume without loss of generality that the rows are sorted so that $i < j$ implies that the lowest column in which a one is present is lower in $i$ than in $j$, and, when it is equal, the highest column in which a one is present is lower in $i$ than in $j$. One such matrix could for example be:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]

We claim that the determinant of such matrix is always 1, 0 or −1. We can prove it by induction. It is clearly true when the matrix has size $1 \times 1$. If it is true for sizes $k \leq n-1$, then let us prove that it is also true for matrices of size $n \times n$. Indeed, if $S_{11} = 0$, then it means that the first column is made of zeros and the determinant must be zero. Otherwise, $S_{11} = 1$, in which case we can subtract the first row from all the rows such that $S_{i1} = 1$, without changing the determinant. In this case the determinant of $S$ is equal to the determinant of a matrix that only has a one in the first column ($S_{11} = 1$), which means that it is equal to the determinant of the $(n-1) \times (n-1)$ matrix made of rows and columns from 2 to $n$. An example of this operation is the following:

\[
\begin{vmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{vmatrix} = \begin{vmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 1 & 1
\end{vmatrix}
\]

Since this smaller matrix has the same properties as the larger one (made of zeros, then ones, and then zeros again), we can use the induction hypothesis to know that its determinant is 1, 0 or −1. This completes the proof. ■

30
Lemma 5

**Proof.** We prove this by showing that the profit of the optimization problem in (9) increases as the difference $D_j - A_j$ increases for any subset of customer types $j \in S \subset \{1, \ldots, J\}$. Consider the optimal solution $\{\alpha_1, \ldots, \alpha_T\}$ of the original system. This solution is feasible for a system in which $D_j - A_j$ is higher for $j \in S$. This implies that, for this feasible solution, $n_j$ is higher for $j \in S$, while all other $n_j$-values remain the same as in the original system. Because $R_j(n_j)$ is increasing in $n_j$, we obtain the desired result. ■

Lemma 6

**Proof.** The retailer’s profit is concave as long as the thresholds $v_t$ are decreasing. It follows from the recursion in Theorem 2 that $w_1 > w_2 > \cdots > w_n > w_{n+1}$. Therefore, there exists $\beta_0$ such that $\beta^{T-1}w_i > w_{i+1}$ for all $\beta > \beta_0$. Then, for $\beta > \beta_0$,

$$v_t = \frac{w_{i+1}}{\beta^i} < \beta^{T-1} \frac{w_i}{\beta^i} \beta^{t-1} = \beta^{T-1} \frac{v_{t-1}}{\beta^{t-1}} = \beta^{(T-1)-(t_i-t_i-1)} v_{t_i-1} < v_{t_i-1},$$

and the result follows. ■

Theorem 5

**Proof.** Let us first consider the myopic case. Given $n$ products being introduced in $t = 1, \ldots, n$, the purchase probability at $t$ is equal to $(1 - \gamma)\gamma^{t-1}$, where $\gamma = F(v_{out})$. As a result,

$$VISITS_{n}^{myop} = \sum_{t=1}^{n} (1 - \gamma)\gamma^{t-1} + n\gamma^n = \sum_{t=0}^{n-1} (t + 1)\gamma^t - \sum_{t=0}^{n} t\gamma^t + n\gamma^n$$

$$= \sum_{t=1}^{n} \gamma^{t-1} = \frac{1 - \gamma^n}{1 - \gamma}$$

This is clearly concave increasing in $n$.

For strategic customers, we have

$$VISITS_{n}^{strat} = \sum_{t=1}^{n} \left( \prod_{j=1}^{t-1} F(w_j^n) - \prod_{j=1}^{t} F(w_j^n) \right) + n \prod_{j=1}^{n} F(w_j^n)$$

$$= \sum_{t=0}^{n-1} (t + 1) \left( \prod_{j=1}^{t} F(w_j^n) \right) - \sum_{t=0}^{n} t \left( \prod_{j=1}^{t} F(w_j^n) \right) + n \prod_{j=1}^{n} F(w_j^n)$$

$$= \sum_{t=0}^{n-1} \left( \prod_{j=1}^{t} F(w_j^n) \right)$$
When adding another product, we have $w_{j+1}^{n+1} = w_j^n$, for all $j = 1, \ldots, n$ (keep in mind that we start the recursion with $w_{n+1}^n = v_{out}$). Thus, by adding the $n + 1$-th product, there is one visit and, if the product was not purchased in the first visit, the additional number of visits is equal to $VISITS_n^{strat}$.

$$VISITS_{n+1}^{strat} = 1 + F(w_1^{n+1})VISITS_n^{strat}$$

and thus $VISITS_{n+1}^{strat} - VISITS_n^{strat} = 1 - \tilde{F}(w_1^{n+1})VISITS_n^{strat}$. We claim that this expression is positive and non-increasing. We show this by induction that $\tilde{F}(w_1^{n+1})VISITS_n^{strat} \leq 1$ and decreasing in $n$. To establish the result, we use that $w_{n+1}^n = w_1^n + E(V - w_1^n)^+ - k$, from Theorem 1. We initiate the recursion for $n = 1$: $VISITS_1^{strat} = 1$ while $\tilde{F}(w_1^n) < 1$, so $\tilde{F}(w_1^n)VISITS_1^{strat} \leq 1$.

Assume the induction property is true for $n - 1 \geq 1$. For $n$,

\[
\tilde{F}(w_1^{n+1})VISITS_n^{strat} = \tilde{F}(w_1^{n+1})(1 + F(w_1^n)VISITS_n^{strat}) = \tilde{F}(w_1^{n+1})\left(1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)}\tilde{F}(w_1^n)VISITS_n^{strat}\right) \leq \tilde{F}(w_1^{n+1})\left(1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)}\right) \text{ using the induction property} \\
\leq \frac{\tilde{F}(w_1^{n+1})}{\tilde{F}(w_1^n)} \leq 1 \text{ because } w_1^{n+1} \geq w_1^n.
\]

We now need to show that $\tilde{F}(w_1^{n+1})VISITS_n^{strat} \geq \tilde{F}(w_1^n)VISITS_n^{strat}$ or equivalently

\[
\frac{\tilde{F}(w_1^{n+1})}{\tilde{F}(w_1^n)} \geq VISITS_n^{strat} = VISITS_n^{strat} - \frac{F(w_1^n)VISITS_n^{strat}}{1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)}VISITS_n^{strat}} = VISITS_n^{strat} - \frac{F(w_1^n)F(w_1^n)VISITS_n^{strat}}{1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)}VISITS_n^{strat}}.
\]

Using the induction property, the right-hand side is no greater than

\[
1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)} \leq 1 + \frac{F(w_1^n)}{\tilde{F}(w_1^n)} = \tilde{F}(w_1^n).
\]

Thus, it is sufficient to show that $\frac{\tilde{F}(w_1^{n+1})}{\tilde{F}(w_1^n)} \leq \frac{\tilde{F}(w_1^{n+1})}{\tilde{F}(w_1^n)}$, which is true if $\frac{\tilde{F}(x+E(V-x)^+-k)}{\tilde{F}(x)}$ is increasing in $x \leq \bar{v}$, because $\bar{v} \geq w_1^n \geq w_1^{n-1}$. This completes the induction. Hence, $VISITS_n^{strat}$ is concave.
Appendix B – Randomized Assortment Changes

In this section, we examine the case of general randomized product rotation policy. In the presence of myopic customers, we show that it is optimal for the retailer to restrict attention to a deterministic product rotation policy, i.e., one with $\alpha_t = 0$ or 1 for all $t = \ldots, T - 1$.

Myopic Customers

We first revisit the sequence of events from the point of view of a single myopic customer. Prior to period $t$, there is uncertainty regarding the action of the retailer: with probability $\alpha_{t-1}$ a new product is introduced and with probability $1 - \alpha_{t-1}$ the same product from the last period is kept on offer. At the beginning of period $t$, the customer learns the realization of $A_{t-1}$ – i.e., whether the product was refreshed or not. At that point, the myopic customer decides whether or not to visit the store (incurring a cost $k$) and, if a visit takes place, whether or not to purchase the product. Because the customer is myopic, her decision is the same as what is described in §3.3 – i.e., the myopic customer only visits when a new product is introduced and purchases the product if her realized valuation is no smaller than the discounted value of the outside option.

Following a similar analysis as in §3.4, the probability that a myopic customer purchases in period $t$ (given that a new product has been introduced) is

$$p_t = \alpha_{t-1} \bar{F}(v_t) \prod_{s=1}^{t-1} (1 - \alpha_{s-1} \bar{F}(v_s)) = \prod_{s=1}^{t-1} (1 - \alpha_{s-1} \bar{F}(v_s)) - \prod_{s=1}^{t} (1 - \alpha_{s-1} \bar{F}(v_s)),$$

where $v_t = \beta^{T-t+1} v_{out}$ is the threshold value in period $t$. The probability that a purchase takes place throughout the selling season is

$$S(\alpha) = \sum_{l=1}^{T} p_l(\alpha) = 1 - \prod_{s=1}^{T} (1 - \alpha_{s-1} \bar{F}(v_s))$$

(we stress the dependence on $\alpha$ in this expression, but we will omit it in the subsequent analysis unless noted). The retailer maximizes $rS(\alpha) - c \sum_{l=1}^{T-1} \alpha_l$. We next show that the optimal rotation policy is deterministic. To that end, we first calculate

$$\frac{\partial S}{\partial \alpha_t} = \bar{F}(v_{t+1}) \prod_{s=t+1}^{T} (1 - \alpha_{s-1} \bar{F}(v_s)).$$

Let $\alpha^*$ be the optimal solution to the retailer’s maximization problem. Suppose that there exist $\alpha^*_n, \alpha^*_m \in (0, 1)$. Let $A = \alpha^*_n + \alpha^*_m$. Let us now consider the optimization problem in which we fix

\[\text{In practice, this may be facilitated by emails, catalogs or other promotional materials indicating that a new product line is in stores.}\]
the policy for all periods except for \( n \) and \( m \) to the values \( \alpha_i^* \) (for \( i \neq n, m \)). Then, \( \alpha_n^* \) and \( \alpha_m^* \) maximize the remaining optimization problem:

\[
\max_{\alpha_n, \alpha_m} rS(\alpha_n, \alpha_m, \{\alpha_i^*, i \neq n, m\}) - c \sum_{i=1, i \neq n, m}^{T-1} \alpha_i^* - c\alpha_n - c\alpha_m,
\]

subject to \( \alpha_n + \alpha_m = A \). Note that the cost remains equal to that in the original solution. We can restate this optimization problem as:

\[
\max_{\alpha_n, \alpha_m} rS(\alpha_n, A - \alpha_n, \{\alpha_i^*, i \neq n, m\}) - c \sum_{i=1}^{T-1} \alpha_i^*.
\]

The derivative of the sales function with respect to \( \alpha_n \) is given by

\[
\bar{F}(v_{n+1}) \prod_{s=1}^{T} \prod_{s \neq n+1, m+1} \left( 1 - \alpha_s^* \bar{F}(v_s) \right) \left( 1 - (A - \alpha_n) \bar{F}(v_{m+1}) \right) - \bar{F}(v_{m+1}) \prod_{s=1}^{T} \prod_{s \neq n+1, m+1} \left( 1 - \alpha_s^* \bar{F}(v_s) \right) \left( 1 - \alpha_n \bar{F}(v_{n+1}) \right).
\]

Let \( K = \prod_{s \neq n+1, m+1}^{T} \left( 1 - \alpha_s^* \bar{F}(v_s) \right) \), a constant with respect to \( \alpha_n \). We now compute the second-order derivative of the sales function with respect to \( \alpha_n \), which is equal to \( 2\bar{F}(v_{n+1}) \bar{F}(v_{m+1})K > 0 \). Therefore, this restricted sales function is convex in \( \alpha_n \), indicating that the optimal \( \alpha_n \) is either \( \max\{-1 + A, 0\} \) or \( \min\{A, 1\} \). If \( \alpha_n = 0 \) or 1, that is a contradiction. If \( \alpha_n = -1 + A \) or \( A \), then \( \alpha_m = A - \alpha_n = 1 \) or 0, respectively – again a contradiction. Suppose now that there exists a single \( \alpha_n^* \in (0, 1) \), while all other periods’ policies are deterministic. Then, if \( S = \{i : \alpha_i^* = 1\} \), we have

\[
rS(\alpha) - c \sum_{l=1}^{T-1} \alpha_l = r \left[ 1 - \prod_{i \in S} F(v_i) \left( 1 - \alpha_n^* \bar{F}(v_{n+1}) \right) \right] - c|S| - c\alpha_n^*.
\]

This function is linear in \( \alpha_n^* \), so we must have that \( \alpha_n^* = 0 \) or 1 – a contradiction. Thus, when customers are myopic, the optimal product rotation policy is deterministic.

**Strategic Customers**

The same sequence of events applies when customers are strategic and the product rotation policy is random. In particular, at the beginning of each period \( t \), the customer learns the realization of \( A_{t-1} \) – i.e., whether the product was refreshed or not – while the future product rotation policy is still unknown to the customer. In the case of strategic consumers, the thresholds \( v_t \) depend on the firm’s product rotation policy \( \alpha_1, \ldots, \alpha_{T-1} \). The analysis quickly becomes intractable. Instead, performed
a numerical study covering over 600 scenarios and determined the retailer’s optimal refreshment policy in each scenario via a grid search of all possible policies. We tested different instances by combining the following parameter values: $T = 5, \beta = 1, c \in [0, 0.2], k \in [0, 0.5], v_{out} \in [0, 0.9]$ and $V \sim U[0, 1]$. In all scenarios, the optimal solution was deterministic: $\alpha_t^* = 0, 1$. This suggests that randomized policies hurt the retailer, even when customers are strategic. Indeed, adding uncertainty into the customer’s buying process seems to reduce the sales probability. The following example illustrates this effect.

Consider the customer’s problem with $T = 3$ and $k = 0$ (so that visits always take place). In period $t = 1$, the customer will always see a new product. Let us compare a policy where the retailer only introduces a product in period $t = 2$, i.e., $\alpha_1 = 1, \alpha_2 = 0$, to one where a product is introduced in periods $t = 2$ and/or $t = 3$ with 50% probability, respectively. That is, $\alpha_1 = \alpha_2 = 0.5$. The retailer incurs the same product rotation cost $c$ under the two policies. In the first scenario, the customer may purchase the product in periods 1 or 2. Let $p_{last}$ be the probability that the customer buys the product in period $t = 2$. In the second scenario, one has to consider the possible product rotation realizations from the perspective of the customer. There are four such possible realizations, each occurring with a probability 0.25:

- No new product is introduced (we call this scenario $NN$). Then, $p_{NN}^2 = p_{NN}^3 = 0$.
- A new product is released only in $t = 3$. Then, $p_{2}^{NN} = 0$ and $p_{3}^{NN} = p_{last}$.
- A new product is released only in $t = 2$. Then, $p_{3}^{YN} = 0$ and $p_{2}^{YN} < p_{last}$ because the customer’s purchase threshold in period $t = 2$ is higher relative to a setting in which there is certainty about the realization of $A_2$ as a refreshed version of the product may still appear in period $t = 3$.
- Two new products are released, in $t = 2$ and $t = 3$. Then, $p_{3}^{YY} = p_{last}$ and $p_{2}^{YY} < p_{last}$.

Overall, the expected sales probability in periods $t = 2, 3$ is $(p_{3}^{YY} + p_{2}^{YY} + p_{2}^{YY} + (1 - p_{2}^{YY})p_{3}^{YY})/4 \leq p_{last}$. Hence, a deterministic product rotation policy yields higher expected profit for the retailer compared to a randomized policy with $\alpha_t = 0.5$. A similar reasoning applies to different randomized policies and longer time horizons.